Ghost-free scalar-fermion interactions

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We discuss a covariant extension of interactions between scalar fields and fermions in a flat space-time. We show, in a covariant theory, how to evade fermionic ghosts appearing because of the extra degrees of freedom behind a fermionic nature even in the Lagrangian with first derivatives. We will give a concrete example of a quadratic theory with up to the first derivative of multiple scalar fields and a Weyl fermion. We examine not only the maximally degenerate condition, which makes the number of degrees of freedom correct, but also a supplementary condition guaranteeing that the time evolution takes place properly. We also show that proposed derivative interaction terms between scalar fields and a Weyl fermion cannot be removed by field redefinitions.

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I. INTRODUCTION

Construction of general theory without ghost degrees of freedom (d.o.f.) has been discussed for a long time. According to Ostrogradsky's theorem, a ghost always appears in a higher (time) derivative theory as long as it is nondegenerate [1] (see also Ref. [2]). If a term with higher derivatives plays an important role in the dynamics, the ghost d.o.f. associated with it must be removed because, otherwise, the dynamics is unstable. This is a different approach from effective field theory, which allows a ghost as long as it appears above the scale we are interested in; that is, a term with higher derivatives can be treated as a perturbation.

Such an Ostrogradsky ghost could be circumvented by introducing the degeneracy of the kinetic matrix, which leads to the existence of a primary constraint and a series of subsequent constraints, which are responsible for eliminating extra ghost d.o.f. associated with higher derivatives. After the recent rediscovery [3–5] of Horndeski theory [6] in the context of the Galileon [7], the pursuit of finding general theory without such ghost d.o.f. has been revived, especially for bosonic d.o.f., in the context of point particles [8–11], their field theoretical application [12],

scalar-tensor theories [4–6,13–20], vector-tensor theories [21–29], and form fields [30–32].

On the other hand, generic theory with fermionic d.o.f. has not yet been investigated well. If we could find such fermionic theories which have not been explored, some applications of them will be expected. One will be fermionic dark matter, the interaction of which could be constrained by the condition of the absence of ghost d.o.f. Another application is inflation and its subsequent reheating, which needs interactions between an inflaton and standard model particles, since their interactions are still unknown. The new interaction may affect the history during the reheating era through particle production. The effect of the interactions can also appear in observables such as non-Gaussianity through the loop corrections of standard model particles, as pointed out in Refs. [33,34]. The construction of a ghost-free (higher derivative) supersymmetric theory is another direction, though some attempts have already been made [35–38].

What is prominent in fermionic theory is that ghosts appear even with no higher derivatives in the usual meaning, i.e., no second derivatives and higher. As mentioned in Ref. [39] for a purely fermionic system, if N fermionic variables carry $2N$ d.o.f. in the phase space, then negative norm states inevitably appear. That is why we usually see that the canonical kinetic term of the fermionic field linearly depends on the derivative of the field like the Dirac Lagrangian. Indeed, in the case of usual Weyl fermions, such unwanted states are evaded by the existence of a sufficient number of primary constraints. In our previous paper [40], as a starting point, we studied point particle theories, the Lagrangian of which contains both

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bosonic and fermionic variables with their first derivatives, and showed that the coexistence of fermionic d.o.f. and bosonic ones allows us to have some extension of their kinetic terms. In the present paper, we apply the analysis done in Ref. [40] to the field system with scalar fields and Weyl fermions, which can be even a preliminary step toward tensor-fermion or any other interesting theories including fermionic d.o.f.

This paper is organized as follows. In Sec. II, following the previous analysis for point particle theories, we give a general formulation of the construction of covariant theory with up to first derivatives of n -scalar fields and N -Weyl fermions. We then derive $4N$ primary constraints by introducing maximally degenerate conditions. In Sec. III, we concretely write down the quadratic theory of n -scalar and one Weyl fermion fields and apply the set of the maximally degenerate conditions, which we have proposed for removing the fermionic ghosts properly. In Sec. IV, we perform Hamiltonian analysis of the quadratic theory and derive a supplementary condition such that all the primary constraints are second class. In Sec. V, we obtain the explicit counterparts in Lagrangian formulation to the conditions in Hamiltonian formulation. In Sec. VI, we show that the obtained theories satisfying the maximally degenerate conditions cannot be mapped into theories of which the Lagrangian linearly depends on the derivative of the fermionic fields. In Sec. VII, we give a summary of our work. In Appendix A, we summarize definitions and identities of the Pauli matrices. In Appendix B, we show the equivalence between the maximally degenerate conditions and the primary constraints obtained in Sec. II. In Appendix C, we extend our analysis in Sec. III to multiple Weyl fermions and derive the primary constraints.

II. GENERAL FORMALISM FOR CONSTRUCTING DEGENERATE LAGRANGIAN

In the present paper, we construct flat space-time theories with n real scalar fields $\phi^a(t, \mathbf{x})$ ($a = 1, \dots, n$),

N Weyl fermionic fields $\psi_I^\alpha(t, \mathbf{x})$, and their Hermitian conjugates $\bar{\psi}_I^{\dot{\alpha}}(t, \mathbf{x})$ ($\alpha, \dot{\alpha} = 1$ or 2 , $I = 1, \dots, N$). Following our previous work [40], we consider the Lagrangian of which the fields carry up to the first derivative. The most general action is symbolized by

$$S = \int d^4x \mathcal{L}[\phi^a, \partial_\mu \phi^a, \psi_I^\alpha, \partial_\mu \psi_I^\alpha, \bar{\psi}_I^{\dot{\alpha}}, \partial_\mu \bar{\psi}_I^{\dot{\alpha}}]. \quad (1)$$

Here, the Lorentzian indices are raised and lowered by the Minkowski metric $\eta_{\mu\nu}$, and the fermionic indices are raised and lowered by the antisymmetric tensors $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}}$. In addition, the capital latin indices ($I, J, K, \dots$) are contracted with the Kronecker delta δ_{IJ} such as $\psi_I^\alpha \psi_{J,\alpha} := \delta_{IJ} \psi_I^\alpha \psi_{J,\alpha}$. Throughout this paper, we use the metric signature $(+, -, -, -)$.

When the Lagrangian (1) consists only of scalar fields, the absence of Ostrogradsky's ghosts is automatically ensured since the Euler-Lagrange equations contain no more than second derivatives with respect to time. On the other hand, second derivatives in fermionic equations of motion are generally dangerous because extra d.o.f. in the fermionic sector immediately lead to negative norm states (see Refs. [39,40] for the detail.). In order to avoid such ghost d.o.f., the system must contain the appropriate number of constraints, which originate from the degeneracy of the kinetic matrix. In this section, we derive degeneracy conditions and a supplementary condition for the Lagrangian (1). The basic treatment of Grassmann algebra and Hamiltonian formulation, of which we are making use, are summarized in, e.g., Refs. [39,40], and we use the left derivative throughout this paper.

We find degeneracy conditions of (1), which yield an appropriate number of primary constraints, eliminating half of the d.o.f. of fermions in phase space. For deriving the conditions, we first take a look at the canonical momenta defined as

$$\pi_{\phi^a}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}, \quad \pi_{\psi_I^\alpha}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_I^\alpha)}, \quad \pi_{\bar{\psi}_I^{\dot{\alpha}}}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_I^{\dot{\alpha}})} = -(\pi_{\psi_I^\alpha}^\mu)^\dagger. \quad (2)$$

Then, the variations of the 0th component of canonical momenta with respect to all canonical variables give the following set of equations,

$$\begin{pmatrix} \delta \pi_{\phi^a} \\ \delta \pi_{\psi_I^\alpha} \\ \delta \pi_{\bar{\psi}_I^{\dot{\alpha}}} \end{pmatrix} = \mathbf{K} \begin{pmatrix} \delta \dot{\phi}^a \\ \delta \dot{\psi}_I^\alpha \\ \delta \dot{\bar{\psi}}_I^{\dot{\alpha}} \end{pmatrix} + \begin{pmatrix} \delta z_a^{(\phi)} \\ \delta z_{\alpha,I}^{(\psi)} \\ \delta z_{\dot{\alpha},I}^{(\bar{\psi})} \end{pmatrix}, \quad (3)$$

where we have omitted the superscript of the 0th component, i.e., $\pi_A \equiv \pi_A^0$, and defined the kinetic matrix,

$$\mathbf{K} = \begin{pmatrix} A_{ab} & \mathcal{B}_{a\beta,J} & \mathcal{B}_{a\dot{\beta},J} \\ \mathcal{C}_{ab,I} & D_{a\beta,IJ} & D_{a\dot{\beta},IJ} \\ \mathcal{C}_{\dot{a}b,I} & D_{\dot{a}\beta,IJ} & D_{\dot{a}\dot{\beta},IJ} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{\dot{\phi}^a \dot{\phi}^b} & -\mathcal{L}_{\dot{\phi}^a \dot{\psi}_J^\beta} & -\mathcal{L}_{\dot{\phi}^a \dot{\bar{\psi}}_J^{\dot{\beta}}} \\ \mathcal{L}_{\dot{\psi}_I^\alpha \dot{\phi}^b} & \mathcal{L}_{\dot{\psi}_I^\alpha \dot{\psi}_J^\beta} & \mathcal{L}_{\dot{\psi}_I^\alpha \dot{\bar{\psi}}_J^{\dot{\beta}}} \\ \mathcal{L}_{\dot{\bar{\psi}}_I^{\dot{\alpha}} \dot{\phi}^b} & \mathcal{L}_{\dot{\bar{\psi}}_I^{\dot{\alpha}} \dot{\psi}_J^\beta} & \mathcal{L}_{\dot{\bar{\psi}}_I^{\dot{\alpha}} \dot{\bar{\psi}}_J^{\dot{\beta}}} \end{pmatrix}, \quad (4)$$

and

$$\begin{pmatrix} \delta z_a^{(\phi)} \\ \delta z_{\alpha,I}^{(\psi)} \\ \delta z_{\dot{\alpha},I}^{(\bar{\psi})} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{\dot{\phi}^a \partial_i \phi^b} & -\mathcal{L}_{\dot{\phi}^a \partial_i \psi_J^\beta} & -\mathcal{L}_{\dot{\phi}^a \partial_i \bar{\psi}_J^{\dot{\beta}}} \\ \mathcal{L}_{\dot{\psi}_I^\alpha \partial_i \phi^b} & \mathcal{L}_{\dot{\psi}_I^\alpha \partial_i \psi_J^\beta} & \mathcal{L}_{\dot{\psi}_I^\alpha \partial_i \bar{\psi}_J^{\dot{\beta}}} \\ \mathcal{L}_{\dot{\bar{\psi}}_I^{\dot{\alpha}} \partial_i \phi^b} & \mathcal{L}_{\dot{\bar{\psi}}_I^{\dot{\alpha}} \partial_i \psi_J^\beta} & \mathcal{L}_{\dot{\bar{\psi}}_I^{\dot{\alpha}} \partial_i \bar{\psi}_J^{\dot{\beta}}} \end{pmatrix} \begin{pmatrix} \delta(\partial_i \phi^b) \\ \delta(\partial_i \psi_J^\beta) \\ \delta(\partial_i \bar{\psi}_J^{\dot{\beta}}) \end{pmatrix} + \begin{pmatrix} \mathcal{L}_{\dot{\phi}^a \phi^b} & -\mathcal{L}_{\dot{\phi}^a \psi_J^\beta} & -\mathcal{L}_{\dot{\phi}^a \bar{\psi}_J^{\dot{\beta}}} \\ \mathcal{L}_{\dot{\psi}_I^\alpha \phi^b} & \mathcal{L}_{\dot{\psi}_I^\alpha \psi_J^\beta} & \mathcal{L}_{\dot{\psi}_I^\alpha \bar{\psi}_J^{\dot{\beta}}} \\ \mathcal{L}_{\dot{\bar{\psi}}_I^{\dot{\alpha}} \phi^b} & \mathcal{L}_{\dot{\bar{\psi}}_I^{\dot{\alpha}} \psi_J^\beta} & \mathcal{L}_{\dot{\bar{\psi}}_I^{\dot{\alpha}} \bar{\psi}_J^{\dot{\beta}}} \end{pmatrix} \begin{pmatrix} \delta \phi^b \\ \delta \psi_J^\beta \\ \delta \bar{\psi}_J^{\dot{\beta}} \end{pmatrix}. \quad (5)$$

Here, we have introduced the shortcut notation,

$$\mathcal{L}_{XY} = \frac{\partial^2 \mathcal{L}}{\partial Y \partial X} = \frac{\partial}{\partial Y} \left(\frac{\partial \mathcal{L}}{\partial X} \right). \quad (6)$$

Here, A_{ab} is a symmetric matrix, while $D_{a\beta,IJ}$, $D_{a\dot{\beta},IJ}$, $D_{\dot{a}\beta,IJ}$, and $D_{\dot{a}\dot{\beta},IJ}$ are antisymmetric matrices under the exchange of the greek indices as

$$\begin{aligned} D_{a\beta,IJ} &= -D_{\beta\alpha,JI}, & D_{a\dot{\beta},IJ} &= -D_{\dot{\beta}\alpha,JI}, \\ D_{\dot{a}\beta,IJ} &= -D_{\beta\dot{a},JI}, & D_{\dot{a}\dot{\beta},IJ} &= -D_{\dot{\beta}\dot{a},JI}, \end{aligned} \quad (7)$$

and $\mathcal{B}$ and $\mathcal{C}$ are Grassmann-odd matrices and related as $\mathcal{C}_{ab,I} = -\mathcal{B}_{ba,I}$ and $\mathcal{C}_{\dot{a}b,I} = -\mathcal{B}_{b\dot{a},I}$. The Hermitian properties of $\pi_{\phi^a}^\mu$ and the anti-Hermitian properties of $\pi_{\psi_I^\alpha}^\mu$, i.e., $(\pi_{\phi^a}^\mu)^\dagger = \pi_{\phi^a}^\mu$ and $(\pi_{\psi_I^\alpha}^\mu)^\dagger = -\pi_{\bar{\psi}_I^{\dot{\alpha}}}^\mu$, lead to

$$\begin{aligned} A_{ab} &= A_{ab}^\dagger, & \mathcal{B}_{a\beta,J} &= -\mathcal{B}_{a\dot{\beta},J}^\dagger, & \mathcal{C}_{ab,I} &= -\mathcal{C}_{\dot{a}b,I}^\dagger, \\ D_{\dot{a}\beta,IJ} &= -D_{a\dot{\beta},IJ}^\dagger, & D_{a\beta,IJ} &= -D_{\dot{a}\beta,IJ}^\dagger. \end{aligned} \quad (8)$$

In the present paper, we assume that the scalar submatrix of the kinetic matrix A_{ab} is nondegenerate, i.e., invertible. This assumption is equivalent to requiring

$$\det A_{ab}^{(0)} \neq 0, \quad (9)$$

where $\det A_{ab}^{(0)}$ is defined by setting all fermionic variables and their derivatives to zero. Then, the first equation (3) can be solved for $\delta \dot{\phi}^b$,

$$\delta \dot{\phi}^b = A^{ba} (\delta \pi_{\phi^a} - B_{a\beta,J} \delta \dot{\psi}_J^\beta - B_{a\dot{\beta},J} \delta \dot{\bar{\psi}}_J^{\dot{\beta}} - \delta z_a^{(\phi)}), \quad (10)$$

where we have defined the inverse of the kinetic matrix in the scalar sector as $A^{ab} = (A^{-1})^{ab}$. Plugging this into the infinitesimal momenta of ψ^α and $\bar{\psi}^{\dot{\alpha}}$, we get

$$\begin{aligned} \delta \pi_{\psi_I^\alpha} &= (D_{a\beta,IJ} - \mathcal{C}_{ab,I} A^{ba} \mathcal{B}_{a\beta,J}) \delta \dot{\psi}_J^\beta \\ &\quad + (D_{a\dot{\beta},IJ} - \mathcal{C}_{ab,I} A^{ba} \mathcal{B}_{a\dot{\beta},J}) \delta \dot{\bar{\psi}}_J^{\dot{\beta}} \\ &\quad + \mathcal{C}_{ab,I} A^{ba} (\delta \pi_{\phi^a} - \delta z_a^{(\phi)}) + \delta z_{\alpha,I}^{(\psi)}, \end{aligned} \quad (11)$$

$$\begin{aligned} \delta \pi_{\bar{\psi}_I^{\dot{\alpha}}} &= (D_{\dot{a}\beta,IJ} - \mathcal{C}_{\dot{a}b,I} A^{ba} \mathcal{B}_{a\beta,J}) \delta \dot{\psi}_J^\beta \\ &\quad + (D_{\dot{a}\dot{\beta},IJ} - \mathcal{C}_{\dot{a}b,I} A^{ba} \mathcal{B}_{a\dot{\beta},J}) \delta \dot{\bar{\psi}}_J^{\dot{\beta}} \\ &\quad + \mathcal{C}_{\dot{a}b,I} A^{ba} (\delta \pi_{\phi^a} - \delta z_a^{(\phi)}) + \delta z_{\dot{\alpha},I}^{(\bar{\psi})}. \end{aligned} \quad (12)$$

In order to obtain the sufficient number of constraints, we need conditions that the velocity terms $\delta \dot{\psi}_I$ and $\delta \dot{\bar{\psi}}_I$ cannot be expressed in terms of the canonical variables. In the present paper, we adopt the “maximally degenerate conditions,”¹ i.e.,

$$D_{a\beta,IJ} - \mathcal{C}_{ab,I} A^{ba} \mathcal{B}_{a\beta,J} = 0, \quad D_{a\dot{\beta},IJ} - \mathcal{C}_{ab,I} A^{ba} \mathcal{B}_{a\dot{\beta},J} = 0, \quad (13)$$

$$D_{\dot{a}\beta,IJ} - \mathcal{C}_{\dot{a}b,I} A^{ba} \mathcal{B}_{a\beta,J} = 0, \quad D_{\dot{a}\dot{\beta},IJ} - \mathcal{C}_{\dot{a}b,I} A^{ba} \mathcal{B}_{a\dot{\beta},J} = 0. \quad (14)$$

Two of these four conditions are equivalent to the others since they are related through Hermitian conjugates,

$$(D_{a\beta,IJ} - \mathcal{C}_{ab,I} A^{ba} \mathcal{B}_{a\beta,J})^\dagger = -(D_{\dot{a}\beta,IJ} - \mathcal{C}_{\dot{a}b,I} A^{ba} \mathcal{B}_{a\beta,J}), \quad (15)$$

$$(D_{a\dot{\beta},IJ} - \mathcal{C}_{ab,I} A^{ba} \mathcal{B}_{a\dot{\beta},J})^\dagger = -(D_{\dot{a}\dot{\beta},IJ} - \mathcal{C}_{\dot{a}b,I} A^{ba} \mathcal{B}_{a\dot{\beta},J}). \quad (16)$$

The maximally degenerate conditions lead to the following 4N primary constraints,

¹There should be other candidates for conditions eliminating ghost d.o.f., but here we adopt the simplest case, in which all the constraints eliminating extra d.o.f. are primary constraints. See Ref. [40] for the other possibilities.

