

Jordanian spin chains for twisted strings in $\text{AdS}_5 \times S^5$

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We study the proposed integrable spin chain formulation of Jordanian deformations of the $\text{AdS}_5 \times S^5$ superstring, realized via Drinfel'd twists. Among these models, we first identify a unique supergravity deformation confined to an $SL(2, \mathbb{R})$ sector and with constant dilaton. We then develop a general framework for closed Drinfel'd twisted spin chains and construct an explicit map to undeformed models with twisted-boundary conditions. Applied to the noncompact $\text{XXX}_{-1/2}$ spin chain, the Jordanian twist breaks the Cartan generator labeling magnon excitations, obstructing the standard Bethe methods. Instead, using the twisted-boundary formulation, we initiate the spectral problem based on a residual root generator both in the continuum limit and for short chains. We find that the ground state is nontrivially deformed, and is in agreement with the classical string result, while our analysis does not capture higher-spin excited states. We also study the asymptotics of the associated Q system, which is well behaved and compatible with the boundary twist. While a full understanding of the spectrum remains open, our work provides concrete steps toward a spectral description of non-Abelian twisted integrable models.

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I. INTRODUCTION

Integrable deformations of superstring theories provide a rich setting for probing the structure of the holographic correspondence beyond the well-established case of type IIB strings on $\text{AdS}_5 \times S^5$. The planar dilatation operator of the $\mathcal{N} = 4$ Super Yang-Mills (SYM) dual acts as the Hamiltonian of a quantum integrable spin chain, whose excitations encode the spectrum of anomalous dimensions. The same spectral problem appears on the string side, where the world sheet theory is described by an integrable sigma model. These two formulations overlap in the limit of long (continuous) spin chains and semiclassical strings, also called the Landau-Lifshitz regime, while their integrability is underpinned by the same quantum group structure and associated R matrices. See [1] for overviews.

Within this framework, homogeneous Yang-Baxter (HYB) deformations [2] offer a controlled way of deforming both the string geometry and the symmetry algebra while preserving integrability. The resulting deformed sigma model is defined by a classical R matrix, which up to equivalence relations corresponds one-to-one

with Drinfel'd twists [3,4], inducing a twisted classical Yangian symmetry structure [5]. When the R matrix satisfies the additional unimodularity condition, the deformed background continues to solve the type IIB supergravity equations of motion [6]. HYB deformations can also be understood as nonlocal canonical transformations on the world sheet [7], and therefore admit an equivalent formulation (at least on shell) to an undeformed string sigma model subject to twisted-boundary conditions [7–11]. For explicit local expressions implementing the twist, see [8,11,12]. Together, these features suggest the existence of a holographic interpretation that is governed by a Drinfel'd twisted integrable structure. On the gauge theory side, the dual deformation is expected to manifest via a (typically noncommutative) star product [13] induced by the Drinfel'd twist [5,14], while the same twist acts on the underlying symmetry algebra of the planar dilatation operator, which then should yield a Drinfel'd-twisted spin chain description. This correspondence has also been illustrated through brane constructions involving successive T- and S-duality transformations [5].

The integrable structure of *Abelian* HYB deformations, which correspond to T-duality-shift-T-duality (TsT) transformations [15], is relatively well understood in the holographic context, see, e.g., [16–21] among many others. Notable examples include the β deformation [22,23] and the dipole deformation [24,25], which break internal and spacetime symmetries, respectively. The twist in this case is of Drinfel'd-Reshetikhin type [26], constructed from commuting symmetry generators of $\mathfrak{psu}(2, 2|4)$, which

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affects the physics of the model only in their nonzero charge sectors. Inclusion of such twists on the canonical XXX spin chain has elucidated the completeness problem of the Bethe root structure and the construction of Q -operators, see, e.g., [27], which are foundational to advanced integrable methods such as the quantum spectral curve.

By contrast, *non-Abelian* HYB deformations, particularly those of Jordanian type, remain much less explored. These deformations necessarily act on a noncompact $\mathfrak{sl}(2, \mathbb{R})$ subalgebra of $\mathfrak{psu}(2, 2|4)$ and are thus natural candidates for studying symmetry-breaking effects in this sector. In particular, they lead to new integrable systems that are relevant for non-AdS holography. At present, however, little is known about their explicit gauge theory dual or spin chain formulation. An exception is the recent work [28], which studied the spectrum of the Jordanian Drinfel'd twisted [29,30] XXX $_{-1/2}$ spin chain relevant to the $\mathfrak{sl}(2, \mathbb{R})$ sector at one loop [31,32]. In [28], using an organization in terms of $\mathfrak{sl}(2, \mathbb{R})$ Cartan states, the spectrum was found to be undeformed—a result that appears to conflict with the classical string energy and its corrections computed for one of the simplest deformed sigma-model solutions [12]. Furthermore, on the string theory side, also the integrable S-matrix framework remains elusive: recent results point to particle production processes involving massive asymptotic states [33], which appear incompatible with Bethe Ansatz approaches based on factorized S matrices.¹ These features indicate substantial challenges for the use of Drinfel'd twisted integrability in non-AdS holography.

In this work, we revisit the spin chain description aligned with Jordanian sigma models, aiming to clarify their physical properties and integrable structure within the context of AdS/CFT. Motivated by the conjectured Drinfel'd twisted extension of the correspondence, we still expect that the continuum (Landau-Lifshitz) limit of the twisted spin chain reproduces the deformed semiclassical string spectrum, in parallel with the undeformed case. However, as the precise formulation of the gauge-theory dual remains unclear, we do not attempt a direct identification of the corresponding spin chain from first principles here. Instead, to make progress, we start directly from the Jordanian-twisted chain and treat it as an integrable “avatar” of the twisted dilatation operator, which we will use to test and organize the spectral structure for holographic Jordanian models. As we will show, this identification is indeed supported in the Landau-Lifshitz regime.

We begin with a review of the sigma-model side, and perform a systematic analysis of the dilaton profiles of all unimodular Jordanian deformations of $\text{AdS}_5 \times S^5$ that have been classified in [35]. Among these, we identify a unique Jordanian twist with constant dilaton and that is nontrivial only in an $\mathfrak{sl}(2, \mathbb{R})$ subalgebra through a single deformation

parameter. In the context of holography, this model is therefore the simplest Jordanian example with a valid supergravity approximation at large 't Hooft coupling. As it turns out, this “minimal” example was already studied from various angles: it can be obtained from combinations of TsT and S dualities [36], allowing a brane construction [5], its semiclassical energy corrections were derived from the spectral curve in [12], and its scattering processes in light-cone gauge was analyzed in [33]. It shares also several important features with the (Abelian) dipole deformation studied in [20,37], and we will therefore compare to this case where appropriate throughout this paper. We will discuss its residual (Schrödinger-type) symmetry structure, and discuss its simplest classical solution (a BMN-like string) both in the deformed and twisted-boundary formulations.

In the rest of this paper, we consider the twisted spin chain. Before specifying to the Jordanian case, we construct a systematic framework for integrability studies of closed Drinfel'd twisted spin chains without specifying Hamiltonian or R matrix. Building on the Hopf-algebraic formulation of Maillet and de Santos [38], we derive universal expressions of the twisted monodromy and transfer matrices, and propose a simple “gluing” prescription that maps spin chains with deformed bulk interactions under periodic boundary conditions to explicit boundary-twist operators on an undeformed chain. This parallels the on-shell equivalence of sigma models with Drinfel'd twists and twisted boundary conditions [8,11].

We then apply our construction to the Jordanian Drinfel'd twist of the $\mathfrak{sl}(2, \mathbb{R})$ -invariant XXX Heisenberg spin chain in the noncompact $s = -1/2$ representation. We examine the modified symmetry structure: the twist breaks the full $\mathfrak{sl}(2, \mathbb{R})$ algebra, leaving only a non-Cartan generator, which we use to study the spectral problem (in contrast to [28]). Notably, the boundary twist is also built from this residual generator. Its eigenbasis departs from the conventional Fock-space picture of magnonlike excitations. Importantly, in this new eigenbasis, integrability techniques such as the Algebraic Bethe ansatz (ABA) appear to be not applicable in the standard way, because there is no suitable pseudovacuum state with nonzero charge under the residual symmetry to initiate it. However, we find that working particularly in the twisted-boundary formulation does enable an initial and nontrivial spectral analysis.

In the continuum or thermodynamic limit (where the spin chain length $J \rightarrow \infty$), we derive the Landau-Lifshitz theory via the coherent state representation and show that the ground state energy shift matches the classical string result. This not only confirms a nontrivial deformation but also provides a first string-spin agreement in this overlapping regime. At finite length, we study the spectral problem at $J = 2$, by reducing the twisted transfer matrix eigenvalue equation to a singular ordinary differential equation (ODE) in the symmetry eigenbasis. Using Frobenius-type methods, we compute the eigenvalue

¹See also [34] for a related model with soft production.

perturbatively in the deformation parameter from recurrence relations. Importantly, this ODE analysis obscures the higher-spin excitations by regularity conditions. Only the ground state explicitly appears and is deformed, and we compute its energy using Baxter's TQ equation.

We further initiate the exact finite-length spectral analysis by studying the asymptotic analytic structure of the TQ and QQ system, providing additional support for the twisted-boundary formulation and setting the first steps to a full spectral curve description in this non-Abelian, non-Cartan twisted context.

The paper is structured as follows. Section II reviews HYB deformations of $\text{AdS}_5 \times S^5$, introduces the minimal Jordanian background, and describes its classical BMN-like solution and associated Landau-Lifshitz limit. Section III develops the Drinfel'd twisted framework, including our gluing construction, and applies it to the Jordanian-twisted $\mathfrak{sl}(2, \mathbb{R})$ chain. In Sec. IV we match the energy of the spin chain to the classical string via coherent states in the thermodynamic Landau-Lifshitz limit, while Sec. V tackles the ground state spectrum of the length $J = 2$ spin chain via Baxter's equation. Section VI studies Q -function asymptotics at finite length and the structure of the QQ system. We conclude in Sec. VII with a discussion and future directions. Appendix A discusses the classical limit of the twisted transfer matrix, and appendix B describes the perturbative method to solve singular ODEs used in the $J = 2$ analysis.

II. JORDANIAN STRING SIGMA MODELS

Here we review the essential elements of HYB deformations, and set the stage for a comparison with the proposed spin chain formulation for its simplest relevant Jordanian subclass.

A. Review of the action and twisted symmetries

The HYB deformations form a large family of integrable deformations of symmetric and semi-symmetric space sigma models. Their world sheet action takes the form [2]

$$S_{\text{WS}} = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left(\frac{\sqrt{|h|} h^{\alpha\beta} - \epsilon^{\alpha\beta}}{2} \right) \times \text{STr}(J_\alpha \hat{d}_- (1 - \eta R_g \hat{d}_-)^{-1} J_\beta), \quad (2.1)$$

where $\frac{\sqrt{\lambda}}{4\pi}$ is the string tension, $h^{\alpha\beta}$ is the world sheet metric, $\epsilon^{\alpha\beta}$ is the antisymmetric tensor with conventions $\epsilon^{\tau\sigma} = -1$, and STr denotes the supertrace on a $\mathbb{Z}_4$ -graded Lie superalgebra $\mathfrak{g} = \text{Lie}(G)$, which we take in this paper to be $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$. We introduced the Maurer-Cartan form $J = g^{-1} dg$, with $g(\tau, \sigma) \in G$ the world sheet embedding, and the operators $\hat{d}_\pm = \mp \frac{1}{2} P^{(1)} + P^{(2)} \pm \frac{1}{2} P^{(3)}$ that act with the projections $P^{(i)}$, $i = 0, 1, 2$, and 3 , onto the

corresponding $\mathbb{Z}_4$ -graded eigenspaces. Last but not least, the key object inducing the deformation is $R_g := \text{Ad}_g^{-1} R \text{Ad}_g$, scaled by a real deformation parameter $\eta \in \mathbb{R}$, where $\text{Ad}_g X := g X g^{-1}$ for $X \in \mathfrak{g}$, and $R: \mathfrak{g} \rightarrow \mathfrak{g}$ is a linear operator satisfying two properties: R is antisymmetric with respect to STr and it solves the classical Yang-Baxter equation (CYBE), i.e., respectively,

$$\begin{aligned} \text{STr}(RX - Y) &= -\text{STr}(X - RY), \\ [RX, RY] - R([RX, Y] + [X, RY]) &= 0, \end{aligned} \quad (2.2)$$

for all $X, Y \in \mathfrak{g}$. Such objects are in one-to-one correspondence (up to equivalence relations) with Drinfel'd twists that are continuously connected to the identity [3,4]. In this context, they explicitly ensure that the sigma model remains classically Lax integrable [2,39]. We further assume that the undeformed model ($\eta = 0$) gives rise to a solution of the type II supergravity equations, as is the case for the $\text{AdS}_5 \times S^5$ superstring that is realized from $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$. A further condition on R is then needed for the deformed model to remain a type II supergravity solution, i.e., R must solve the linear “unimodularity” constraint [6]

$$[T_A, RT^A] = 0, \quad (2.3)$$

where $\{T_A\}$ and $\{T^A\}$ with $A = 1, \dots, \dim(G)$ are dual bases with respect to the supertrace.²

We can equivalently represent R via the twofold graded wedge product $r = -\frac{1}{2} R^{AB} T_A \wedge T_B$. Our focus in this paper is on rank-2 unimodular Jordanian deformations,³ which are of the form [35,40,41]

$$r_{\text{Jor}} = \mathbf{h} \wedge \mathbf{e} - \frac{i}{2} (\mathbf{Q}_1 \wedge \mathbf{Q}_1 + \mathbf{Q}_2 \wedge \mathbf{Q}_2), \quad (2.4)$$

where $\mathbf{h}, \mathbf{e}$ are bosonic and $\mathbf{Q}_1, \mathbf{Q}_2$ fermionic generators in $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$ that are such that $[\mathbf{h}, \mathbf{e}] = \mathbf{e}$, $[\mathbf{Q}_i, \mathbf{e}] = 0$, $[\mathbf{h}, \mathbf{Q}_i] = \frac{1}{2} (\mathbf{Q}_i - \epsilon_{ij} a \mathbf{Q}_j)$, with $a \in \mathbb{C}$ a free parameter, and $\{\mathbf{Q}_i, \mathbf{Q}_j\} = -i \delta_{ij} \mathbf{e}$, for $i, j = 1, 2$. Note that $\mathbf{e}$ acts as a raising operator with respect to $\mathbf{h}$.⁴ Importantly, for Jordanian models, the fermionic tail built from $\mathbf{Q}_i$ is

²Given a basis of generators T_A for $\mathfrak{g}$, with structure constants defined by $[T_A, T_B] = F_{AB}^C T_C$, let the invariant metric be $K_{AB} = \text{STr}(T_A T_B)$, and its inverse K^{AB} used to define the dual basis $T^A = K^{AB} T_B$. Writing $RT^A = R_B^A T^B$, the unimodularity conditions become $R^{AB} F_{AB}^C = 0$, with $R^{AB} = R^A_C K^{CB}$.

³The rank denotes the number of bosonic generators in the construction of the r wedge product.

⁴In order to preserve reality conditions, this requires that $\mathfrak{g}$ have a noncompact $\mathfrak{sl}(2, \mathbb{R})$ subalgebra.

necessary to ensure unimodularity. For $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$, the inequivalent Jordanian R -operators (including higher-rank ones) admitting unimodular extensions have been classified in [35]. More details can be found therein.

Another important subclass of HYB models is based on the Abelian r matrices

$$r_{ab} = \mathfrak{t}_1 \wedge \mathfrak{t}_2, \quad (2.5)$$

where $\mathfrak{t}_1, \mathfrak{t}_2$ are commuting generators of $\mathfrak{psu}(2, 2|4)$. These deformations are equivalent to TsT transformations [23] over the isometries generated by $\mathfrak{t}_1$ and $\mathfrak{t}_2$ [15] and they are trivially unimodular. Their integrability in the context of $\text{AdS}_5/\text{CFT}_4$ have been discussed for various examples. The simplest case is the Lunin-Maldacena background [23] obtained from an r wedge product over two Cartan charges of $SO(6)$, and thus deforming the S^5 , dual to the β deformation of $N = 4$ SYM [22]. Other interesting examples are the Hashimoto-Itzhaki-Maldacena-Russo background [42], from two commuting generators in $SO(2, 4)$, and the Schrödinger background [25, 43], from one (null) generator in $SO(2, 4)$ and one Cartan in $SO(6)$, both deforming AdS_5 and thus introducing noncommutativity in the dual field theory [13] (for the latter in terms of a dipole deformation, see [24, 25]). Their integrability studies are based on Abelian Drinfel'd-Reshetikhin (DR) twists [26], see [16] and [20] for the β and dipole deformation, respectively.

1. Manifest symmetries

The residual (manifest) symmetries of the deformed sigma model (2.1) are generated by the subalgebra $\mathfrak{k} = \text{span}(T_{\bar{A}}) \subset \mathfrak{psu}(2, 2|4)$ defined by

$$\text{ad}_{T_{\bar{A}}} R(X) = \text{Rad}_{T_{\bar{A}}}(X), \quad \forall X \in \mathfrak{g}, \quad (2.6)$$

with $\text{ad}_X Y := [X, Y]$. This guarantees invariance of the action (2.1) under left multiplication

$$g \rightarrow \exp(\epsilon T_{\bar{A}})g, \quad (2.7)$$

with ϵ a global symmetry parameter. For all Jordanian R operators (2.4), it then follows that $\mathfrak{k}$ always generates a global symmetry.⁵ This fact will be important in several parts of our analysis.

2. Classical integrability and twisted symmetries

Provided the conditions (2.2) are met, HYB deformations preserve classical world sheet integrability. This means that their equations of motion can be represented by a one-parameter family of flat Lax connection 1-forms

⁵It is straightforward to show this using $R(X) = \text{STr}_2(r(1 \otimes X))$ for $X \in \mathfrak{g}$ as well as the above commutation relations.

$\mathcal{L}(z)$, which ensures the construction of an infinite family of conserved charges, and that these charges Poisson commute [2, 39]. In conformal gauge, the Lax can be written as⁶

$$\mathcal{L}_{\pm}(z) = A_{\pm}^{(0)} + zA_{\pm}^{(1)} + z^{\pm 2}A_{\pm}^{(2)} + z^{-1}A_{\pm}^{(3)}, \quad z \in \mathbb{C}, \quad (2.8)$$

where $A^{(i)} := P^{(i)}A$ and

$$A_{\pm} = (1 \pm \eta R_g \hat{d}_{\pm})^{-1} J_{\pm}. \quad (2.9)$$

An interesting property of *homogeneous* Yang-Baxter deformations is that, on shell, they are classically equivalent to the undeformed sigma model up to a twist in the boundary conditions [7–12] (see also [44] for early considerations in the Jordanian context). This is due to the fact that the deformation can be understood as a nonlocal canonical transformation on the world sheet [7], which is precisely realized as the map $J_{\pm} \rightarrow A_{\pm}$. Formally, up to gauge transformations generated by $\mathfrak{g}^{(0)}$, one may therefore identify

$$\tilde{J}_{\pm} := \tilde{g}^{-1} \partial_{\pm} \tilde{g} \equiv h A_{\pm} h^{-1} - \partial_{\pm} h h^{-1}, \quad (2.10)$$

where $\tilde{g} \in G$ represents the collection of fields of the undeformed sigma model with twisted-boundary conditions and $h \in G^{(0)} = SO(1, 4) \times SO(5)$. Periodic-boundary solutions to the equations of motion of the fields g of the deformed sigma model (2.1) are then mapped to twisted-boundary solutions of the fields $\tilde{g}$ of the undeformed sigma model, i.e., (2.1) at $\eta = 0$. We can translate between the two formulations with the “twist field” $\mathcal{F}(\tau, \sigma) \in F$ as

$$g = \mathcal{F} \tilde{g} h, \quad (2.11)$$

with $F \subset G$ the Lie group of $\mathfrak{k} = \text{Im}(R)$. Explicitly, we can write the twist of the boundary conditions of $\tilde{g}$ as [12]

$$\tilde{G}(\tau, \sigma = 2\pi) = W \tilde{G}(\tau, \sigma = 0) W^T, \quad (2.12)$$

where $\tilde{G} = \tilde{g} K \tilde{g}^T$ is the gauge-invariant collection of fields, with K the $G^{(0)} = SO(1, 4) \times SO(5)$ invariant, and $W = \mathcal{F}^{-1}(\tau, 2\pi) \mathcal{F}(\tau, 0) \in F$ the object that encodes the twisting.

⁶In the rest of this Section, we adapt conformal gauge to keep notation light. One may, however, translate the formulas back to prior gauge fixing by replacing one-form components $V_{\pm}$ with $V_{\pm} \rightarrow P_{(\pm)}^{\alpha\beta} V_{\beta}$ and $V_{\pm} \rightarrow P_{(-)\alpha}^{\beta} V_{\beta}$ with $P_{(\pm)}^{\alpha\beta} = \frac{1}{2} \left(\sqrt{|h|} h^{\alpha\beta} \pm \epsilon^{\alpha\beta} \right)$ while raising and lowering world sheet indices with $\sqrt{|h|} h^{\alpha\beta}$.