$$\Phi_{\psi_I^\alpha} \equiv \pi_{\psi_I^\alpha} - F_{\alpha,I}(\phi^a, \pi_{\phi^a}, \partial_i \phi^a, \psi_J^\beta, \partial_i \psi_J^\beta, \bar{\psi}_J^\beta, \partial_i \bar{\psi}_J^\beta) = 0, \quad (17)$$

$$\begin{aligned} \Phi_{\bar{\psi}_I^\alpha} &= -(\Phi_{\psi_I^\alpha})^\dagger \\ &= \pi_{\bar{\psi}_I^\alpha} - G_{\alpha,I}(\phi^a, \pi_{\phi^a}, \partial_i \phi^a, \psi_J^\beta, \partial_i \psi_J^\beta, \bar{\psi}_J^\beta, \partial_i \bar{\psi}_J^\beta) = 0, \end{aligned} \quad (18)$$

with $G_{\alpha,I} = -(F_{\alpha,I})^\dagger$. The explicit proof of the equivalence between Eqs. (17) and (18) and Eqs. (13) and (14) is shown in Appendix B. As discussed in Ref. [40], the existence of these primary constraints (17) and (18) based on the maximally degenerate conditions is enough to remove the extra d.o.f. in the fermionic sector. However, if these primary constraints yield secondary constraints, we have an even smaller number of physical d.o.f. Here, let us consider

that we have the maximum number of physical d.o.f. in the maximally degenerate Lagrangian, as we have with usual Weyl fields. Then, we need to check that no secondary constraints appear by examining the consistency conditions of the primary constraints. To this end, we first define the total Hamiltonian density as

$$\mathcal{H}_T = \mathcal{H} + \Phi_{\psi_I^\alpha} \lambda_I^\alpha + \Phi_{\bar{\psi}_I^\alpha} \bar{\lambda}_I^\alpha, \quad (19)$$

where the Hamiltonian and the total Hamiltonian are given by

$$H = \int d^3x \mathcal{H}, \quad H_T = \int d^3x \mathcal{H}_T. \quad (20)$$

Now, we would like to calculate the Poisson bracket, defined as

$$\begin{aligned} \{\mathcal{F}(t, \mathbf{x}), \mathcal{G}(t, \mathbf{y})\} &= \int d^3z \left[\frac{\delta \mathcal{F}(t, \mathbf{x})}{\delta \phi^a(t, \mathbf{z})} \frac{\delta \mathcal{G}(t, \mathbf{y})}{\delta \pi_{\phi^a}(t, \mathbf{z})} - \frac{\delta \mathcal{F}(t, \mathbf{x})}{\delta \pi_{\phi^a}(t, \mathbf{z})} \frac{\delta \mathcal{G}(t, \mathbf{y})}{\delta \phi^a(t, \mathbf{z})} \right. \\ &\quad \left. + (-)^{\varepsilon_{\mathcal{F}}} \left(\frac{\delta \mathcal{F}(t, \mathbf{x})}{\delta \psi_I^\alpha(t, \mathbf{z})} \frac{\delta \mathcal{G}(t, \mathbf{y})}{\delta \pi_{\psi_I^\alpha}(t, \mathbf{z})} + \frac{\delta \mathcal{F}(t, \mathbf{x})}{\delta \pi_{\psi_I^\alpha}(t, \mathbf{z})} \frac{\delta \mathcal{G}(t, \mathbf{y})}{\delta \psi_I^\alpha(t, \mathbf{z})} + \frac{\delta \mathcal{F}(t, \mathbf{x})}{\delta \bar{\psi}_I^\alpha(t, \mathbf{z})} \frac{\delta \mathcal{G}(t, \mathbf{y})}{\delta \pi_{\bar{\psi}_I^\alpha}(t, \mathbf{z})} + \frac{\delta \mathcal{F}(t, \mathbf{x})}{\delta \pi_{\bar{\psi}_I^\alpha}(t, \mathbf{z})} \frac{\delta \mathcal{G}(t, \mathbf{y})}{\delta \bar{\psi}_I^\alpha(t, \mathbf{z})} \right) \right]. \end{aligned} \quad (21)$$

The Poisson brackets between the canonical variables are given by

$$\{\phi^a(t, \mathbf{x}), \pi_{\phi^b}(t, \mathbf{y})\} = \delta^a_b \delta^3(\mathbf{x} - \mathbf{y}), \quad (22)$$

$$\{\psi_I^\alpha(t, \mathbf{x}), \pi_{\psi_J^\beta}(t, \mathbf{y})\} = -\delta^\alpha_\beta \delta_{IJ} \delta^3(\mathbf{x} - \mathbf{y}), \quad (23)$$

$$\{\bar{\psi}_I^\alpha(t, \mathbf{x}), \pi_{\bar{\psi}_J^\beta}(t, \mathbf{y})\} = -\delta^\alpha_\beta \delta_{IJ} \delta^3(\mathbf{x} - \mathbf{y}), \quad (24)$$

while other Poisson brackets are zero. Then, the Poisson brackets between the primary constraints are

$$\{\Phi_{\psi_I^\alpha}(t, \mathbf{x}), \Phi_{\psi_J^\beta}(t, \mathbf{y})\} = \frac{\delta F_{\alpha,I}(t, \mathbf{x})}{\delta \psi_J^\beta(t, \mathbf{y})} + \frac{\delta F_{\beta,J}(t, \mathbf{y})}{\delta \psi_I^\alpha(t, \mathbf{x})} + \int d^3z \left[\frac{\delta F_{\alpha,I}(t, \mathbf{x})}{\delta \phi^a(t, \mathbf{z})} \frac{\delta F_{\beta,J}(t, \mathbf{y})}{\delta \pi_{\phi^a}(t, \mathbf{z})} - \frac{\delta F_{\alpha,I}(t, \mathbf{x})}{\delta \pi_{\phi^a}(t, \mathbf{z})} \frac{\delta F_{\beta,J}(t, \mathbf{y})}{\delta \phi^a(t, \mathbf{z})} \right], \quad (25)$$

$$\{\Phi_{\psi_I^\alpha}(t, \mathbf{x}), \Phi_{\bar{\psi}_J^\beta}(t, \mathbf{y})\} = \frac{\delta F_{\alpha,I}(t, \mathbf{x})}{\delta \bar{\psi}_J^\beta(t, \mathbf{y})} + \frac{\delta G_{\beta,J}(t, \mathbf{y})}{\delta \psi_I^\alpha(t, \mathbf{x})} + \int d^3z \left[\frac{\delta F_{\alpha,I}(t, \mathbf{x})}{\delta \phi^a(t, \mathbf{z})} \frac{\delta G_{\beta,J}(t, \mathbf{y})}{\delta \pi_{\phi^a}(t, \mathbf{z})} - \frac{\delta F_{\alpha,I}(t, \mathbf{x})}{\delta \pi_{\phi^a}(t, \mathbf{z})} \frac{\delta G_{\beta,J}(t, \mathbf{y})}{\delta \phi^a(t, \mathbf{z})} \right], \quad (26)$$

$$\{\Phi_{\bar{\psi}_I^\alpha}(t, \mathbf{x}), \Phi_{\bar{\psi}_J^\beta}(t, \mathbf{y})\} = \frac{\delta G_{\alpha,I}(t, \mathbf{x})}{\delta \bar{\psi}_J^\beta(t, \mathbf{y})} + \frac{\delta G_{\beta,J}(t, \mathbf{y})}{\delta \bar{\psi}_I^\alpha(t, \mathbf{x})} + \int d^3z \left[\frac{\delta G_{\alpha,I}(t, \mathbf{x})}{\delta \phi^a(t, \mathbf{z})} \frac{\delta G_{\beta,J}(t, \mathbf{y})}{\delta \pi_{\phi^a}(t, \mathbf{z})} - \frac{\delta G_{\alpha,I}(t, \mathbf{x})}{\delta \pi_{\phi^a}(t, \mathbf{z})} \frac{\delta G_{\beta,J}(t, \mathbf{y})}{\delta \phi^a(t, \mathbf{z})} \right]. \quad (27)$$

Then, the time evolution of the primary constraints are

$$\begin{pmatrix} \dot{\Phi}_{\psi_I^\alpha}(t, \mathbf{x}) \\ \dot{\Phi}_{\bar{\psi}_I^\alpha}(t, \mathbf{x}) \end{pmatrix} = \begin{pmatrix} \{\Phi_{\psi_I^\alpha}(t, \mathbf{x}), H_T\} \\ \{\Phi_{\bar{\psi}_I^\alpha}(t, \mathbf{x}), H_T\} \end{pmatrix} = \begin{pmatrix} \{\Phi_{\psi_I^\alpha}(t, \mathbf{x}), H\} \\ \{\Phi_{\bar{\psi}_I^\alpha}(t, \mathbf{x}), H\} \end{pmatrix} + \int d^3y \mathbf{C}_{IJ}(t, \mathbf{x}, \mathbf{y}) \begin{pmatrix} \lambda_J^\beta(t, \mathbf{y}) \\ \bar{\lambda}_J^\beta(t, \mathbf{y}) \end{pmatrix} \approx 0, \quad (28)$$

where

$$\mathbf{C}_{IJ}(t, \mathbf{x}, \mathbf{y}) = \begin{pmatrix} \{\Phi_{\psi_I^\alpha}(t, \mathbf{x}), \Phi_{\psi_J^\beta}(t, \mathbf{y})\} & \{\Phi_{\psi_I^\alpha}(t, \mathbf{x}), \Phi_{\bar{\psi}_J^\beta}(t, \mathbf{y})\} \\ \{\Phi_{\bar{\psi}_I^\alpha}(t, \mathbf{x}), \Phi_{\psi_J^\beta}(t, \mathbf{y})\} & \{\Phi_{\bar{\psi}_I^\alpha}(t, \mathbf{x}), \Phi_{\bar{\psi}_J^\beta}(t, \mathbf{y})\} \end{pmatrix}. \quad (29)$$

In order not to have secondary constraints, all the Lagrange multipliers should be fixed by the equation (28). This can be realized if the coefficient matrix of the Lagrange multipliers (29) has the inverse after integrating over $\mathbf{y}$, and then the primary constraints (17) and (18) are second class. In this case, the number of d.o.f. is

$$\text{d.o.f.} = \frac{2 \times n(\text{boson}) + 2 \times 4N(\text{fermions}) - 4N(\text{constraints})}{2} = n(\text{boson}) + 2N(\text{fermions}), \quad (30)$$

as desired. To find explicit expressions of these obtained conditions, one needs a concrete Lagrangian form. Hereafter, we will consider the most simplest case $N = 1$. The extension to multiple Weyl fields can be done in the same way, although the analysis will be tedious. A brief introduction for constructing Lagrangians for arbitrary N is given in Appendix C.

III. DEGENERATE SCALAR-FERMION THEORIES

Since the fermionic fields obey the Grassmann algebra, one can dramatically simplify the analysis by restricting the number of fermionic fields. Hereafter, we confine the theory to n scalar fields and one Weyl field to simplify the analysis. In this section, we construct the most general scalar-fermion theory of which the Lagrangian contains up to quadratic in first derivatives of scalar and fermionic fields. To this end, we first construct Lorentz scalars and vectors without any derivatives, which consist of the scalar fields ϕ^a and the fermionic fields ψ^α , $\bar{\psi}^{\dot{\alpha}}$. The fermionic indices can be contracted with the building block matrices, $\varepsilon_{\alpha\beta}$, $\varepsilon_{\dot{\alpha}\dot{\beta}}$, $\sigma_{\alpha\dot{\alpha}}^\mu$, $(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}$, and $(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}$,² and we have three possibilities,

$$\Psi = \psi^\alpha \psi_\alpha, \quad \bar{\Psi} = \bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}, \quad J^\mu = \bar{\psi}^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^\mu \psi^\alpha, \quad (31)$$

where $\Psi^\dagger = \bar{\Psi}$. Note that any contractions of $(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}$ and $(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}$ with the fermionic fields always vanish due to the Grassmann properties.³ One can also construct a scalar quantity from the square of J^μ contracted with the Minkowski metric. However, it reduces to

$$\eta_{\mu\nu} J^\mu J^\nu = -2\Psi\bar{\Psi}, \quad (32)$$

where we have used (A4). Furthermore, the square of Ψ vanishes because of the fact that $\Psi^2 \propto \psi^1 \psi^2 \psi^1 \psi^2$ and the Grassmann property, i.e., $\Psi^2 = \bar{\Psi}^2 = 0$, while $\Psi\bar{\Psi}$ is

²See Appendix A for the detailed definition of each matrix.

³For $N \neq 1$, $(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}$ and $(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}$ can be contracted with different Weyl fields, and therefore rank-2 tensors exist. See Appendix C for the detail.

nonzero. This implies that an arbitrary function $\mathcal{A}(\phi^a, \Psi, \bar{\Psi})$ can be expanded in terms of Ψ and $\bar{\Psi}$ as

$$\begin{aligned} \mathcal{A}(\phi^a, \Psi, \bar{\Psi}) &= a_0(\phi^a) + a_1(\phi^a)\Psi + a_2(\phi^a)\bar{\Psi} \\ &\quad + a_3(\phi^a)\Psi\bar{\Psi}. \end{aligned} \quad (33)$$

When the arbitrary function $\mathcal{A}$ is real, i.e., $\mathcal{A} = \mathcal{A}^\dagger$, then a_0 and a_3 are also real, and $a_1^* = a_2$. The inverse of $\mathcal{A}(\phi^a, \Psi, \bar{\Psi})$ is given by

$$\begin{aligned} \mathcal{A}^{-1}(\phi^a, \Psi, \bar{\Psi}) &= a_0^{-1}(\phi^a) \left(1 - \frac{a_1(\phi^a)}{a_0(\phi^a)}\Psi - \frac{a_2(\phi^a)}{a_0(\phi^a)}\bar{\Psi} \right. \\ &\quad \left. + \frac{2a_1(\phi^a)a_2(\phi^a) - a_0(\phi^a)a_3(\phi^a)}{a_0(\phi^a)^2}\Psi\bar{\Psi} \right), \end{aligned} \quad (34)$$

and the condition that $\mathcal{A}(\phi^a, \Psi, \bar{\Psi})$ has an inverse is simply $a_0(\phi^a) \neq 0$.

As we will show in the following subsection, the most general action up to quadratic in first derivatives of the scalar fields and the Weyl fermion can be written as

$$S = \int d^4x (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2), \quad (35)$$

where

$$\mathcal{L}_0 = P_0, \quad (36)$$

$$\begin{aligned} \mathcal{L}_1 &= P_a^{(1)} \partial_\mu \phi^a J^\mu + P^{(2)} (\bar{\psi}^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \psi^\alpha - \partial_\mu \bar{\psi}^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^\mu \psi^\alpha) \\ &\quad + P^{(3)} J^\mu \psi_\alpha \partial_\mu \psi^\alpha + P^{(3)\dagger} J^\mu \partial_\mu \bar{\psi}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}, \end{aligned} \quad (37)$$

$$\begin{aligned} \mathcal{L}_2 &= \frac{1}{2} V_{ab}^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b + S_{aa}^{\mu\nu} \partial_\mu \phi^a \partial_\nu \psi^\alpha + \partial_\mu \phi^a \partial_\nu \bar{\psi}^{\dot{\alpha}} (S_{aa}^{\mu\nu})^\dagger \\ &\quad + \frac{1}{2} W_{\alpha\beta}^{\mu\nu} \partial_\mu \psi^\alpha \partial_\nu \psi^\beta + \frac{1}{2} \partial_\nu \bar{\psi}^{\dot{\beta}} \partial_\mu \bar{\psi}^{\dot{\alpha}} (W_{\alpha\beta}^{\mu\nu})^\dagger \\ &\quad + \partial_\mu \bar{\psi}^{\dot{\alpha}} Q_{\alpha\dot{\alpha}}^{\mu\nu} \partial_\nu \psi^\alpha, \end{aligned} \quad (38)$$

and

$$\begin{aligned}
V_{ab}^{\mu\nu} &= V_{ab}\eta^{\mu\nu}, \\
S_{a\alpha}^{\mu\nu} &= S_a^{(1)}\eta^{\mu\nu}\psi_\alpha + S_a^{(2)}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta, \\
W_{\alpha\beta}^{\mu\nu} &= W_1\eta^{\mu\nu}\varepsilon_{\alpha\beta} + W_2\eta^{\mu\nu}\psi_\alpha\psi_\beta + W_3(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta} \\
&\quad + W_4[\psi_\alpha(\sigma^{\mu\nu}\varepsilon)_{\beta\gamma} + \psi_\beta(\sigma^{\mu\nu}\varepsilon)_{\alpha\gamma}]\psi^\gamma, \\
Q_{\alpha\dot{\alpha}}^{\mu\nu} &= Q_1\eta^{\mu\nu}\psi_\alpha\bar{\psi}_{\dot{\alpha}} + Q_2(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta\bar{\psi}_{\dot{\alpha}} - Q_2^\dagger\psi_\alpha\bar{\psi}^{\dot{\beta}}(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\beta}\dot{\alpha}} \\
&\quad + Q_3(\sigma^{\rho\mu}\varepsilon)_{\alpha\beta}\psi^\beta\bar{\psi}^{\dot{\beta}}(\varepsilon\bar{\sigma}^{\rho\nu})_{\dot{\beta}\dot{\alpha}}. \quad (39)
\end{aligned}$$

Here, P_0 , $P^{(i)}$, V , $S^{(i)}$, W_i , and Q_i are arbitrary functions of ϕ^a , Ψ , and $\bar{\Psi}$. These arbitrary functions satisfy the following properties:

$$\begin{aligned}
V_{ab} &= V_{ba}, & V_{ab} &= V_{ab}^\dagger, \\
P_0 &= P_0^\dagger, & P_a^{(1)} &= P_a^{(1)\dagger}, & P^{(2)} &= -P^{(2)\dagger}, \\
Q_1 &= Q_1^\dagger, & Q_3 &= Q_3^\dagger. \quad (40)
\end{aligned}$$

A. Construction of Lagrangian up to quadratic in the derivatives

In this subsection, we derive the most general Lagrangian (35)–(40) one by one.

1. No derivative term

The Lagrangian without any derivatives should be the arbitrary function of the form in (33); thus,

$$\mathcal{L}_0 = P_0(\phi^a, \Psi, \bar{\Psi}), \quad (41)$$

where P_0 is an arbitrary function of ϕ^a , Ψ , and $\bar{\Psi}$, and the Hermitian property of the Lagrangian requires $P_0 = P_0^\dagger$.

2. Linear terms in the derivatives

The candidate for the Lagrangian including terms proportional to the first derivatives is

$$\begin{aligned}
\mathcal{L}_1 &= P_a^{(1)}\partial_\mu\phi^a J^\mu + P^{(2)}\bar{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu\psi^\alpha + P^{(2)\dagger}\partial_\mu\bar{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu\psi^\alpha \\
&\quad + P^{(3)}J^\mu\psi_\alpha\partial_\mu\psi^\alpha + P^{(3)\dagger}J^\mu\partial_\mu\bar{\psi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}, \quad (42)
\end{aligned}$$

where $P^{(i)}$ are arbitrary functions of ϕ^a , Ψ , and $\bar{\Psi}$ and the Hermitian property of the Lagrangian requires $P_a^{(1)} = P_a^{(1)\dagger}$. The second and third terms can be decomposed as

$$\begin{aligned}
&P^{(2)}\bar{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu\psi^\alpha + P^{(2)\dagger}\partial_\mu\bar{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu\psi^\alpha \\
&= \frac{1}{2}(P^{(2)} + P^{(2)\dagger})\partial_\mu J^\mu \\
&\quad + \frac{1}{2}(P^{(2)} - P^{(2)\dagger})(\bar{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu\psi^\alpha - \partial_\mu\bar{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu\psi^\alpha). \quad (43)
\end{aligned}$$

If we integrate the first term in the right-hand side by parts, it becomes

$$\begin{aligned}
&-\frac{1}{2}(P^{(2)} + P^{(2)\dagger})_{,\phi^a}\partial_\mu\phi^a J^\mu + (P^{(2)} + P^{(2)\dagger})_{,\Psi}J^\mu\psi_\alpha\partial_\mu\psi^\alpha \\
&\quad + (P^{(2)} + P^{(2)\dagger})_{,\bar{\Psi}}J^\mu\partial_\mu\bar{\psi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}, \quad (44)
\end{aligned}$$

where $P_{,z} \equiv \partial P/\partial z$ and $P_{,z}^\dagger \equiv \partial P^\dagger/\partial z$. Thus, the real part of $P^{(2)}$ can be absorbed into the other terms in $\mathcal{L}_1$, and thus we can impose $P^{(2)} = -P^{(2)\dagger}$ without loss of generality.

3. Quadratic terms in the derivatives

Next, the Lagrangian containing two derivatives is

$$\begin{aligned}
\mathcal{L}_2 &= \frac{1}{2}V_{ab}^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b + S_{a\alpha}^{\mu\nu}\partial_\mu\phi^a\partial_\nu\psi^\alpha + \partial_\mu\phi^a\partial_\nu\bar{\psi}^{\dot{\alpha}}(S_{a\alpha}^{\mu\nu})^\dagger \\
&\quad + \frac{1}{2}W_{\alpha\beta}^{\mu\nu}\partial_\mu\psi^\alpha\partial_\nu\psi^\beta + \frac{1}{2}\partial_\nu\bar{\psi}^{\dot{\beta}}\partial_\mu\bar{\psi}^{\dot{\alpha}}(W_{\alpha\beta}^{\mu\nu})^\dagger \\
&\quad + \partial_\mu\bar{\psi}^{\dot{\alpha}}Q_{\alpha\dot{\alpha}}^{\mu\nu}\partial_\nu\psi^\alpha, \quad (45)
\end{aligned}$$

where $V_{ab}^{\mu\nu}$, $S_{a\alpha}^{\mu\nu}$, $W_{\alpha\beta}^{\mu\nu}$, and $Q_{\alpha\dot{\alpha}}^{\mu\nu}$ consist of ϕ^a , ψ^α , $\bar{\psi}^{\dot{\alpha}}$, and constant matrices. The coefficient $V_{ab}^{\mu\nu}$ should take the form

$$V_{ab}^{\mu\nu} = V_{ab}^{(1)}\eta^{\mu\nu} + V_{ab}^{(2)}J^\mu J^\nu, \quad (46)$$

where $V_{ab}^{(i)}$ are arbitrary functions of ϕ^a , Ψ , and $\bar{\Psi}$. However, the identity (A6) leads to

$$J^\mu J^\nu = -\frac{1}{2}\eta^{\mu\nu}\Psi\bar{\Psi}. \quad (47)$$

Therefore, the $V_{ab}^{(2)}$ term can be absorbed into the $V_{ab}^{(1)}$ term, and we can generally set $V_{ab}^{(2)} = 0$. As a result, we have

$$V_{ab}^{\mu\nu} = V_{ab}\eta^{\mu\nu}, \quad (48)$$

where V_{ab} is an arbitrary real function of ϕ^a , Ψ , and $\bar{\Psi}$, which is symmetric under the exchange of a and b .

Second, the general form of $S_{a\alpha}^{\mu\nu}$ is

$$\begin{aligned}
S_{a\alpha}^{\mu\nu} &= S_a^{(1)}\eta^{\mu\nu}\psi_\alpha + S_a^{(2)}J^\mu\bar{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\nu + S_a^{(3)}J^\nu\bar{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu \\
&\quad + S_a^{(4)}J^\mu J^\nu\psi_\alpha, \quad (49)
\end{aligned}$$

where $S_a^{(i)}$ are arbitrary functions of ϕ^a , Ψ , and $\bar{\Psi}$. By using (A5) and (A6), the second and third terms can be rewritten as

$$J^\mu\bar{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\nu = \frac{1}{2}\bar{\Psi}\eta^{\mu\nu}\psi_\alpha + \bar{\Psi}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta, \quad (50)$$

$$J^\nu\bar{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu = \frac{1}{2}\bar{\Psi}\eta^{\mu\nu}\psi_\alpha - \bar{\Psi}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta. \quad (51)$$

Therefore, the second and third terms can be compactly expressed as $S_a^{(2)}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta$, which is antisymmetric under

μ and ν , by absorbing the symmetric remaining parts into $S_a^{(1)}$. The last term with $S_a^{(4)}$ is proportional to $\psi^1\psi^1$ or $\psi^2\psi^2$ in a component expression; thus, this automatically vanishes. As a result, we can assign

$$S_{aa}^{\mu\nu} = S_a^{(1)}\eta^{\mu\nu}\psi_\alpha + S_a^{(2)}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta. \quad (52)$$

Third, the general form of $W_{\alpha\beta}^{\mu\nu}$ is

$$\begin{aligned} W_{\alpha\beta}^{\mu\nu} = & W_1\eta^{\mu\nu}\varepsilon_{\alpha\beta} + W_2\eta^{\mu\nu}\psi_\alpha\psi_\beta + \tilde{W}_3(\tilde{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu)(\tilde{\psi}^{\dot{\beta}}\sigma_{\beta\dot{\beta}}^\nu) \\ & + \tilde{W}_4(\tilde{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\nu)(\tilde{\psi}^{\dot{\beta}}\sigma_{\beta\dot{\beta}}^\mu) + W_5J^\mu\psi_\alpha\tilde{\psi}^{\dot{\beta}}\sigma_{\beta\dot{\beta}}^\nu \\ & + W_6J^\nu\psi_\alpha\tilde{\psi}^{\dot{\beta}}\sigma_{\beta\dot{\beta}}^\mu + W_7J^\mu J^\nu\psi_\alpha\psi_\beta. \end{aligned} \quad (53)$$

We note that $J^\mu\psi_\beta\tilde{\psi}^{\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\nu \simeq -J^\nu\psi_\alpha\tilde{\psi}^{\dot{\beta}}\sigma_{\beta\dot{\beta}}^\mu$ and $J^\nu\psi_\beta\tilde{\psi}^{\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\mu \simeq -J^\mu\psi_\alpha\tilde{\psi}^{\dot{\beta}}\sigma_{\beta\dot{\beta}}^\nu$ hold as long as they are contracted with $\partial_\mu\psi^\alpha\partial_\nu\psi^\beta$. Similarly to the case of $S^{(2)}$ and $S^{(3)}$, we can rewrite $\tilde{W}_3$, $\tilde{W}_4$, W_5 , and W_6 as

$$(\tilde{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu)(\tilde{\psi}^{\dot{\beta}}\sigma_{\beta\dot{\beta}}^\nu) = \frac{1}{2}\tilde{\Psi}\eta^{\mu\nu}\varepsilon_{\alpha\beta} - \tilde{\Psi}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}, \quad (54)$$

$$(\tilde{\psi}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\nu)(\tilde{\psi}^{\dot{\beta}}\sigma_{\beta\dot{\beta}}^\mu) = \frac{1}{2}\tilde{\Psi}\eta^{\mu\nu}\varepsilon_{\alpha\beta} + \tilde{\Psi}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}. \quad (55)$$

$$J^\mu\psi_\alpha\tilde{\psi}^{\dot{\beta}}\sigma_{\beta\dot{\beta}}^\nu = \frac{1}{2}\tilde{\Psi}\eta^{\mu\nu}\psi_\alpha\psi_\beta + \tilde{\Psi}\psi_\alpha(\sigma^{\mu\nu}\varepsilon)_{\beta\gamma}\psi^\gamma, \quad (56)$$

$$J^\nu\psi_\alpha\tilde{\psi}^{\dot{\beta}}\sigma_{\beta\dot{\beta}}^\mu = \frac{1}{2}\tilde{\Psi}\eta^{\mu\nu}\psi_\alpha\psi_\beta - \tilde{\Psi}\psi_\alpha(\sigma^{\mu\nu}\varepsilon)_{\beta\gamma}\psi^\gamma. \quad (57)$$

Thus, the symmetric parts of $\tilde{W}_3$, $\tilde{W}_4$ and W_5 , W_6 can be expressed in terms of $\eta^{\mu\nu}\varepsilon_{\alpha\beta}$ and $\eta^{\mu\nu}\psi_\alpha\psi_\beta$. On the other hand, antisymmetric parts of them are taken into account by $(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}$ and $\psi_\alpha(\sigma^{\mu\nu}\varepsilon)_{\beta\gamma}\psi^\gamma$. As before, the last term with W_7 is proportional to $\psi^1\psi^1$ in a component expression; thus, this automatically vanishes. As a result, we can express $W_{\alpha\beta}^{\mu\nu}$ as

$$\begin{aligned} W_{\alpha\beta}^{\mu\nu} = & W_1\eta^{\mu\nu}\varepsilon_{\alpha\beta} + W_2\eta^{\mu\nu}\psi_\alpha\psi_\beta + W_3(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta} \\ & + W_4[\psi_\alpha(\sigma^{\mu\nu}\varepsilon)_{\beta\gamma} + \psi_\beta(\sigma^{\mu\nu}\varepsilon)_{\alpha\gamma}]\psi^\gamma. \end{aligned} \quad (58)$$

The W_4 term is manifestly symmetrized by making use of the fact that $\psi_\alpha(\sigma^{\mu\nu}\varepsilon)_{\beta\gamma}\psi^\gamma \simeq \psi_\beta(\sigma^{\mu\nu}\varepsilon)_{\alpha\gamma}\psi^\gamma$ in the Lagrangian, which holds again since they are contracted with $\partial_\mu\psi^\alpha\partial_\nu\psi^\beta$.

Finally, let us take a look at $Q_{\alpha\dot{\alpha}}^{\mu\nu}$. The general form is given by

$$\begin{aligned} Q_{\alpha\dot{\alpha}}^{\mu\nu} = & Q_1\eta^{\mu\nu}\psi_\alpha\tilde{\psi}_{\dot{\alpha}} + \tilde{Q}_2(\tilde{\psi}^{\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\mu)(\sigma_{\beta\dot{\alpha}}^\nu\psi^\beta) + \tilde{Q}_3(\tilde{\psi}^{\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\nu)(\sigma_{\beta\dot{\alpha}}^\mu\psi^\beta) + Q_4J^\mu\sigma_{\alpha\dot{\alpha}}^\nu + Q_5J^\nu\sigma_{\alpha\dot{\alpha}}^\mu \\ & + Q_6J^\mu J^\nu\psi_\alpha\tilde{\psi}_{\dot{\alpha}} + Q_7J^\mu\psi_\alpha\psi^\beta\sigma_{\beta\dot{\alpha}}^\nu + Q_8J^\nu\psi_\alpha\psi^\beta\sigma_{\beta\dot{\alpha}}^\mu + Q_9J^\mu\tilde{\psi}_{\dot{\alpha}}\tilde{\psi}^{\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\nu + Q_{10}J^\nu\tilde{\psi}_{\dot{\alpha}}\tilde{\psi}^{\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\mu. \end{aligned} \quad (59)$$

The last five terms, $Q_{6,7,8,9,10}$, contain more than one ψ^1 , ψ^2 , $\tilde{\psi}^1$, or $\tilde{\psi}^2$, and therefore they automatically vanish. We can rewrite $\tilde{Q}_2$, $\tilde{Q}_3$, Q_4 , and Q_5 as

$$(\tilde{\psi}^{\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\mu)(\sigma_{\beta\dot{\alpha}}^\nu\psi^\beta) = -\frac{1}{2}\eta^{\mu\nu}\psi_\alpha\tilde{\psi}_{\dot{\alpha}} + (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta\tilde{\psi}_{\dot{\alpha}} - \psi_\alpha(\varepsilon\tilde{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\tilde{\psi}^{\dot{\beta}} - 2(\sigma^{\rho\mu}\varepsilon)_{\alpha\beta}\psi^\beta(\varepsilon\tilde{\sigma}^{\rho\nu})_{\dot{\alpha}\dot{\beta}}\tilde{\psi}^{\dot{\beta}}, \quad (60)$$

$$(\tilde{\psi}^{\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\nu)(\sigma_{\beta\dot{\alpha}}^\mu\psi^\beta) = -\frac{1}{2}\eta^{\mu\nu}\psi_\alpha\tilde{\psi}_{\dot{\alpha}} - (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta\tilde{\psi}_{\dot{\alpha}} + \psi_\alpha(\varepsilon\tilde{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\tilde{\psi}^{\dot{\beta}} - 2(\sigma^{\rho\mu}\varepsilon)_{\alpha\beta}\psi^\beta(\varepsilon\tilde{\sigma}^{\rho\nu})_{\dot{\alpha}\dot{\beta}}\tilde{\psi}^{\dot{\beta}}. \quad (61)$$

$$J^\mu\sigma_{\alpha\dot{\alpha}}^\nu = \frac{1}{2}\eta^{\mu\nu}\psi_\alpha\tilde{\psi}_{\dot{\alpha}} + (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta\tilde{\psi}_{\dot{\alpha}} + \psi_\alpha(\varepsilon\tilde{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\tilde{\psi}^{\dot{\beta}} - 2(\sigma^{\rho\mu}\varepsilon)_{\alpha\beta}\psi^\beta(\varepsilon\tilde{\sigma}^{\rho\nu})_{\dot{\alpha}\dot{\beta}}\tilde{\psi}^{\dot{\beta}}, \quad (62)$$

$$J^\nu\sigma_{\alpha\dot{\alpha}}^\mu = \frac{1}{2}\eta^{\mu\nu}\psi_\alpha\tilde{\psi}_{\dot{\alpha}} - (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta\tilde{\psi}_{\dot{\alpha}} - \psi_\alpha(\varepsilon\tilde{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\tilde{\psi}^{\dot{\beta}} - 2(\sigma^{\rho\mu}\varepsilon)_{\alpha\beta}\psi^\beta(\varepsilon\tilde{\sigma}^{\rho\nu})_{\dot{\alpha}\dot{\beta}}\tilde{\psi}^{\dot{\beta}}. \quad (63)$$

Note that the last terms are symmetric under the replacement of μ and ν , as shown in (A8). Thus, four functions from $Q_{1,4,5}$ and $\tilde{Q}_{2,3}$ are independent. We adopt $\eta^{\mu\nu}\psi_\alpha\tilde{\psi}_{\dot{\alpha}}$, $(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta\tilde{\psi}_{\dot{\alpha}}$, $\psi_\alpha(\varepsilon\tilde{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\tilde{\psi}^{\dot{\beta}}$, and $(\sigma^{\rho\mu}\varepsilon)_{\alpha\beta}\psi^\beta(\varepsilon\tilde{\sigma}^{\rho\nu})_{\dot{\alpha}\dot{\beta}}\tilde{\psi}^{\dot{\beta}}$ as independent functions, and then we have

$$Q_{\alpha\dot{\alpha}}^{\mu\nu} = Q_1\eta^{\mu\nu}\psi_\alpha\tilde{\psi}_{\dot{\alpha}} + Q_2(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta\tilde{\psi}_{\dot{\alpha}} - Q_2^\dagger\psi_\alpha\tilde{\psi}^{\dot{\beta}}(\varepsilon\tilde{\sigma}^{\mu\nu})_{\dot{\beta}\dot{\alpha}} + Q_3(\sigma^{\rho\mu}\varepsilon)_{\alpha\beta}\psi^\beta\tilde{\psi}^{\dot{\beta}}(\varepsilon\tilde{\sigma}^{\rho\nu})_{\dot{\beta}\dot{\alpha}}. \quad (64)$$

Since the first and the fourth terms are Hermite, the coefficients are required to be real.

Collecting all the results, we obtain the most general Lagrangian (35)–(40).

B. Degeneracy conditions

In the previous subsection, we have obtained the most general Lagrangian n -scalar and one Weyl fields which contains up to quadratic in first derivatives. As explained earlier, the Euler-Lagrange equations, in general, contain

second derivatives of the fermionic fields. Thus, one should choose the arbitrary functions appearing in the Lagrangian with care in order to have the correct number of d.o.f. In this subsection, we derive the degeneracy conditions for the Lagrangian (38) with (39). (Note that the linear Lagrangian in derivatives (37) is irrelevant to the degeneracy conditions.) The maximally degenerate conditions (13) and (14) lead to

$$D_{\alpha\beta} - C_{ab}A^{ba}B_{\alpha\beta} = W_1\varepsilon_{\alpha\beta} + (W_2 + S_b^{(1)}V^{ba}S_a^{(1)})\psi_\alpha\psi_\beta = 0, \quad (65)$$

$$D_{\alpha\dot{\beta}} - C_{ab}A^{ba}B_{\alpha\dot{\beta}} = -(Q_1 + S_b^{(1)}V^{ba}(S_a^{(1)})^\dagger)\psi_\alpha\bar{\psi}_{\dot{\beta}} - Q_3(\sigma^{i0}\varepsilon)_{\alpha\gamma}\psi^\gamma\bar{\psi}^{\dot{\gamma}}(\varepsilon\bar{\sigma}^{i0})_{\dot{\gamma}\dot{\beta}} = 0, \quad (66)$$

where $V^{ab} = (V^{-1})^{ab}$. In general, they give us four conditions,⁴

$$W_1 = 0, \quad (W_2 + S_b^{(1)}V^{ba}S_a^{(1)})\psi_\alpha\psi_\beta = 0, \quad (Q_1 + S_b^{(1)}V^{ba}(S_a^{(1)})^\dagger)\psi_\alpha\bar{\psi}_{\dot{\beta}} = 0, \quad Q_3\psi^\gamma\bar{\psi}^{\dot{\gamma}} = 0. \quad (67)$$

The functions included in $\mathcal{L}_2$ are simplified after inserting these four conditions:

$$\begin{aligned} V_{ab}^{\mu\nu} &= V_{ab}\eta^{\mu\nu}, \\ S_{aa}^{\mu\nu} &= S_a^{(1)}\eta^{\mu\nu}\psi_\alpha + S_a^{(2)}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta, \\ W_{\alpha\beta}^{\mu\nu} &= -S_b^{(1)}V^{ba}S_a^{(1)}\eta^{\mu\nu}\psi_\alpha\psi_\beta + W_3(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta} + W_4[\psi_\alpha(\sigma^{\mu\nu}\varepsilon)_{\beta\gamma} + \psi_\beta(\sigma^{\mu\nu}\varepsilon)_{\alpha\gamma}]\psi^\gamma, \\ Q_{\alpha\dot{\alpha}}^{\mu\nu} &= -S_b^{(1)}V^{ba}(S_a^{(1)})^\dagger\eta^{\mu\nu}\psi_\alpha\bar{\psi}_{\dot{\alpha}} + Q_2(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi^\beta\bar{\psi}_{\dot{\alpha}} - Q_2^\dagger\psi_\alpha\bar{\psi}^{\dot{\beta}}(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\beta}\dot{\alpha}}. \end{aligned} \quad (68)$$

IV. HAMILTONIAN FORMULATION

In the previous section, we derived the Lagrangian satisfying the degeneracy conditions which lead to four primary constraints, as explicitly shown in the following. They suggest that the extra d.o.f. associated with fermionic Ostrogradsky's ghosts are removed; however, one needs to make sure of the condition that no more (secondary) constraints arise in order to have four physical d.o.f. in the phase space. Following Sec. II, we derive such a condition.