For Jordanian models with R matrix (2.4), an explicit local expression for W is given by [11,12]

$$W_{\text{for}} = \exp(\mathbf{Q}(\mathbf{h} - \mathbf{q}\mathbf{e})),$$

$$\mathcal{F}_{\text{for}} = \exp\left(\frac{2\pi\eta}{\sqrt{\lambda}}Y_{\mathbf{h}}\mathbf{e}\right) \exp\left(\log\left(1 - \frac{2\pi\eta}{\sqrt{\lambda}}Y_{\mathbf{e}}\right)\mathbf{h}\right), \quad (2.13)$$

with

$$\mathbf{Q} := \log\left(\frac{1 - \frac{2\pi\eta}{\sqrt{\lambda}}Y_{\mathbf{e}}(\tau, 0)}{1 - \frac{2\pi\eta}{\sqrt{\lambda}}Y_{\mathbf{e}}(\tau, 2\pi)}\right), \quad \mathbf{q} := \frac{Y_{\mathbf{h}}(\tau, 2\pi) - Y_{\mathbf{h}}(\tau, 0)}{Y_{\mathbf{e}}(\tau, 2\pi) - Y_{\mathbf{e}}(\tau, 0)}, \quad (2.14)$$

and $Y_{\mathbf{e}} = \text{STr}(\mathbf{e}Y)$, $Y_{\mathbf{h}} = \text{STr}(\mathbf{h}Y)$ projections of the Lie-algebra value field⁷

$$Y(\tau, \sigma) := \frac{\sqrt{\lambda}}{2\pi} P^T \left(\int_0^\sigma d\sigma' \hat{d}_{\tilde{g}^{-1}} (\partial_\tau \tilde{g} \tilde{g}^{-1})(\tau, \sigma') \right. \\ \left. + \int_0^\tau d\tau' \hat{d}_{\tilde{g}^{-1}} (\partial_\sigma \tilde{g} \tilde{g}^{-1})(\tau', 0) \right) + Y(0, 0), \quad (2.15)$$

with $\hat{d}_{\tilde{g}^{-1}} := \text{Ad}_{\tilde{g}} \hat{d} \text{Ad}_{\tilde{g}}^{-1}$ and P^T a projector on $\mathfrak{f}^*$, the dual of $\mathfrak{f} = \text{Im}(R)$. One may include additional contributions from fermionic degrees of freedom associated to the supercharges $\mathbf{Q}_i$, but it was shown in [12] that such twists are equivalent, after field redefinitions, to (2.13) for classical solutions (i.e., ones with trivial fermionic dynamics). Furthermore, in on-shell sectors where $\mathbf{Q} \neq 0$ (which will be the case for the solution that we will study in Section II C), an additional field redefinition can actually relate W_{for} to $W'_{\text{for}} = \exp(\mathbf{Q}\mathbf{h})$. Interestingly, we find especially the latter expression at large λ to be consistent with the spin chain formulation (cf. Sec. III D, VI and Appendix A).

For Abelian models with R matrix (2.5), the local expression for W is much simpler and given by [8,10,11]

$$W_{\text{ab}} = \exp\left(-\frac{2\pi\eta}{\sqrt{\lambda}}(\mathbf{Q}_1\mathbf{t}_2 - \mathbf{Q}_2\mathbf{t}_1)\right),$$

$$\mathcal{F}_{\text{ab}} = \exp\left(\frac{2\pi\eta}{\sqrt{\lambda}}(Y_1\mathbf{t}_2 - Y_2\mathbf{t}_1)\right), \quad (2.16)$$

where $Y_i = \text{STr}(\mathbf{t}_i Y)$ and $\mathbf{Q}_{1,2}$ are the Noether charges corresponding to the $\mathbf{t}_{1,2}$ isometries

⁷Here we rescale $Y(\tau, \sigma)$ with $\frac{\sqrt{\lambda}}{2\pi}$ compared to [11,12] to match with conventional expressions for conserved Noether charges [as in, e.g., (2.17)].

$$\mathbf{Q}_i = \text{STr}(\mathbf{t}_i(Y(\tau, 2\pi) - Y(\tau, 0)))$$

$$= \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \text{STr}(\mathbf{t}_i \text{Ad}_{\tilde{g}} \tilde{J}_\tau^{(2)}). \quad (2.17)$$

Importantly, both W_{for} and W_{ab} are constant and thus encode conserved quantities (charges) of the respective models. The Jordanian $\mathbf{Q}$, $\mathbf{q}$ charges do not, however, have an immediate interpretation as Noether charges for a continuous symmetry, in contrast to the $\mathbf{Q}_{1,2}$ appearing in the Abelian twists. Furthermore note that generally the $\mathfrak{psu}(2, 2|4)$ Noether symmetries of the undeformed model are explicitly broken by the twist in the boundary conditions. The subalgebra of residual symmetries $\tilde{\mathfrak{t}} = \text{span}(T_{\tilde{\lambda}})$ of the twisted-boundary model, which is determined by $[T_{\tilde{\lambda}}, \tilde{\mathfrak{f}}] = 0$, is itself a subalgebra of the residual symmetries $\mathfrak{t}$ of the deformed model, and their local Noether charges coincide on the on-shell mapping [11,12].

Next to the subset of manifest residual Noether symmetries, HYB deformations have an infinite set of (hidden) twisted classical Yangian symmetries [5]. The monodromy matrix that generates the conserved charges can indeed be written as⁸

$$\mathcal{T}(z) = \overleftarrow{\mathcal{P}} \exp\left(-\int_0^{2\pi} d\sigma \mathcal{L}_\sigma^g(z)\right)$$

$$= \mathcal{F}(\tau, 2\pi) \mathcal{T}_{\eta=0}(z) \mathcal{F}^{-1}(\tau, 0)$$

where $\mathcal{L}^g(z) = g\mathcal{L}(z)g^{-1} - dg g^{-1}$ and $\mathcal{T}_{\eta=0}(z)$ is the monodromy matrix of the undeformed sigma model built from the (gauge-transformed) Lax connection in terms of projections of the currents $J_\pm$. We can thus understand the original ($\eta = 0$) classical Yangian symmetry to be twisted by $\mathcal{F}$. At large λ we get

$$\mathcal{F}_{\text{for}} = 1 - \frac{2\pi\eta}{\sqrt{\lambda}}(Y_{\mathbf{e}}\mathbf{h} - Y_{\mathbf{h}}\mathbf{e}) + \mathcal{O}(\lambda^{-1})$$

$$= 1 - \frac{2\pi\eta}{\sqrt{\lambda}}\iota_Y r_{\text{for}} + \mathcal{O}(\lambda^{-1}), \quad (2.18)$$

$$\mathcal{F}_{\text{ab}} = 1 - \frac{2\pi\eta}{\sqrt{\lambda}}(Y_2\mathbf{t}_1 - Y_1\mathbf{t}_2) + \mathcal{O}(\lambda^{-1})$$

$$= 1 - \frac{2\pi\eta}{\sqrt{\lambda}}\iota_Y r_{\text{ab}} + \mathcal{O}(\lambda^{-1}), \quad (2.19)$$

with the contraction ι_Y defined as $\iota_Y r := \text{STr}_2(r(1 \otimes Y))$. This precisely represents a classical Drinfel'd twist to leading order [3,5] with ι_Y the action of the operators (charges) on the classical fields $\tilde{g}$. We thus see explicitly how the deformed sigma-model inherits a twisted

⁸To show the second equality one can use the fact that on-shell $A_\pm \equiv h^{-1}\tilde{J}_\pm h + h^{-1}\partial_\pm h = \mathcal{G}J_\pm \mathcal{G}^{-1} - \partial_\pm \mathcal{G} \mathcal{G}^{-1}$ with $\mathcal{G} = h^{-1}\tilde{g}^{-1}g = g^{-1}\mathcal{F}g$.

Hopf-algebra structure at the classical level. In a holographic interpretation, the twisting may be interpreted as a $\frac{1}{\sqrt{\lambda}}$ correction.

At various instances in this paper, we will absorb the 't Hooft coupling λ in the (sigma-model) deformation parameter η as

$$\xi := \frac{2\pi\eta}{\sqrt{\lambda}}. \quad (2.20)$$

In particular, $\xi \in \mathbb{R}$ will then denote the deformation parameter in the Drinfel'd twist, which will be deforming the relevant spin chain model, as we will discuss in Sec. III.

B. Singling out the “minimal” Jordanian model

In this Section, we briefly analyse the dilaton Φ of all rank-2 unimodular Jordanian deformations of $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$. We focus on identifying those backgrounds for which Φ is a constant across target space, motivated from the simple fact that this implies a constant string coupling g_s , such that the classical supergravity approximation from the world sheet is valid.

For HYB models, the dilaton Φ is given by [6]

$$e^{-2\Phi} = e^{-2\Phi_0} \text{sdet}(1 + \eta R_g \hat{d}_+), \quad (2.21)$$

where Φ_0 is the constant dilaton of the undeformed $\text{AdS}_5 \times S^5$ background. We will compute Φ for all rank-2 bosonic Jordanian models of the $\mathfrak{so}(2, 4)$ subalgebra that admit unimodular extensions in $\mathfrak{psu}(2, 2|4)$, as listed in [35] (cf. Table 2 or 6 therein for notation and definitions). For this purpose, we will take the following (bosonic) coset representatives: For $\bar{R}_1, \bar{R}_{1'}, \bar{R}_2, \bar{R}_{2'}, \bar{R}_3, \bar{R}_4, \bar{R}_5, \bar{R}_{5'}$, we use

$$g = g_a \cdot g_{\bar{s}}, \quad g_a = e^{TH_T + VH_V + \Theta H_\Theta} e^{Pp_1} e^{\log(Z)D}, \quad (2.22)$$

where $g_{\bar{s}} \in SO(6)$ parametrizes the round S^5 sphere, and

$$\begin{aligned} H_T &= \frac{1}{2}(p_0 - k_0 - p_3 - k_3), & H_V &= \mathbf{e} = p_0 + p_3, \\ H_\Theta &= J_{12}, \end{aligned} \quad (2.23)$$

are commuting generators of $\mathfrak{psu}(2, 2|4)$, with D, J_{ij}, p_i , and k_i , $i = 0, 1, 2$, and 3 , the standard dilatation, rotation, translation, and special conformal generators of the conformal $\mathfrak{so}(2, 4)$ subalgebra. For $\bar{R}_6, \bar{R}_{6'}$, we instead take

$$g = g'_a \cdot g_{\bar{s}}, \quad g'_a = e^{VH'_V + \Theta H_\Theta} e^{P_1 p_1 + P_2 p_3} e^{\log(Z)D}, \quad (2.24)$$

with $H'_V = \mathbf{e} = p_0$. Both g_a and g'_a are faithful representatives of $SO(2, 4)/SO(1, 4)$, with local coordinates $X^m = (T, V, \Theta, P, Z)$ and $X^m = (V, \Theta, P_1, P_2, Z)$, $m = 1, \dots, 5$, respectively. They are adapted in such a way that shifting V is a manifest isometry ($\mathbf{e}$) of the corresponding

backgrounds. We then obtain

$$\bar{R}_1, \bar{R}_{1'}: e^{-2(\Phi - \Phi_0)} = 1 - \frac{\eta^2(1 + 2b)^2 \sin^2 2T}{Z^4}, \quad (2.25)$$

$$\bar{R}_2, \bar{R}_{2'}: e^{-2(\Phi - \Phi_0)} = 1 - \frac{16\eta^2 \cos^4 T}{Z^4}, \quad (2.26)$$

$$\bar{R}_3, \bar{R}_4: e^{-2(\Phi - \Phi_0)} = 1 - \frac{\eta^2 \sin^2 2T}{Z^4}, \quad (2.27)$$

$$\bar{R}_5, \bar{R}_{5'}: e^{-2(\Phi - \Phi_0)} = 1 - \frac{16\eta^2 b^2}{Z^4}, \quad (2.28)$$

$$\bar{R}_6: e^{-2(\Phi - \Phi_0)} = \left(1 - \frac{4\eta^2((1 + a^2)P_1^2 + P_2^2 + Z^2)}{Z^4}\right) \mathcal{D}, \quad (2.29)$$

$$\bar{R}_{6'}: e^{-2(\Phi - \Phi_0)} = \left(1 - \frac{4\eta^2(P_1^2 + P_2^2 + Z^2)}{Z^4}\right) \mathcal{D}, \quad (2.30)$$

where a, b are free, real parameters, and $\mathcal{D}$ is a complicated function depending on the sphere parametrization, but that, importantly, is independent of the coordinates P_1, P_2 as well as the free parameter a entering the definition of the Jordanian $\bar{R}_6$ matrix.

We thus conclude that only $\bar{R}_1, \bar{R}_{1'}$ on $b = -\frac{1}{2}$ and $\bar{R}_5, \bar{R}_{5'}$ on $b = 0$ yield constant dilatons. All of these share the same bosonic $\mathbf{h}$ and $\mathbf{e}$ generator, namely $\mathbf{h} = \frac{D - J_{03}}{2} + aJ_{12}$ and $\mathbf{e} = p_0 + p_3$, with $a = 0$ for $\bar{R}_1, \bar{R}_5$, and generic a for $\bar{R}_{1'}, \bar{R}_{5'}$. Notably, turning on a corresponds to an additional TsT deformation along the residual bosonic isometries J_{12} and $\mathbf{e}$ of the $a = 0$ deformed background, see, e.g., [45] for more comments. Furthermore, the $a = 0$ case preserves 12 supercharges, the maximal number found in [35], while turning on a breaks all supersymmetry. Both cases are precisely the ones for which the spectrum has been studied semiclassically using spectral curve methods in [12] and [45], respectively.

We therefore define the “minimal” Jordanian model to be the unimodular rank-2 model with

$$\mathbf{h} = \frac{D - J_{03}}{2}, \quad \mathbf{e} = p_0 + p_3, \quad (2.31)$$

ensuring both a valid supergravity background with constant dilaton, as well as maximal preserved supersymmetry. For its unimodular extension, see [35]. Since we will focus on a classical bosonic string solution in what follows, let us give here the resulting deformed target space metric and B field extracted from (2.1) with the above choice, and in the coordinates (2.22). One gets

$$ds^2 = \frac{dZ^2 + dP^2 + P^2 d\Theta^2 - 2dTdV}{Z^2} - \frac{(4Z^4 + \eta^2)(Z^2 + P^2)}{4Z^6} dT^2 + ds_{S^5}^2, \\ B = \eta \frac{PdP \wedge dT}{2Z^4} - \eta d\left(\frac{dT}{4Z^2}\right), \quad (2.32)$$

with $ds_{S^5}^2$ the metric of the round S^5 . The full type IIB supergravity background is also supported by a nontrivial F_3 and F_5 flux; see, e.g., Eq. 7 of [45], after sending $a \rightarrow 0$ and $\eta \rightarrow \eta/\sqrt{2}$, for explicit expressions in these coordinates and [36,41,46] for original constructions. The spacetime was shown to be geodesically complete in [12] and exhibits features of a Schrödinger Sch_2 geometry [47]. The residual AdS Noether symmetries of the deformed model are

$$\mathfrak{t}_a = \text{span}(D + J_{03}, k_0 + k_3, p_0 - p_3, p_0 + p_3, J_{12}) \\ \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{u}(1)^2, \quad (2.33)$$

where the $\mathfrak{sl}(2, \mathbb{R})$ subalgebra encodes the nonrelativistic CFT characteristics of Schrödinger-invariant field theories and is therefore a convenient language to describe the residual symmetry structure [48,49]. The $\mathfrak{u}(1)^2$ part are the two central elements $H_V = \mathbf{e} = p_0 + p_3$ and $H_\Theta = J_{12}$, while $H_T = \frac{1}{2}(p_0 - k_0 - p_3 - k_3) \in \mathfrak{sl}(2, \mathbb{R})$ is the only timelike Cartan element up to inner automorphisms.⁹ In the deformed bulk spacetime, the isometric coordinate T thus serves as global time (with conjugate energy E generated by H_T), while Θ is angular (associated to AdS spin) and V is a noncompact null direction with conjugate null momentum M (also “central mass” in the Schrödinger language). Interestingly, in the nonrelativistic CFT setup, $D + J_{03}$ defines nonrelativistic scaling dimensions Δ . The generator $k_0 + k_3$ acts as a lowering operator and is used to define nonrelativistic primary operators, while the generator $p_0 - p_3$ defines a state-operator map, relating eigenstates of the global time Hamiltonian H_T (bulk strings) to operators diagonalized by $D + J_{03}$ [48]. An important takeaway for the holographic spectral problem is thus that one should expect that the relevant symmetry generator to be diagonalized is not the original dilatation operator D , but rather the twisted dilatation $D + J_{03}$.¹⁰

⁹On the other hand, the residual (AdS) Noether symmetries of the twisted-boundary model $\tilde{\mathfrak{t}}_a$ share the same $\mathfrak{sl}(2, \mathbb{R})$ algebra and central J_{12} with $\mathfrak{t}_a$, but it does not contain the central $\mathbf{e} = p_0 + p_3$. We can instead understand the latter (at least to first order in λ^{-1}) as being manifested in the twist W_{tor} (2.13).

¹⁰Nevertheless, as emphasized in [20] and references therein, in the related dipole model a genuine nonrelativistic field theory interpretation arises only after dimensional reduction along nonlocal directions of the deformed four-dimensional quantum field theory dual, and the same should be expected for the Jordanian case.

1. Comments on extensions

Although it is outside the aim of this paper, one may also consider rank-2 bosonic Jordanian models of the full $\mathfrak{so}(2, 4) \oplus \mathfrak{so}(6)$ subalgebra admitting unimodular extensions, as well as the higher-rank unimodular Jordanian deformations given in [35].¹¹ However, as explained in [35], the extensions to $\mathfrak{so}(6)$ can also be interpreted as additional TsT deformations on top of the rank-2 bosonic $\mathfrak{so}(2, 4)$ solutions.¹² For the higher-rank cases, we simply find, by restricting to the bosonic sector, that the only rank-4 ones that would give rise to a constant dilaton are R_7 on $b = -\frac{1}{2}$ and $a = 0$ and R_9 on $d_+ = b_- = 0$ and $d_- = \frac{b_+}{1+b_+^2}$ with $b_+ \in \mathbb{R}$ free.¹³ The only rank-6 cases with constant dilaton would be R_{13} on $b = -\frac{1}{2}$ but $a \in \mathbb{R}$ free and R_{14} without any further constraints on the parameters other than those listed in Eq. (2.14) of [35]. Interestingly, all cases are ultimately built on $\mathfrak{h} = \frac{D-J_{03}}{2}$ and $\mathbf{e} = p_0 + p_3$, which supports the universality of the minimal Jordanian R operator defined in (2.31).

C. Classical solution in $AdS_3 \times S^1$ and $SL(2, \mathbb{R})$ Landau-Lifshitz limit

After singling out the minimal Jordanian model (2.31), let us now specify a pointlike string solution in the bosonic target space background (2.32). The simplest case is already nontrivial in an $AdS_3 \times S^1$ subspace of $AdS_5 \times S^5$ [12] (see also [20])

$$T = -\kappa\tau, \quad V = \frac{\eta^2}{4}m\tau, \quad Z = \sqrt{\frac{\kappa}{m}}, \quad \psi = \omega\tau, \quad (2.34)$$

where ψ is an angle parametrizing a big circle in S^5 and κ, m, ω are constant parameters. The other coordinates are trivial: in particular, $X_2 := \frac{P \cos \Theta}{Z} = 0$ and $X_3 := \frac{P \sin \Theta}{Z} = 0$ in addition to the vanishing of the other two isometric angles (ψ_1, ψ_2) of S^5 . Hence this solution has no AdS spin ($S = 0$) in Θ and only one angular momentum $J = J_3$ along ψ (with angular frequency ω), while the other

¹¹For these cases, one cannot simply restrict to an $\mathfrak{sl}(2, \mathbb{R})$ subsector and in general one would instead have to match with calculations of the full Drinfel'd twisted $\mathfrak{psu}(2, 2|4)$ spin chain.

¹²A key example would be the Schrödinger/dipole deformation considered in [20], which is obtained by a TsT along a null generator $\mathbf{e} = p_0 + p_3$ and a Cartan generator in $SO(6)$ (the R charge).

¹³Using an inner automorphism, R_7 on $b = -\frac{1}{2}$ and $a = 0$ can be seen as a special case of R_9 on $d_+ = b_- = 0$ and $d_- = \frac{b_+}{1+b_+^2}$, i.e., for $b_+ = 0$.

momenta vanish ($J_1 = J_2 = 0$). The parameters are related to the Cartan Noether charges as¹⁴

$$\begin{aligned} E &= \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \text{STr}(H_T \text{Ad}_g A_\tau^{(2)}) = \sqrt{\lambda} \kappa, \\ M &= \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \text{STr}(H_V \text{Ad}_g A_\tau^{(2)}) = \sqrt{\lambda} m, \\ S &= \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \text{STr}(H_\Theta \text{Ad}_g A_\tau^{(2)}) = 0, \\ J &= \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \text{STr}(H_\psi \text{Ad}_g A_\tau^{(2)}) = \sqrt{\lambda} \omega, \end{aligned} \quad (2.35)$$

with H_ψ the generator of rotations along ψ . The Virasoro constraint then relates them as

$$\Delta = \sqrt{J^2 + \frac{\eta^2 M^2}{4}} = \sqrt{J^2 + \lambda \frac{\xi^2 M^2}{16\pi^2}}, \quad (2.36)$$

where we implemented $\Delta \equiv E$ and (2.20), since we have in mind the holographic interpretation discussed in the previous Section, where Δ is the (twisted) conformal dimension of the operator that should be dual to the solution (2.34) with global energy E . Following [20] we call this solution “BMN-like,” as it deforms the BMN vacuum (BPS operator) $\Delta = J$. One may interpret it also as a higher-order correction in λ .

On shell, the periodic BMN-like solution (2.34) is equivalent to a solution of the undeformed model with twisted-boundary conditions, as reviewed in Sec. II A, propagating in the background (2.32) at $\eta = 0$. For the Jordanian deformation (2.31), the mapping is nontrivial only in the AdS sector. Following the transformations (2.10) and (2.11) in the parametrization (2.22) (with fields X^m for g_a and $\tilde{X}^m$ for $\tilde{g}_a$) one then finds [12]

$$\tilde{T} = -\kappa\tau, \quad \tilde{V} = 0, \quad \tilde{Z} = \sqrt{\frac{\kappa}{m}} \exp\left(-\frac{\eta}{2} m\sigma\right), \quad \psi = \omega\tau, \quad (2.37)$$

with all other fields still vanishing. This solution satisfies the twisted-boundary conditions (2.12) with the twist charges $\mathbf{Q}$ and $\mathbf{q}$ defining W_{Jor} taking the values

$$\mathbf{Q} = -2\pi\eta m = -\xi M, \quad \mathbf{q} = 0. \quad (2.38)$$

As in the deformed, periodic case, this solution is nontrivial only in an $AdS_3 \times S^1$ subspace. In this formulation the Cartan Noether charges are $\tilde{E} = E = \sqrt{\lambda} \kappa$ and $\tilde{S} = S = 0$, and the charge M appears in $\mathbf{Q}$ (cf. also footnote 9). The angular momenta are still $J_1 = J_2 = 0$ and $J = J_3 = \sqrt{\lambda} \omega$ and the Virasoro constraint remains the same.