The momenta can be directly calculated from their definition,

$$\pi_{\phi^a} = P_a^{(1)}J^0 + V_{ab}\dot{\phi}^b + S_{aa}^{0\nu}\partial_\nu\psi^a + \partial_\nu\bar{\psi}^{\dot{\alpha}}(S_{aa}^{0\nu})^\dagger, \quad (69)$$

$$\begin{aligned} \pi_{\psi^a} &= -P^{(2)}\bar{\psi}^{\dot{\alpha}}\sigma_{a\dot{\alpha}}^0 - P^{(3)}J^0\psi_a - S_{aa}^{0\nu}\partial_\nu\phi^a \\ &\quad + W_{a\beta}^{0\nu}\partial_\nu\psi^\beta - \partial_\nu\bar{\psi}^{\dot{\beta}}Q_{a\dot{\beta}}^{0\nu}, \end{aligned} \quad (70)$$

⁴As far as we seek a healthy theory where we have no ghost not only in the background but also in the perturbations, these four conditions are reasonable; e.g., even if the background evolution is like $\psi_\alpha\psi_\beta \propto \varepsilon_{\alpha\beta}$ as a result of the equations of motion, the perturbations on it will face serious instabilities.

$$\pi_{\bar{\psi}^{\dot{\alpha}}} = -(\pi_{\psi^a})^\dagger. \quad (71)$$

Solving (69) for $\dot{\phi}^a$ and plugging it into (70) and (71), we indeed obtain primary constraints (17) and (18), where F_a and $G_{\dot{\alpha}}$ are given by

$$\begin{aligned} F_a &= -P^{(2)}\bar{\psi}^{\dot{\alpha}}\sigma_{a\dot{\alpha}}^0 - P^{(3)}J^0\psi_a \\ &\quad - S_{aa}^{0\nu}V^{ab}[\pi_{\phi^b} - P_b^{(1)}J^0 - S_{b\beta}^{0i}\partial_i\psi^\beta - \partial_i\bar{\psi}^{\dot{\beta}}(S_{b\beta}^{0i})^\dagger] \\ &\quad - S_{aa}^{i0}\partial_i\phi^a + W_{a\beta}^{0i}\partial_i\psi^\beta - \partial_i\bar{\psi}^{\dot{\beta}}Q_{a\dot{\beta}}^{i0}, \end{aligned} \quad (72)$$

$$G_{\dot{\alpha}} = -(F_a)^\dagger. \quad (73)$$

Then, plugging these expression into (25) and (26) and picking up the bosonic parts, we obtain

$$\begin{aligned} &\{\Phi_{\psi^a}(t, \mathbf{x}), \Phi_{\psi^b}(t, \mathbf{y})\}^{(0)} \\ &= 2(S_a^{(2)})^{(0)}\partial_i\phi^a(\sigma^{0i}\varepsilon)_{a\beta}\delta(\mathbf{x} - \mathbf{y}) \\ &\quad + (\sigma^{0i}\varepsilon)_{a\beta}\left[W_3^{(0)}(\mathbf{x})\frac{\partial}{\partial x^i}\delta(\mathbf{x} - \mathbf{y}) + W_3^{(0)}(\mathbf{y})\frac{\partial}{\partial y^i}\delta(\mathbf{x} - \mathbf{y})\right], \end{aligned} \quad (74)$$

$$\{\Phi_{\psi^\alpha}(t, \mathbf{x}), \Phi_{\bar{\psi}^\beta}(t, \mathbf{y})\}^{(0)} = -2(P^{(2)})^{(0)} \sigma_{\alpha\beta}^0 \delta(\mathbf{x} - \mathbf{y}), \quad (75)$$

where (0) represents their bosonic parts. When we calculate the time derivative of the primary constraints, we integrate the product of the constraints matrix and the Lagrange multipliers over $\mathbf{y}$. Focusing on the second term in (74),

$$\begin{aligned} & \int d^3\mathbf{y} (\sigma^{0i}\epsilon)_{\alpha\beta} \left[W_3^{(0)}(\mathbf{x}) \frac{\partial}{\partial x^i} \delta(\mathbf{x} - \mathbf{y}) + W_3^{(0)}(\mathbf{y}) \frac{\partial}{\partial y^i} \delta(\mathbf{x} - \mathbf{y}) \right] \lambda^\beta(t, \mathbf{y}) \\ &= - \int d^3\mathbf{y} (\sigma^{0i}\epsilon)_{\alpha\beta} \frac{\partial}{\partial y^i} (W_3^{(0)}(\mathbf{y})) \delta(\mathbf{x} - \mathbf{y}) \lambda^\beta(t, \mathbf{y}) \end{aligned} \quad (76)$$

holds as far as we drop total derivative terms, where we have used the fact that $(\partial/\partial x^i)\delta(\mathbf{x} - \mathbf{y}) = -(\partial/\partial y^i)\delta(\mathbf{x} - \mathbf{y})$. Using this result and the following identities,

$$\{\Phi_{\bar{\psi}^\alpha}(t, \mathbf{x}), \Phi_{\bar{\psi}^\beta}(t, \mathbf{y})\} = -\{\Phi_{\psi^\alpha}(t, \mathbf{x}), \Phi_{\psi^\beta}(t, \mathbf{y})\}^\dagger, \quad \{\Phi_{\bar{\psi}^\alpha}(t, \mathbf{x}), \Phi_{\psi^\beta}(t, \mathbf{y})\} = -\{\Phi_{\psi^\alpha}(t, \mathbf{x}), \Phi_{\bar{\psi}^\beta}(t, \mathbf{y})\}^\dagger, \quad (77)$$

we can write the bosonic part of the constraint matrix (29) as

$$\mathbf{C}^{(0)}(t, \mathbf{x}, \mathbf{y}) = \begin{pmatrix} (2S_a^{(2)} \partial_i \phi^a - \partial_i W_3)(\sigma^{0i}\epsilon)_{\alpha\beta} & -2P^{(2)} \sigma_{\alpha\beta}^0 \\ -2P^{(2)} \sigma_{\beta\alpha}^0 & -(2S_a^{(2)\dagger} \partial_i \phi^a - \partial_i W_3^\dagger)(\epsilon \bar{\sigma}^{0i})_{\dot{\alpha}\dot{\beta}} \end{pmatrix}^{(0)} \delta(\mathbf{x} - \mathbf{y}). \quad (78)$$

After performing the integration in (28), we can define the matrix $\mathbf{J}_H^{(0)}(t, \mathbf{x})$, thanks to the delta function, as

$$\mathbf{J}_H^{(0)}(t, \mathbf{x}) = \int d^3\mathbf{y} \mathbf{C}^{(0)}(t, \mathbf{x}, \mathbf{y}). \quad (79)$$

If this matrix $\mathbf{J}_H^{(0)}(t, \mathbf{x})$ is invertible, i.e., the determinant is nonzero,⁵

$$\det \mathbf{J}_H^{(0)}(t, \mathbf{x}) \neq 0, \quad (80)$$

all the Lagrange multipliers introduced in the total Hamiltonian (19) are determined by solving the simultaneous equation (28). In this case, the theory does not have secondary constraints, and all the (primary) constraints are second class. Therefore, the number of d.o.f. is $n + 2$ as counted in (30), and the extra d.o.f. are properly removed.

V. EULER-LAGRANGE EQUATIONS

In this section, we show that the degeneracy conditions and the supplementary condition obtained in the Hamiltonian formulation can be also derived in the Lagrangian formulation. In addition, we show that if the supplementary condition is satisfied, the equations of motion for fermions can always be solved in terms of the first derivative of fermionic fields.

⁵Allowing the theory to have solutions with $\phi^a = \phi^a(t)$ requires nonvanishing $P^{(2)}$.

A. Equations of motion for fermions with the maximally degenerate conditions

We derive the equations of motion for the fields and see the EOMs for fermions become the first-order (non-linear) differential equations after applying the maximally degenerate conditions.

The variations with respect to the scalar and Weyl fields yield a set of Euler-Lagrange equations,

$$\begin{pmatrix} V_{ab}^{00} & S_{a\beta}^{00} & -(S_{a\beta}^{00})^\dagger \\ -S_{ba}^{00} & W_{\alpha\beta}^{00} & -Q_{\alpha\beta}^{00} \\ (S_{ba}^{00})^\dagger & Q_{\beta\alpha}^{00} & -(W_{\alpha\beta}^{00})^\dagger \end{pmatrix} \begin{pmatrix} \ddot{\phi}^b \\ \ddot{\psi}^\beta \\ \ddot{\bar{\psi}}^{\dot{\beta}} \end{pmatrix} = \begin{pmatrix} E_a \\ \mathcal{E}_\alpha \\ \mathcal{E}_{\dot{\alpha}} \end{pmatrix}, \quad (81)$$

where the coefficient matrix in the left-hand side corresponds the kinetic matrix in (4) and we have defined

$$\begin{aligned} E_a &= \frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu (P_a^{(1)} J^\mu) - \partial_\mu V_{ab}^{\mu\nu} \partial_\nu \phi^b \\ &\quad - \partial_\mu S_{aa}^{\mu\nu} \partial_\nu \psi^\alpha - \partial_\nu \bar{\psi}^{\dot{\alpha}} \partial_\mu (S_{aa}^{\mu\nu})^\dagger \\ &\quad - V_{ab}^{ij} \partial_i \partial_j \phi^b - S_{aa}^{ij} \partial_i \partial_j \psi^\alpha - \partial_i \partial_j \bar{\psi}^{\dot{\alpha}} (S_{aa}^{ij})^\dagger, \end{aligned} \quad (82)$$

$$\begin{aligned} \mathcal{E}_\alpha &= \frac{\partial \mathcal{L}}{\partial \psi^\alpha} + \partial_\mu (P^{(2)} \bar{\psi}^{\dot{\alpha}}) \sigma_{\alpha\dot{\alpha}}^\mu + \partial_\mu (P^{(3)} J^\mu \psi_\alpha) \\ &\quad + \partial_\mu S_{aa}^{\mu\nu} \partial_\nu \phi^a - \partial_\mu W_{\alpha\beta}^{\mu\nu} \partial_\nu \psi^\beta + \partial_\nu \bar{\psi}^{\dot{\alpha}} \partial_\mu Q_{\alpha\dot{\alpha}}^{\nu\mu} \\ &\quad + S_{aa}^{ij} \partial_i \partial_j \phi^a - W_{\alpha\beta}^{ij} \partial_i \partial_j \psi^\beta + 2 \partial_i \bar{\psi}^{\dot{\alpha}} Q_{\alpha\dot{\alpha}}^{(0i)} \\ &\quad + \partial_i \partial_j \bar{\psi}^{\dot{\alpha}} Q_{\alpha\dot{\alpha}}^{ij}, \end{aligned} \quad (83)$$

where $(0i)$ means symmetrized with respect to 0 and i , and $\mathcal{E}_{\dot{\alpha}} = -(\mathcal{E}_{\alpha})^{\dagger}$. Solving (82) for $\ddot{\phi}$, we obtain

$$\ddot{\phi}^a = V^{ab}[E_b - S_{ba}^{00}\ddot{\psi}^{\alpha} - \ddot{\psi}^{\dot{\alpha}}(S_{ba}^{00})^{\dagger}]. \quad (84)$$

Eliminating $\ddot{\phi}$ from the second lines of (81), we find the equations of motion for fermions,

$$\begin{aligned} [W_{a\beta}^{00} + S_{ba}^{00}V^{ba}S_{a\beta}^{00}]\ddot{\psi}^{\beta} - [Q_{a\beta}^{00} + S_{ba}^{00}V^{ba}(S_{a\beta}^{00})^{\dagger}]\ddot{\psi}^{\dot{\beta}} \\ = \mathcal{E}_{\alpha} + S_{ba}^{00}V^{ba}E_a. \end{aligned} \quad (85)$$

The condition that the dependence of the second derivative of fermions vanishes is exactly the same as the degeneracy conditions (65), (66), and the third line of (81) is also reduced to the Hermitian conjugate. Imposing the maximally degenerate conditions, we obtain the first-order differential equations for ψ^{α} and $\bar{\psi}^{\dot{\alpha}}$,

$$\mathcal{Y}_{\alpha} \equiv \mathcal{E}_{\alpha} + S_{aa}^{00}V^{ab}E_b = 0, \quad (86)$$

$$\mathcal{Y}_{\dot{\alpha}} \equiv -(\mathcal{Y}_{\alpha})^{\dagger} = \mathcal{E}_{\dot{\alpha}} - (S_{aa}^{00})^{\dagger}V^{ab}E_b = 0, \quad (87)$$

and by substituting (82) and (83), $\mathcal{Y}_{\alpha}$ is written as

$$\begin{aligned} \mathcal{Y}_{\alpha} = \frac{\partial \mathcal{L}}{\partial \psi^{\alpha}} + \partial_{\mu}(P^{(2)}\bar{\psi}^{\dot{\alpha}})\sigma_{\alpha\dot{\alpha}}^{\mu} + \partial_{\mu}(P^{(3)}J^{\mu}\psi_{\alpha}) + \partial_{\mu}S_{a\alpha}^{\nu\mu}\partial_{\nu}\phi^a - \partial_{\mu}W_{a\beta}^{\mu\nu}\partial_{\nu}\psi^{\beta} + \partial_{\nu}\bar{\psi}^{\dot{\alpha}}\partial_{\mu}Q_{a\alpha}^{\nu\mu} \\ + S_a^{(1)}V^{ab}\psi_{\alpha}\left[\frac{\partial \mathcal{L}}{\partial \phi^b} - \partial_{\mu}(P_b^{(1)}J^{\mu}) - \partial_{\mu}V_{bc}\partial^{\mu}\phi^c - \partial_{\mu}S_{b\beta}^{\mu\nu}\partial_{\nu}\psi^{\beta} - \partial_{\nu}\bar{\psi}^{\dot{\beta}}\partial_{\mu}(S_{b\beta}^{\mu\nu})^{\dagger}\right]. \end{aligned} \quad (88)$$

We note that the dependence on $\partial_i\bar{\psi}^{\dot{\alpha}}$ and the second spatial derivatives of the fermions vanish after applying (68), and they have become the first-order differential equations for both time and spatial derivatives.

As we saw, thanks to the maximally degenerate conditions, one can remove the second derivative terms of the Weyl field in the Euler-Lagrange equations by combining the equations of motion for the scalar fields, suggesting that the number of initial conditions to solve the EOMs is appropriate. Thus, we have confirmed even in the Lagrangian formulation that the extra d.o.f. do not appear in the theory with the maximally degenerate conditions. Note that the second derivative terms of the Weyl field in the equations of motion for the scalar fields (84) can be removed by using the time derivative of (86) and (87), as discussed in Ref. [40].

B. Solvable condition for nonlinear equations including first derivatives

So far, we have seen that the maximally degenerate conditions lead to the first- (second-)order differential equations of motion for the Weyl (scalar) fields. As discussed in Ref. [40], it is not clear whether the equations of motion for fermions can be solved or not due to the Grassmann properties. Here, we derive a condition, under which the equations of motion even nonlinear in the time derivatives can be explicitly solved. We start with the simplest example in a point particle system, and then we extend this analysis to the theory we focus on in the present paper.

1. Example: Two Grassmann-odd variables

Suppose that we have a system composed of bosons q^i and two fermionic variables θ_1 and θ_2 . Due to the

maximally degenerate conditions, the equations of motion for the fermionic variables should be the first-order differential equations after appropriately combining the bosonic equations of motion. Let us assume that we have already done this process, and we can generally write down the reduced equations of motion for fermionic variables as

$$\begin{aligned} a_1\dot{\theta}_1 + a_2\dot{\theta}_2 + a_3\dot{\theta}_1\dot{\theta}_2 &= a_4, \\ b_1\dot{\theta}_1 + b_2\dot{\theta}_2 + b_3\dot{\theta}_1\dot{\theta}_2 &= b_4, \end{aligned} \quad (89)$$

where a_1, a_2, b_1 , and b_2 (a_3, a_4, b_3 , and b_4) are Grassmann-even (Grassmann-odd) functions depending on θ_{α} , q^i , and $\dot{q}^i$. We first assume at least one of $a_1^{(0)}, a_2^{(0)}, b_1^{(0)}$, and $b_2^{(0)}$ is nonzero. For instance, if $a_1^{(0)} \neq 0$, we can solve the first equation of (89) for $\dot{\theta}_1$:

$$\dot{\theta}_1 = a_1^{-1}a_4 + a_1^{-1}(-a_2 + a_3a_1^{-1}a_4)\dot{\theta}_2 =: A_1 + A_2\dot{\theta}_2. \quad (90)$$

Then, plugging this into the second equation of (89), we have

$$(b_1A_2 + b_2 + b_3A_1)\dot{\theta}_2 = b_4 - b_1A_1. \quad (91)$$

Thus, if the uniqueness condition,

$$(b_1A_2 + b_2 + b_3A_1)^{(0)} = [a_1^{-1}(-b_1a_2 + a_1b_2)]^{(0)} \neq 0, \quad (92)$$

is satisfied, one can solve the set of nonlinear equation (89). In the other case, where the equations of motion have no linear terms in $\dot{\theta}_{\alpha}$, there is no way to express each time derivative in terms of nonderivative variables, and the

equations of motion are unsolvable. Therefore, the nonzero determinant of the coefficient matrix of the linear terms in $\dot{\theta}_\alpha$ is the necessary and sufficient condition for the equations to be uniquely solved for the time derivatives of the fermionic variables.

2. More general argument

In this subsection, we will extend the previous analysis to our degenerate theory and show the equivalence between the condition (80) and the condition that the fermionic equations (86) and (87) are uniquely solved.