1. Landau-Lifshitz map

We now want to map the twisted-boundary model in the $S = J_1 = J_2 = 0$ sector to the $SL(2, \mathbb{R})$ Landau-Lifshitz (LL) Hamiltonian of [50] and evaluate it on the twisted solution (2.37). For this purpose, we found it useful to relate the global Jordanian-adapted coordinates (T, V, Θ, P, Z) for the AdS_5 metric in (2.32) at $\eta = 0$ (also called Schrödinger coordinates) to the global cylindrical coordinates of AdS_5 . This is most conveniently done through their relation to the embedding coordinates in $\mathbb{R}^{2,4}$

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 - X_5^2 = -1, \quad (2.39)$$

which was worked out in Appendix A of [12], see Eq. (A.2) therein. The standard cylindrical parametrization is defined as

$$\begin{aligned} X_0 + iX_5 &= \cosh \rho e^{it}, & X_1 + iX_4 &= \sinh \rho \cos \theta e^{i\phi_1}, \\ X_2 + iX_3 &= \sinh \rho \sin \theta e^{i\phi_1}. \end{aligned} \quad (2.40)$$

The $SL(2, \mathbb{R})$ (or $S = J_1 = J_2 = 0$) sector that we are interested in is obtained by considering the consistent truncation $\theta = 0$. Indeed, then $X_2 = X_3 = 0$, as is the case for the BMN-like solution (2.34) or (2.37). The $SL(2, \mathbb{R})$ Landau-Lifshitz variables, which define the two-dimensional elliptic hyperboloid $\eta^{ij} \ell_i \ell_j = -1$ with $\eta^{ij} = \text{diag}(-1, +1, +1)$, $i, j = 0, 1, 2$, are then

$$\begin{aligned} \ell_0 &= \cosh 2\rho = X_0^2 + X_1^2 + X_4^2 + X_5^2, \\ \ell_1 &= \sinh 2\rho \cos 2\varphi = 2X_0X_1 + 2X_4X_5, \\ \ell_2 &= \sinh 2\rho \sin 2\varphi = 2X_1X_5 - 2X_0X_4, \end{aligned} \quad (2.41)$$

where $\varphi := \frac{1}{2}(t - \phi_1)$. In terms of the Jordanian-adapted variables (on the truncation $P = 0$), they read

¹⁴In contrast to classical solutions in undeformed AdS_5 , which are characterized by the energy E and two spins S_1, S_2 , here the solutions are characterized by energy E , momentum M , and one spin $S = S_2$, as adapted to the Cartan subalgebra of the residual isometries (2.33).

$$\begin{aligned}
\ell_0 &= \frac{3(Z^4 + 4V^2 + 2) + (Z^4 - 4V^2 - 2) \cos 2T - 4VZ^2 \sin 2T}{8Z^2}, \\
\ell_1 &= \frac{(Z^4 - 4V^2 - 2) \cos T \sin T + 2VZ^2 \cos 2T}{\sqrt{2}Z^2}, \\
\ell_2 &= -\frac{(Z^4 + 4V^2 + 2) + 3(Z^4 - 4V^2 - 2) \cos 2T - 12VZ^2 \sin 2T}{8Z^2}.
\end{aligned} \tag{2.42}$$

The $SL(2, \mathbb{R})$ LL theory is obtained by eliminating the coordinate ψ associated to the (angular) momentum J from the (undeformed) $AdS_3 \times S^1$ sigma model by world sheet gauge fixing, and taking the large charge $J \rightarrow \infty$ limit while keeping the effective coupling $\tilde{\lambda} = \frac{\lambda}{J^2}$ fixed. In terms of the ℓ_i variables, the $SL(2, \mathbb{R})$ LL Hamiltonian is [50,51] (see also the review [52])

$$H_{s\text{-LL}} = \frac{\tilde{\lambda}J}{8} \int_0^{2\pi} \frac{d\sigma}{2\pi} \eta^{ij} \partial_\sigma \ell_i \partial_\sigma \ell_j, \tag{2.43}$$

and on the twisted solution (2.37) it evaluates to

$$H_{s\text{-LL}} = \frac{\eta^2 M^2}{8J} = \frac{\lambda}{32\pi^2} \frac{\xi^2 M^2}{J}. \tag{2.44}$$

This expression matches with the shift $\Delta^{(1)}$ of the dimension $\Delta = J + \Delta^{(1)}$ of this solution (2.36) when expanded to first order in $1/J$ while keeping η and M fixed.

III. DRINFEL'D TWISTED SPIN CHAINS

A. Hopf algebra elements

We start this Section by recalling key elements of a (quasitriangular) Hopf algebra as a quantum group that are relevant for our construction. For more details we refer to [53]. In the standard case, the Hopf algebra $\mathcal{A} = U(\mathfrak{g})$ is the universal enveloping algebra associated with a Lie algebra $\mathfrak{g}$. The Hopf structure extends the Lie algebra symmetry to composite (multiparticle) systems via the coproduct $\Delta: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, which in the canonical case acts on elements $X \in \mathfrak{g}$ as

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \Delta(1) = 1 \otimes 1, \tag{3.1}$$

and is comultiplicative

$$\Delta(XY) = \Delta(X)\Delta(Y), \quad \text{for } X, Y \in \mathcal{A}. \tag{3.2}$$

Iterating the coproduct $J - 1$ times extends the action of symmetries to the tensor product $\mathcal{A}^{\otimes J}$

$$\Delta^{J-1}(X) = \sum_{j=1}^J X_j, \quad X_j = 1^{\otimes j-1} \otimes X \otimes 1^{\otimes J-j}, \quad X \in \mathfrak{g}, \tag{3.3}$$

which lends itself naturally to systems such as J -sited spin chains. In addition, the Hopf algebra is equipped with a co-unit $\epsilon: \mathcal{A} \rightarrow \mathbb{C}$ and an antipode map $s: \mathcal{A} \rightarrow \mathcal{A}$ acting as

$$\begin{aligned}
\epsilon(1) &= 1, & \epsilon(X) &= 0, \\
\epsilon(XY) &= \epsilon(X)\epsilon(Y), & s(X) &= -X.
\end{aligned} \tag{3.4}$$

One also defines the opposite coproduct $\Delta^{\text{op}} := P \circ \Delta$, where $P(X \otimes Y) = Y \otimes X$ is a permutation of tensor spaces.

In the quasitriangular case, the Hopf algebra is further equipped with a universal $\mathcal{R}$ matrix, $\mathcal{R}: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, which is an invertible element satisfying the quasicommutativity condition

$$\Delta^{\text{op}}(X) = \mathcal{R}\Delta(X)\mathcal{R}^{-1}, \tag{3.5}$$

as well as quasitriangularity relations with the standard leg-ordering convention

$$(\Delta \otimes 1)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (1 \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}. \tag{3.6}$$

These imply the famous Yang-Baxter equation (YBE)

$$\mathcal{R}_{ij}\mathcal{R}_{ik}\mathcal{R}_{jk} = \mathcal{R}_{jk}\mathcal{R}_{ik}\mathcal{R}_{ij}, \tag{3.7}$$

in which the subscripts i, j, k label the factors in the J -fold tensor product space $\mathcal{A}^{\otimes J}$ on which $\mathcal{R}$ acts. Upon choosing a representation in which each copy of $\mathcal{A}$ acts on a vector space V , the universal $\mathcal{R}$ -matrix becomes an operator R on $V \otimes V$, typically depending on a free spectral parameter $u \in \mathbb{C}$. This yields the familiar $R_{ij}(u)$ and Lax operators of integrable physical models, which play a central role in solving such systems.

1. Drinfel'd twists

Starting from a solution $\mathcal{R}$ to the YBE, one may apply certain transformations to generate new solutions. A key solution-generating technique is applying a Drinfel'd

twist¹⁵ $\mathcal{F}$, which is an invertible element of $\text{End}(\mathcal{A} \otimes \mathcal{A})$ that is normalized $(\epsilon \otimes 1)\mathcal{F} = (1 \otimes \epsilon)\mathcal{F} = 1$ and satisfies the cocycle condition [3]

$$(\mathcal{F} \otimes 1)(\Delta \otimes 1)(\mathcal{F}) = (1 \otimes \mathcal{F})(1 \otimes \Delta)(\mathcal{F}). \quad (3.8)$$

The twisted quasitriangular Hopf algebra is then equipped with a twisted coproduct

$$\Delta_{\mathcal{F}}(X) = \mathcal{F}\Delta(X)\mathcal{F}^{-1}, \quad (3.9)$$

and twisted universal $\mathcal{R}$ matrix, which also satisfies the YBE (3.7),

$$\mathcal{R}^{\mathcal{F}} = \mathcal{F}^{\text{op}}\mathcal{R}\mathcal{F}^{-1}. \quad (3.10)$$

Here we introduced the notation $X^{\text{op}} := PXP$ for any endomorphism $X: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$. It will be convenient to further introduce

$$\begin{aligned} 2X_{1\dots j-1,j} &:= (\Delta^{j-2} \otimes 1)(X), & X_{1,2\dots j} &:= (1 \otimes \Delta^{j-2})(X), \\ X_{1\dots j-1,j}^{\text{op}} &:= (\Delta^{j-2} \otimes 1)(X^{\text{op}}) =: X_{j,1\dots j-1}, \\ X_{1,2\dots j}^{\text{op}} &:= (1 \otimes \Delta^{j-2})(X^{\text{op}}) =: X_{2\dots j,1} \end{aligned} \quad (3.11)$$

for any $X \in \text{End}(\mathcal{A} \otimes \mathcal{A})$. Iterating $J-1$ times $\Delta_{\mathcal{F}}$ then gives the expression [38]

$$\Delta_{\mathcal{F}}^{J-1}(X) = \Omega_{1\dots J}\Delta^{J-1}(X)\Omega_{1\dots J}^{-1}, \quad (3.12)$$

with the global intertwiner $\Omega_{1\dots J} \in \text{End}(\mathcal{A}^{\otimes J})$ defined as

$$\Omega_{1\dots J} := \mathcal{F}_{12}\mathcal{F}_{12,3} \cdots \mathcal{F}_{12\dots J-1,J}. \quad (3.13)$$

The cocycle condition further implies that we can alternatively write [38]

$$\Omega_{1\dots J} = \mathcal{F}_{(J-1)J}\mathcal{F}_{(J-2),(J-1)J} \cdots \mathcal{F}_{1,23\dots J}. \quad (3.14)$$

Useful identities for later finally are

$$\begin{aligned} \Omega_{1\dots J} &= \Omega_{1\dots J-1}\mathcal{F}_{12\dots J-1,J} \\ &= \Omega_{2\dots J}\mathcal{F}_{1,23\dots J}, \end{aligned} \quad (3.15)$$

which “strip off” intertwiners of certain lengths.

Having in mind deformations, we will consider twists continuously connected to the identity

$$\mathcal{F} = 1 \otimes 1 + \xi\mathcal{F}^{(1)} + \mathcal{O}(\xi^2), \quad (3.16)$$

with $\xi \in \mathbb{R}$. The co-cycle condition (3.8) implies at $\mathcal{O}(\xi)$ that $\mathcal{F}^{(1)}$ must satisfy a comptability condition with Δ . At $\mathcal{O}(\xi^2)$, its antisymmetric part $r_{12} = \frac{1}{2}(\mathcal{F}_{12}^{(1)} - \mathcal{F}_{21}^{(1)})$ satisfies the CYBE $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ [4]. This matches the defining properties (2.2) of integrable HYB deformations in tensor product form. As in Sec. II A, we will consider two types of Drinfel’d twists. The Abelian, or DR twists [26], are built from Abelian r matrices (2.5)

$$\mathcal{F}_{\text{ab}} = \exp(\xi \mathbf{t}_1 \wedge \mathbf{t}_2), \quad (3.17)$$

with $\mathbf{t}_1, \mathbf{t}_2$ commuting generators of $\mathfrak{g}$. These are arguably well understood, including in the context of AdS/CFT, particularly when $\mathbf{t}_{1,2}$ are compact Cartan generators, see, e.g., [16–21,54]. The non-Abelian Jordanian twists can instead be represented as [29,30]

$$\mathcal{F}_{\text{jor}} = \exp(\mathbf{h} \otimes \sigma), \quad \sigma := \log(1 + \xi \mathbf{e}), \quad (3.18)$$

where $\mathbf{h}, \mathbf{e} \in \mathfrak{g}$ with $[\mathbf{h}, \mathbf{e}] = \mathbf{e}$, as before. Jordanian twists come in various forms [55], all of which are related by co-boundary twists, which provide different representations of the same (twisted) Hopf algebra. This includes r -symmetric versions [4] for which $\mathcal{F}_{21}^{(1)} = -\mathcal{F}_{12}^{(1)}$ (as opposed to the above non- r -symmetric form). Also further extended Jordanian Drinfel’d twists exist [40,56]. We will however not consider or need any of these different forms or generalizations here.

Our main focus in this paper will be the understanding of the spectrum of the simplest Jordanian-twisted spin chains that should correspond to the string state sector laid out in Sec. II C, i.e., the $\mathfrak{sl}(2, \mathbb{R})$ -invariant XXX_{-1/2} spin chain. Before doing that, we will give two essential aspects for this study: General expressions for the deformed monodromy and transfer matrix in Sec. III B, as well as an explicit and general map of a Drinfel’d twisted system to an undeformed system with twisted-boundary conditions for the one-site algebra operators in Sec. III C.

B. Transfer matrix after general twists

A central object underlying many of the techniques used to solve integrable systems such as spin chains is the transfer matrix. It is constructed by introducing an auxiliary space, labeled “a,” in a representation V_a of $\mathfrak{g}$, and taking the iterated canonical coproduct of $R_{aj}(u)$, where j labels a physical site in a representation V_j . Specifically, in the ordering

$$V_a \otimes V_1 \otimes \cdots \otimes V_j \otimes \cdots \otimes V_J, \quad (3.19)$$

one can define the monodromy matrix as

¹⁵Based on the conjecture of [5], we purposefully use the same notation $\mathcal{F}$ as for the twist in Sec. II A, although here it will act on Hilbert space states rather than classical fields.

$$T_a(u) := (1_a \otimes \Delta^{J-1})(R(u)) = R_{aJ}(u) \cdots R_{a2}(u) R_{a1}(u), \quad (3.20)$$

where (3.6) was used. The transfer matrix is then obtained by tracing over the auxiliary space

$$\hat{\tau}(u) := \text{Tr}_a T_a(u), \quad (3.21)$$

and (by virtue of the RTT relations) it encodes a one-parameter family of commuting operators, typically including the momentum (shift) operator and Hamiltonian of closed chains, at least when $\mathfrak{a}$ is taken to be in the same representation as the physical sites.

Let us now give a useful expression of the monodromy and transfer matrix after applying a Drinfel'd twist. The deformed monodromy matrix is then constructed by iteratively applying $\Delta_{\mathcal{F}}$ to $R^{\mathcal{F}}$

$$T_a^{\mathcal{F}}(u) := (1_a \otimes \Delta_{\mathcal{F}}^{J-1})(R^{\mathcal{F}}(u)) = R_{aJ}^{\mathcal{F}}(u) \cdots R_{a2}^{\mathcal{F}}(u) R_{a1}^{\mathcal{F}}(u). \quad (3.22)$$

Using (3.12), we may rewrite this as a similarity transformation in the physical space by the global intertwiner

$$T_a^{\mathcal{F}}(u) = \Omega_{1 \dots J} T_a^{\star}(u) \Omega_{1 \dots J}^{-1}, \quad (3.23)$$

where $T_a^{\star}(u) := (1_a \otimes \Delta^{J-1})(R^{\mathcal{F}}(u))$. We will call $T_a^{\star}(u)$ the monodromy matrix in the *twisted-boundary* picture (for more justification of this name, see the next Section) or just “twisted monodromy.” Using the definition of $R^{\mathcal{F}}$, the compatibility of the coproduct, as well as the notation of (3.11), one can simply write¹⁶

$$\begin{aligned} T_a^{\star}(u) &= (1_a \otimes \Delta^{J-1})(\mathcal{F}^{\text{op}})(1_a \otimes \Delta^{J-1}) \\ &\quad \times (R)(1_a \otimes \Delta^{J-1})(\mathcal{F}^{-1}) \\ &= \mathcal{F}_{1 \dots J, a} T_a(u) \mathcal{F}_{a, 1 \dots J}^{-1}, \end{aligned} \quad (3.24)$$

so that ultimately for any Drinfel'd twist we have

$$T_a^{\mathcal{F}}(u) = \Omega_{1 \dots J} \mathcal{F}_{1 \dots J, a} T_a(u) \mathcal{F}_{a, 1 \dots J}^{-1} \Omega_{1 \dots J}^{-1}, \quad (3.25)$$

without additional assumptions. We are not aware of a derivation as presented in this Section, although all the ingredients are present in [38]. The formula (3.25) is equivalent to Eq. (3.30) of [20] when specified to the case of Abelian DR twists (see also [18,57] for alternative

derivations) and Eq. (3.33) of [28] for the case of Jordanian twists of the type (3.18). In Sec. III D we will however show that our (3.25) takes a particularly nice form in the case of Jordanian twists.

C. Mapping to twisted-boundary conditions of one-site operators

Closed Drinfel'd twisted spin chains are understood to have two equivalent formulations: (i) a model of spins with *deformed* interactions and *periodic-boundary* conditions along a one-dimensional chain, or (ii) a model of spins with *undeformed* interactions but with *twisted-boundary* conditions. Indeed, consider, e.g., models with nearest-neighbor interactions, such as the Heisenberg XXX spin chain, for which the Hamiltonian density $h \in \text{End}(\mathcal{A} \otimes \mathcal{A})$ is a two-sited operator and with

$$R_{12}(u = u_0) = P_{12}, \quad h_{12} := P_{12} \left(\frac{d}{du} R_{12}(u) \right)_{u=u_0}, \quad (3.26)$$

for some $u_0 \in \mathbb{C}$. The deformed interactions are then governed by

$$H^{\mathcal{F}} := \sum_{j=1}^J h_{j,j+1}^{\mathcal{F}}, \quad h_{j,j+1}^{\mathcal{F}} := \mathcal{F}_{j,j+1} h_{j,j+1} \mathcal{F}_{j,j+1}^{-1}, \quad (3.27)$$

with periodic-boundary conditions $h_{J,J+1}^{\mathcal{F}} \equiv h_{J,1}^{\mathcal{F}}$. Under a similarity transformation Ω of the Hilbert space of states, this is equivalent to an undeformed interacting Hamiltonian

$$H^{\star} := \Omega^{-1} H^{\mathcal{F}} \Omega = \sum_{j=1}^{J-1} h_{j,j+1} + S^{-1} h_{J,1} S, \quad (3.28)$$

with twisted-boundary conditions $h_{J,J+1} = S^{-1} h_{J,1} S$. In the case of Abelian and Jordanian twists of the type (3.17) and (3.18), one can show that $S = \mathcal{F}_{J,1}^{-1} \Omega$ and $\Omega = \Omega_{1 \dots J}$, the global intertwiner, see, e.g., [20,28].

In this Section, we will instead propose a general method of deriving the twisted-boundary conditions for one-site operators on $\mathcal{A} = U(\mathfrak{g})$.¹⁷ On the one hand, we have the deformed picture, where the action of one-site operators $X \in \mathcal{A}$ is

$$\Delta_{\mathcal{F}}^{J-1}(X) = \sum_{j=1}^J \Omega_{1 \dots j} X_j \Omega_{1 \dots j}^{-1}, \quad (3.29)$$

with periodic-boundary $X_{J+i} \equiv X_i$. On the other hand, we will have the twisted-boundary picture where

¹⁶Note that one should pay attention to the fact that we are working over the tensor-product space (3.19) here.

¹⁷See [27] for obtaining twisted-boundary conditions of one-site operators in the case of constant twists using the transformation of the deformed to the undeformed Hamiltonian.

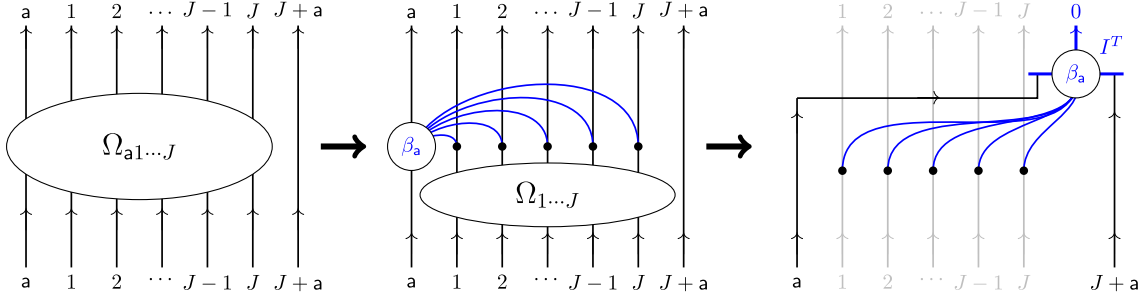


FIG. 1. Illustration of the gluing procedure leading to twisted-boundary conditions.

$$\Delta^{J-1}(X) = \sum_{j=1}^J X_j, \quad (3.30)$$

but with the price to pay that $X_{J+i} \neq X_i$. One may travel between the two formulations using the nonlocal (all-site) similarity transform $X_j \rightarrow X_j^* \equiv \Omega_{1\dots J}^{-1} X_j \Omega_{1\dots J}$.

The question that we want to answer now is: How do the periodic-boundary conditions map to an explicit formula for the twisted-boundary conditions?

In order to systematically implement the boundary twist induced by the Drinfel'd twist, we begin by extending the spin chain through the introduction of two auxiliary spaces: one before and one after the physical chain, labeled $\mathbf{a}$ and $J + \mathbf{a}$, respectively. That is, we consider the extended open chain

$$V_{\mathbf{a}} \otimes V_1 \otimes \dots \otimes V_J \otimes \dots \otimes V_J \otimes V_{J+\mathbf{a}}. \quad (3.31)$$

The purpose of this extension is to realize the eventual twisted periodicity as an operator insertion *localized* at the boundary site $J + \mathbf{a}$.¹⁸

Before explicitly performing the procedure, let us first sketch the idea of each step, which we also illustrate in Fig. 1. (i) We will deform the chain by applying the Drinfel'd twist on the chain from $\mathbf{a}$ to J .¹⁹ Since the twist $\mathcal{F}$ is a two-site operator, this will ensure (at the end of the day) that every physical site contributes to the total boundary deformation. In particular, the resulting boundary twist will be composed from multiple $\mathcal{F}$ factors of which one leg is consistently anchored at the auxiliary space $\mathbf{a}$ and the other leg connects to each physical site j , such that the entire physical chain contributes to its overall

expression. (ii) To isolate the physical effect of the boundary twist, we then apply a similarity transformation that removes the “internal” twist factors—those connecting pairs of physical sites—and retains only the twist components linking $\mathbf{a}$ to the physical chain. This will result in a configuration where the twist is effectively localized at $J + \mathbf{a}$. (iii) Finally, to recover the canonical operator action up to a boundary twist, we glue together the auxiliary spaces $\mathbf{a}$ and $J + \mathbf{a}$ via a suitable projection I^T . The resulting space, denoted “0,” represents the defect site that carries the entire effect of the twisted periodicity. See (3.60) and (3.61) for explicit forms of the resulting boundary operator.