The equations of motion for fermions (86) and (87) include up to first time derivatives, and due to the Grassmann properties, they can be therefore generally written as [see also (88)]

$$\mathcal{V}_\alpha = \sum_n^2 \sum_m^2 \mathcal{X}_{\alpha\beta_1\cdots\beta_n\dot{\gamma}_1\cdots\dot{\gamma}_m} \dot{\psi}^{\beta_1} \cdots \dot{\psi}^{\beta_n} \dot{\bar{\psi}}^{\dot{\gamma}_1} \cdots \dot{\bar{\psi}}^{\dot{\gamma}_m} = 0, \quad (93)$$

$$\begin{aligned} \mathcal{V}_{\dot{\alpha}} &= -(\mathcal{V}_\alpha)^\dagger \\ &= -\sum_n^2 \sum_m^2 \dot{\bar{\psi}}^{\dot{\gamma}_m} \cdots \dot{\bar{\psi}}^{\dot{\gamma}_1} \dot{\psi}^{\beta_n} \cdots \dot{\psi}^{\beta_1} (\mathcal{X}_{\alpha\beta_1\cdots\beta_n\dot{\gamma}_1\cdots\dot{\gamma}_m})^\dagger = 0, \end{aligned} \quad (94)$$

where $\mathcal{X}_{\alpha\beta_1\cdots\beta_n\dot{\gamma}_1\cdots\dot{\gamma}_m}$ consists of ψ^α , $\bar{\psi}^{\dot{\alpha}}$, ϕ^a , $\dot{\phi}^a$, and their spatial derivatives. If these equations can be solved for $\dot{\psi}^\alpha$ and $\dot{\bar{\psi}}^{\dot{\alpha}}$, they should have the forms

$$\begin{aligned} \dot{\psi}^\alpha &= (\dot{\bar{\psi}}^{\dot{\alpha}})^\dagger \\ &= \sum_{k,l,m,n} C_{\beta_1\cdots\beta_k\dot{\beta}_1\cdots\dot{\beta}_l\delta_1\cdots\delta_m\dot{\delta}_1\cdots\dot{\delta}_n}^{\alpha i_1\cdots i_m j_1\cdots j_n} \psi^{\beta_1} \cdots \psi^{\beta_k} \bar{\psi}^{\dot{\beta}_1} \cdots \bar{\psi}^{\dot{\beta}_l} \partial_{i_1} \psi^{\delta_1} \cdots \partial_{i_m} \psi^{\delta_m} \partial_{j_1} \bar{\psi}^{\dot{\delta}_1} \cdots \partial_{j_n} \bar{\psi}^{\dot{\delta}_n}, \end{aligned} \quad (95)$$

where the coefficients $C_{\beta_1\cdots}^{\alpha i_1\cdots}$ consist of ϕ^a , $\dot{\phi}^a$, and $\partial_i \phi^a$ and has no dependence on ψ^α and $\bar{\psi}^{\dot{\alpha}}$.⁶ The nonlinear equations (93) and (94) can be solved for the first time derivatives of the fermion if we could uniquely determine the coefficients $C_{\beta_1\cdots}^{\alpha i_1\cdots}$ by substituting (95) into them. After the substitution, the EOMs should become trivial at each order of $(\psi^\gamma, \bar{\psi}^{\dot{\gamma}})$, $(\partial_i \psi^\gamma, \partial_i \bar{\psi}^{\dot{\gamma}})$, $(\psi^\gamma \psi^\delta \bar{\psi}^{\dot{\gamma}}, \psi^\gamma \bar{\psi}^{\dot{\gamma}} \bar{\psi}^{\dot{\delta}})$, and so on. At the lowest order, where the terms are linear in ψ^γ and $\bar{\psi}^{\dot{\gamma}}$, we have

$$\begin{pmatrix} \mathcal{X}_{\alpha\dot{\beta}}^{(0)} & \mathcal{X}_{\alpha\dot{\beta}}^{(0)} \\ -(\mathcal{X}_{\alpha\dot{\beta}}^{(0)})^* & -(\mathcal{X}_{\alpha\dot{\beta}}^{(0)})^* \end{pmatrix} \begin{pmatrix} C_{\gamma}^{\beta} & C_{\dot{\gamma}}^{\beta} \\ (C_{\gamma}^{\beta})^* & (C_{\dot{\gamma}}^{\beta})^* \end{pmatrix} \begin{pmatrix} \psi^\gamma \\ \bar{\psi}^{\dot{\gamma}} \end{pmatrix} = - \begin{pmatrix} \left(\frac{\partial \mathcal{X}_a}{\partial \psi^\gamma}\right)^{(0)} & \left(\frac{\partial \mathcal{X}_a}{\partial \bar{\psi}^{\dot{\gamma}}}\right)^{(0)} \\ -\left(\frac{\partial \mathcal{X}_a}{\partial \psi^\gamma}\right)^{(0)*} & -\left(\frac{\partial \mathcal{X}_a}{\partial \bar{\psi}^{\dot{\gamma}}}\right)^{(0)*} \end{pmatrix} \begin{pmatrix} \psi^\gamma \\ \bar{\psi}^{\dot{\gamma}} \end{pmatrix}. \quad (96)$$

Therefore, if the matrix

$$\mathbf{J}_L^{(0)}(t, \mathbf{x}) = - \begin{pmatrix} \mathcal{X}_{\alpha\dot{\beta}}^{(0)} & \mathcal{X}_{\alpha\dot{\beta}}^{(0)} \\ -(\mathcal{X}_{\alpha\dot{\beta}}^{(0)})^* & -(\mathcal{X}_{\alpha\dot{\beta}}^{(0)})^* \end{pmatrix} \quad (97)$$

has a nonzero determinant, we can uniquely determine C_{γ}^{β} and $C_{\dot{\gamma}}^{\beta}$ from (96). Similarly, the coefficients associated with the spatial derivative terms $(\partial_i \psi^\gamma, \partial_i \bar{\psi}^{\dot{\gamma}})$ are determined by

$$\begin{pmatrix} \mathcal{X}_{\alpha\dot{\beta}}^{(0)} & \mathcal{X}_{\alpha\dot{\beta}}^{(0)} \\ -(\mathcal{X}_{\alpha\dot{\beta}}^{(0)})^* & -(\mathcal{X}_{\alpha\dot{\beta}}^{(0)})^* \end{pmatrix} \begin{pmatrix} C_{\gamma}^{\beta i} & C_{\dot{\gamma}}^{\beta i} \\ (C_{\gamma}^{\beta i})^* & (C_{\dot{\gamma}}^{\beta i})^* \end{pmatrix} \begin{pmatrix} \partial_i \psi^\gamma \\ \partial_i \bar{\psi}^{\dot{\gamma}} \end{pmatrix} = - \begin{pmatrix} \left(\frac{\partial \mathcal{X}_a}{\partial (\partial_i \psi^\gamma)}\right)^{(0)} & \left(\frac{\partial \mathcal{X}_a}{\partial (\partial_i \bar{\psi}^{\dot{\gamma}})}\right)^{(0)} \\ -\left(\frac{\partial \mathcal{X}_a}{\partial (\partial_i \psi^\gamma)}\right)^{(0)*} & -\left(\frac{\partial \mathcal{X}_a}{\partial (\partial_i \bar{\psi}^{\dot{\gamma}})}\right)^{(0)*} \end{pmatrix} \begin{pmatrix} \partial_i \psi^\gamma \\ \partial_i \bar{\psi}^{\dot{\gamma}} \end{pmatrix}. \quad (98)$$

Once the coefficients at the linear order in the fermionic fields are determined, the coefficients at the cubic order can be determined by the equation

$$\begin{pmatrix} \mathcal{X}_{\alpha\dot{\beta}}^{(0)} & \mathcal{X}_{\alpha\dot{\beta}}^{(0)} \\ -(\mathcal{X}_{\alpha\dot{\beta}}^{(0)})^* & -(\mathcal{X}_{\alpha\dot{\beta}}^{(0)})^* \end{pmatrix} \begin{pmatrix} C_{\gamma\delta\dot{\eta}}^{\beta} & C_{\gamma\dot{\delta}\dot{\eta}}^{\beta} \\ (C_{\gamma\delta\dot{\eta}}^{\beta})^* & (C_{\gamma\dot{\delta}\dot{\eta}}^{\beta})^* \end{pmatrix} \begin{pmatrix} \psi^\gamma \psi^\delta \bar{\psi}^{\dot{\eta}} \\ \psi^\gamma \bar{\psi}^{\dot{\delta}} \bar{\psi}^{\dot{\eta}} \end{pmatrix} = - \begin{pmatrix} \mathcal{G}_{\alpha\gamma\delta\dot{\eta}} & \mathcal{G}_{\alpha\gamma\dot{\delta}\dot{\eta}} \\ -(\mathcal{G}_{\alpha\gamma\delta\dot{\eta}})^* & -(\mathcal{G}_{\alpha\gamma\dot{\delta}\dot{\eta}})^* \end{pmatrix} \begin{pmatrix} \psi^\gamma \psi^\delta \bar{\psi}^{\dot{\eta}} \\ \psi^\gamma \bar{\psi}^{\dot{\delta}} \bar{\psi}^{\dot{\eta}} \end{pmatrix}. \quad (99)$$

⁶ $(k+l+m+n)$ is a finite odd number.

Here, the coefficient matrices $\mathcal{G}_{\alpha\gamma\delta\dot{\eta}}$ and $\mathcal{G}_{\alpha\gamma\dot{\delta}\dot{\eta}}$ are expressed in terms of $\mathcal{X}_{\alpha\beta_1\cdots\beta_n\dot{\gamma}_1\cdots\dot{\gamma}_m}$ and their derivatives with respect to ψ^α , $\bar{\psi}^{\dot{\alpha}}$, and their spatial derivatives, as well as the linear coefficients C^β_γ , $C^\beta_{\dot{\gamma}}$, $C^{\beta i}_\gamma$, and $C^{\beta i}_{\dot{\gamma}}$. Again, we can solve the above equations for $C^\beta_{\gamma\delta\dot{\eta}}$ and $C^\beta_{\gamma\dot{\delta}\dot{\eta}}$ and successively determine all the coefficients appearing in (95) with the same procedure as long as the matrix $\mathbf{J}_L^{(0)}$ has an inverse.

Now, it is straightforward to calculate the matrix $\mathbf{J}_L^{(0)}$ from the concrete expression of the equations of motion (86) and (87). The definition of $\mathbf{J}_L^{(0)}$ is rewritten with $\mathcal{Y}_\alpha$ and $\mathcal{Y}_{\dot{\alpha}}$ as

$$\mathbf{J}_L^{(0)}(t, \mathbf{x}) = - \begin{pmatrix} \left(\frac{\partial \mathcal{Y}_\alpha}{\partial \dot{\psi}^\beta}\right)^{(0)} & \left(\frac{\partial \mathcal{Y}_\alpha}{\partial \dot{\bar{\psi}}^{\dot{\beta}}}\right)^{(0)} \\ \left(\frac{\partial \mathcal{Y}_{\dot{\alpha}}}{\partial \dot{\psi}^\beta}\right)^{(0)} & \left(\frac{\partial \mathcal{Y}_{\dot{\alpha}}}{\partial \dot{\bar{\psi}}^{\dot{\beta}}}\right)^{(0)} \end{pmatrix}, \quad (100)$$

and the components are explicitly given by

$$\begin{aligned} \left(\frac{\partial \mathcal{Y}_\alpha}{\partial \dot{\psi}^\beta}\right)^{(0)} &= - \left(\frac{\partial \mathcal{Y}_{\dot{\alpha}}}{\partial \dot{\bar{\psi}}^{\dot{\beta}}}\right)^{\dagger(0)} \\ &= - \left(2(S_a^{(2)})^{(0)} \partial_i \phi^a - \partial_i W_3^{(0)}\right) (\sigma^{0i} \varepsilon)_{\alpha\beta}, \end{aligned} \quad (101)$$

$$\left(\frac{\partial \mathcal{Y}_{\dot{\alpha}}}{\partial \dot{\bar{\psi}}^{\dot{\beta}}}\right)^{(0)} = - \left(\frac{\partial \mathcal{Y}_\alpha}{\partial \dot{\psi}^\beta}\right)^{\dagger(0)} = 2(P^{(2)})^{(0)} \sigma_{\alpha\dot{\beta}}^0. \quad (102)$$

It is manifest that $\mathbf{J}_L^{(0)}$ agrees with the matrix (79) with (78), obtained in Hamiltonian formulation. Therefore, the condition that the fermionic equations can be uniquely solved in the Lagrange formulation is equivalent to the condition that all the primary constraints are second class.

VI. FIELD REDEFINITION

So far, we have successfully obtained the most general quadratic Lagrangian for n -scalar and one Weyl fields satisfying the maximally degenerate conditions including up to first derivatives. However, one needs to carefully check whether the obtained theories can be mapped into a known theory, which trivially satisfies the degeneracy conditions.⁷ For an example, let us consider the Lagrangian containing a canonical scalar field and a Weyl field,

⁷Degeneracy of fermionic fields is always necessary to avoid negative norm states even if it includes only first derivatives. This fact is in sharp contrast with the cases of bosonic fields, which can be healthy as far as the Lagrangian does not include second or higher derivatives even if it is nondegenerate. When the Lagrangian does not have mixing between bosonic and fermionic fields, the maximally degenerate conditions (13) and (14) are equivalent to the degeneracy of the kinetic matrix of the fermionic fields, $\det D = 0$. In a special case where each field has no mixing with the others, all the components of the submatrix D vanish; i.e., the kinetic matrix is trivially degenerate. One of the simplest examples is a no-derivative interacting system of a canonical scalar field and a Weyl field like (103), which is linear in the first derivative of fermions.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{i}{2}(\bar{\psi}^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \psi^\alpha - \partial_\mu \bar{\psi}^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^\mu \psi^\alpha). \quad (103)$$

Note that the kinetic matrix of (103) is trivially degenerate since $\mathcal{B} = \mathcal{C} = D = 0$. Under the following field redefinition,

$$\phi = \varphi - \frac{i}{2}\Psi + \frac{i}{2}\bar{\Psi}, \quad (104)$$

where φ is a new scalar field, while keeping the Weyl field the same, the Lagrangian is transformed as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \varphi)^2 - i\partial^\mu \varphi (\psi^\alpha \partial_\mu \psi_\alpha - \bar{\psi}_{\dot{\alpha}} \partial_\mu \bar{\psi}^{\dot{\alpha}}) \\ &\quad + \frac{i}{2}(\bar{\psi}^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \psi^\alpha - \partial_\mu \bar{\psi}^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^\mu \psi^\alpha) \\ &\quad - \frac{1}{2}(\psi^\alpha \partial_\mu \psi_\alpha)^2 + (\psi^\alpha \partial_\mu \psi_\alpha)(\bar{\psi}_{\dot{\alpha}} \partial^\mu \bar{\psi}^{\dot{\alpha}}) - \frac{1}{2}(\bar{\psi}_{\dot{\alpha}} \partial_\mu \bar{\psi}^{\dot{\alpha}})^2. \end{aligned} \quad (105)$$

As one can see, the submatrices $\mathcal{B}$, $\mathcal{C}$, and D in the kinetic matrix become nonzero due to the transformation (104), and the mapped Lagrangian apparently yields the second-order differential equations for the fermionic fields. One can, however, easily check that this mapped Lagrangian satisfies the maximally degenerate conditions (65) and (66) as well as the supplementary condition (80); therefore, the number of d.o.f. is the same as the original Lagrangian (103).⁸ This is because the field redefinition (104) is an invertible transformation, keeping the number of d.o.f. under the transformation [41].

Now, we would like to know whether our degenerate Lagrangian can be mapped into theories of which the kinetic matrix is trivially degenerate by the covariant field redefinition $(\phi^a, \psi^\alpha, \bar{\psi}^{\dot{\alpha}}) \rightarrow (f^A(\phi, \psi, \bar{\psi}), \eta^\Lambda(\phi, \psi, \bar{\psi}), \bar{\eta}^{\dot{\Lambda}}(\phi, \psi, \bar{\psi}))$,⁹ i.e., if it is possible that

$$\mathbf{K} = \begin{pmatrix} A_{ab} & \mathcal{B}_{a\beta,J} & \mathcal{B}_{a\dot{\beta},J} \\ C_{ab,I} & D_{a\beta,IJ} & D_{a\dot{\beta},IJ} \\ C_{\dot{a}b,I} & D_{\dot{a}\beta,IJ} & D_{\dot{a}\dot{\beta},IJ} \end{pmatrix} \xrightarrow{\text{field redefinition}} \begin{pmatrix} \tilde{A}_{ab} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (106)$$

⁸The Hamiltonian analysis of this Lagrangian has already been examined in Ref. [40].

⁹One can consider field redefinitions including derivatives of the scalar and fermionic fields, but, for instance, redefinitions like $(\phi, \psi, \bar{\psi}) \rightarrow (f(\phi, \partial\phi, \psi, \bar{\psi}), \eta(\phi, \psi, \bar{\psi}), \bar{\eta}(\phi, \psi, \bar{\psi}))$, $(\phi, \psi, \bar{\psi}) \rightarrow (f(\phi, \psi, \bar{\psi}), \eta(\phi, \psi, \bar{\psi}, \partial\psi), \bar{\eta}(\phi, \psi, \bar{\psi}, \partial\bar{\psi}))$, $(\phi, \psi, \bar{\psi}) \rightarrow (f(\phi, \psi, \bar{\psi}), \eta(\phi, \psi, \bar{\psi}, \partial\bar{\psi}), \bar{\eta}(\phi, \psi, \bar{\psi}, \partial\psi))$, or $(\phi, \psi, \bar{\psi}) \rightarrow (f(\phi, \psi, \bar{\psi}, \partial\psi, \partial\bar{\psi}), \eta(\phi, \psi, \bar{\psi}), \bar{\eta}(\phi, \psi, \bar{\psi}))$ result in the appearance of the higher derivatives, implying that they might not keep the Lagrangian including just up to the first derivatives. To introduce such transformations, we also need to check the invertibility of them.

where $\tilde{A}_{ab}$ is the kinetic matrix of the redefined scalar fields f^A . We note that, under the field redefinition, the Lagrangians, (36), (37), and (38), are not mixed since the transformation does not include the derivatives of the fields. From the definition of the kinetic matrix, the quadratic terms in derivatives in the Lagrangian (38) can be responsible for the kinetic matrix, while linear or lower ones (37), (36) are obviously not. Furthermore, among the quadratic terms, antisymmetrically contracted (with respect to Lorentzian indices of the derivatives) ones are irrelevant. The only contribution to the kinetic matrix is coming from the symmetrically contracted ones in $\mathcal{L}_2$. When all the submatrices except $\tilde{A}_{ab}$ in the kinetic matrix vanish, the corresponding symmetrically contracted covariant terms in the Lagrangian also should vanish simultaneously because of the covariance of the theory. As a result, we can just focus on the symmetrically contracted terms in $\mathcal{L}_2$ in this section. Picking up these, we rewrite the relevant Lagrangian with the degeneracy conditions as

$$\begin{aligned} \mathcal{L}_{\text{rel}} = & \frac{1}{2} \left[V_{ac} \partial_\mu \phi^c - \frac{1}{2} (S_a^{(1)} \partial_\mu \Psi + (S_a^{(1)})^\dagger \partial_\mu \bar{\Psi}) \right] \\ & \times V^{ab} \left[V_{bd} \partial^\mu \phi^d - \frac{1}{2} (S_b^{(1)} \partial^\mu \Psi + (S_b^{(1)})^\dagger \partial^\mu \bar{\Psi}) \right]. \end{aligned} \quad (107)$$

If the coefficients V_{ab} and $S_a^{(1)}$ are constants, the square brackets $[\dots]$ can be redefined as the derivatives of new scalar fields. Then, the nontrivial terms in (107), which yield second time derivatives of the fermionic fields in Euler-Lagrange equations, can be removed just as a nontrivial Lagrangian (105) is reduced to a trivial Lagrangian (103) by the transformation (104). Under the field redefinition, $(\phi^a, \psi^\alpha, \bar{\psi}^{\dot{\alpha}}) \rightarrow (f^A(\phi, \psi, \bar{\psi}), \eta^\Lambda(\phi, \psi, \bar{\psi}), \bar{\eta}^{\dot{\Lambda}}(\phi, \psi, \bar{\psi}))$, the above Lagrangian is rewritten in terms of the new variables as

$$\begin{aligned} \mathcal{L}_{\text{rel}} = & \frac{1}{2} \mathcal{G}_A^a V_{ab} \mathcal{G}_B^b \partial_\mu f^A \partial^\mu f^B - \mathcal{G}_A^a V_{ab} \mathcal{G}_\Lambda^b \partial_\mu f^A \partial^\mu \eta^\Lambda - \mathcal{G}_A^a V_{ab} \bar{\mathcal{G}}_\Lambda^b \partial_\mu f^A \partial^\mu \bar{\eta}^{\dot{\Lambda}} \\ & - \frac{1}{2} \mathcal{G}_\Lambda^a V_{ab} \mathcal{G}_\Sigma^b \partial_\mu \eta^\Lambda \partial^\mu \eta^\Sigma - \mathcal{G}_\Lambda^a V_{ab} \bar{\mathcal{G}}_\Sigma^b \partial_\mu \eta^\Lambda \partial^\mu \bar{\eta}^{\dot{\Sigma}} - \frac{1}{2} \bar{\mathcal{G}}_\Lambda^a V_{ab} \bar{\mathcal{G}}_\Sigma^b \partial_\mu \bar{\eta}^{\dot{\Lambda}} \partial^\mu \bar{\eta}^{\dot{\Sigma}}, \end{aligned} \quad (108)$$

where we have defined

$$\mathcal{G}_A^a = \phi_{,f^A}^a - \frac{1}{2} V^{ab} [S_b^{(1)} \Psi_{,f^A} + (S_b^{(1)})^\dagger \bar{\Psi}_{,f^A}], \quad (109)$$

$$\mathcal{G}_\Lambda^a = \phi_{,\eta^\Lambda}^a - \frac{1}{2} V^{ab} [S_b^{(1)} \Psi_{,\eta^\Lambda} + (S_b^{(1)})^\dagger \bar{\Psi}_{,\eta^\Lambda}], \quad (110)$$