Let us now explicitly apply the procedure as described above. We consider twist the chain and consider the deformed action

$$B^{\mathcal{F}}(X) := (\Delta_{\mathcal{F}}^J \otimes 1_{J+\mathbf{a}})(X \otimes 1_{J+\mathbf{a}} + 1 \otimes X_{J+\mathbf{a}}). \quad (3.32)$$

Using the definitions (3.3), (3.12) and the identities (3.15), one can write

$$\begin{aligned} B^{\mathcal{F}}(X) &= \Omega_{\mathbf{a}\dots J} \left(X_{\mathbf{a}} + \sum_{j=1}^J X_j \right) \Omega_{\mathbf{a}\dots J}^{-1} + X_{J+\mathbf{a}} \\ &= \Omega_{1\dots J} \beta_{\mathbf{a}} (X_{\mathbf{a}} + \Delta^{J-1}(X)) \beta_{\mathbf{a}}^{-1} \Omega_{1\dots J}^{-1} + X_{J+\mathbf{a}}, \end{aligned} \quad (3.33)$$

where we lightened notation by introducing $\beta_{\mathbf{a}} := \mathcal{F}_{\mathbf{a},1\dots J}$. Now, we perform a similarity transformation by $\Omega_{1\dots J} \beta_{J+\mathbf{a}}$ as $B^{\mathcal{F}}(X) \rightarrow B^*(X) = (\Omega_{1\dots J} \beta_{J+\mathbf{a}})^{-1} B^{\mathcal{F}}(X) (\Omega_{1\dots J} \beta_{J+\mathbf{a}})$ so that

$$B^*(X) = \beta_{J+\mathbf{a}}^{-1} \beta_{\mathbf{a}} (X_{\mathbf{a}} + \Delta^{J-1}(X)) \beta_{\mathbf{a}}^{-1} \beta_{J+\mathbf{a}} + \beta_{J+\mathbf{a}}^{-1} X_{J+\mathbf{a}} \beta_{J+\mathbf{a}}, \quad (3.34)$$

where $\beta_{J+\mathbf{a}} := \mathcal{F}_{J+\mathbf{a},1\dots J}$. This has removed the internal physical twists present in $\Omega_{1\dots J}$.

We will now consider the gluing of the auxiliary spaces $V_{\mathbf{a}}$ and $V_{J+\mathbf{a}}$ and projection onto the single defect space V_0 .

¹⁸We use the some notation $\mathbf{a}$ as in the previous Section to make a simple comparison with the twisted transfer matrix later on.

¹⁹Due to the (in general) non-Abelian nature of Drinfel'd twists, the ordering in which we deform the chain (from $\mathbf{a}$ to J , or from 1 to $J + \mathbf{a}$) and map the effect to the boundary (at $J + \mathbf{a}$, or at $\mathbf{a}$) is important. We will comment more on this point later for the explicit example of Jordanian twists.

For this purpose, let us introduce two maps²⁰

$$I: V_0 \rightarrow V_a \otimes V_{J+a}, \quad I^T: V_a \otimes V_{J+a} \rightarrow V_0. \quad (3.36)$$

We then assume the following gluing condition

$$I^T \beta_a = I^T \beta_{J+a}, \quad \beta_a I = \beta_{J+a} I. \quad (3.37)$$

Concretely, this ensures that when we project back to the single defect space V_0 , the same overall twist $\beta_a = \mathcal{F}_{a,1\dots J}$ acts regardless of whether it was applied to the first or second auxiliary leg. Performing the gluing on (3.34) we then find the canonical coproduct up to the defect term

$$\begin{aligned} B(X) &= I^T B^*(X) I \\ &= I^T X_a I + I^T \Delta^{J-1}(X) I + I^T \beta_{J+a}^{-1} X_{J+a} \beta_{J+a} I. \end{aligned} \quad (3.38)$$

We thus have two copies of the operator algebra acting on the 0 site: one un-twisted, appearing “before” site 1, and one twisted, appearing “after” site J . This action can only be consistent when we identify the twisted-boundary condition

$$X_{J+a} \equiv \beta_a X_a \beta_a^{-1}. \quad (3.39)$$

In other words, the operator on the defect site is the original generator conjugated by the total boundary-twist $\beta_a = \mathcal{F}_{a,1\dots J}$. In the next Section, we will give an explicit expression of (3.39) for the $\mathfrak{sl}(2, \mathbb{R})$ Jordanian twist (3.18), and we will demonstrate that it passes several consistency checks. Nonetheless, at this stage, let us already mention two interesting observations: Firstly, the total Jordanian boundary twist will be expressed in terms of a symmetry operator of the resulting model, and the boundary conditions will respect the $\mathfrak{sl}(2, \mathbb{R})$ algebra. This would not have been the case if we (naively) pulled the twist from site 1 to $J + a$ and measured at a .²¹ We believe this difference is

²⁰In an orthonormal basis $|e_a\rangle$ of V , we can for instance define

$$\begin{aligned} I &:= \sum_a |e_a\rangle_a \otimes |e_a\rangle_{J+a} \langle e_a|_0, \\ I^T &:= \sum_a |e_a\rangle_0 \langle e_a|_a \otimes |e_a\rangle_{J+a}. \end{aligned} \quad (3.35)$$

In principle, one may also introduce coefficients λ_a in the definition of I and I^T to allow for a weighted identification. By construction we have $I^T(|e_a\rangle_a \otimes |e_b\rangle_{J+a}) = \delta_{ab} |e_a\rangle$, implementing the formal gluing identification by projecting onto the (diagonal) subspace where the two auxiliary sites agree. The opposite I embeds a defect space vector $\psi_0 = \sum_a \langle e_a | \psi \rangle |e_a\rangle_0$ as a diagonal tensor, $I(\psi_0) = \sum_a \langle e_a | \psi \rangle |e_a\rangle_a \otimes |e_a\rangle_{J+a}$. We can interpret the latter as an entangled state between V_a and V_{J+a} .

²¹We would however find the same (consistent) formulas as given in (3.39) when pulling the twisting from $J + a$ to 1 by means of the opposite coproduct and measuring at a .

rooted in the underlying (leg-ordering) axioms of the quasitriangular Hopf algebra and in the fact that the twist (3.18) is not r symmetric (unlike Abelian twists) such that its tensor legs are on unequal footing. It would be interesting to find a derivation that makes this distinction more explicit. Secondly, we note that the above formula (3.39) is consistent with the form of the twisted monodromy matrix (3.24), obtained from the iterated coproduct of the twisted R matrix (which encodes a permutation), after the similarity transformation. In both cases, the object that first acts on the Hilbert space is $\beta_a^{-1} = \mathcal{F}_{a,1\dots J}^{-1}$.

D. Applied to the Jordanian-twisted $\mathfrak{sl}(2, \mathbb{R})$ spin chain

Given the BMN-like string solution of the Jordanian sigma model presented in II C, which is nontrivial only in an $\text{AdS}_3 \times S^1$ subspace of $\text{AdS}_5 \times S^5$, the relevant spin chain we want to study and twist is the noncompact $\mathfrak{sl}(2, \mathbb{R})$ -invariant XXX_{-1/2} spin chain [31,32,58] (see also [59,60]). In this Section we will clarify the residual symmetry of this model in both the deformed and twisted-boundary picture, and we will give explicit formulas for the twisted monodromy matrix and boundary conditions of the one-site algebra generators.

We denote the generators of $\mathfrak{sl}(2, \mathbb{R})$ by $X = \{\mathfrak{h}, \mathfrak{e}, \mathfrak{f}\}$, with the commutation relations

$$[\mathfrak{h}, \mathfrak{e}] = \mathfrak{e}, \quad [\mathfrak{h}, \mathfrak{f}] = -\mathfrak{f}, \quad [\mathfrak{e}, \mathfrak{f}] = -2\mathfrak{h}. \quad (3.40)$$

Here, $\mathfrak{h}$ is the Cartan generator, while $\mathfrak{e}$ and $\mathfrak{f}$ are, respectively, raising and lowering operators with respect to the spectrum of $\mathfrak{h}$. Unless otherwise stated, from here on out we will take $\mathcal{F} \equiv \mathcal{F}_{\text{for}}$ as given in (3.18). The twisted coproducts of the $\mathfrak{sl}(2, \mathbb{R})$ generators can then be computed explicitly as

$$\Delta_{\mathcal{F}}(\mathfrak{h}) = \mathfrak{h} \otimes \exp(-\sigma) + 1 \otimes \mathfrak{h}, \quad (3.41)$$

$$\Delta_{\mathcal{F}}(\mathfrak{e}) = \mathfrak{e} \otimes \exp(\sigma) + 1 \otimes \mathfrak{e} = \mathfrak{e} \otimes 1 + 1 \otimes \mathfrak{e} + \xi \mathfrak{e} \otimes \mathfrak{e}, \quad (3.42)$$

$$\begin{aligned} \Delta_{\mathcal{F}}(\mathfrak{f}) &= \mathfrak{f} \otimes \exp(-\sigma) + 1 \otimes \mathfrak{f} - 2\xi \mathfrak{h} \otimes \mathfrak{h} \exp(-\sigma) \\ &\quad - \xi(\mathfrak{h} - \mathfrak{h}^2) \otimes (\exp(-\sigma) - \exp(-2\sigma)). \end{aligned} \quad (3.43)$$

From this, one finds on $U(\mathfrak{sl}(2))$ that

$$\Delta_{\mathcal{F}}(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma, \quad (3.44)$$

meaning that σ is primitive (or canonical) with respect to the twisted coproduct. It furthermore implies the following factorization identities for Jordanian twists of the type (3.18)

$$(\Delta \otimes 1)(\mathcal{F}) = \exp(\Delta(\mathfrak{h}) \otimes \sigma) = \mathcal{F}_{13} \mathcal{F}_{23}, \quad (3.45)$$

$$(1 \otimes \Delta_{\mathcal{F}})(\mathcal{F}) = \exp(\mathfrak{h} \otimes \Delta_{\mathcal{F}}(\sigma)) = \mathcal{F}_{12}\mathcal{F}_{13}, \quad (3.46)$$

Equation (3.45) allows us to simplify the global intertwiner (3.13) and write

$$\begin{aligned} \Omega_{1\dots J} &= \mathcal{F}_{12}(\mathcal{F}_{13}\mathcal{F}_{23}) \cdots (\mathcal{F}_{1J}\mathcal{F}_{2J} \cdots \mathcal{F}_{(J-1)J}) \\ &= (\mathcal{F}_{12}\mathcal{F}_{13} \cdots \mathcal{F}_{1J})(\mathcal{F}_{23}\mathcal{F}_{24} \cdots \mathcal{F}_{2J}) \cdots \mathcal{F}_{(J-1)J}. \end{aligned} \quad (3.47)$$

For Abelian DR twists, where $[\mathcal{F}_{ij}, \mathcal{F}_{kl}] = 0$ for all tensor sites, the above of course simplifies even further, as the ordering does not matter.

1. Residual symmetries

As mentioned earlier, the Hamiltonian of the Jordanian-twisted spin chain admits two equivalent formulations. In the deformed picture, one works with $H^{\mathcal{F}}$ given in (3.27), while in the twisted-boundary picture one works with $H^{\star} = \Omega^{-1}H^{\mathcal{F}}\Omega$ given in (3.28), with $S = \mathcal{F}_{J1}\Omega$ and $\Omega = \Omega_{1\dots J}$, see [28].²² Importantly, the universal enveloping Lie algebra $U(\mathfrak{sl}(2, \mathbb{R}))$ is represented differently in these cases: In the deformed picture, the action of $U(\mathfrak{sl}(2, \mathbb{R}))$ is twisted via the twisted coproduct $\Delta_{\mathcal{F}}$, while in the twisted-boundary picture one uses the canonical coproduct Δ to define the algebra action.

The residual $\mathfrak{sl}(2, \mathbb{R})$ symmetries of the models may be derived following Sec. 3.3 of [28] and which we recap here. The undeformed periodic XXX_{-1/2} spin chain Hamiltonian is [58,61] (see also [59])

$$H = \sum_{j=1}^J h_{j,j+1}, \quad h_{j,j+1} = \psi(\mathbb{J}_{j,j+1} + 1) - \psi(0), \quad (3.48)$$

with $\psi(x)$ the digamma function and $\mathbb{J}$ a two-site operator defined implicitly through $\mathbb{J}(\mathbb{J} + 1) = \Delta(X) \cdot \Delta(X) = \mathbb{C}_2$, with $\mathbb{C}_2$ the two-site quadratic Casimir, and the dot product defined through the Cartan-Killing form. This Hamiltonian is $\mathfrak{sl}(2, \mathbb{R})$ invariant

$$[\Delta^{J-1}(X), h_{j,j+1}] = 0, \quad \forall X \in \mathfrak{sl}(2, \mathbb{R}), \quad j = 1, \dots, J. \quad (3.49)$$

For the deformed Hamiltonian $H_{\mathcal{F}}$ one then finds

²²The two formulations are thus related by the same nonlocal (all-site) similarity transform of the Hilbert space that relates also the deformed $T_{\mathfrak{a}}^{\mathcal{F}}(u)$ and twisted-boundary $T_{\mathfrak{a}}^{\star}(u)$ monodromy, as presented in Sec. III B, as well as the periodic and twisted-boundary conditions of one-site operators (and states), as presented in Sec. III C.

$$\begin{aligned} [\Delta_{\mathcal{F}}^{J-1}(X), H^{\mathcal{F}}] &= \sum_{j=1}^{J-1} \Omega[\Delta_{\mathcal{F}}^{J-1}(X), h_{j,j+1}] \Omega^{-1} \\ &\quad + [\Delta_{\mathcal{F}}^{J-1}(X), h_{J,J+1}^{\mathcal{F}}] \\ &= [\Delta_{\mathcal{F}}^{J-1}(X), h_{J,J+1}^{\mathcal{F}}], \end{aligned} \quad (3.50)$$

since $h_{j,j+1}^{\mathcal{F}} = \Omega h_{j,j+1} \Omega^{-1}$ for $j = 1, \dots, J-1$ [28]. Thus, the only potential violation of $\mathfrak{sl}(2, \mathbb{R})$ symmetry arises at the end of the chain. A generator $X \in \mathfrak{sl}(2, \mathbb{R})$ survives as an ordinary symmetry if there exists some $X' \in \mathfrak{sl}(2, \mathbb{R})$ such that, under the shift operator $U := P_{12} \cdots P_{J-1,J}$, one has

$$U^{-1} \Delta_{\mathcal{F}}^{J-1}(X) U = \Delta_{\mathcal{F}}^{J-1}(X'). \quad (3.51)$$

Indeed, in that case $[\Delta_{\mathcal{F}}^{J-1}(X), h_{J,J+1}^{\mathcal{F}}] = U[\Delta_{\mathcal{F}}^{J-1}(X'), h_{J-1,J}^{\mathcal{F}}]U^{-1} = 0$ and so $\Delta_{\mathcal{F}}^{J-1}(X)$ commutes with the full Hamiltonian.

Crucially, the operator $\sigma = \log(1 + \xi \mathfrak{e}) \in U(\mathfrak{sl}(2, \mathbb{R}))$ is primitive with respect to $\Delta_{\mathcal{F}}$, and one can check immediately that it satisfies the shift-equivariance condition (3.51). Although nonstandard, it is a residual symmetry of both formulations

$$[\Delta_{\mathcal{F}}^{J-1}(\sigma), H^{\mathcal{F}}] = 0, \quad \text{and} \quad [\Delta^{J-1}(\sigma), H^{\star}] = 0. \quad (3.52)$$

Accordingly, in the deformed picture it would be natural to diagonalize

$$\hat{Q} := \Delta_{\mathcal{F}}^{J-1}(\sigma) = \sum_{j=1}^J \sigma_j, \quad (3.53)$$

whereas in the twisted-boundary formulation, given the fact that $\Delta^{J-1}(\sigma) = \log(1 + \xi \Delta^{J-1}(\mathfrak{e}))$, it would be convenient to diagonalize the sum of raising operators

$$\hat{M} := \Delta^{J-1}(\mathfrak{e}) = \sum_{j=1}^J \mathfrak{e}_j. \quad (3.54)$$

However, nontrivial eigenstates of the $\hat{Q}$ operator turn out to be involved to work with as they do not admit a regular expansion in ξ , while $\hat{M}$ eigenstates have simple expressions. One may also verify explicitly that $\mathfrak{e}$ is a residual symmetry generator. In fact, Eq. (3.42) implies

$$\begin{aligned} U \Delta_{\mathcal{F}}^{J-1}(\mathfrak{e}) U^{-1} &= U \left(\sum_{j=1}^J \mathfrak{e}_j + \xi \sum_{j \neq j'}^J \mathfrak{e}_j \mathfrak{e}_{j'} \right. \\ &\quad \left. + \cdots \xi^{J-1} \mathfrak{e}_1 \mathfrak{e}_2 \cdots \mathfrak{e}_J \right) U^{-1} = \Delta_{\mathcal{F}}^{J-1}(\mathfrak{e}), \end{aligned} \quad (3.55)$$

such that $\mathbf{e}$ satisfies (3.51). While $\Delta_{\mathcal{F}}(\mathbf{e})$ is again non-standard, the spectrum of $\hat{M} = \Delta^{J-1}(\mathbf{e})$ defines an ordinary conserved charge M in the twisted-boundary formulation. Importantly, the other $\mathfrak{sl}(2, \mathbb{R})$ generators, the Cartan $\mathbf{h}$ and the lowering $\mathbf{f}$, do not satisfy (3.51). This is consistent with the Jordanian sigma model with (2.4), which always has $\mathbf{e}$ as a residual symmetry.

In the upcoming Sections, we will therefore abandon diagonalizing the Cartan generator $\mathbf{h}$. This will represent a rather radical departure from more familiar spin chains, where the energy spectrum can be organized by its eigenvalues, i.e., in a Fock space of magnonlike excitations. Instead, we will work in an eigenbasis of the raising operator $\mathbf{e}$. These $\mathbf{e}$ eigenstates can be interpreted as (noncompact) coherent states with respect to the usual Fock space at each site, see, e.g., [62] for such a construction. The eigenbasis of $\hat{M}$ is thus a tensor product of such states across the chain. While it aligns naturally with the residual symmetry structure, this basis is overcomplete and continuous, rather than orthonormal and discrete. It will thus be a significant change in how we label and organize our spin chain states.²³

2. Twisted transfer matrix and twisted-boundary conditions

We can now evaluate the general expressions for the twisted-monodromy matrix (3.24) and twisted-boundary condition (3.39) explicitly for the Jordanian twist (3.18). In particular, we have

$$\mathcal{F}_{1\dots J, \mathbf{a}} = (1_{\mathbf{a}} \otimes \Delta^{J-1})(\mathcal{F}^{\text{op}}) = \exp(\sigma_{\mathbf{a}} \otimes \Delta^{J-1}(\mathbf{h})), \quad (3.56)$$

$$\beta_{\mathbf{a}} = \mathcal{F}_{\mathbf{a}, 1\dots J} = (1_{\mathbf{a}} \otimes \Delta^{J-1})(\mathcal{F}) = \exp(\mathbf{h}_{\mathbf{a}} \otimes \Delta^{J-1}(\sigma)). \quad (3.57)$$

From the above discussion, it is then obvious that $\beta_{\mathbf{a}}$ is expressible in terms of the symmetry operator $\hat{M}$, and thus we can write²⁴

$$T_{\mathbf{a}}^*(u) = \exp(\sigma_{\mathbf{a}} \otimes \Delta^{J-1}(\mathbf{h})) T_{\mathbf{a}}(u) \times \exp(-\mathbf{h}_{\mathbf{a}} \otimes \log(1 + \xi \hat{M})), \quad (3.58)$$

and

²³Similar considerations were also needed in the dipole deformation of [20], realized via an Abelian DR twist based on a root generator $\mathbf{e} \in \mathfrak{sl}(2, \mathbb{R})$, as well as a Cartan of the $SO(6)$ R symmetry.

²⁴Note that we may also express the overall twist as $\mathcal{F}_{\mathbf{a}, 1\dots J} = \text{Ad}_{\Omega_{1\dots J}}^{-1} \exp(\mathbf{h}_{\mathbf{a}} \otimes \Delta_{\mathcal{F}}^{J-1}(\sigma)) = \Omega_{1\dots J}^{-1} (\prod_{j=1}^J \mathcal{F}_{\mathbf{a}, j}) \Omega_{1\dots J}$ such that the expression (3.58) clearly matches with Eq. (3.33) of [28].

$$X_{J+\mathbf{a}} \equiv \exp(\mathbf{h}_{\mathbf{a}} \otimes \log(1 + \xi \hat{M})) X_{\mathbf{a}} \times \exp(-\mathbf{h}_{\mathbf{a}} \otimes \log(1 + \xi \hat{M})), \quad (3.59)$$

for $X \in \mathfrak{sl}(2, \mathbb{R})$. Note that here we see explicitly that the inclusion of the “auxiliary” defect site in the derivation of the boundary conditions is necessary such that the overall twist can be expressed in terms of the total symmetry operator $\hat{M}$ on the full physical chain from site 1 to J . Explicitly, the twisted-boundary conditions read

$$\mathbf{e}_{J+\mathbf{a}} \equiv \mathbf{e}_{\mathbf{a}}(1 + \xi \hat{M}), \quad \mathbf{h}_{J+\mathbf{a}} = \mathbf{h}_{\mathbf{a}}, \quad \mathbf{f}_{J+\mathbf{a}} \equiv \mathbf{f}_{\mathbf{a}}(1 + \xi \hat{M})^{-1}, \quad (3.60)$$

which can be shown using the identity $\text{Ad}_{e^x} = e^{\text{ad}_x}$ for $X \in \mathcal{A}^{\otimes J}$. An important check of (3.60) is that the $X_{J+\mathbf{a}}$ generators should satisfy the $\mathfrak{sl}(2, \mathbb{R})$ algebra (3.40). This is indeed the case here. Note that when the physical spin chain is in an eigenstate of the operator $\hat{M}$ we can replace in (3.60) its action with the corresponding eigenvalue M .