$\bar{\mathcal{G}}_\Lambda^a = -(\mathcal{G}_\Lambda^a)^\dagger$, and they satisfy $\mathcal{G}_A^a = (\mathcal{G}_A^a)^\dagger$. Now, let us seek the field redefinition such that the quadratic terms in the first derivative of η^Λ and $\bar{\eta}^{\dot{\Lambda}}$, the second line of (108), are removed. This becomes possible if we find a transformation realizing

$$\mathcal{G}_\Lambda^a = 0. \quad (111)$$

To find such a transformation, we first expand V^{ab} and $S_a^{(1)}$ as

$$V^{ab} = v^{ab(0)} + v^{ab(1)} \Psi + (v^{ab(1)})^* \bar{\Psi} + v^{ab(2)} \Psi \bar{\Psi}, \quad S_a^{(1)} = s_a^{(0)} + s_a^{(1)} \bar{\Psi}, \quad (112)$$

where $v^{ab(i)}$ and $s_a^{(i)}$ depend only on ϕ^a . Then, Eq. (111) can be rewritten as

$$\phi_{,\eta^\Lambda}^a = \frac{1}{2} \left[v^{ab(0)} s_b^{(0)} \Psi_{,\eta^\Lambda} + v^{ab(0)} (s_b^{(0)})^* \bar{\Psi}_{,\eta^\Lambda} + (v^{ab(0)} s_b^{(1)} + (v^{ab(1)})^* s_b^{(0)}) \Psi_{,\eta^\Lambda} \bar{\Psi} + (v^{ab(0)} (s_b^{(1)})^* + v^{ab(1)} (s_b^{(0)})^*) \Psi \bar{\Psi}_{,\eta^\Lambda} \right] \quad (113)$$

$$\begin{aligned} = & \frac{1}{2} \left[v^{ab(0)} s_b^{(0)} \Psi_{,\eta^\Lambda} + v^{ab(0)} (s_b^{(0)})^* \bar{\Psi}_{,\eta^\Lambda} + \Re(v^{ab(0)} s_b^{(1)} + (v^{ab(1)})^* s_b^{(0)}) (\Psi \bar{\Psi})_{,\eta^\Lambda} \right. \\ & \left. + i \Im(v^{ab(0)} s_b^{(1)} + (v^{ab(1)})^* s_b^{(0)}) (\Psi_{,\eta^\Lambda} \bar{\Psi} - \Psi \bar{\Psi}_{,\eta^\Lambda}) \right]. \end{aligned} \quad (114)$$

It is obvious that the first three terms can have their integral forms, but the last term of (114), proportional to $\Psi_{,\eta^\Lambda} \bar{\Psi} - \Psi \bar{\Psi}_{,\eta^\Lambda}$, cannot be integrated as we will see below. Let us explicitly see the transformation where ϕ^a coincides with the integral of the right-hand side of (114) except the last term. Such a transformation is realized through

$$f^A = f^A \left(\phi^b - \frac{1}{2} [A^b \Psi + (A^b)^* \bar{\Psi} + B^b \Psi \bar{\Psi}] \right), \quad \eta^\Lambda = \eta^\Lambda(\phi^b, \psi^\beta, \bar{\psi}^{\dot{\beta}}), \quad \bar{\eta}^{\dot{\Lambda}} = \bar{\eta}^{\dot{\Lambda}}(\phi^b, \psi^\beta, \bar{\psi}^{\dot{\beta}}), \quad (115)$$

where A^b and B^b are functions of ϕ^a , determined properly in the following. For keeping the equivalence between the former and the latter Lagrangians, we assume the field redefinition invertible. The assumption of the invertible transformation requires the bosonic part of the whole Jacobian matrix,

$$\text{Jacobian matrix} = \begin{pmatrix} \frac{\partial f^A}{\partial \phi^b} & \frac{\partial f^A}{\partial \psi^\beta} & \frac{\partial f^A}{\partial \bar{\psi}^{\dot{\beta}}} \\ \frac{\partial \eta^\Lambda}{\partial \phi^b} & \frac{\partial \eta^\Lambda}{\partial \psi^\beta} & \frac{\partial \eta^\Lambda}{\partial \bar{\psi}^{\dot{\beta}}} \\ \frac{\partial \bar{\eta}^{\dot{\Lambda}}}{\partial \phi^b} & \frac{\partial \bar{\eta}^{\dot{\Lambda}}}{\partial \psi^\beta} & \frac{\partial \bar{\eta}^{\dot{\Lambda}}}{\partial \bar{\psi}^{\dot{\beta}}} \end{pmatrix}, \quad (116)$$

to have a nonzero determinant. The off-diagonal bosonic parts of the Jacobian matrix are inevitably zero because of the Grassmann-odd property. Therefore, the above requirement is equivalent to

$$\det \left(\frac{\partial f^A}{\partial \phi^b} \right)^{(0)} \neq 0, \quad \text{and} \quad \det \begin{pmatrix} \frac{\partial \eta^\Lambda}{\partial \psi^\beta} & \frac{\partial \eta^\Lambda}{\partial \bar{\psi}^{\dot{\beta}}} \\ \frac{\partial \bar{\eta}^{\dot{\Lambda}}}{\partial \psi^\beta} & \frac{\partial \bar{\eta}^{\dot{\Lambda}}}{\partial \bar{\psi}^{\dot{\beta}}} \end{pmatrix}^{(0)} \neq 0. \quad (117)$$

Since we have assumed the transformation is invertible, we can invert the definition of f^A in (115) for ϕ^a ,

$$\phi^a = \frac{1}{2} [A^a \Psi + (A^a)^* \bar{\Psi} + B^a \Psi \bar{\Psi}] + \phi^{a(0)}(f), \quad (118)$$

where $\phi^{a(0)}(f)$ is the inverse function of f^A in (115). The derivative of (118) with respect to η^Λ gives

$$\begin{aligned} \phi_{,\eta^\Lambda}^a &= \frac{1}{2} \left[A^a \Psi_{,\eta^\Lambda} + (A^a)^* \bar{\Psi}_{,\eta^\Lambda} + B^a (\Psi_{,\eta^\Lambda} \bar{\Psi} + \Psi \bar{\Psi}_{,\eta^\Lambda}) \right] + \frac{1}{2} \phi_{,\eta^\Lambda}^b \left[A_{,\phi^b}^a \Psi + (A_{,\phi^b}^a)^* \bar{\Psi} + B_{,\phi^b}^a \Psi \bar{\Psi} \right] \\ &= \frac{1}{2} \left[A^a \Psi_{,\eta^\Lambda} + (A^a)^* \bar{\Psi}_{,\eta^\Lambda} + \left(B^a + \frac{1}{2} A^b (A^a)^*_{,\phi^b} \right) \Psi_{,\eta^\Lambda} \bar{\Psi} + \left(B^a + \frac{1}{2} (A^b)^* A_{,\phi^b}^a \right) \Psi \bar{\Psi}_{,\eta^\Lambda} \right], \end{aligned} \quad (119)$$

where we have recursively solved the first line and have used $\Psi_{,\eta^\Lambda} \Psi = (1/2)(\Psi \Psi)_{,\eta^\Lambda} = 0$ and $\bar{\Psi}_{,\eta^\Lambda} \bar{\Psi} = (1/2)(\bar{\Psi} \bar{\Psi})_{,\eta^\Lambda} = 0$ to find the second line without $\phi_{,\eta^\Lambda}^b$. Now, comparing (113) and (119), we can easily determine the coefficients A^a and B^a as

$$A^a = v^{ab(0)} s_b^{(0)}, \quad (120)$$

$$B^a = \Re[C^a], \quad \text{where } C^a = v^{ab(0)} s_b^{(1)} + (v^{ab(1)})^* s_b^{(0)} - \frac{1}{2} (v^{ab(0)} (s_b^{(0)})^*)_{,\phi^c} v^{cd(0)} s_d^{(0)}. \quad (121)$$

Introducing the transformation to the definitions (109) and (110), we then have

$$\mathcal{G}_\Lambda^a = -\frac{i}{2} \Im[C^a] (\Psi_{,\eta^\Lambda} \bar{\Psi} - \Psi \bar{\Psi}_{,\eta^\Lambda}), \quad (122)$$

$$\mathcal{G}_A^a = \left(\delta_b^a + \frac{1}{2} (A^a_{,\phi^b} \Psi + (A^*)^a_{,\phi^b} \bar{\Psi}) + \frac{1}{2} (B^a_{,\phi^b} + \Re[A^a_{,\phi^c} (A^*)^c_{,\phi^b}]) \Psi \bar{\Psi} \right) \phi_{,f^A}^{b(0)} - \frac{i}{2} \Im[C^a] (\Psi_{,f^A} \bar{\Psi} - \Psi \bar{\Psi}_{,f^A}). \quad (123)$$

Since we cannot pick up the imaginary part of C^a as one can see from (121), we still have nonvanishing $\mathcal{G}_\Lambda^a$. We cannot remove the dependence on $\Psi_{,\eta^\Lambda} \bar{\Psi} - \Psi \bar{\Psi}_{,\eta^\Lambda}$ in any way as we expected from the expression of (114). Nevertheless, the quadratic derivative interactions of fermions in (108) is the square of (122) (and its Hermitian), and thus they accidentally vanish due to the Grassmann properties,

$$\mathcal{G}_\Lambda^a \mathcal{G}_\Sigma^b \propto (\Psi_{,\eta^\Lambda} \bar{\Psi} - \Psi \bar{\Psi}_{,\eta^\Lambda}) (\Psi_{,\eta^\Sigma} \bar{\Psi} - \Psi \bar{\Psi}_{,\eta^\Sigma}) = 0, \quad (124)$$

$$\mathcal{G}_\Lambda^a \bar{\mathcal{G}}_\Sigma^b \propto (\Psi_{,\eta^\Lambda} \bar{\Psi} - \Psi \bar{\Psi}_{,\eta^\Lambda}) (\Psi_{,\eta^\Sigma} \bar{\Psi} - \Psi \bar{\Psi}_{,\eta^\Sigma}) = 0; \quad (125)$$

whereas, the cross term of the redefined scalar field and fermionic field cannot be removed because

$$\mathcal{G}_A^a V_{ab} \mathcal{G}_\Lambda^b \partial_\mu f^A \partial^\mu \eta^\Lambda = -\frac{i}{2} v_{ab}^{(0)} \phi_{,f^A}^{a(0)} \Im[C^b] (\Psi_{,\eta^\Lambda} \bar{\Psi} - \Psi \bar{\Psi}_{,\eta^\Lambda}) \partial_\mu f^A \partial^\mu \eta^\Lambda \neq 0. \quad (126)$$

As a result, the final Lagrangian with the field redefinition (115) with (120) and (121) is given by

$$\mathcal{L}_{\text{rel}} = \frac{1}{2} \mathcal{G}_A^a V_{ab} \mathcal{G}_B^b \partial_\mu f^A \partial^\mu f^B - \mathcal{G}_A^a V_{ab} \mathcal{G}_\Lambda^b \partial_\mu f^A \partial^\mu \eta^\Lambda - \mathcal{G}_A^a V_{ab} \bar{\mathcal{G}}_\Lambda^b \partial_\mu f^A \partial^\mu \bar{\eta}^\Lambda. \quad (127)$$

Therefore, we conclude that the fermionic derivative interactions like $\partial_\mu \eta^\Lambda \partial^\mu \eta^\Sigma$ can be eliminated by the field redefinition, but the cross terms between scalar fields and fermions like $\partial_\mu f^A \partial^\mu \eta^\Lambda$ cannot be removed. Such cross terms indicate that the Euler-Lagrange equations apparently contain the second derivatives of the fermionic field, while they are reduced to first-order equations because the maximally degenerate conditions are satisfied,

$$D_{\Lambda\Sigma} - \mathcal{C}_{\Lambda B} A^{BA} \mathcal{B}_{A\Sigma} = \mathcal{G}_B^a V_{ab} \mathcal{G}_\Lambda^b A^{BA} \mathcal{G}_A^c V_{cd} \mathcal{G}_\Sigma^d \propto \mathcal{G}_\Lambda^b \mathcal{G}_\Sigma^d = 0, \quad (128)$$

$$D_{\Lambda\dot{\Sigma}} - \mathcal{C}_{\Lambda B} A^{BA} \mathcal{B}_{A\dot{\Sigma}} = \mathcal{G}_B^a V_{ab} \mathcal{G}_\Lambda^b A^{BA} \mathcal{G}_A^c V_{cd} \bar{\mathcal{G}}_\Sigma^d \propto \mathcal{G}_\Lambda^b \bar{\mathcal{G}}_\Sigma^d = 0. \quad (129)$$

To summarize, the kinetic matrix cannot be transformed to the form in (106) with any field redefinition, but it can be reduced to

$$\mathbf{K} = \begin{pmatrix} A_{ab} & \mathcal{B}_{a\beta,J} & \mathcal{B}_{a\dot{\beta},J} \\ \mathcal{C}_{ab,I} & D_{a\beta,IJ} & D_{a\dot{\beta},IJ} \\ \mathcal{C}_{\dot{a}b,I} & D_{\dot{a}\beta,IJ} & D_{\dot{a}\dot{\beta},IJ} \end{pmatrix} \xrightarrow{\text{field redefinition}} \begin{pmatrix} \tilde{A}_{ab} & \tilde{\mathcal{B}}_{a\beta,J} & \tilde{\mathcal{B}}_{a\dot{\beta},J} \\ \tilde{\mathcal{C}}_{ab,I} & 0 & 0 \\ \tilde{\mathcal{C}}_{\dot{a}b,I} & 0 & 0 \end{pmatrix}. \quad (130)$$

This fact indicates that the cross terms are really newly found derivative interactions.

The crucial difference between the purely bosonic degenerate system (such as Horndeski theory [4–6] and Degenerate Higher-Order Scalar-Tensor (DHOST) theories [17,20]) and our scalar-fermion degenerate system is the highest derivatives appearing in the Lagrangian. This is because the Lagrangian containing first derivatives of the fermionic fields in general yields second-order differential equations of motion, which is a signal of fermionic (Ostrogradsky's) ghosts. Therefore, this situation corresponds to the bosonic Lagrangian containing up to second derivatives. On the analogy of the conformal and disformal transformation in the scalar-tensor theory, the fermionic version of conformal (disformal) transformations does not

involve any derivatives of the fermionic field. Thus, the conformal transformation, which mixes the metric and the Weyl field, takes the following form¹⁰:

$$\bar{g}_{\mu\nu} = A(\Psi, \bar{\Psi}) g_{\mu\nu}. \quad (131)$$

¹⁰A candidate for the disformal transformation including the Weyl field is $\bar{g}_{\mu\nu} = A(\Psi, \bar{\Psi}) g_{\mu\nu} + B(\Psi, \bar{\Psi}) J_\mu J_\nu$, where J_μ is now appropriately mapped with tetrad fields e_μ^a , defined by $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$, from a flat tangent space. However, one can easily show that the arbitrary function B can be absorbed into the conformal factor A by using (47). On the other hand, the identity (C3) tells us that the disformal transformation with multiple Weyl fields is independent from the conformal transformation.

Such a transformation will be significantly useful in finding new theories of “tensor-fermion theories,” and we will leave this interesting issue to future work.

VII. SUMMARY AND DISCUSSION

As usually discussed in the Lagrangian composed of bosonic d.o.f., we have to avoid the appearance of ghosts even in boson-fermion coexisting Lagrangians. Fermions easily suffer from the fermionic ghost as pointed out in Ref. [39]; i.e., fermionic d.o.f. should be constrained by the same number of constraints as the (physical) d.o.f. That is true even when we additionally have bosonic d.o.f.

Our purpose is the construction of covariant derivative interactions between fermions or between scalar fields and fermions, free from fermionic ghosts. We have given the prescription to find new interactions in the Lagrangian with up to the first derivative of fields, made up of a set of conditions imposed on the Lagrangian. One of the conditions is the maximally degenerate condition which produces the sufficient number of primary constraints. In this paper, all of the constraints are simply assumed to be second class. If we have gauge invariance, that is, first class constraints, the number of primary constraints can be smaller. However, it should be notice that, after the gauge fixing, all of the primary constraints can be turned into second class. In the Lagrangian formulation, the equations of motion for fermions can be reduced to the first-order differential equations by imposing the maximally degenerate condition, which should be solved for the first derivative of each fermionic variable. Whether they are solvable is not *a priori* because of the Grassmann-odd nature, and the requirement is exactly the same with the invertibility of the constraint matrix.

The coefficients of the derivative of fields in the Lagrangian are complicated but classified thanks to the covariance, and they become much simpler in case of one Weyl field with multiple scalar fields. One of the prominent features which we can learn from the concrete analysis is that the first condition, that is, the maximally degenerate condition, affects the symmetric part of the Lagrangian with respect to the space-time indices of the derivatives, but the supplementary second condition affects the anti-symmetric part.

The proposed Lagrangian is possible to be used as a model for the interaction between scalar fields and a Weyl field, but some may ask what happens when we perform field redefinitions. Transformation including derivatives seems to introduce cubic or higher nonlinear derivative terms as well as second or higher derivatives in the Lagrangian. In a simple setup where the whole Lagrangian is expressed by the proposed quadratic Lagrangian, the quadratic terms in the derivative of fermions are absorbed by certain invertible field redefinitions, but the derivative interaction terms between the derivative of the

scalar fields and the fermions remain, which suggests that they are strictly new terms. As shown in Ref. [10], in a bosonic case, any healthy theory even with arbitrary higher derivative terms can be reduced to a nondegenerate Lagrangian written in terms of (unconstrained) variables with at most the first-order time derivatives through canonical transformation. On the other hand, in a fermionic case, any healthy theories require constraints to reduce extra d.o.f. of fermionic variables. Then, it would be interesting to investigate what kind of a simpler Lagrangian can be generally found through canonical transformation from healthy fermionic theories even with arbitrary higher derivative terms.

There are several directions for applying the method we have developed: inclusion of nonlinear derivative interactions, second or higher derivatives, gauge invariance and/or gravity, and so on. Some of these issues will be discussed in future publications.