As per illustration, let us also compare to the DR twist considered in [20]. In that case, one would find $\mathcal{F}_{\mathbf{a}, 1\dots J}^{\text{ab}} = \exp(\xi \mathbf{e}_{\mathbf{a}} \hat{R} - \xi R_{\mathbf{a}} \hat{M})$ with $\hat{R} := \Delta^{J-1} R_j$ the total R charge. In the $\mathfrak{sl}(2, \mathbb{R})$ sector, this implies $X_{J+\mathbf{a}} = \exp(\xi \hat{R} \mathbf{e}_{\mathbf{a}}) X_{\mathbf{a}} \exp(-\xi \hat{R} \mathbf{e}_{\mathbf{a}})$ because R acts as trivially (as the identity) on $\mathbf{a}$. The boundary conditions (3.39) then become

$$\mathbf{e}_{J+\mathbf{a}} \equiv \mathbf{e}_{\mathbf{a}}, \quad \mathbf{h}_{J+\mathbf{a}} \equiv \mathbf{h}_{\mathbf{a}} - \xi \hat{R} \mathbf{e}_{\mathbf{a}}, \quad \mathbf{f}_{J+\mathbf{a}} \equiv \mathbf{f}_{\mathbf{a}} - 2\xi \hat{R} \mathbf{h}_{\mathbf{a}} + \xi^2 \hat{R}^2 \mathbf{e}_{\mathbf{a}}, \quad (3.61)$$

with $\hat{R} = J$ when evaluated on the physical spin chain, which matches with [20].

Several consistency checks thus support our method presented in Sec. III C to deriving twisted-boundary conditions for Drinfel'd twisted spin chains. In particular, next to consistency with the transfer matrix, the boundary conditions (3.59) are given in terms of the residual symmetry operator $\hat{M}$ and respect the $\mathfrak{sl}(2, \mathbb{R})$ algebra. In Appendix A, we also analyse the classical limit of the twisted transfer matrix of (3.58). Its overall twisting (denoted by $\mathcal{W}$) is structurally analogous to the twist appearing in the boundary conditions of the one-site operators. In particular, we show that the eigenvalues of the classical $\mathcal{W}$ depend only on the $\hat{M}$ charge, with no contribution from the $\Delta^{J-1}(\mathbf{h})$ operator on the physical sites, in analog to the form of the boundary conditions in (3.59). We also comment in Appendix A on connections of $\mathcal{W}$ and (3.60) to the classical sigma-model picture, where similar simplifications of the twisted-boundary conditions arise. Finally, in Sec. VI we find consistency with the large u asymptotics of the Q functions of the underlying integrable system, which generally are known to encode the boundary conditions.

IV. LANDAU-LIFSHITZ MODEL FROM THE TWISTED SPIN CHAIN

In Sec. II C, we considered a classical twisted-string solution of the Jordanian sigma model in the $S = J_1 = J_2 = 0$ sector, and mapped it to the semiclassical $SL(2, R)$ LL Hamiltonian. Here, we will show how the same LL Hamiltonian arises directly from the spin chain description described in Sec. III D. We begin by reviewing how the untwisted $SL(2, R)$ LL model is obtained from the classical continuum limit of the $\mathfrak{sl}(2, R)$ spin chain, following [50,51] (generalizing the seminal work [63]). Then, we show how the Jordanian Drinfel'd twist with (3.60) leads to an LL model with corresponding twisted-boundary conditions. Finally, we demonstrate that the classical energy of the LL configuration from the twisted spin chain matches that of the twisted-string solution on the sigma-model side, thus providing a first direct link between the two descriptions.

A. $\mathfrak{sl}(2, \mathbb{R})$ coherent states and Landau-Lifshitz limit

1. Review of the bulk model

The classical limit of the $\mathfrak{sl}(2, \mathbb{R})$ spin chain is typically obtained via the coherent state formalism, which captures semiclassical dynamics by associating to each site an $\mathfrak{sl}(2, \mathbb{R})$ coherent state, defined as eigenstates of the lowering operator f

$$|\mathbf{n}\rangle = \sqrt{1 - |\vartheta|^2} e^{\vartheta e} |0\rangle, \quad \langle \mathbf{n}| = \langle 0| e^{\bar{\vartheta} f} \sqrt{1 - |\vartheta|^2}, \quad (4.1)$$

with $|0\rangle$ the lowest-weight state with $h|0\rangle = -\frac{1}{2}|0\rangle$ and $f|0\rangle = 0$. The parameter ϑ labels a point on a two-dimensional elliptic hyperboloid.²⁵ In fact, it is a key property of these states that the expectation values of the $\mathfrak{sl}(2, \mathbb{R})$ generators

$$n_0 := -2\langle \mathbf{n}|h|\mathbf{n}\rangle, \quad n_+ := -2\langle \mathbf{n}|e|\mathbf{n}\rangle, \quad n_- := -2\langle \mathbf{n}|f|\mathbf{n}\rangle, \quad (4.2)$$

which evaluate to

$$n_0 = \frac{1 + |\vartheta|^2}{1 - |\vartheta|^2}, \quad n_+ = -\frac{2\vartheta}{1 - |\vartheta|^2}, \quad n_- = -\frac{2\bar{\vartheta}}{1 - |\vartheta|^2}, \quad (4.3)$$

satisfy the constraint

$$\vec{\mathbf{n}} \cdot \vec{\mathbf{n}} = \eta^{ij} n_i n_j = -n_0^2 + n_+ n_- = -1. \quad (4.4)$$

The vector $\vec{\mathbf{n}} := (n_0, n_1, n_2)$ with $n_{\pm} = n_1 \pm i n_2$ is thus constrained to the two-dimensional elliptic hyperboloid.

²⁵In particular, in terms of the parametrization (2.41) it is $\vartheta = -\tanh(\rho) e^{2i\varphi}$.

Importantly, these coherent states are normalized, $\langle \mathbf{n}|\mathbf{n}\rangle = 1$, but they are nonorthogonal and overcomplete, see [51,62] for more details.

To construct the classical spin chain Hamiltonian, one evaluates the quantum Hamiltonian density on a product of neighboring coherent states, yielding [50,51]

$$h_{j,j+1}^{\text{cl}} := \langle \mathbf{n}_j | \mathbf{n}_{j+1} | h_{j,j+1} | \mathbf{n}_j | \mathbf{n}_{j+1} \rangle \\ = \log \left(1 + \frac{(\vec{\mathbf{n}}_j - \vec{\mathbf{n}}_{j+1}) \cdot (\vec{\mathbf{n}}_j - \vec{\mathbf{n}}_{j+1})}{4} \right), \quad (4.5)$$

where $\vec{\mathbf{n}}_j = \{n_0, n_1, n_2\}_j$ and the dot product is defined as in (4.4). In the following we will drop the vector notation. To obtain the full classical action, however, one must use the coherent state path integral with (4.5) as the time evolution operator [51,64]. This involves discretizing total time T into N intervals $\delta t = T/N$ and inserting resolutions of the identity in terms of coherent states at each step. Importantly, the overlap between (the nonorthogonal) coherent states then introduces a nontrivial phase which will give rise to a Wess-Zumino like term. In the time-continuum (classical) limit $\delta t \rightarrow 0$ one then arrives at $Z = \int [\mathbf{d}\mathbf{n}] e^{iS_{\text{cl}}}$ with

$$S_{\text{cl}} = \int dt \sum_{j=1}^J \left(i \langle \dot{\mathbf{n}}_j | \mathbf{n}_j \rangle - \frac{\lambda}{8\pi^2} h_{j,j+1}^{\text{cl}} \right), \quad (4.6)$$

and $\frac{\lambda}{8\pi^2}$ the (effective) spin chain coupling (which originally arises at one-loop from the planar dilatation operator in $\mathcal{N} = 4$ SYM). At this point, the semiclassical coherent states $\mathbf{n}_j(t)$ have become dynamical variables that remain constrained to the hyperboloid.

Passing to the field theory (long chain) continuum limit is achieved by letting the number of sites $J \rightarrow \infty$, while interpreting the site index j as the discretization of a spatial coordinate $\sigma = \frac{2\pi j}{J} \in [0, 2\pi]$. The spin vector then becomes a field: $\mathbf{n}_j(t) \rightarrow \mathbf{n}(t, \sigma)$ and $\mathbf{n}_{j+1} - \mathbf{n}_j \rightarrow \frac{2\pi}{J} \partial_{\sigma} \mathbf{n}$. This gives

$$h_{j,j+1}^{\text{cl}} = \log \left(1 + \frac{(\mathbf{n}_j - \mathbf{n}_{j+1}) \cdot (\mathbf{n}_j - \mathbf{n}_{j+1})}{4} \right) \\ = \frac{\pi^2}{J^2} \partial_{\sigma} \mathbf{n} \cdot \partial_{\sigma} \mathbf{n} + \mathcal{O}(J^{-4}), \quad (4.7)$$

so that taking $J \rightarrow \infty$ while keeping $\tilde{\lambda} = \lambda/J^2$ fixed yields the LL sigma model (for noncompact ferromagnets) [50,64,65]

$$S_{\text{c-LL}} = \frac{J}{2\pi} \int dt \int_0^{2\pi} d\sigma \left(\mathcal{L}_{\text{WZ}}(\mathbf{n}) - \frac{\tilde{\lambda}}{8} \partial_{\sigma} \mathbf{n} \cdot \partial_{\sigma} \mathbf{n} + \mathcal{O}(J^{-1}) \right), \quad (4.8)$$

with the Wess-Zumino term

$$\mathcal{L}_{\text{WZ}}(\mathbf{n}) = i\langle \partial_t \mathbf{n} | \mathbf{n} \rangle = -\frac{1}{2} \int_0^1 dz \varepsilon^{ijk} n_i \partial_z n_j \partial_t n_k, \quad (4.9)$$

and total Hamiltonian

$$H_{\text{C-LL}} = \frac{J\tilde{\lambda}}{16\pi} \int_0^{2\pi} d\sigma [-(\partial_\sigma n_0)^2 + \partial_\sigma n_+ \partial_\sigma n_-], \quad (4.10)$$

which is equivalent to the effective Hamiltonian (2.43) of the string sigma model in the $SL(2, \mathbb{R})$ sector under the identification $\ell_i = n_i$. The fields $\mathbf{n}(t, \sigma)$ satisfy the Poisson brackets

$$\begin{aligned} \{n_0(\sigma), n_\pm(\sigma')\} &= \pm 2in_\pm(\sigma)\delta(\sigma - \sigma'), \\ \{n_+(\sigma), n_-(\sigma')\} &= -4in_0(\sigma)\delta(\sigma - \sigma'), \end{aligned} \quad (4.11)$$

which are the classical counterparts of the $\mathfrak{sl}(2, \mathbb{R})$ commutation relations (3.40) for the spin operators $\{\mathbf{h}, \mathbf{e}, \mathbf{f}\}$ under the relation (4.2) and $\{\cdot, \cdot\} \rightarrow -i[\cdot, \cdot]$. The bulk equations of motion of the LL model (4.8) finally read

$$\begin{aligned} \partial_t n_0 &= \frac{i\tilde{\lambda}}{4} (n_+ \partial_\sigma^2 n_- - n_- \partial_\sigma^2 n_+), \\ \partial_t n_\pm &= \pm \frac{i\tilde{\lambda}}{2} (n_\pm \partial_\sigma^2 n_0 - n_0 \partial_\sigma^2 n_\pm), \end{aligned} \quad (4.12)$$

which must be solved under the constraint (4.4) and, in the untwisted case, periodic boundary conditions, as directly inherited from the periodic spin chain.

2. Jordanian twisted-boundary LL model

We will now take the LL limit of the noncompact Jordanian $\mathfrak{sl}(2, \mathbb{R})$ spin chain. One could in principle consider doing this both in the deformed or the twisted-boundary picture. However, in the infinite-dimensional $s = -1/2$ representation of $\mathfrak{sl}(2, \mathbb{R})$, the Jordanian Drinfel'd twist (3.18) is an infinite series of differential operators and correspondingly it will be highly cumbersome to work with the overcomplete and nonorthonormal coherent states.²⁶ We therefore prefer to work with the twisted-boundary picture, which in addition must also straightforwardly connect to the twisted sigma model of Sec. II C. The LL action from the bulk spin chain then remains undeformed, but we should instead take the classical continuous limit of the twisted-boundary conditions (3.60). A similar route was followed for the nondiagonal DR twist considered in [20].

²⁶This is in contrast to the two-dimensional compact representation, for which the Jordanian twist truncates at first order in ξ , since then $\mathbf{e}^2 = 0$. A deformed LL model obtained from a Jordanian-type Drinfel'd twist in the compact representation was studied in [34].

To evaluate the (quantum) twisted-boundary conditions in the coherent state representation, we also assign a state $|\mathbf{n}_a\rangle$ to the defect space. We can then immediately evaluate (3.60) in the product state $|\mathbf{N}\rangle := |\mathbf{n}_a\rangle \otimes \bigotimes_{j=1}^J |\mathbf{n}_j\rangle$. Defining

$$\begin{aligned} (n_0)_{J+a} &:= -2\langle \mathbf{N} | \mathbf{h}_{J+a} | \mathbf{N} \rangle, & (n_+)_{J+a} &:= -2\langle \mathbf{N} | \mathbf{e}_{J+a} | \mathbf{N} \rangle, \\ (n_-)_{J+a} &:= -2\langle \mathbf{N} | \mathbf{f}_{J+a} | \mathbf{N} \rangle, \end{aligned} \quad (4.13)$$

then gives

$$(n_0)_{J+a} = (n_0)_a, \quad (n_\pm)_{J+a} = (n_\pm)_a (1 + \xi M_{\text{cl}})^{\pm 1}, \quad (4.14)$$

with the classical value M_{cl} of the symmetry operator $\hat{M}$ defined as

$$M_{\text{cl}} := \sum_{j=1}^J \langle \mathbf{n}_j | \mathbf{e}_j | \mathbf{n}_j \rangle = -\frac{1}{2} \sum_{j=1}^J (n_+)_j, \quad (4.15)$$

where we used that, by construction, the coherent states satisfy $\langle \mathbf{n}_j | \mathbf{e}_j^s = (-\frac{n_+}{2})^s \langle \mathbf{n}_j |$ for any power s . Recall furthermore that $\langle \mathbf{n} | \mathbf{n} \rangle = 1$ at each site. In the classical and (two-dimensional) field theory continuum limit this then finally becomes

$$M_{\text{C-LL}} := -\frac{J}{4\pi} \int_0^{2\pi} d\sigma' n_+(t, \sigma'), \quad (4.16)$$

so that the LL twisted-boundary conditions read

$$n_0(t, 2\pi) = n_0(t, 0), \quad n_\pm(t, 2\pi) = n_\pm(t, 0) (1 + \xi M_{\text{C-LL}})^{\pm 1}. \quad (4.17)$$

Since $\hat{M}$ was a residual symmetry of the spin chain [recall Eq. (3.52) or (3.55)] and governs the twisted-boundary conditions, it makes sense to also demand that solutions to the LL equations of motion be such that $M_{\text{C-LL}}$ remains conserved. This will in fact require one to study only a subsector of solutions: demanding $\partial_t M_{\text{C-LL}} = 0$ for $\xi M_{\text{C-LL}} \neq 0$ introduces the following additional constraint

$$n_0(t, 0) \partial_\sigma n_+(t, 0) - n_+(0) \partial_\sigma n_0(0) = 0. \quad (4.18)$$

It seems that the continuum limit thus spoiled the manifest M conservation of this charge. We believe it would be interesting to understand how to restore it in the LL theory, and expect this to be related to appropriate global treatment of the hyperboloid fields and WZ term. This may also provide an understanding for the apparent loss of $\mathbf{e}$ symmetry of the Jordanian string sigma model in the twisted picture, cf. footnote 9. We will however find that for the simplest possible ground state deformations, (4.18) is trivially satisfied.

B. Solutions and energy shift

We are now interested in twisted solutions of the equations of motions (4.12), subject to the hyperbolic constraint (4.4), with fixed (minimal) energy and momentum $M_{\text{C-LL}}$.

In the undeformed, periodic theory, the simplest ground state is the constant configuration $n_0 = 1$ and $n_{\pm} = 0$, which trivially has zero energy $H_{\text{C-LL}} = 0$ and momentum $M_{\text{C-LL}} = 0$. However, we can also consider ground state configurations carrying nonzero momentum $M_{\text{C-LL}} = M$. To maintain vanishing energy while allowing nonzero M , we take a constant configuration with $n_+ = -\frac{2M}{J}$ as required by (4.16). Imposing the hyperbolic constraint (4.4) then leads to two distinct possibilities (up to overall factors). Firstly, we can have

$$n_0 = 1, \quad n_+ = -\frac{2M}{J}, \quad n_- = 0, \quad (4.19)$$

which has a smooth $M \rightarrow 0$ limit to the trivial vacuum. This is in contrast to a second solution with momentum M , given by

$$n_0 = 0, \quad n_+ = -\frac{2M}{J}, \quad n_- = \frac{J}{2M}. \quad (4.20)$$

As we will see, the solution (4.20) is the relevant undeformed solution in the case of the Jordanian deformation, while (4.19) was the appropriate undeformed solution in the dipole-deformed case [20]. In both cases, the fields $n_{\pm}$ are generically complex-valued for real M , necessitating an analytic continuation of the LL model in which n_+ and n_- are treated as independent complex variables constrained to lie on the complexified hyperboloid. Despite the presence of complex fields, the energy and consequently anomalous scaling dimensions remain real (zero in this case).

Now we turn to the Jordanian twisted LL model and look for deformations of the ground state solutions with nonzero momentum M , which are subjected to the twisted-boundary conditions (4.17). For this purpose, we consider the ansatz $n_0(t, \sigma) = c \in \mathbb{R}$ constant, which is justified by the solutions (4.19) and (4.20) and the fact that n_0 remains periodic in the twisted model. Demanding nonzero momentum, and thus nonzero n_+ , we can furthermore use the hyperboloid constraint (4.4) to remove n_- from the system of equations as $n_- = \frac{c^2-1}{n_+}$. The equation of motion for n_0 (4.12) then becomes

$$(c^2 - 1)((\partial_{\sigma} n_+)^2 - n_+ \partial_{\sigma}^2 n_+) = 0, \quad (4.21)$$

which is solved by $c = 1$ or

$$n_+(\sigma, t) = e^{\sigma \alpha_1(t)} \alpha_2(t), \quad (4.22)$$

where $\alpha_{1,2}(t)$ are two *a priori* arbitrary functions of t . In the case $c = 1$ there is no consistent twisted solution of the full

system of equations of motion as well as (4.16) and the constraint (4.18). We thus continue with (4.22). Imposing the twisted-boundary conditions for n_+ (4.17) then fixes

$$\alpha_1(t) = \frac{\log(1 + \xi M_{\text{C-LL}})}{2\pi} + im, \quad m \in \mathbb{Z}, \quad (4.23)$$

while requiring fixed $M_{\text{C-LL}} = M$ for the momentum from (4.16) imposes

$$\alpha_2(t) = -\frac{2 \log(1 + \xi M) + 4\pi im}{\xi J}, \quad (4.24)$$

such that the solution for n_+ has no t dependence. To have a well-defined $\xi \rightarrow 0$ limit in which we recover $n_+ = -\frac{2M}{J}$ as above we furthermore need to demand $m = 0$. Inserting the resulting solution for n_+ in the remaining equations of motions (4.12) for nonzero ξ and M then demands $c = 0$. Ultimately we thus have

$$\begin{aligned} n_0(t, \sigma) &= 0, & n_+(t, \sigma) &= -\frac{2 \log(1 + \xi M)}{\xi J} (1 + \xi M)^{\sigma/2\pi}, \\ n_-(t, \sigma) &= -\frac{1}{n_+(t, \sigma)}, \end{aligned} \quad (4.25)$$

coinciding with (4.20) in the undeformed limit. The constraint (4.18) and remaining twisted-boundary conditions for n_0 and n_- are furthermore trivially satisfied. Evaluating (4.10) on this solution then gives the following nontrivial energy shift of the ground state

$$\begin{aligned} H_{\text{C-LL}} &= \frac{J\tilde{\lambda}}{32\pi^2} \log(1 + \xi M)^2 = \frac{J\tilde{\lambda}}{32\pi^2} \log\left(1 + \frac{2\pi\eta}{\tilde{\lambda}^{1/2}J} M\right)^2 \\ &= \frac{\eta^2 M^2}{8J} + \mathcal{O}(J^{-2}), \end{aligned} \quad (4.26)$$

where we employed the identification (2.20) and expanded the result at large J but fixed η , M and $\tilde{\lambda}$. This matches precisely the classical string theory result given in (2.44) corresponding to the energy shift of the twisted scaling dimension (2.36) expanded at large J but fixed η , M . Here we thus have a nontrivial overlap between the energy of the sigma-model BMN-like solution and the twisted-boundary spin chain description in the coherent state and supergravity approximation.

It is interesting to note that the undeformed ground state configuration does not have a well-defined $M \rightarrow 0$ limit. Related observations have been found at the Jordanian sigma-model side, where no solutions were found with vanishing M . Furthermore, the corresponding spectral curve of the BMN-like solution has two branch cuts whose size is proportional to ξM [12], and attempts to find no-cut but twisted solutions did not materialize.

V. SPECTRUM OF THE $J=2$ TWISTED SPIN CHAIN

In this Section, we study the spectral problem of the Jordanian-twisted $\mathfrak{sl}(2, \mathbb{R})$ spin chain at length $J=2$ by directly diagonalizing the transfer matrix and using Baxter's TQ equation.

The traditional method to determine the spectrum is the ABA [66], which involves constructing the monodromy matrix using a two-dimensional ($s = +1/2$) auxiliary space representation of $\mathfrak{sl}(2, \mathbb{R})$. From its expression, one derives the Bethe equations that encode the spectrum of the corresponding transfer matrix. While the Hamiltonian of the noncompact $\mathfrak{sl}(2, \mathbb{R})$ spin chain will not be included in its spectral expansion, the eigenvectors of the transfer matrix constructed from the ABA can still be shown to diagonalize the Hamiltonian [59,67]. In addition, the transfer matrix eigenvalues can be used as input to Baxter's equation, from which physical quantities such as the energy can be extracted, see, e.g., [67].

In our case, however, we cannot directly apply the ABA. A key aspect of this work is to take the noncompact spin chain in an eigenstate of the residual $\hat{M}$ symmetry operator. In this basis, a suitable reference (or “pseudovacuum”) state required for the ABA is not available. Without the ABA, we will instead try to directly solve the eigenvalue problem of the transfer matrix, aiming to uncover at least some of the “integrable data.” For this purpose, we restrict to the small $J=2$ chain: Since the physical sites are in a noncompact representation, the $\mathfrak{sl}(2, \mathbb{R})$ generators act as differential operators, and the eigenvalue problem becomes a PDE with as many independent variables as the number of sites of the spin chain. Although this rapidly becomes intractable for longer chains,²⁷ it remains manageable for $J=2$.