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APPENDIX A: IDENTITIES OF PAULI MATRICES

In this Appendix, we summarize notations and identities of Pauli matrices [42]. We have used the following matrices defined as

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta} \sigma_{\dot{\beta}\beta}^{\mu}, \quad (\text{A1})$$

$$\begin{aligned} (\sigma^{\mu\nu})_{\alpha}^{\beta} &= \frac{1}{4} (\sigma_{\alpha\dot{\alpha}}^{\mu} \bar{\sigma}^{\nu\dot{\alpha}\beta} - \sigma_{\alpha\dot{\alpha}}^{\nu} \bar{\sigma}^{\mu\dot{\alpha}\beta}), \\ (\bar{\sigma}^{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}} &= \frac{1}{4} (\bar{\sigma}^{\mu\dot{\alpha}\alpha} \sigma_{\alpha\dot{\beta}}^{\nu} - \bar{\sigma}^{\nu\dot{\alpha}\alpha} \sigma_{\alpha\dot{\beta}}^{\mu}), \end{aligned} \quad (\text{A2})$$

$$(\sigma^{\mu\nu}\epsilon)_{\alpha\beta} = (\sigma^{\mu\nu})_{\alpha}^{\gamma} \epsilon_{\gamma\beta}, \quad (\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\gamma}} (\bar{\sigma}^{\mu\nu})^{\dot{\gamma}}_{\dot{\beta}}. \quad (\text{A3})$$

Note that the last two matrices are symmetric under the exchange of the fermionic indices, i.e., $(\sigma^{\mu\nu}\epsilon)_{\alpha\beta} = (\sigma^{\mu\nu}\epsilon)_{\beta\alpha}$ and $(\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} = (\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\beta}\dot{\alpha}}$. They satisfy the following useful properties:

$$\begin{aligned} \sigma_{\alpha\dot{\alpha}}^{\mu} \bar{\sigma}_{\dot{\beta}\beta}^{\nu} &= -2\delta_{\alpha}^{\beta} \delta_{\dot{\alpha}}^{\dot{\beta}}, & \sigma_{\alpha\dot{\alpha}}^{\mu} \sigma_{\mu\dot{\beta}\beta} &= -2\epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{\alpha\beta}, \\ \bar{\sigma}^{\mu\dot{\alpha}\alpha} \bar{\sigma}_{\mu}^{\dot{\beta}\beta} &= -2\epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta}, \end{aligned} \quad (\text{A4})$$

$$\sigma_{\alpha\dot{\alpha}}^{\mu} \sigma_{\dot{\beta}\beta}^{\nu} - \sigma_{\alpha\dot{\alpha}}^{\nu} \sigma_{\dot{\beta}\beta}^{\mu} = 2[(\sigma^{\mu\nu}\epsilon)_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} + (\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} \epsilon_{\alpha\beta}], \quad (\text{A5})$$

$$\sigma_{\alpha\dot{\alpha}}^{\mu} \sigma_{\dot{\beta}\beta}^{\nu} + \sigma_{\alpha\dot{\alpha}}^{\nu} \sigma_{\dot{\beta}\beta}^{\mu} = -\eta^{\mu\nu} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} - 4(\sigma^{\mu\rho}\epsilon)_{\alpha\beta} (\epsilon\bar{\sigma}_{\rho}^{\nu})_{\dot{\alpha}\dot{\beta}}. \quad (\text{A6})$$

In addition, one can easily derive the following useful identities:

$$(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\sigma_{\nu\gamma} = \frac{1}{2}(\varepsilon_{\alpha\gamma}\sigma_{\beta\dot{\gamma}}^{\mu} + \varepsilon_{\beta\gamma}\sigma_{\alpha\dot{\gamma}}^{\mu}), \quad (\text{A7})$$

$$(\sigma^{\mu\rho}\varepsilon)_{\alpha\beta}(\varepsilon\bar{\sigma}_{\rho}^{\nu})_{\dot{\gamma}\dot{\delta}} = -\frac{1}{8}(\sigma_{\alpha\dot{\gamma}}^{\mu}\sigma_{\beta\dot{\delta}}^{\nu} + \sigma_{\alpha\dot{\delta}}^{\mu}\sigma_{\beta\dot{\gamma}}^{\nu} + \sigma_{\alpha\dot{\gamma}}^{\nu}\sigma_{\beta\dot{\delta}}^{\mu} + \sigma_{\alpha\dot{\delta}}^{\nu}\sigma_{\beta\dot{\gamma}}^{\mu}), \quad (\text{A8})$$

$$\begin{aligned} &(\sigma^{\mu\rho}\varepsilon)_{\alpha\beta}(\sigma_{\rho}^{\nu}\varepsilon)_{\gamma\dot{\delta}} \\ &= \frac{1}{4}[-\eta^{\mu\nu}(\varepsilon_{\alpha\gamma}\varepsilon_{\beta\dot{\delta}} + \varepsilon_{\beta\gamma}\varepsilon_{\alpha\dot{\delta}}) + (\sigma^{\mu\nu}\varepsilon)_{\alpha\gamma}\varepsilon_{\beta\dot{\delta}} \\ &+ (\sigma^{\mu\nu}\varepsilon)_{\beta\gamma}\varepsilon_{\alpha\dot{\delta}} + (\sigma^{\mu\nu}\varepsilon)_{\alpha\dot{\delta}}\varepsilon_{\beta\gamma} + (\sigma^{\mu\nu}\varepsilon)_{\beta\dot{\delta}}\varepsilon_{\alpha\gamma}]. \end{aligned} \quad (\text{A9})$$

The contraction of the Weyl indices is given as

$$\varepsilon^{\alpha\dot{\beta}}\sigma_{\alpha\dot{\alpha}}^{\mu}\sigma_{\beta\dot{\beta}}^{\nu} = -2(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} + \eta^{\mu\nu}\varepsilon_{\dot{\alpha}\dot{\beta}}, \quad (\text{A10})$$

$$\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma_{\alpha\dot{\alpha}}^{\mu}\sigma_{\beta\dot{\beta}}^{\nu} = -2(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta} + \eta^{\mu\nu}\varepsilon_{\alpha\beta}. \quad (\text{A11})$$

Therefore, any contraction of the building block matrices, $\varepsilon_{\alpha\beta}$, $\varepsilon_{\dot{\alpha}\dot{\beta}}$, $\sigma_{\alpha\dot{\alpha}}^{\mu}$, $(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}$, and $(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}$, with respect to the Lorentzian or fermionic indices reduces to the uncontracted combination of them. Here, Eq. (A8) implies $(\sigma^{\mu\rho}\varepsilon)_{\alpha\beta}(\varepsilon\bar{\sigma}_{\rho}^{\nu})_{\dot{\gamma}\dot{\delta}}$ is symmetric under the exchange of μ and ν and traceless,

$$(\sigma^{\mu\rho}\varepsilon)_{\alpha\beta}(\varepsilon\bar{\sigma}_{\rho\mu})_{\dot{\gamma}\dot{\delta}} = 0, \quad (\text{A12})$$

through (A4). The matrices and their complex conjugates are related through

$$(\sigma_{\alpha\dot{\beta}}^{\mu})^* = \sigma_{\beta\dot{\alpha}}^{\mu}, \quad ((\sigma^{\mu\nu}\varepsilon)_{\alpha\beta})^* = (\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\beta}\dot{\alpha}}. \quad (\text{A13})$$

APPENDIX B: MAXIMALLY DEGENERATE CONDITIONS AND PRIMARY CONSTRAINTS

As discussed in the Sec. II, we need $4N$ constraints making the number of the d.o.f. of fermions half to avoid negative norm states in a n -scalar and N -fermion Lagrangian with up to first derivatives. For the purpose, we have adopted the maximally degenerate conditions, Eqs. (13) and (14), having $4N$ primary constraints for the fermions, Eqs. (17) and (18). Here, we show that, if we have $4N$ primary constraints for the fermions, the maximally degenerate conditions are automatically satisfied.

From the definition of the canonical momenta of fermions, we generally have

$$\pi_{\psi_I^{\alpha}} = F_{\alpha,I}(\phi^a, \pi_{\phi^a}, \partial_i\phi^a, \psi_J^{\beta}, \partial_i\psi_J^{\beta}, \bar{\psi}_J^{\dot{\beta}}, \partial_i\bar{\psi}_J^{\dot{\beta}}, \dot{\psi}_J^{\beta}, \dot{\bar{\psi}}_J^{\dot{\beta}}), \quad (\text{B1})$$

after we locally solve the canonical momenta of the scalar fields for $\dot{\phi}^a$, which can be justified by our assumption that $\partial\pi_a/\partial\dot{\phi}^b = A_{ab}$ has an inverse. Taking the derivative of (B1) with respect to $\dot{\psi}_J^{\beta}$ and $\dot{\bar{\psi}}_J^{\dot{\beta}}$, we have

$$\begin{aligned} \mathcal{L}_{\dot{\psi}_I^{\alpha}\dot{\bar{\psi}}_J^{\dot{\beta}}} &= \mathcal{L}_{\dot{\phi}^a\dot{\psi}_J^{\beta}} \frac{\partial F_{\alpha,I}}{\partial\pi_{\phi^a}} + \frac{\partial F_{\alpha,I}}{\partial\dot{\psi}_J^{\beta}} \Leftrightarrow \frac{\partial F_{\alpha,I}}{\partial\dot{\psi}_J^{\beta}} \\ &= D_{\alpha\beta,II} + \mathcal{B}_{\alpha\beta,J} \frac{\partial F_{\alpha,I}}{\partial\pi_{\phi^a}} = D_{\alpha\beta,II} - \mathcal{C}_{ab,I}A^{ba}\mathcal{B}_{\alpha\beta,J}, \end{aligned} \quad (\text{B2})$$

$$\begin{aligned} \mathcal{L}_{\dot{\bar{\psi}}_I^{\dot{\alpha}}\dot{\psi}_J^{\beta}} &= \mathcal{L}_{\dot{\phi}^a\dot{\bar{\psi}}_J^{\dot{\beta}}} \frac{\partial F_{\alpha,I}}{\partial\pi_{\phi^a}} + \frac{\partial F_{\alpha,I}}{\partial\dot{\bar{\psi}}_J^{\dot{\beta}}} \Leftrightarrow \frac{\partial F_{\alpha,I}}{\partial\dot{\bar{\psi}}_J^{\dot{\beta}}} \\ &= D_{\alpha\dot{\beta},IJ} + \mathcal{B}_{\alpha\dot{\beta},J} \frac{\partial F_{\alpha,I}}{\partial\pi_{\phi^a}} = D_{\alpha\dot{\beta},IJ} - \mathcal{C}_{ab,I}A^{ba}\mathcal{B}_{\alpha\dot{\beta},J}, \end{aligned} \quad (\text{B3})$$

where we have used the relations derived from the derivative of (B1) with respect to $\dot{\phi}^a$,

$$\mathcal{L}_{\dot{\psi}_I^{\alpha}\dot{\phi}^a} = \mathcal{L}_{\dot{\phi}^b\dot{\phi}^a} \frac{\partial F_{\alpha,I}}{\partial\pi_{\phi^b}} \Leftrightarrow \mathcal{C}_{aa,I} = A_{ba} \frac{\partial F_{\alpha,I}}{\partial\pi_{\phi^b}}, \quad (\text{B4})$$

in the last equalities. Thus, Eqs. (B2) and (B3) tell us that no dependence of $F_{\alpha,I}$ on the time derivative of the fermions means that we have $4N$ primary constraints for the fermions. Thus, Eqs. (B2) and (B3) suggest that the maximally degeneracy conditions, Eqs. (13) and (14), are not only the sufficient condition for the existence of $4N$ primary constraints for fermions but also the necessary condition for that.

APPENDIX C: SCALAR-FERMION THEORIES QUADRATIC IN FIRST DERIVATIVES OF MULTIPLE WEYL FIELDS

In this Appendix, we extend our analysis in Sec. III to the case of multiple Weyl fields and construct the most general scalar-fermion theory of which the Lagrangian contains up to quadratic in first derivatives of scalar and fermionic fields. As in Sec. III, we first construct Lorentz-invariant scalar quantities with no derivatives, which only contain scalar fields ϕ^a and fermionic fields ψ_I^{α} , $\bar{\psi}_I^{\dot{\alpha}}$. Since the fermionic indices can be contracted with $\varepsilon_{\alpha\beta}$, $\varepsilon_{\dot{\alpha}\dot{\beta}}$, $\sigma_{\alpha\dot{\alpha}}^{\mu}$, $(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}$, and $(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}$, we have five possibilities,

$$\begin{aligned} \Psi_{IJ} &= \psi_I^{\alpha}\psi_{J,\alpha}, & \bar{\Psi}_{IJ} &= \bar{\psi}_{I,\dot{\alpha}}\bar{\psi}_J^{\dot{\alpha}}, & J_{IJ}^{\mu} &= \bar{\psi}_I^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^{\mu}\psi_J^{\alpha}, \\ K_{IJ}^{\mu\nu} &= \psi_I^{\alpha}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi_J^{\beta}, & \bar{K}_{IJ}^{\mu\nu} &= \bar{\psi}_I^{\dot{\alpha}}(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\bar{\psi}_J^{\dot{\beta}}, \end{aligned} \quad (\text{C1})$$

which satisfy the following properties,

$$\begin{aligned}\Psi_{IJ} &= \Psi_{JI}, & K_{IJ}^{\mu\nu} &= -K_{JI}^{\mu\nu}, & K_{IJ}^{\mu\nu} &= -K_{IJ}^{\nu\mu}, \\ (\Psi_{IJ})^\dagger &= \bar{\Psi}_{JI}, & (J_{IJ}^\mu)^\dagger &= J_{JI}^\mu, & (K_{IJ}^{\mu\nu})^\dagger &= \bar{K}_{JI}^{\mu\nu},\end{aligned}\quad (\text{C2})$$

and

$$J_{IJ}^\mu J_{KL}^\nu = -\frac{1}{2}\eta^{\mu\nu}\bar{\Psi}_{IK}\Psi_{JL} - 2\bar{K}_{IK}^{\lambda\nu}K_{JL}^\lambda + \bar{\Psi}_{IK}K_{JL}^{\mu\nu} - \Psi_{JL}\bar{K}_{IK}^{\mu\nu}, \quad (\text{C3})$$

$$J_{IJ}^\mu J_{\mu KL} = \eta_{\mu\nu}J_{IJ}^\mu J_{KL}^\nu = -2\bar{\Psi}_{IK}\Psi_{JL}, \quad (\text{C4})$$

$$K_{IJ}^{\mu\nu}J_{\nu KL} = \frac{1}{2}(J_{KI}^\mu\Psi_{JL} - J_{KJ}^\mu\Psi_{IL}), \quad (\text{C5})$$

$$K_{IJ}^{\mu\nu}J_{\nu KL}J_{\mu MN} = -\bar{\Psi}_{KM}(\Psi_{IN}\Psi_{JL} - \Psi_{JN}\Psi_{IL}), \quad (\text{C6})$$

$$\bar{K}_{IJ}^{\mu\nu}J_{\nu KL} = \frac{1}{2}(-J_{IL}^\mu\bar{\Psi}_{JK} + J_{JL}^\mu\bar{\Psi}_{IK}), \quad (\text{C7})$$

$$\bar{K}_{IJ}^{\mu\nu}J_{\nu KL}J_{\mu MN} = -\Psi_{NL}(\bar{\Psi}_{MJ}\bar{\Psi}_{KI} - \bar{\Psi}_{MI}\bar{\Psi}_{KJ}), \quad (\text{C8})$$

$$\begin{aligned}K_{IJ}^{\mu\nu}K_{\nu KL}^\lambda &= \frac{1}{4}\eta^{\mu\lambda}(-\Psi_{IL}\Psi_{JK} + \Psi_{IK}\Psi_{JL}) + \frac{1}{4}(K_{IL}^{\mu\lambda}\Psi_{JK} \\ &\quad - K_{IK}^{\mu\lambda}\Psi_{JL} - K_{JL}^{\mu\lambda}\Psi_{IK} + K_{JK}^{\mu\lambda}\Psi_{IL}),\end{aligned}\quad (\text{C9})$$

$$\begin{aligned}K_{IJ}^{\mu\nu}\bar{K}_{\nu KL}^\lambda &= \frac{1}{8}(J_{KI}^\mu J_{LJ}^\lambda - J_{LI}^\mu J_{KJ}^\lambda - J_{KJ}^\mu J_{LI}^\lambda + J_{LJ}^\mu J_{KI}^\lambda) \\ &= K_{IJ}^{\lambda\nu}\bar{K}_{\nu KL}^\mu,\end{aligned}\quad (\text{C10})$$

$$K_{IJ}^{\mu\nu}\bar{K}_{\nu\mu KL} = \frac{1}{4}(J_{KI}^\mu J_{\mu LJ} - J_{LI}^\mu J_{\mu KJ}) = 0, \quad (\text{C11})$$

where we have used (A4)–(A9). In general, any quantities without derivatives and Weyl indices (Weyl indices are appropriately contracted) are written as

$$\mathcal{A}^{\mu_1\mu_2\cdots\mu_n} = \mathcal{A}^{\mu_1\mu_2\cdots\mu_n}(\eta_{\mu\nu}, \phi^a, \Psi_{IJ}, \bar{\Psi}_{IJ}, J_{IJ}^\mu, K_{IJ}^{\mu\nu}, \bar{K}_{IJ}^{\mu\nu}), \quad (\text{C12})$$

where n is zero or a natural number. In addition, from the above identities, we know that all scalar quantities without derivatives can be expressed only by ϕ^a , Ψ_{IJ} , and $\bar{\Psi}_{IJ}$. We also see that the space-time index of vector quantities is described by J_{IJ}^μ , and those of second-rank tensors are described by $\eta^{\mu\nu}$, $K_{IJ}^{\mu\nu}$, $\bar{K}_{IJ}^{\mu\nu}$, and $K_{IJ}^{\mu\lambda}\bar{K}_{\lambda}^\nu{}_{KL}$, i.e.,

$$\begin{aligned}\mathcal{F} &= \mathcal{F}(\phi^a, \Psi_{IJ}, \bar{\Psi}_{IJ}), & \mathcal{G}^\mu &= \mathcal{G}_{(1)}^{IJ}J_{IJ}^\mu, \\ \mathcal{H}^{\mu\nu} &= \mathcal{H}_{(1)}\eta^{\mu\nu} + \mathcal{H}_{(2)}^{IJ}K_{IJ}^{\mu\nu} + \mathcal{H}_{(3)}^{IJ}\bar{K}_{IJ}^{\mu\nu} + \mathcal{H}_{(4)}^{IJKL}K_{IJ}^{\mu\lambda}\bar{K}_{\lambda}^\nu{}_{KL},\end{aligned}\quad (\text{C13})$$

where $\mathcal{F}$, $\mathcal{G}^\mu$, and $\mathcal{H}^{\mu\nu}$ are an arbitrary scalar, vector, and second-rank tensor, respectively. $\mathcal{G}_{(1)}$, $\mathcal{H}_{(1)}$, $\mathcal{H}_{(2)}$, and so on are scalar quantities just like $\mathcal{F}$. Though further investigation is needed for the construction of arbitrary functions with Weyl indices, we can formally write down the most general action with arbitrary functions, and this is given by

$$S = \int d^4x (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2), \quad (\text{C14})$$

where

$$\begin{aligned}\mathcal{L}_0 &= P_0, \\ \mathcal{L}_1 &= P^{(1)\mu}_a \partial_\mu \phi^a + P^{(2)\mu,I}_\alpha \partial_\mu \psi_I^\alpha + \partial_\mu \bar{\psi}_I^\alpha (P^{(2)\mu,I}_\alpha)^\dagger \\ \mathcal{L}_2 &= \frac{1}{2}V_{ab}^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b + S_{aa}^{\mu\nu,I} \partial_\mu \phi^a \partial_\nu \psi_I^\alpha + \partial_\mu \phi^a \partial_\nu \bar{\psi}_I^\alpha (S_{aa}^{\mu\nu,I})^\dagger, \\ &\quad + \frac{1}{2}W_{\alpha\beta}^{\mu\nu,IJ} \partial_\mu \psi_I^\alpha \partial_\nu \psi_J^\beta + \frac{1}{2}\partial_\nu \bar{\psi}_J^\beta \partial_\mu \bar{\psi}_I^\alpha (W_{\alpha\beta}^{\mu\nu,IJ})^\dagger + \partial_\mu \bar{\psi}_I^\alpha Q_{\alpha\dot{\alpha}}^{\mu\nu,IJ} \partial_\nu \psi_J^\alpha,\end{aligned}\quad (\text{C15})$$

and P_0 , $P^{(1)\mu}_a$, $P^{(2)\mu,I}_\alpha$, $V_{ab}^{\mu\nu}$, $S_{aa}^{\mu\nu,I}$, $W_{\alpha\beta}^{\mu\nu,IJ}$, and $Q_{\alpha\dot{\alpha}}^{\mu\nu,IJ}$ are appropriately contracted arbitrary functions of Lorentz scalar quantities, Pauli matrices, and fermionic fields. The properties of these coefficients are

$$\begin{aligned}(P_0)^\dagger &= P_0, & (P^{(1)\mu}_a)^\dagger &= P^{(1)\mu}_a, & (V_{ab}^{\mu\nu})^\dagger &= V_{ab}^{\mu\nu}, & (Q_{\alpha\dot{\alpha}}^{\mu\nu,IJ})^\dagger &= Q_{\beta\dot{\alpha}}^{\nu\mu,JI}, \\ V_{ab}^{\mu\nu} &= V_{ba}^{\nu\mu}, & W_{\alpha\beta}^{\mu\nu,IJ} &= -W_{\beta\alpha}^{\nu\mu,JI}.\end{aligned}\quad (\text{C16})$$