Working in an $\hat{M}$ eigenbasis is in contrast to standard approaches, where one works in an eigenstate of the Cartan of $\mathfrak{sl}(2, \mathbb{R})$. In this Section, we will therefore first show how to obtain the transfer matrix eigenvalue of the undeformed ($J=2$) chain in the $\hat{M}$ basis instead of the $\Delta(\mathbf{h})$ basis. This will provide useful structure and a benchmark for the Jordanian-twisted case, to which we then apply the identical procedure. Subsequently, we will substitute its deformed transfer-eigenvalue into Baxter's expression for the ground state energy.

Finally, note that one might also consider a more hands-on approach and directly diagonalize the deformed or twisted-boundary Hamiltonian of, respectively, (3.27) and (3.28). However, this is not immediately straightforward in the $\hat{M}$ eigenbasis, since the $\Delta^{J-1}(\mathbf{e})$ operator does not explicitly appear in either Hamiltonian. We will report more on this aspect elsewhere [68].

²⁷This difficulty was one of the motivations behind the construction of the quantum separation of variables method for noncompact $\mathfrak{sl}(2, \mathbb{R})$ spin chains [59].

A. Eigenvalue problem of the undeformed chain

As usual, to construct the monodromy and transfer matrix, we take the finite-dimensional $s = +1/2$ representation of $\mathfrak{sl}(2, \mathbb{R})$ for the auxiliary space $\mathbf{a}$. In this case, the generators satisfying the commutation relations (3.40), can be represented as in (A2). Consider then the $J=2$ monodromy matrix built from the R matrix (or Lax matrix) defined as

$$R_{aj}(u) := u1_{aj} + 2iX_a \cdot S_j = \begin{pmatrix} u + ih_j & -if_j \\ ie_j & u - ih_j \end{pmatrix}, \quad (5.1)$$

where the dot product is defined through the Cartan-Killing form, the operators $X_a = \{\mathbf{h}_a, \mathbf{e}_a, \mathbf{f}_a\}$ are as in (A2), while $S_j = \{\mathbf{h}_j, \mathbf{e}_j, \mathbf{f}_j\}$ are the physical spin operators at site j in the $s = -1/2$ representation, which we represent as²⁸

$$\mathbf{e}_j = -\partial_j, \quad \mathbf{h}_j = -z_j \partial_j - \frac{1}{2}, \quad \mathbf{f}_j = -z_j^2 \partial_j - z_j, \quad (5.2)$$

with z_j the representation space parameters at each site $j = 1, 2$. In general, one then writes

$$T_a(u) = R_{a2}(u)R_{a1}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

$$\hat{\tau}(u) = \text{Tr}_a(T_a(u)) = A(u) + D(u), \quad (5.3)$$

where A, B, C, D are built out of the differential spin S operators. In the undeformed case, we have

$$\hat{\tau}(u) = 2u^2 + \mathbf{e}_2 \mathbf{f}_1 + \mathbf{f}_2 \mathbf{e}_1 - 2\mathbf{h}_2 \mathbf{h}_1. \quad (5.4)$$

The differential operator structure of the transfer matrix implies that we should work within a functional representation of $\mathfrak{sl}(2, \mathbb{R})$, where states are realized as functions of the continuous site parameters z_1 and z_2 . As per the illustration, the undeformed case with $s = -1/2$, where one can use an eigenbasis of the total Cartan operator $\Delta(\mathbf{h}) = \mathbf{h}_1 + \mathbf{h}_2$, there is a highest-weight module with basis states

$$|k_1, k_2\rangle := z_1^{k_1} z_2^{k_2}, \quad k_1, k_2 \in \mathbb{N}, \quad (5.5)$$

with eigenvalue $-1 - k_1 - k_2$ and with $|0, 0\rangle$ annihilated by $\Delta(\mathbf{e})$. These are thus regular polynomials in z_j that have a discrete spectrum under $\Delta(\mathbf{h})$. Note that they are also easily generalizable to longer chains. One can then straightforwardly apply the ABA on the above basis. In fact for $J=2$, the state $|0, 0\rangle$ is a suitable pseudovacuum, satisfying $C(u)|0, 0\rangle = 0$.²⁹ Since $\hat{\tau}(u)$ commutes with $\Delta(\mathbf{h})$, one can diagonalize it sector by sector in $\Delta(\mathbf{h})$ eigenvalues [31].

²⁸This representation has a quadratic Casimir (A1) equal to $-\frac{1}{4}$.

²⁹Under automorphisms of the algebra, this is of course equivalent to a $B(u)$ annihilation.

Under the Jordanian twist, however, this eigensystem becomes nondiagonalizable [28], as $\Delta(\mathfrak{h})$ is not a symmetry, and the weight structure is obscured in terms of generalized eigenvectors. The above states (5.5) are obviously not eigenstates of $\hat{M} = \mathbf{e}_1 + \mathbf{e}_2$, which is a raising operator with respect to $\Delta(\mathfrak{h})$. In certain deformations (e.g., the Abelian dipole [20] or Jordanian twists), $\hat{M}$ is the only conserved quantity in the $\mathfrak{sl}(2, \mathbb{R})$ sector, making it natural and advantageous to diagonalize the transfer matrix in an $\hat{M}$ eigenbasis from the start.

Hence, we instead consider eigenfunctions of the (non-compact) differential operator $\hat{M} = \mathbf{e}_1 + \mathbf{e}_2 = -\partial_1 - \partial_2$, which take the form

$$\Psi_M(z_1, z_2) := e^{-Mx}g(z), \quad \text{with} \quad x := \frac{z_1 + z_2}{2},$$

$$\text{and} \quad z := z_2 - z_1, \quad (5.6)$$

and have eigenvalue M . As we will see, these functional states will generally not be polynomial. Eigenstates of a root generator such as here naturally arise in the discrete series of $\mathfrak{sl}(2, R)$, see, e.g., [62,69], but they can exist also in the principal series.³⁰

One can check that there are no consistent and nontrivial solutions of $C(u)\Psi_M = 0$ (or $B(u)\Psi_M = 0$) in the Hilbert space of eigenfunctions, so that the ABA is not straightforwardly applicable. Instead we aim to directly solve

$$\hat{\tau}(u)\Psi_M = \tau(u)\Psi_M, \quad (5.7)$$

with $\tau(u)$ the eigenvalue of the transfer matrix. Since $\hat{\tau}(u)$ is a polynomial in the spectral parameter u , we can rewrite this as an eigenvalue equation for each term in the u expansion (i.e., three equations at $J = 2$). Writing $\tau(u) = \alpha u^2 + \beta u + \chi$ we then immediately see from (5.4) that $\alpha = 2$ and $\beta = 0$, while the constant u^0 term yields the following ODE for $g(z)$

$$4z^2g''(z) + 8zg'(z) + (4\chi + 2 - M^2z^2)g(z) = 0. \quad (5.8)$$

Its general solution is given by

$$g(z) = c_1 \text{SJ}\left(-\frac{1}{2}(1 + i\sqrt{1 + 4\chi}); -\frac{i}{2}Mz\right) + c_2 \text{SY}\left(-\frac{1}{2}(1 + i\sqrt{1 + 4\chi}); -\frac{i}{2}Mz\right), \quad (5.9)$$

where $\text{SJ}(n; y)$ and $\text{SY}(n; y)$ are the spherical Bessel functions of the first and second kind and $c_{1,2}$ are

³⁰For the eigenvalue problem, it will be enough to simply consider (5.6), but it would be interesting to have a more systematic identification of the underlying module structure to clarify the spectral behavior of the twisted model.

integration constants. To ensure the well definedness of the eigenfunction Ψ_M and the action of spin operators, we further impose regularity and single valuedness of Ψ_M across the representation space. The solution involving SY diverges at $z = 0$ and we thus excluded it, setting $c_2 = 0$ (and effectively one can then also set $c_1 = 1$). The remaining function $\text{SJ}(n; y)$ is regular at $y = 0$, but it is single valued only when $n \in \mathbb{N}$. Imposing this condition then leads to the quantization

$$\chi = -\frac{1}{2} - n(n + 1), \quad n \in \mathbb{N}. \quad (5.10)$$

Since the transfer matrix includes conserved charges in its spectral expansion, the eigenvalue χ is a conserved charge associated to the u^0 operator in $\hat{\tau}(u)$. Importantly, this operator can be written as

$$\hat{q}_2 := \mathbf{e}_2 \mathbf{f}_1 + \mathbf{f}_2 \mathbf{e}_1 - 2\mathbf{h}_2 \mathbf{h}_1 = -\frac{1}{2} - \mathbb{C}_2, \quad (5.11)$$

where $\mathbb{C}_2 := \Delta(\mathbf{S}^2) = \Delta(\mathbf{S}) \cdot \Delta(\mathbf{S})$ is the two-site Casimir and we used $s = -1/2$. The integer n appearing in the solution (5.10) is thus simply the quantization of total spin, $\mathbb{C}_2 = n(n + 1)$. Recalling the expressions of the Hamiltonian of the undeformed chain, given in (3.48), and using the fact that on each module $\mathbb{C}_2 = \mathbb{J}(\mathbb{J} + 1) = n(n + 1)$ one has $\mathbb{J} = n$, the ground state of the spin chain clearly corresponds to $n = 0$, for which $\chi = -\frac{1}{2}$. The corresponding eigenfunction with nonvanishing M is

$$\Psi_{\text{GS},M}(z) = e^{-Mx} \text{SJ}\left(0; -\frac{i}{2}Mz\right) = \frac{2}{Mz} e^{-Mx} \sinh\left(\frac{Mz}{2}\right). \quad (5.12)$$

We can expand it around $z = 0$ to find

$$\Psi_{\text{GS},M}(z) = e^{-Mx} \left(1 + \frac{M^2 z^2}{24} + \frac{M^4 z^4}{1920} + \frac{M^6 z^6}{322560} + \mathcal{O}(z^7)\right), \quad (5.13)$$

which will provide an important consistency check for the Jordanian case that we discuss now.

B. Eigenvalue problem of Jordanian-twisted spin chain

We now proceed analogously for the Jordanian twisted spin chain. It will be convenient to do so in the twisted-boundary picture, where the nonlocal similarity transformation $\Omega_{1\dots J}$ has been absorbed in a rotation of the Hilbert space. The corresponding twisted monodromy is given in (3.58), with importantly $\hat{M}$ acting initially on the states. We write

$$T_a^\star(u) = \begin{pmatrix} 1 & \xi\Delta(h) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \begin{pmatrix} (1 + \xi\hat{M})^{-1/2} & 0 \\ 0 & (1 + \xi\hat{M})^{1/2} \end{pmatrix}, \quad (5.14)$$

and

$$\hat{\tau}_\xi(u) := \text{Tr} T_a^\star(u) = (A(u) + \xi\Delta(h)C(u))(1 + \xi\hat{M})^{-1/2} + D(u)(1 + \xi\hat{M})^{1/2}, \quad (5.15)$$

with A, B, C, D as obtained from (5.1) for two sites, as before. We then again consider the corresponding eigenvalue problem on the $\hat{M}$ eigenstates Ψ_M defined in (5.6). The initial action of $(1 + \xi\hat{M})^{\pm 1/2}$ can be immediately replaced by the constants $(1 + \xi M)^{\pm 1/2}$. As before, we then expand both the twisted transfer matrix and its eigenvalue in the spectral parameter and solve the eigenvalue equation order by order in u . Writing $\tau_\xi(u) = \alpha_\xi u^2 + \beta_\xi u + \chi_\xi$, one then finds at $\mathcal{O}(u^2)$ an algebraic equation that immediately sets

$$\alpha_\xi = \frac{2 + \xi M}{\sqrt{1 + \xi M}}. \quad (5.16)$$

The $\mathcal{O}(u^2)$ term of the transfer matrix eigenvalue thus clearly gets a ξM correction. On the other hand, there is no $\mathcal{O}(u^1)$ term, and thus $\beta_\xi = 0$ as in the undeformed case. Finally, the $\mathcal{O}(u^0)$ term yields a differential equation for $g(z)$ of the form

$$p_1(z)g(z) + p_2(z)g'(z) + p_3(z)g''(z) - 8\xi z^2 g'''(z) = 0, \quad (5.17)$$

where

$$\begin{aligned} p_1(z) &:= 4 + M(2\xi - Mz(-2\xi + z(2 + \xi M))) \\ &\quad + 8\sqrt{1 + \xi M}\chi_\xi, \\ p_2(z) &:= 2(8z + (-4 + Mz(4 + Mz))\xi), \\ p_3(z) &:= 4z((2z - 6\xi + Mz\xi)). \end{aligned} \quad (5.18)$$

For nonzero ξ , this differential equation is significantly more complicated than in the undeformed case: it is now a third-order ODE, rather than second-order, due to the additional action of $\Delta(h)$ induced by the Jordanian twist. The ODE has a singularity of the second kind at $z = 0$ [that is, a z^2 coefficient in front of the $g'''(z)$ term], and exhibits a turning point behavior around $\xi = 0$, where the order of the ODE changes. In Appendix B, we give a general explanation of how to “perturbatively” construct solutions to such singular ODEs as formal series around $z = 0$, using our case as an illustrative example. As it turns out, the form of the solution can be written as

$$g(z) = g_1(z) + \log(z)g_2(z) + \log^2(z)g_3(z), \quad (5.19)$$

with $g_{1,2,3}(z)$ regular expansions around $z = 0$, whose coefficients are determined through recurrence relations found from solving Eq. (B11) with (B12), and with the values $g_{1,2,3}(0)$ playing the role of integration constants. We have computed the expansion coefficients explicitly until $\mathcal{O}(z^8)$ in *Mathematica*. As in the undeformed case, we again need to impose regularity and single-valuedness at $z = 0$, such that we have to set $g_2(0) = g_3(0) = 0$. Recursively, we can then immediately conclude that this imposes $g_2(z) = g_3(z) = 0$. We are thus left with a single regular expansion around $z = 0$, given by $g_1(z)$, to which we can apply the usual Frobenius methods. We will set the last integration constant to $g_1(0) = 1$.

At this stage, we already observe an interesting turning point behavior in the differential equation: In the undeformed case, the indicial equation that determines the leading power p of the solution to (5.8) in the context of Frobenius’s method reads

$$4 + 8p(1 + p) + 8\chi = 0. \quad (5.20)$$

Requiring regularity corresponds to demanding $p \in \mathbb{N}$, which precisely coincides with the quantization condition for χ given in (5.10). In other words, the power p coincides with the energy excitation number n above the ground state. However, in the deformed case, the indicial equation for (5.17) reads

$$-8\xi p^3 = 0, \quad (5.21)$$

where p will be the leading exponent of $g_1(z)$ in (5.19). For generic and fixed ξ , the only admissible Frobenius index is $p = 0$, meaning that the deformed eigenfunction $g_1(z)$ must behave as a constant around $z = 0$, and analytic continuation to excited states with $p > 0$ is obstructed in this analysis, even as $\xi \rightarrow 0$. In other words, within the Frobenius treatment of the eigenvalue equation at fixed ξ , we find that the Jordanian deformation connects smoothly only to the undeformed *ground state* solution, and not to the undeformed excited states of higher total spin $n = p > 0$. This is an immediate consequence of the discrete order change of the ODE induced by the additional action of the twist. While our analysis thus suggests a lifting of the excited states after the twist, this should more properly be

interpreted as a limitation of the perturbative Frobenius method for turning point problems, and the lack of a nonperturbative closed form analysis of the ODE. We will give a few further comments on these points in a discussion at the end of this Section.

Let us now solve along the lines of Frobenius, considering the ansatz

$$g_1(z) = \sum_{n=0}^{\infty} g_n z^n, \quad (5.22)$$

and substitute it into the ODE (5.17). After some straightforward algebra, we then obtain the recurrence relation

$$g_n = \frac{1}{8\xi n^3} ((2(2n^2 - 2n + 1)(2 + \xi M) + 8\sqrt{1 + \xi M}\chi_\xi)g_{n-1} + 2\xi(n-1)M^2g_{n-2} - (M^2(2 + \xi M))g_{n-3}), \quad (5.23)$$

which is induced by $g_0 = 1$ and $g_{-1} = g_{-2} = 0$. The first three terms explicitly read

$$g_1(z) = 1 + \frac{z}{4\xi} \left(2 + \xi M + 4\sqrt{1 + \xi M}\chi_\xi \right) + \frac{z^2}{128\xi^2} (9\xi^2 M^2 + 24\chi_\xi \sqrt{1 + \xi M}(2 + \xi M) + (1 + \xi M)(20 + 16\chi_\xi^2)) + \mathcal{O}(z^3), \quad (5.24)$$

and one can in principle obtain arbitrarily many expansion coefficients recursively from (5.23).

We have thus obtained a perturbative solution to the differential equation as a regular expansion around $z = 0$. However, in contrast to the undeformed case, we did not yet obtain a constraint on the eigenvalue χ_ξ . Such a constraint arises by requiring the solution to have a smooth limit as $\xi \rightarrow 0$: Since ξ plays the role of a deformation parameter, we must also demand that $g_1(z)$ remains regular in this limit.³¹ However, looking at the expansion (5.24), we observe that it is effectively an expansion in z/ξ , with coefficients depending polynomially on χ_ξ . As a result, regularity at $\xi = 0$ requires that χ_ξ itself admits a regular expansion in ξ , of the form

$$\chi_\xi = \sum_{n=0}^{\infty} c_n \xi^n, \quad (5.25)$$

with the coefficients c_n fixed order by order in an expansion of small ξ .

Consider therefore the $\mathcal{O}(z)$ term in (5.24), whose expansion around $\xi = 0$ reads

$$\begin{aligned} & \frac{1}{4\xi} \left(2 + \xi M + 4\sqrt{1 + \xi M}\chi_\xi \right) \\ &= \frac{2 + 4c_0}{4\xi} + \frac{M(1 + 2c_0) + 4c_1}{4} + \mathcal{O}(\xi). \end{aligned} \quad (5.26)$$

In order for this expression to be regular at $\xi = 0$, the pole must vanish, which imposes $c_0 = -\frac{1}{2}$. This corresponds precisely to the undeformed ground state solution, i.e., (5.10) at $n = 0$, as expected from the analysis of the indicial equation, Eq. (5.20).

Similarly, expanding the $\mathcal{O}(z^2)$ term of (5.24) gives

$$\frac{c_1}{4\xi} + \frac{3M^2 + 16Mc_1 + 8c_1^2 + 16c_2}{64} + \mathcal{O}(\xi), \quad (5.27)$$

which sets $c_1 = 0$. One can proceed analogously, order by order in z , to fix (in principle) arbitrary coefficients of χ_ξ . Explicitly, using the solution of $g_1(z)$ to $\mathcal{O}(z^8)$, we can fix χ_ξ to $\mathcal{O}(\xi^7)$, giving

$$\begin{aligned} \chi_\xi = & -\frac{1}{2} - \frac{\xi^2 M^2}{48} + \frac{\xi^3 M^3}{48} - \frac{667\xi^4 M^4}{34560} + \frac{307\xi^5 M^5}{17280} \\ & - \frac{286211\xi^6 M^6}{17418240} + \mathcal{O}(\xi^7). \end{aligned} \quad (5.28)$$

Notice in particular that this is an expansion in ξM . This can in fact be expected from the string relation (2.36). In particular, for eigenstates with vanishing momenta M , the spectrum will thus not change. The same behavior is observed in [20]. Finally, inserting the expansion for χ_ξ into the regular solution (5.24) yields

$$\begin{aligned} g_1(z) = & 1 + \frac{M^2 z^2}{24} + \frac{M^4 z^4}{1920} + \frac{M^6 z^6}{322560} + \mathcal{O}(\xi^0, z^7) \\ & + \xi M \left(\frac{Mz}{24} + \frac{M^3 z^3}{960} + \frac{M^5 z^5}{107520} + \mathcal{O}(\xi^0, z^6) \right) \\ & + \xi^2 M^2 \left(-\frac{Mz}{48} + \frac{M^2 z^2}{1080} - \frac{M^3 z^3}{1920} + \frac{M^4 z^4}{60480} + \mathcal{O}(\xi^0, z^5) \right) \\ & + \mathcal{O}(\xi^3). \end{aligned} \quad (5.29)$$

This expansion is organized as a double expansion in ξ and z , where the highest power of z at each order in ξ is constrained by the structure of the original $g_1(z)$ expansion in z/ξ and the imposed regularity at $\xi = 0$. For example, the $\mathcal{O}(\xi^0, z^7)$ term depends on the as-yet undetermined coefficient c_7 of (5.25), and similarly for higher order terms.

Crucially, the $\mathcal{O}(\xi^0)$ part of (5.29) exactly matches the expansion of the undeformed ground state eigenfunction (5.13). This provides a nontrivial consistency check because the $\mathcal{O}(\xi^0)$ contributions in (5.29) arise cumulatively from all orders in the expansion of χ_ξ , such as the constant terms in (5.26) and (5.27). In particular, the

³¹We have verified that swapping the order of demanding regularity in z and ξ did not influence our conclusions.

coefficients c_n in (5.25) were not chosen to reproduce the undeformed solution—rather, they are uniquely fixed by the requirement of regularity at $\xi = 0$.

In summary, we have solved the eigenvalue problem of the $J = 2$ Jordanian spin chain perturbatively in the twisted-boundary picture, with the $\mathcal{O}(u^0)$ term of the transfer matrix eigenvalue as an expansion in ξM

$$\tau_\xi(u) = u^2 \frac{2 + \xi M}{\sqrt{1 + \xi M}} - \frac{1}{2} - \frac{\xi^2 M^2}{48} + \frac{\xi^3 M^3}{48} - \frac{667\xi^4 M^4}{34560} + \frac{307\xi^5 M^5}{17280} + \mathcal{O}(\xi^6), \quad (5.30)$$

as well as the eigenfunction

$$\Psi_M(z_1, z_2) = e^{-M \frac{z_1 + z_2}{2}} g_1(z_2 - z_1), \quad (5.31)$$

with $g_1(z)$ given as an expansion in ξM and z in (5.29). Both expansions can be computed in principle to arbitrary order, as they all result from the recurrence relations (5.23), but the complete solution should be understood as a formal series (an infinite-degree polynomial). We however do not exclude the existence of a closed form solution.

As in the undeformed case, the u^0 coefficient of the transfer matrix defines an operator that commutes with the Hamiltonian. In the twisted setting, we denote this operator as $\hat{q}_{2,\xi}$, with eigenvalue χ_ξ (5.28) on the $\hat{M}$ eigenstates. Its explicit form is

$$\begin{aligned} \hat{q}_{2,\xi} = & (\mathbf{e}_1 \mathbf{f}_2 - \mathbf{h}_1 \mathbf{h}_2 + \xi \Delta(\mathbf{h})(\mathbf{e}_1 \mathbf{h}_2 + \mathbf{h}_1 \mathbf{e}_2)) \\ & \times (1 + \xi \Delta(\mathbf{e}))^{-1/2} + (\mathbf{e}_2 \mathbf{f}_1 - \mathbf{h}_1 \mathbf{h}_2)(1 + \xi \Delta(\mathbf{e}))^{1/2}, \end{aligned} \quad (5.32)$$

it reduces to $\hat{q}_2 = -\frac{1}{2} - \mathbb{C}_2$ (5.11) in the limit $\xi \rightarrow 0$, with $\mathbb{C}_2$ the two-site Casimir. One may thus think of it as containing a “twisted two-site Casimir” that knows about the boundary conditions, although its precise physical or algebraic interpretation is not immediately obvious to us.

1. Discussion

We conclude with a discussion of our results in this Section and the previous one. First let us relate to the recent work [28], which studied the spectral problem of the same model considered here. To match our setup with theirs, one must perform the algebra automorphism $\mathbf{h} \rightarrow -J^3$, $\mathbf{e} \rightarrow J^-$, $\mathbf{f} \rightarrow J^+$ and also $\xi \rightarrow 2\xi$. Under this transformation, the lowering operator $\Delta^{J-1}(J^-)$ becomes a residual symmetry. One can then consider its eigenstates in the $J = 2$ sector, in which case (5.6) is modified to

$$\Psi_M = \frac{1}{z_1 z_2} \exp^{\frac{M}{2}(\frac{1}{z_1} + \frac{1}{z_2})} g\left(\frac{1}{z_2} - \frac{1}{z_1}\right), \quad (5.33)$$

where the function $g(z)$ with $z = \frac{1}{z_2} - \frac{1}{z_1}$ satisfies the same differential equation (5.19), with the same deformed eigenvalue (5.28).

In [28], the ABA for the Jordanian-twisted $J = 2$ spin chain was applied in the deformed picture by considering the usual $\mathfrak{sl}(2, \mathbb{R})$ states as defined in (5.5) (but in the lowest-weight equivalent representation). Although $\Delta(\mathbf{h})$ is not a residual symmetry of the chain, it was observed that the maximal value of $\Delta(\mathbf{h})$ within a subsector is preserved. This is due to the form of the deformed transfer matrix, which is constructed using a twisted R matrix, where the Jordanian twist acts with the lowering operator J^- first on the physical chain. In any given subsector with a fixed maximal value of $\Delta(\mathbf{h})$, it was then found that the deformation contributes only to the strictly upper-triangular part of the transfer matrix, and that its spectrum is not deformed. Similar arguments led to the same conclusion for the Hamiltonian of the spin chain.

In contrast, our analysis finds that the eigenvalue of the transfer matrix does depend on ξ , showing that the Jordanian twist *does* affect the spectrum. Crucially, our computation differs from the one of [28] by a key aspect mentioned throughout, i.e., we work with $\hat{M} = \Delta(\mathbf{e})$ eigenstates of a residual symmetry. For $M \neq 0$, these states exist only in the noncompact representation of $\mathfrak{sl}(2, \mathbb{R})$, and do not admit a finite (maximal) value of $\Delta(\mathbf{h})$. As a result, we cannot restrict the eigenvalue problem to an algebraic problem with finite size matrices, because non-trivial $\Delta^{J-1}(J^-)$ eigenstates (5.33) are formal infinite series over the Fock basis states $|k_1, k_2\rangle$ as in (5.5) (but in the lowest-weight equivalent representation). The only $\Delta^{J-1}(J^-)$ eigenstates that are finite linear combinations of $|k_1, k_2\rangle$ can only be those with an $M = 0$ eigenvalue: Since $\Delta^{J-1}(J^-)$ is a lowering operator, one could consider

$$\Psi = \sum_{k_1=0, k_2=0}^{N_{\max}} c_{k_1, k_2} |k_1, k_2\rangle, \quad (5.34)$$

with some coefficients c_{k_1, k_2} and a finite upper cutoff $N_{\max}$, but acting with J^- will of course never give the same states back. Hence, demanding $\Delta^{J-1}(J^-)\Psi = M\Psi$ immediately forces $M = 0$ or $\Psi = 0$. A trivial example is the lowest-weight $|0, 0\rangle$, which trivially has $M = 0$, as well as $|0, 1\rangle - |1, 0\rangle$, etc. As we found in this Section, the eigenvalue of the transfer matrix (5.30) is crucially a function of the combination ξM , and thus states in a sector of vanishing charge $M = 0$ do not exhibit a deformed spectrum.

Similar conclusions were reached in the case of the nondiagonal Abelian twist (the dipole deformation) of [20]. Although the Jordanian twist considered here is more intricate due to its non-Abelian nature, both deformations preserve the same residual symmetries in the $\mathfrak{sl}(2, \mathbb{R})$ subsector. While the explicit form of the eigenstate and

eigenvalue thus differ, the dependence on the combination ξM is common. As mentioned, this is consistent with the classical string relation (2.36) (which is in fact the same in both models). Conversely, a truncated analysis at fixed maximal value of $\Delta(h)$ would also yield an undeformed spectrum in the dipole case.

Nevertheless, it remains unclear whether the Jordanian twisted Hamiltonian is fully diagonalizable over the complete Hilbert space. In the undeformed case, the spectrum is organized via the two-site Casimir and by working in either the $\hat{M}$ or $\Delta(h)$ eigenbases, both of which eventually label the total spin modules. In the deformed case, however, only $\hat{M}$ remains, while C_2 and $\Delta(h)$ are nonconserved.³² Within ODE analysis the Jordanian deformation connects smoothly only to the undeformed ground state. No Frobenius expansion led to excited (higher-spin) solutions. We believe that obtaining a closed form solution to the eigenvalue differential equation would help to understand whether we can recover the full spectrum.³³ While we do obtain a consistent subsector of eigenfunctions with $\hat{M}$ charge, it is in particular not clear whether these form a complete basis for the Hilbert space or if nondiagonalizable sectors (Jordan blocks) persist. It would thus also be valuable to investigate whether an alternative basis or algebraic perspective can reincorporate the “missing” excitations (like $|0, 1\rangle + |1, 0\rangle$, etc.) in the deformed setting.

All in all, it is obvious that the undeformed limit is a subtle point.³⁴ Interestingly, a similar discontinuity was in fact observed at the sigma-model side, where the diagonalization of the monodromy matrix of the Jordanian sigma model does not connect smoothly to the undeformed case due to a change of order in the asymptotic expansion of the quasimomenta [11,12].

To end, let us mention that it is not straightforward to extract the energy spectrum of the spin chain from the twisted Hamiltonian to arbitrarily high order in ξ —in fact, its action on the $\hat{M}$ eigenstates is complicated by the nonlocal intertwiner present in its boundary term. In the dipole deformation of [20] a way out of this was to use the light-ray representation of [70] extracted from the corresponding noncommutative dipole deformation of $N = 4$ SYM obtained via the Seiberg-Witten map. Unlike their case, such a formulation for the Jordanian

twist is an *a priori* open problem. Alternatively, however, one can study the Baxter’s equation representation of the spin chain.

C. Energy from Baxter’s equation

In this Section, we compute the energy of the deformed ground state of the Jordanian-twisted $J = 2$ spin chain using our result (5.30). As we find ourselves in the situation where the traditional Bethe ansatz techniques cannot provide a full description of the model, we have to resort to more advanced methods for this purpose. One such approach is the quantum separation of variables (qSoV), pioneered by Sklyanin [71], which has been successfully applied to many cases where the ABA is not applicable [59–61]. While a complete implementation of the qSoV formalism is beyond the scope of this work, it nonetheless provides a practical way to extract the ground state energy, and may give indications to future applications of qSoV.

For $\mathfrak{sl}(2, \mathbb{R})$ based systems, the central object is the following Baxter TQ relation

$$\Delta^+(u)Q(u-i) + \Delta^-(u)Q(u+i) = \tau(u)Q(u), \quad (5.35)$$

where we used the notation $f^{[k]}(u) := f(u + iks)$ and $f^\pm(u) = f^{[\pm 1]}(u)$. Here $\tau(u)$ is the transfer matrix eigenvalue, $\Delta(u)$ a model specific coefficient, and $Q(u)$ is Baxter’s Q function [72], defined up to multiplication by i periodic functions in u . In the undeformed, $\mathfrak{sl}(2, \mathbb{R})$ XXX_s spin chain with $s = -1/2$,³⁵ Baxter’s equation reads [59,60]

$$(u - i/2)^J Q(u - i) + (u + i/2)^J Q(u + i) = \tau(u)Q(u). \quad (5.36)$$

This form was originally obtained from reformulations of the Bethe ansatz equations, but it can also be derived directly via representation theory of Yangian algebras, where it arises as consequences of fusion relations, see [27,73] and [69]. Importantly, Drinfel’d twists preserve the level-0 algebra and the regularity of the R matrix. It has also been shown explicitly that the form (5.36) remains valid for chains deformed by Abelian twisted-boundary conditions [19,20,73,74], but with modified analytic structure for $\tau(u)$ and $Q(u)$. All of these models can be formulated as closed undeformed spin chains that are Yangian invariant in the bulk, while their boundary conditions break the Yangian symmetry.³⁶ It is therefore expected that Baxter’s equation is formally of the same form in all models based on rational

³²It would however be interesting to explore whether a twisted version of the Casimir (5.32) could play a role in the deformed spectrum.

³³A closed form to the ODE in the dipole case was obtained in [20], which showed a smooth connection to the undeformed higher-spin states labeled by S therein.

³⁴A possible reason for this may be the equivalence of Jordanian sigma models with a non-Abelian T-duality transformation over the Borel subalgebra of $\mathfrak{sl}(2, R)$ after deformation-dependent field redefinitions, and it would be interesting to understand that also in this context.

³⁵Note that compared to [59,60], where the Casimir is $C = s(s-1)$, one should send $s \rightarrow -s$.

³⁶This also includes the undeformed XXX chain with periodic boundary conditions [75].

$\mathfrak{sl}(2, \mathbb{R})$ R matrices, i.e., with the same coefficients $\Delta(u)$, even when the boundary breaks the symmetry. While we are not aware of a general algebraic proof, [69] and references therein support this expectation.

In the case of the Jordanian-twisted chain, we are as of yet not in a position to derive both $Q(u)$ and $\tau(u)$ from first principles (that is, in the absence of ABA and a known complete analytic structure in u). Instead, we reverse engineer: As argued, we assume that the TQ equation (5.36) retains its form in the twisted-boundary picture, i.e., with the same coefficients $\Delta(u)$, but with modified analytic structure of $\tau_\xi(u)$ and $Q_\xi(u)$

$$(u - i/2)^J Q_\xi(u - i) + (u + i/2)^J Q_\xi(u + i) = \tau_\xi(u) Q_\xi(u). \quad (5.37)$$

We then determine $Q_\xi(u)$ from the explicit result (5.30) of the transfer matrix $\tau_\xi(u)$ for the deformed ground state. The Baxter equation thus serves here merely as a reformulation of the result and may offer a starting point for generalizations. We will however use that the derived $Q_\xi(u)$ functions around $u = 0$ typically encode the energy of the $XXX_{-1/2}$ Hamiltonian as (see, e.g., [67])

$$E = \frac{i\lambda}{8\pi^2} \partial_u \left(\log \frac{Q_\xi(u + i/2)}{Q_\xi(u - i/2)} \right)_{u=0}, \quad (5.38)$$

while the $u \rightarrow \infty$ asymptotic regime encodes the boundary conditions (to which we will come back to in the next section).

By demanding regularity at $u = 0$ we take a general polynomial ansatz for the Q_ξ function, where the coefficients may depend on ξ . Following the structure of the eigenvalue (5.30), which is a polynomial of degree two in u with coefficients that are an infinite series in ξM , we can thus take

$$Q_\xi(u) = \sum_{k=0}^{\infty} \sum_{n=0}^k c_{n,k} u^n (\xi M)^k. \quad (5.39)$$

One can then simply solve Baxter's equation as an expansion in ξ for the unknown coefficients $c_{n,k}$. A solution for the deformed ground state then reads³⁷

³⁷Here, for simplicity of the presentation, we have set all remaining free coefficients of the polynomial ansatz to $c_{n,k} = 1$, since their explicit value did not affect the resulting energy E_{GS} given below.

$$\begin{aligned} Q_\xi(u) = & 1 + \xi M + \xi^2 M^2 \left(1 - \frac{u^2}{24} \right) + \xi^3 M^3 \\ & + \xi^4 M^4 \left(1 - \frac{329u^2}{8640} + \frac{u^4}{1920} \right) \\ & + \xi^5 M^5 \left(1 - \frac{31u^2}{8640} - \frac{u^4}{1920} \right) + \mathcal{O}(\xi^6 M^6), \end{aligned} \quad (5.40)$$

with energy

$$\begin{aligned} E_{GS} = & \frac{\lambda}{\pi^2} \left(\frac{\xi^2 M^2}{96} - \frac{\xi^3 M^3}{96} + \frac{\xi^4 M^4 131}{13824} \right. \\ & \left. - \frac{\xi^5 M^5 59}{6912} + \mathcal{O}(\xi^6 M^6) \right). \end{aligned} \quad (5.41)$$

We can in principle compute these to arbitrary order in ξ . As before, we clearly see a deformation that depends only on the combination ξM . Although with a lacking Seiberg-Witten map and gauge theory interpretation, we did not benchmark this result (as it was done in [20]), we hope it provides useful input for future studies. We will come back to this in a follow up [68].

VI. ASYMPTOTICS AND TWISTS OF THE Q FUNCTIONS AT ALL LENGTHS

As we were able to diagonalize the transfer matrix only for the $J = 2$ case, a complete construction of the Q_ξ functions for arbitrary chain length J is not straightforward at this point. In particular, we lack a direct derivation of the full spectrum or the corresponding analytic structure that would normally support Baxter's equation. Nevertheless, following the same logic as in the previous section, we can still extract meaningful information from the asymptotic $u \rightarrow \infty$ structure of the twisted transfer matrix (3.58) at any J . See (5.15) for its formal expression in the $+1/2$ representation of $\mathfrak{a}$, but generalized to arbitrary physical sites. In particular, we will focus here on the $u \rightarrow \infty$ regime of the Q functions, which typically encodes spectral charges and twisted-boundary conditions of the chain.

Let us thus consider (5.15) at any length $J \geq 2$, acting on $\hat{M}$ eigenstates. By induction, one can show that the asymptotics of its eigenvalue around $u \rightarrow \infty$ reads³⁸

³⁸One can prove this starting from the inductive hypothesis that the elements $A(u)$, $C(u)$, $D(u)$ of the undeformed monodromy matrix $T(u) = R_{aJ} \cdots R_{a1}$ at length J read

$$\begin{aligned} A(u) &= u^J + iu^{J-1} \Delta^{J-1}(\mathbf{h}) + \mathcal{O}(u^{J-2}), \\ C(u) &= iu^{J-1} \hat{M} + \mathcal{O}(u^{J-2}), \\ D(u) &= u^J - iu^{J-1} \Delta^{J-1}(\mathbf{h}) + \mathcal{O}(u^{J-2}), \end{aligned} \quad (6.1)$$

which is easily verified for $J = 1, 2$ using the definition of the R matrix (5.1), and $J' = J + 1$ when assuming the above holds for general J .

$$\begin{aligned}\hat{\tau}_\xi(u) &= \frac{2 + \xi\hat{M}}{\sqrt{1 + \xi\hat{M}}} u^J + \mathcal{O}(u^{J-2}), \\ \tau_\xi(u) &= \frac{2 + \xi M}{\sqrt{1 + \xi M}} u^J + \mathcal{O}(u^{J-2}),\end{aligned}\quad (6.2)$$

where crucially, after a nontrivial cancelation, there is no remaining $\Delta^{J-1}(\mathbf{h})$ or $\mathcal{O}(u^{J-1})$ contribution. Asymptotically this implies constant twisted behavior, as we will explain. This is in stark contrast to the dipole deformation [20], where the twist introduces a nontrivial contribution at $\mathcal{O}(u^{J-1})$, which ultimately results in involved nonanalytic (in particular nonpowerlike, square-root exponential) behavior of the Q_ξ functions, cf. Eq. (6.17) therein.

The leading and subleading terms in the asymptotic expansion of $\tau_\xi(u)$ are sufficient to extract the (effective) leading $u \rightarrow \infty$ asymptotics of the Q_ξ function. Noting that, up to an overall factor, $\tau_\xi(u)$ has the same asymptotic structure as its undeformed counterpart, it is natural to consider the following ansatz (see also [27,76]) around $u \rightarrow \infty$

$$Q_\xi(u) = e^{u\phi}[\text{power-series}] = e^{u\phi}(u^\alpha + \beta u^{\alpha-1} + \mathcal{O}(u^{\alpha-2})), \quad (6.3)$$

where ϕ , α , and β are parameters to be fixed from Baxter's equation (5.37). After cancelling an overall $e^{u\phi}$ factor, the leading behavior of (5.37) at $u \rightarrow \infty$ reads

$$\left(e^{i\phi} + e^{-i\phi} - \frac{2 + \xi M}{\sqrt{1 + \xi M}} \right) u^{J+\alpha} + \mathcal{O}(u^{J+\alpha-1}) = 0, \quad (6.4)$$

which gives a simple equation for ϕ , solved by

$$\phi = \pm \frac{i}{2} \ln(1 + \xi M) + 2\pi n, \quad \forall n \in \mathbb{N}. \quad (6.5)$$

The subleading term then reads

$$\begin{aligned} & \left(e^{i\phi} \left(\frac{iJ}{2} + i\alpha + \beta \right) - e^{-i\phi} \left(\frac{iJ}{2} + i\alpha - \beta \right) \right. \\ & \quad \left. - \frac{2 + \xi M}{\sqrt{1 + \xi M}} \beta \right) u^{J+\alpha-1} + \mathcal{O}(u^{J+\alpha-2}) = 0. \end{aligned} \quad (6.6)$$

Upon the result for ϕ in (6.5) the terms proportional to β cancel, so that this equation is satisfied only for $\alpha = -\frac{J}{2}$. In conclusion, depending on the sign for ϕ , we have two possible choices for the leading asymptotics of the $Q_\xi(u)$

function, which we denote by $\mathbf{Q}_{1,2}$. Explicitly³⁹

$$\begin{aligned}\mathbf{Q}_1(u) &= (1 + \xi M)^{\frac{i}{2}u} (u^{-\frac{J}{2}} + \mathcal{O}(u^{-\frac{J}{2}-1})), \\ \mathbf{Q}_2(u) &= (1 + \xi M)^{-\frac{i}{2}u} (u^{-\frac{J}{2}} + \mathcal{O}(u^{-\frac{J}{2}-1})).\end{aligned}\quad (6.7)$$

Baxter's Q function is then generically a linear combination of $\mathbf{Q}_1$ and $\mathbf{Q}_2$.

Baxter Q functions are not isolated objects: they satisfy additional algebraic constraints known as the **QQ** relations that encode the same integrable (TQ) structure from a more algebraic point of view [76]. In the case of $\mathfrak{sl}(2, \mathbb{R})$ -symmetric models, the **QQ** relations read

$$\mathbf{Q}_1^+ \mathbf{Q}_2^- - \mathbf{Q}_1^- \mathbf{Q}_2^+ = \mathbf{Q}_0 \mathbf{Q}_{12}, \quad (6.8)$$

where we recall, for $s = -1/2$, $f^\pm := f(u \mp \frac{i}{2})$. The functions $\mathbf{Q}_0$ and $\mathbf{Q}_{12}$ are additional, auxiliary Q functions encoding details of our model. One can however generically set $\mathbf{Q}_{12} = 1$ by a rescaling (also referred to as gauge transformation [76,77]). Using the asymptotic behavior of $\mathbf{Q}_1$ and $\mathbf{Q}_2$, we can thus also determine $\mathbf{Q}_0$ at large u

$$\begin{aligned}\mathbf{Q}_0 &\sim u^{-J} \frac{\xi M}{\sqrt{1 + \xi M}} = u^{-J} \left(\sqrt{1 + \xi M} - \frac{1}{\sqrt{1 + \xi M}} \right), \\ &u \rightarrow \infty.\end{aligned}\quad (6.9)$$

Interestingly, the asymptotics of $\mathbf{Q}_0$ and the overall exponential behavior of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ match with the expressions obtained for spin chains with twisted-boundary conditions determined by a constant twist matrix $\beta_a = e^{2i\phi h_a}$ [19,27,73,74,76], which are of the form

$$\begin{aligned}\mathbf{e}_{J+a} &= \mathbf{e}_a(1 + \xi M), & \mathbf{h}_{J+a} &= \mathbf{h}_a, \\ \mathbf{f}_{J+a} &= \mathbf{f}_a(1 + \xi M)^{-1},\end{aligned}\quad (6.10)$$

as were derived from the twisted XXX-Hamiltonian in [27], cf. (2.16) and (2.21) therein. While the subleading behavior of our Q functions should differ from the case of constant twists, it is nevertheless interesting to note that the above

³⁹Note that we do not reproduce the Q asymptotics of the undeformed model, which depends on the integer Cartan charge S_1 of $\mathfrak{sl}(2, \mathbb{R})$ that corresponds to an AdS spin in the $SL(2, \mathbb{R})$ sector of the string sigma model. This may be expected: First, the Baxter equation is a difference equation that does not necessarily have smooth dependence on continuous parameters such as ξ or M . This is what happens for Abelian twists too, see, e.g., [19,20]. Second, as explained in Secs. II B (also II C) and III D, the Jordanian deformation explicitly breaks the symmetry associated to S_1 . The fact that such a charge does not appear in the Q asymptotics also happens in the classical spectral curve of the Jordanian string [12], where the asymptotics of the quasimomenta are determined only by the twist charge, energy E and a spin S_2 [one outside of the $SL(2, \mathbb{R})$ sector studied here], rather than E, S_1, S_2 .

expression coincides with the Jordanian twisted-boundary conditions (3.60) once $\hat{M}$ is replaced by its eigenvalue. We consider this as additional support of (3.60) and more generally (3.39). Furthermore, the twist charge ϕ has an expression that is very reminiscent of the sigma model twist charge (2.14), which appears in the leading asymptotics of the classical spectral curve of the Jordanian string [12] (at least at large λ and after field redefinitions, see the discussion in that Section). At last, we note that the expression $\beta_a = e^{2i\phi h_a}$ also coincides with the classical twist of the twisted monodromy (A7).

The asymptotic resemblance to a chain with constant twist is interesting. However, even without explicitly analyzing the subleading behavior, which should be twisted by genuine operators, we know that Q functions derived for constant twists are not compatible with our story. In fact, spin chains with constant twists such as $\beta_a = e^{2i\phi h_a}$ in [19] would preserve only a $U(1)$ symmetry associated to the Cartan operator $\Delta^{J-1}(\mathbf{h})$, while the $\hat{M} = \Delta^{J-1}(\mathbf{e})$ symmetry would be broken, yet exactly the opposite happens for the Jordanian chain.

VII. CONCLUSIONS

In this work, we studied the proposed spin chain description of a Homogeneous Yang-Baxter deformation of the $\text{AdS}_5 \times S^5$ superstring sigma model of Jordanian type, focusing on its formulation via Drinfel'd twists of the associated R matrix. By analyzing the dilaton profiles of all possible Jordanian deformations that yield type IIB supergravity backgrounds, we first identified a “minimal” Jordanian deformation⁴⁰ with constant dilaton, a property that suggests a valid supergravity approximation and that is nontrivial only in a $SL(2, \mathbb{R})$ sector. To understand the string-spin correspondence in this setup, we studied a BMN-like string solution that is nontrivial only in an $\text{AdS}_3 \times S^1$ subspace of $\text{AdS}_5 \times S^5$. The relevant spin chain in this case is the Jordanian Drinfel'd twist of the $\mathfrak{sl}(2, \mathbb{R})$ -invariant $\text{XXX}_{-1/2}$ Heisenberg spin chain.

Beyond this motivating example, we approached the topic of closed Drinfel'd twisted spin chains more generally, deriving universal formulas for their monodromy and transfer matrices. We further formalized the expectation that all closed Drinfel'd twisted models are equivalent to undeformed models with twisted-boundary conditions, by proposing a new method to derive the twist on one-site algebra generators when traversing the spin chain. Our framework reproduces the Abelian Drinfel'd-Reshetikhin

case (see, e.g., [18,20,27,57]) and passes several nontrivial checks. Crucially, we extended this understanding to the non-Abelian Jordanian case and identified its boundary-twist operator, which is consistent both with the classical string expression of [11,12] at large 't Hooft coupling and the asymptotics of the Q system of the length J spin chain. To our knowledge, the Jordanian twisted-boundary conditions we obtain are the first for Drinfel'd twists that are neither (successively) Abelian nor r symmetric. Such twists belong to an equivalence class connected via co-boundary Drinfel'd twists [30,55]. While these transformations are expected to preserve physical observables, it would be interesting to understand how they affect the gluing realization of the spin chain and the sigma-model formulation.

In the rest of the paper we used the twisted-boundary formulation to initiate the spectral analysis of the Jordanian $\mathfrak{sl}(2, \mathbb{R})$ - $\text{XXX}_{-1/2}$ spin chain. A central aspect is that the twist breaks the full $\mathfrak{sl}(2, \mathbb{R})$ symmetry, and in particular its Cartan generator. Instead, we used the residual symmetry generated by $\hat{M} = \Delta^{J-1}(\mathbf{e})$, with $\mathbf{e}$ a root generator of $\mathfrak{sl}(2, \mathbb{R})$, to study the spectrum. Using the coherent state representation, we obtained a first string-spin agreement by comparing the ground state energy shift in the twisted spin chain with the classical energy shift of the BMN-like string solution in the continuum (large J) limit. At short lengths ($J = 2$), we computed the spectrum of the twisted transfer matrix perturbatively in the representation space parameter and in the deformation parameter. Interestingly, within our differential analysis we found that the Jordanian twist connects only the ground state of the undeformed chain, while the magnon excitations associated to higher total spin appear to be hidden in the deformed spectrum. It would be very interesting to find an explicit understanding on how the magnons are repackaged in the twisted picture. We eventually used the $J = 2$ transfer matrix eigenvalue of the deformed ground state as an input to Baxter's equation to obtain its energy.

In the zero-charge sector of $\hat{M}$, we observed that the transfer matrix spectrum remains undeformed. This appears consistent with the work [28] on the same model, where a trivial spectrum was obtained when relying on a Fock space decomposition, which should effectively truncate part of the Hilbert space to $\hat{M}$ charge $M = 0$. This supports the interpretation of Drinfel'd twists as deformations that only affect sectors of the spectrum with nonzero charge under the symmetry used to construct them. However, we are not yet in a position to propose a change of basis of the full Fock space to eigenfunctions of the Hamiltonian. In fact, without a smooth connection to the magnon states, it is unclear that our deformed $\hat{M}$ eigenfunctions span the whole Hilbert space. It is thus possible that the Hamiltonian remains nondiagonalizable in other sectors of the Hilbert space, as supported by [28].

Looking forward, the rather simple asymptotic behavior of Baxter's Q functions and QQ system at general length J

⁴⁰The only other possible Jordanian backgrounds with constant dilaton profiles are necessarily nontrivial in subalgebras of $\mathfrak{psu}(2, 2|4)$ that are of higher rank than $\mathfrak{sl}(2, \mathbb{R})$. In general it would be interesting to extend our analysis to the full $\mathfrak{psu}(2, 2|4)$, especially considering the unimodular fermionic extension of the minimal model, and investigate higher-rank cases as well.

suggests a possible treatment via the quantum spectral curve and separation of variables program. A crucial next step is to determine the full analytic structure of the Q functions and to obtain closed-form expressions in ξ , at all length J . Understanding the fate of the magnon states and their Bethe root structure would prove key for this. We leave this for future work. It would then be particularly interesting to take the large J limit of the spectrum and compare it with the classical string prediction and its semiclassical corrections, as computed in [12], to further support the string-spin correspondence in the overlapping continuum/semiclassical regime and the use of Baxter's equation.

Another intriguing open problem is the formulation of an integrable (quantum) S matrix for the Jordanian string sigma model. Recent work has observed particle production involving massive asymptotic states [33], which appears to challenge the use of integrable methods based on factorized S matrices. This analysis was performed after adopting an “alternative” light-cone gauge-fixing based on the residual symmetries [78]. However, our results, along with those of [28], suggest that the notion of a fixed number of asymptotic particles is not well defined in the deformed setting. After adopting the Hilbert space according to residual symmetries, the discrete excited states are not straightforwardly realized in the deformed model. This supports the interpretation that Jordanian deformations fundamentally alter the structure of the asymptotic Hilbert space. Additionally, [28] explicitly observed that the original Fock states in the undeformed chain mix after the deformation. In the recent work [34], production processes involving massless and soft particles were also found in the continuum limit of a deformed spin chain of Jordanian type. There, it was proposed that a consistent scattering framework may exist by absorbing soft particles into the asymptotic Hilbert space via Faddeev-Kulish coherent states [79]. It would be interesting to explore whether such a construction can accommodate the massive particle production observed in the Jordanian sigma model using coherent $\hat{M}$ eigenstates as we employed here.

More broadly, the approach developed in this work may provide new insights into the construction of a dual deformed gauge theory, in which the spin chain Hamiltonian should arise as the one-loop anomalous dimension of a twisted scaling operator, and the relevant basis of gauge-invariant operators. This would not only enhance our understanding of non-AdS holography and its possible nonrelativistic field theory realizations, but also provide valuable structural input for the Q system, similar to what has been achieved in previous integrability constructions. It is still a widely open problem for Jordanian models due to technical complications from noncommutative calculus, but significant progress has been made in the case of (successive) Abelian twists of the Poincaré subalgebra of AdS in [21].

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DATA AVAILABILITY

The data that support the findings of this article are openly available through the current accepted publication in PRD. No other data was used to construct the findings of the current article. All “data” that is not explicitly present in the article can be reproduced using the methods described in the article.

APPENDIX A: COMMENTS ON THE CLASSICAL LIMIT OF TWISTED TRANSFER MATRIX

In this Appendix, we discuss the classical limit of the twisted transfer matrix (3.58), and comment on its relation to the twisted-boundary conditions, as well as connections to the would-be sigma-model dual of the Jordanian-twisted spin chain considered in this paper.

For this purpose, it will be sufficient to take the auxiliary space $\mathfrak{a}$ in the two-dimensional spin-1/2 representation of $\mathfrak{sl}(2, \mathbb{R})$, with⁴¹

$$h_a = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f_a = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \quad (A2)$$

Consider then the twisted monodromy (3.58)

$$T_a^*(u) = \begin{pmatrix} 1 & \xi \Delta^{J-1}(h) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \times \begin{pmatrix} (1 + \xi \hat{M})^{-1/2} & 0 \\ 0 & (1 + \xi \hat{M})^{1/2} \end{pmatrix}, \quad (A3)$$

and the twisted transfer matrix $\hat{\tau}_\xi(u) := \text{Tr}(T_a^*(u))$.

⁴¹The $\mathfrak{sl}(2, \mathbb{R})$ Casimir

$$C = h^2 - \frac{1}{2}(\mathbf{e}\mathbf{f} + \mathbf{f}\mathbf{e}) = s(s+1) \quad (A1)$$

is equal to $\frac{3}{4}$ in this representation (i.e., $s = \frac{1}{2}$).

To start the spectral analysis (particularly in the context of the ABA), one typically aims to rewrite $\hat{\tau}_\xi(u)$ as

$$\hat{\tau}_\xi(u) = \text{Tr}_a(\mathcal{W}_a T_a(u)), \quad (\text{A4})$$

where we call $\mathcal{W}_a$ the twist matrix, which acts on the auxiliary space. When $\mathcal{W}_a$ is diagonalizable, the change in the (Bethe) spectrum of the twisted transfer matrix can then be directly read off in terms of the eigenvalues of $\mathcal{W}_a$ (see, e.g., [18,80,81]).

However, to relate the Jordanian twisted case (A3) to (A4), observe that this is only possible when $\hat{\tau}_\xi(u)$ acts on (energy) eigenstates Ψ_M of the $\hat{M} = \Delta^{J-1}(\mathbf{e})$ operator, as we also do throughout the main text, so that $\hat{M}\Psi_M = M\Psi_M$. On such states, we can write

$$\mathcal{W}_a = \begin{pmatrix} (1 + \xi M)^{-1/2} & \xi(1 + \xi M)^{-1/2} \Delta^{J-1}(\mathbf{h}) \\ 0 & (1 + \xi M)^{1/2} \end{pmatrix}. \quad (\text{A5})$$

However, this object still contains the operator $\Delta^{J-1}(\mathbf{h})$, so $\mathcal{W}_a$ remains nondiagonalizable as an operator on the full spin chain. In the classical limit, we may instead replace $\Delta^{J-1}(\mathbf{h})$ by its expectation value in the state Ψ_M , i.e., $H := \langle \Psi_M | \Delta^{J-1}(\mathbf{h}) | \Psi_M \rangle$. This turns the twist $\mathcal{W}_a$ into an ordinary 2×2 matrix, which can now be trivially diagonalized by a change of basis in the auxiliary space. Its eigenvalues are $(1 + \xi M)^{\pm 1/2}$, and the corresponding eigenvectors

$$\mathbf{v}_a^- = (1, 0), \quad \mathbf{v}_a^+ = (M^{-1}H, 1), \quad (\text{A6})$$

resulting exactly in

$$\mathcal{W}_a = e^{-\mathbf{h}_a \log(1 + \xi M)}. \quad (\text{A7})$$

This is consistent with the twisted-boundary conditions (3.60) of the spin operators derived in Sec. III C, as well as the asymptotics of the Q functions discussed in Sec. VI, which have been used for the study of the semiclassical coherent states in Sec. IV. In particular, these boundary conditions are governed by the symmetry operator $\hat{M}$ but not by $\Delta^{J-1}(\mathbf{h})$.

Interestingly, for $M = 0$ eigenstates, the eigenvectors (A6) become ill defined. In that case, the classical twist $\mathcal{W}_a$ obtains a Jordan block structure and is not diagonalizable. This aligns with the observation that the spectrum is undeformed in such sectors, as seen from the ξM expansion in (5.30) (see also [28]).

Finally, it is interesting to note that a similar classical freedom to diagonalize $\mathcal{W}_a$ appears in the Jordanian sigma-model analysis. There, in the undeformed but twisted formulation, the sigma-model fields obey twisted-boundary conditions governed by W as in (2.13). This same object appears in the conserved monodromy of the twisted model,

whose diagonalized form has an asymptotic expansion in terms of local charges [the sigma-model equivalent of (A4), see [11,12,45] for more details]. A sigma-model field redefinition in fact allows one to remove the twist charge $\mathbf{q}$ in W [which may be seen as the classical sigma-model analog of $\Delta^{J-1}(\mathbf{h})$]. This results in a diagonal twist $W' = \exp(\mathbf{Q}\mathbf{h})$, which depends only on the conserved charge $\mathbf{Q}$ [which may be seen as the sigma-model analog of $\Delta^{J-1}(\mathbf{e})$ at large λ]. See Eq. (2.13) and the surrounding discussions below, as well as Secs. 4.3 and 7 of [12] for further details. Importantly, this diagonalization is only possible in sectors of classical sigma-model solutions where $\mathbf{Q} \neq 0$. When $\mathbf{Q} = 0$ (which we may understand as $M = 0$ eigenstates of the spin chain), the twist is not diagonalizable but instead becomes of Jordan form. Similarly, the twisted-boundary conditions (3.60) are consistent with $W' = \exp(\mathbf{Q}\mathbf{h})$ at large λ .

It would be interesting to understand whether these transformations—field redefinitions in the sigma model and basis changes of the auxiliary space in the spin chain—can be interpreted as Hopf algebra isomorphisms, possibly induced by co-boundary Drinfel'd twists [30,55].

APPENDIX B: SINGULAR ODES AND THEIR FORMAL RESOLUTION

In this Appendix, we present a method for obtaining the formal solution $g(z)$ to ODEs of the form (5.17), which are singular at some point $z = \lambda$. A standard reference for this purpose is the book [82], but we will follow the algorithm and terminology of [83] here.

The first step is to recast (5.17) as a first order ODE in vector form. We let $\mathbf{v}(z) := (g(z), g'(z), g''(z))^T$ and rewrite (5.17) as the generic system

$$\frac{d\mathbf{v}(z)}{dz} = \frac{A(z)}{(z - \lambda)^{1+p[A]}} \mathbf{v}(z), \quad (\text{B1})$$

that one also denotes formally as $[A]$. Here, $p[A]$ is an integer called the ‘‘Poincaré’’ rank of $[A]$, determined by requiring that the matrix $A(z)$ is analytic at $z = \lambda$, i.e., it has a regular polynomial expansion, with a constant leading term. When $p[A] < 0$, the point $z = \lambda$ is a regular point. When $p[A] = 0$, it is a singularity of the first kind, and for $p[A] \geq 1$, it is a singularity of the second kind. In our case, we have $\lambda = 0$ and

$$A(z) = \sum_{n=0}^2 A_n z^n = \frac{1}{8\xi} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p_1(z) & p_2(z) & p_3(z) \end{pmatrix}, \quad (\text{B2})$$

with $p_i(z)$ as in Eqs. (5.18), and $p[A] = 1$. Explicitly, the coefficient matrices are

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2+\xi M+16\sqrt{1+\xi M}\chi_\xi}{4\xi} & -1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{M^2}{4} & \frac{2+\xi M}{\xi} & -3 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{M^2(2+\xi M)}{8\xi} & \frac{M^2}{4} & \frac{2+\xi M}{2\xi} \end{pmatrix}. \quad (\text{B3})$$

1. Regular and irregular singularities

Having singularities of the “first” and “second” kinds classifies the form of the differential system, but it does not determine the actual analytical behavior of its solution. There are strong theorems of asymptotic analysis [82] that guarantee that the generic differential system (B1) always admits a formal solution around such singularities, at least as an asymptotic trans-series. Depending on the nature of the asymptotic expansion of the formal solution near $z = \lambda$, one further distinguishes between regular and irregular singular points.

Regular singularities. In this case, the formal solution is of the form

$$\mathbf{v}(z) = P(z - \lambda)(z - \lambda)^\Lambda \mathbf{c}, \quad (\text{B4})$$

with $P(z - \lambda)$ a regular asymptotic matrix expansion around $z = \lambda$, Λ a constant matrix, and $\mathbf{c}$ a vector of integration constants. These solutions can be obtained by (matrix versions of) Frobenius’s method.

Irregular singularities. In this case, the formal solution is of the form

$$\mathbf{v}(z) = P(t)t^\Lambda e^{q(1/t)} \mathbf{c}, \quad (\text{B5})$$

where P , Λ , and $\mathbf{c}$ are as above, $t = (z - \lambda)^{1/s}$, with $s \in \mathbb{N}_0$ called the branching exponent, and $q(1/t)$ is a polynomial in $1/t$, called the exponential part, capturing the instantonic behavior.

Singularities of the first kind are always regular, while singularities of the second kind can be either regular or irregular singularities. Our case is precisely of the second kind, thereby making the analysis more subtle. However, as we will now explain, one can also play with “gauge transformations” of the system to try to map singularities of the second kind to singularities of the first kind. To simplify the upcoming formulas, we will take $\lambda = 0$ from now on, but having nonzero λ should be easily reinstatable.

a. Gauge transformations

In certain cases, it is possible to simplify the singular structure of the system $[A]$ by means of a so-called “gauge transformation.” Given (B1), one can consider the

transformation $\mathbf{v}(z) = T(z)\mathbf{w}(z)$ with $T(z)$ an invertible matrix, and map it to a new, equivalent system

$$\frac{d\mathbf{w}(z)}{dz} = \frac{B(z)}{z^{1+p[B]}} \mathbf{w}(z),$$

$$B(z) := z^{1+p[B]} \left(\frac{T^{-1}(z)A(z)T(z)}{z^{1+p[A]}} - T^{-1}(z)T'(z) \right), \quad (\text{B6})$$

with $p[B]$ again such that $B(z) = \sum_{n=0}^N B_n z^n$ for some positive integer N . We will denote this system as $[B] = T[A]$. Importantly, while both systems are equivalent in terms of their solutions, the Poincaré rank is generally not invariant under such transformations. Specifically, a $T(z)$ with appropriately chosen behavior around $z = 0$ can raise or lower the rank by modifying the singular structure via its derivative $T'(z)$. Constant $T(z)$ do, however, leave the rank invariant.

b. Moser reducibility

Given the generic system $[A]$ in (B1), there exists a criterion, due to Moser, for the reducibility of the system to one with lower Poincaré rank. In particular,

$$\exists T(z) \quad \text{such that } p[T[A]] < p[A]$$

$$\Leftrightarrow z^{\text{rank}(A_0)} \det(\alpha 1 - A_0 z^{-1} - A_1)|_{z=0} = 0, \quad (\text{B7})$$

for some generic α . As it turns out, our system (B2) has $\text{rank}(A_0) = 1$ and thus satisfies this criterion. Since $p[A] = 1$, we are guaranteed that there exists a $T(z)$ such that $p[T[A]] = 0$, giving us a singularity at $z = 0$ of the first kind, and a regular solution of the form (B4). In practice, one can find such a $T(z)$ by first doing a constant gauge transformation to put A_0 in its Jordan form, and then by performing a “shearing transform” $T(z) = \text{diag}(z^{k_1}, z^{k_2}, z^{k_3})$ with k_i integers chosen such that the rank reduces. After this, one usually does an additional constant gauge transform to put also the leading term B_0 of $[B] = T[A]$ in Jordan form. The system can then be solved by a Frobenius-like method. In our case, we find that $k_1 = 0$ and $k_2 = k_3 = 1$ works.

c. Matrix Frobenius method (or Sauvage’s method)

Since we are in the regular singular case, we will use a Frobenius-like method, known as Sauvage’s method, in the matrix case. Looking at the form of the solution (B4), the idea is to build the coefficients of the expansion of $P(z)$ around $z = 0$ using recurrence relations. However, before proceeding, we must distinguish two cases. Consider a system with a singularity of the first kind

$$\frac{d\mathbf{w}(z)}{dz} = \frac{B(z)}{z} \mathbf{w}(z), \quad (\text{B8})$$

and assume without loss of generality that B_0 is in Jordan form.

Resonant case. If some of the eigenvalues of B_0 differ with another by a positive integer, we are in the so-called resonant case. We then need to perform additional shearing transforms $T(z) = \text{diag}(1, \dots, 1, z, 1, \dots, 1)$ with z in the entry corresponding to the eigenvalue that differs from the other by a positive integer. This will essentially reduce this eigenvalue by 1, and we repeat this until we are out of the resonant case. In our case, after reducing the system from the second to the first kind and transforming the leading B_0 matrix to Jordan form, we find

$$B_0 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (\text{B9})$$

so that we are precisely in the resonant case. We transform to a nonresonant case by performing the shearing transform $T(z) = \text{diag}(1, 1, z)$.

Nonresonant case. When no eigenvalue differs from another by a positive integer, we can finally solve the system. Consider again the generic system (B8) but with B_0 now such that one is in the nonresonant case. The solution is then given by

$$\mathbf{w}(z) = P(z)z^{B_0}\mathbf{c}, \quad \text{with} \quad P(z) = \sum_{n=0}^{\infty} P_n z^n, \quad (\text{B10})$$

with, as usual, $\mathbf{c}$ a vector of integration constants, and the coefficients of $P(z)$ given by the solutions of

$$P_0 = 1, \quad (B_0 - n1)P_n - P_n B_0 = - \sum_{l=0}^{n-1} B_{n-l} P_l. \quad (\text{B11})$$

The condition that no eigenvalue differs by positive integer is what guarantees that (B11) has a unique solution for all n . It is a Sylvester equation, which has well-studied solutions, and one can solve it through recurrence relations using, e.g., the `LyapunovSolve` method in *Mathematica*.

After transforming the resonant case (B9) with the shearing transform, we find the following leading matrix in Jordan form

$$B_0 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}, \quad (\text{B12})$$

which is a three-dimensional Jordan block. This is what causes both the $\log(z)$ and $\log^2(z)$ in (5.19) to appear. The -2 eigenvalues of B_0 further imply that (B10) leads with z^{-2} , which is eventually shifted to z^0 by the successive shearing and Jordan transforms that bring us back to the original system $[A]$, reproducing finally (5.19).

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