The maximally degenerate conditions, Eqs. (13) and (14), are

$$D_{\alpha\beta,IJ} - \mathcal{C}_{ab,IA}{}^{ba}\mathcal{B}_{\alpha\beta,J} = W_{\alpha\beta}^{00,IJ} + S_{ba}^{00,I}(V^{00})^{-1ba}S_{\alpha\beta}^{00,J} = 0, \quad (\text{C17})$$

$$D_{\alpha\dot{\beta},IJ} - \mathcal{C}_{ab,IA}{}^{ba}\mathcal{B}_{\alpha\dot{\beta},J} = -Q_{\alpha\dot{\beta}}^{00,JI} - S_{ba}^{00,I}(V^{00})^{-1ba}(S_{\alpha\dot{\beta}}^{00,J})^\dagger = 0. \quad (\text{C18})$$

We can write down the momenta as

$$\pi_{\phi^a} = P_a^{(1)0} + V_{ab}^{0\nu} \partial_\nu \phi^b + S_{aa}^{0\nu,I} \partial_\nu \psi_I^\alpha + \partial_\nu \bar{\psi}_I^\alpha (S_{aa}^{0\nu,I})^\dagger, \quad (\text{C19})$$

$$\pi_{\psi^\alpha}^I = -P_\alpha^{(2)0,I} - S_{aa}^{0\nu,I} \partial_\mu \phi^a + W_{\alpha\beta}^{0\nu,IJ} \partial_\nu \psi_J^\beta - \partial_\mu \bar{\psi}^{\dot{\beta}} Q_{\alpha\dot{\beta}}^{\mu 0,IJ}, \quad (\text{C20})$$

$$\pi_{\bar{\psi}^{\dot{\alpha}}}^I = -(\pi_{\psi^\alpha}^I)^\dagger. \quad (\text{C21})$$

As in Sec. III, we assume $\det(V_{ab}^{00})^{(0)} \neq 0$. Substituting (C19) into (C20) for eliminating $\dot{\phi}^a$, we obtain the $4N$ primary constraints,

$$\begin{aligned} \Phi_{\psi^\alpha}^I &= \pi_{\psi^\alpha}^I + P_\alpha^{(2)0,I} - S_{aa}^{00,I} (V^{00})^{-1 ab} P_b^{(1)0} + S_{aa}^{00,I} (V^{00})^{-1 ab} \pi_{\phi^b} - [S_{aa}^{00,I} (V^{00})^{-1 ab} V_{bc}^{0i} - S_{ca}^{i0,I}] \partial_i \phi^c \\ &\quad - [S_{aa}^{00,I} (V^{00})^{-1 ab} S_{b\beta}^{0i,J} + W_{\alpha\beta}^{0i,IJ}] \partial_i \psi_J^\beta + \partial_i \bar{\psi}_J^{\dot{\beta}} [S_{aa}^{00,I} (V^{00})^{-1 ab} (S_{b\beta}^{0i,J})^\dagger + Q_{\alpha\dot{\beta}}^{i0,IJ}], \\ \Phi_{\bar{\psi}^{\dot{\alpha}}}^I &= -(\Phi_{\psi^\alpha}^I)^\dagger. \end{aligned} \quad (\text{C22})$$

APPENDIX D: A SYSTEMATIC WAY TO FIND INDIVIDUAL FUNCTIONS

We show a systematic way to construct the concrete expression for the coefficients of the derivatives in (C15), while in the case of $N = 1$, we have constructed them one by one. Our prescription proposes a minimal set of functions categorized with respect to each tensorial type, but we do not, at this moment, consider the elimination of the redundant functions of which the origin is in the invariance of a Lagrangian under the addition of total derivatives when the proposed functions are used as the coefficients in the Lagrangian. In Sec. III A 2, we concretely reduced the redundant functions coming from the total derivatives in the $N = 1$ case. We also leave the introduction of the symmetries (C16) here. Let us first introduce a new notation. Since any coefficients, in general, have space-time indices and Weyl indices, we label each with

(indices for space-time|indices for Weyl components),
(D1)

according to the tensorial type. We do not include the labels of scalar fields, $a, b, \dots$, and fermions, $I, J, \dots$, in the notation since they are not related to any symmetry so far and are supposed to be arbitrarily contracted with functions of ϕ^a , Ψ_{IJ} , and $\bar{\Psi}_{IJ}$. With this notation, we can classify the candidates for the coefficients with respect to the tensorial type, e.g., $\{\eta^{\mu\nu} \psi_{\alpha,I}, (\sigma^{\mu\nu} \varepsilon)_{\alpha\beta} \psi_I^\beta\} \subset (\mu\nu|\alpha)$. Then, $S_{aa}^{\mu\nu,I}$ should be composed of $(\mu\nu|\alpha)$ tensors, and similar statements also apply to other coefficients as

$$\begin{aligned} P_0 &\in (|), \\ P^{(1)\mu}_a &\in (\mu|), \quad P^{(2)\mu,I}_a \in (\mu|\alpha), \\ V_{ab}^{\mu\nu} &\in (\mu\nu|), \quad S_{aa}^{\mu\nu,I} \in (\mu\nu|\alpha), \quad W_{\alpha\beta}^{\mu\nu,IJ} \in (\mu\nu|\alpha\beta), \\ Q_{\alpha\dot{\beta}}^{\mu\nu,IJ} &\in (\mu\nu|\alpha\dot{\beta}). \end{aligned} \quad (\text{D2})$$

Thus, our goal in this Appendix is to find the most general form of (D2). Let us begin with focusing on the quantities without the Weyl fields as well as the scalar fields, given by

$$\text{no space-time index: } \{1, \varepsilon_{\alpha\beta}, \varepsilon_{\dot{\alpha}\dot{\beta}}\}, \quad (\text{D3})$$

$$\text{one space-time index: } \{\sigma_{\alpha\dot{\alpha}}^\mu\}, \quad (\text{D4})$$

$$\begin{aligned} \text{two space-time indices: } \{&\eta^{\mu\nu}, (\sigma^{\mu\nu} \varepsilon)_{\alpha\beta}, (\varepsilon \bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}, \\ &(\sigma^{\mu\nu} \varepsilon)_{\alpha\beta} (\varepsilon \bar{\sigma}^{\rho\nu})_{\dot{\gamma}\dot{\delta}}\}. \end{aligned} \quad (\text{D5})$$

Here, we considered tensors with up to two space-time indices, which are needed to construct (D2), and classified them according to the number of the space-time indices. Other tensors can be obtained by combining and contracting elements of (D4) and (D5), e.g., $\sigma_{\nu\bullet} (\sigma^{\mu\nu} \varepsilon)_{\bullet\bullet}$. It is, however, shown by using (A4)–(A9) that any combination of tensors in (D4) and (D5) with any contraction of space-time indices is reduced to the linear combination of (D4) and (D5) with coefficients of (D3), just as $\sigma_{\nu\bullet} (\sigma^{\mu\nu} \varepsilon)_{\bullet\bullet}$ is a linear combination of $\varepsilon_{\bullet\bullet} \sigma_{\alpha_1}^\mu$ according to (A7). What we should consider next is the contractions of Weyl indices with $\varepsilon^{\alpha\beta}$ and $\varepsilon^{\dot{\alpha}\dot{\beta}}$ for each combination of elements in (D3)–(D5). If the contraction is taken for an index of (D3), it does not produce new types of tensors; e.g., $\varepsilon_{\alpha\beta} \sigma_{\alpha_1}^\mu \times \varepsilon^{\alpha\alpha_1}$ is just reduced to $-\delta_{\beta}^{\alpha_1} \sigma_{\alpha_1}^\mu$. The contraction of two Weyl indices of each element in (D5) vanishes as they are symmetric. Therefore, in order to find the most general form of (D2), Weyl indices of each combination of elements in (D3)–(D5) should not be contracted with epsilon tensors but with the Weyl fields ψ^α and $\bar{\psi}^{\dot{\alpha}}$.

We begin with the quantities with no space-time index such as $(|\alpha)$, $(|\dot{\alpha})$, $(|\alpha\beta)$, $(|\alpha\dot{\beta})$, and $(|\dot{\alpha}\dot{\beta})$. These are simply obtained by multiplying the building block (D3) by ψ^α and $\bar{\psi}^{\dot{\alpha}}$. All the possible combinations for each are listed in Table I. [Column 1 is composed of an element in (D3) and

TABLE I. For the sequence with no space-time index.

Type	Column 1	Column 2
$(\alpha\rangle)$	$\psi_{\alpha,I}$	
$(\dot{\alpha}\rangle)$	$\bar{\psi}_{\dot{\alpha},I}$	
$(\alpha\beta\rangle)$	$\varepsilon_{\alpha\beta}$	$\psi_{\alpha,I}\psi_{\beta,J}$
$(\alpha\dot{\beta}\rangle)$		$\psi_{\alpha,I}\bar{\psi}_{\dot{\beta},J}$
$(\dot{\alpha}\dot{\beta}\rangle)$	$\varepsilon_{\dot{\alpha}\dot{\beta}}$	$\bar{\psi}_{\dot{\alpha},I}\bar{\psi}_{\dot{\beta},J}$

TABLE II. For the sequence with one space-time index.

Type	Column 1	Column 2
(μ)	J_{IJ}^μ	
$(\mu \alpha)$	$\sigma_{\alpha\dot{\beta}}^\mu \bar{\psi}_{\dot{\beta},I}$	$J_{IJ}^\mu \psi_{\alpha K}$
$(\mu \dot{\alpha})$	$\sigma_{\beta\dot{\alpha}}^\mu \bar{\psi}_{\dot{\alpha},I}$	$J_{IJ}^\mu \bar{\psi}_{\dot{\alpha} K}$

TABLE III. For the sequence with two space-time indices.

Type	Column 1	Column 2	Column 3
$(\mu\nu)$	$\eta^{\mu\nu}, K_{IJ}^{\mu\nu}, \bar{K}_{IJ}^{\mu\nu}, K_{IJ}^{\mu\rho} \bar{K}_{\rho}{}^\nu{}_{KL}$		
$(\mu\nu \alpha)$	$(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi_I^\beta, (\sigma^{\mu\rho}\varepsilon)_{\alpha\beta}\psi_I^\beta \bar{K}_{\rho}{}^\nu{}_{JK}$	$\eta^{\mu\nu}\psi_{\alpha I}, K_{IJ}^{\mu\nu}\psi_{\alpha K}, \bar{K}_{IJ}^{\mu\nu}\psi_{\alpha K}, K_{IJ}^{\mu\rho} \bar{K}_{\rho}{}^\nu{}_{KL}\psi_{\alpha M}$	
$(\mu\nu \dot{\alpha})$	$(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\bar{\psi}_I^{\dot{\beta}}, K_{IJ}^{\mu\rho}(\varepsilon\bar{\sigma}_{\rho}{}^\nu)_{\dot{\alpha}\dot{\beta}}\bar{\psi}_K^{\dot{\beta}}$	$\eta^{\mu\nu}\bar{\psi}_{\dot{\alpha} I}, K_{IJ}^{\mu\nu}\bar{\psi}_{\dot{\alpha} K}, \bar{K}_{IJ}^{\mu\nu}\bar{\psi}_{\dot{\alpha} K}, K_{IJ}^{\mu\rho} \bar{K}_{\rho}{}^\nu{}_{KL}\bar{\psi}_{\dot{\alpha} M}$	
$(\mu\nu \alpha\beta)$	$(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}, (\sigma^{\mu\rho}\varepsilon)_{\alpha\beta} \bar{K}_{\rho}{}^\nu{}_{IJ}$	$(\sigma^{\mu\nu}\varepsilon)_{\alpha\gamma}\psi_I^\gamma\psi_{\beta J}, (\sigma^{\mu\rho}\varepsilon)_{\alpha\gamma}\psi_I^\gamma \bar{K}_{\rho}{}^\nu{}_{JK}\psi_{\beta L},$ $(\sigma^{\mu\nu}\varepsilon)_{\beta\gamma}\psi_I^\gamma\psi_{\alpha J}, (\sigma^{\mu\rho}\varepsilon)_{\beta\gamma}\psi_I^\gamma \bar{K}_{\rho}{}^\nu{}_{JK}\psi_{\alpha L}$	$\eta^{\mu\nu}\psi_{\alpha I}\psi_{\beta J}, K_{IJ}^{\mu\nu}\psi_{\alpha K}\psi_{\beta L}, \bar{K}_{IJ}^{\mu\nu}\psi_{\alpha K}\psi_{\beta L},$ $K_{IJ}^{\mu\rho} \bar{K}_{\rho}{}^\nu{}_{KL}\psi_{\alpha M}\psi_{\beta N}, \eta^{\mu\nu}\varepsilon_{\alpha\beta}, K_{IJ}^{\mu\rho}\varepsilon_{\alpha\beta},$ $\bar{K}_{IJ}^{\mu\rho}\varepsilon_{\alpha\beta}, K_{IJ}^{\mu\rho} \bar{K}_{\rho}{}^\nu{}_{KL}\varepsilon_{\alpha\beta}$
$(\mu\nu \alpha\dot{\beta})$	$(\sigma^{\mu\rho}\varepsilon)_{\alpha\gamma}\psi_I^\gamma(\varepsilon\bar{\sigma}_{\rho}{}^\nu)_{\dot{\beta}\dot{\gamma}}\bar{\psi}_J^{\dot{\gamma}}$	$(\sigma^{\mu\nu}\varepsilon)_{\alpha\gamma}\psi_I^\gamma\bar{\psi}_{\dot{\beta} J}, (\sigma^{\mu\rho}\varepsilon)_{\alpha\gamma}\psi_I^\gamma \bar{K}_{\rho}{}^\nu{}_{JK}\bar{\psi}_{\dot{\beta} L},$ $(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\beta}\dot{\gamma}}\bar{\psi}_I^{\dot{\gamma}}\psi_{\alpha J}, K_{IJ}^{\mu\rho}(\varepsilon\bar{\sigma}_{\rho}{}^\nu)_{\dot{\beta}\dot{\gamma}}\bar{\psi}_K^{\dot{\gamma}}\psi_{\alpha L}$	$\eta^{\mu\nu}\psi_{\alpha I}\bar{\psi}_{\dot{\beta} J}, K_{IJ}^{\mu\nu}\psi_{\alpha K}\bar{\psi}_{\dot{\beta} L},$ $\bar{K}_{IJ}^{\mu\nu}\psi_{\alpha K}\bar{\psi}_{\dot{\beta} L}, K_{IJ}^{\mu\rho} \bar{K}_{\rho}{}^\nu{}_{KL}\psi_{\alpha M}\bar{\psi}_{\dot{\beta} N}$
$(\mu\nu \dot{\alpha}\dot{\beta})$	$(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}, K_{IJ}^{\mu\rho}(\varepsilon\bar{\sigma}_{\rho}{}^\nu)_{\dot{\alpha}\dot{\beta}}$	$(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\gamma}}\bar{\psi}_I^{\dot{\gamma}}\bar{\psi}_{\dot{\beta} J}, K_{IJ}^{\mu\rho}(\varepsilon\bar{\sigma}_{\rho}{}^\nu)_{\dot{\alpha}\dot{\gamma}}\bar{\psi}_K^{\dot{\gamma}}\bar{\psi}_{\dot{\beta} L},$ $(\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\beta}\dot{\gamma}}\bar{\psi}_I^{\dot{\gamma}}\bar{\psi}_{\dot{\alpha} J}, K_{IJ}^{\mu\rho}(\varepsilon\bar{\sigma}_{\rho}{}^\nu)_{\dot{\beta}\dot{\gamma}}\bar{\psi}_K^{\dot{\gamma}}\bar{\psi}_{\dot{\alpha} L}$	$\eta^{\mu\nu}\bar{\psi}_{\dot{\alpha} I}\bar{\psi}_{\dot{\beta} J}, K_{IJ}^{\mu\nu}\bar{\psi}_{\dot{\alpha} K}\bar{\psi}_{\dot{\beta} L},$ $\bar{K}_{IJ}^{\mu\nu}\bar{\psi}_{\dot{\alpha} K}\bar{\psi}_{\dot{\beta} L}, K_{IJ}^{\mu\rho} \bar{K}_{\rho}{}^\nu{}_{KL}\bar{\psi}_{\dot{\alpha} M}\bar{\psi}_{\dot{\beta} N},$ $\eta^{\mu\nu}\varepsilon_{\dot{\alpha}\dot{\beta}}, K_{IJ}^{\mu\rho}\varepsilon_{\dot{\alpha}\dot{\beta}}, \bar{K}_{IJ}^{\mu\rho}\varepsilon_{\dot{\alpha}\dot{\beta}}, K_{IJ}^{\mu\rho} \bar{K}_{\rho}{}^\nu{}_{KL}\varepsilon_{\dot{\alpha}\dot{\beta}}$

the Weyl fields, and a cell in column 2 is made up of the combination of the upper left elements.] The quantities only with one space-time index such as $(\mu|)$, $(\mu|\alpha)$, and $(\mu|\dot{\alpha})$ can be similarly obtained by multiplying the building block (D4) by the Weyl fields, shown in Table II. (A cell in column 2 is composed of the combination of the upper left element and $(|\alpha\rangle)$, $(|\dot{\alpha}\rangle)$.) In a similar way, Table III is

obtained from the building block (D5) as the list of the quantities with two space-time indices. [A cell in column 3 is composed of the combination of the upper left elements with no Weyl indices and $(|\alpha\beta\rangle)$, $(|\alpha\dot{\beta}\rangle)$, $(|\dot{\alpha}\dot{\beta}\rangle)$.]

We can write down all the coefficients in (C15) with the linear combinations of the elements listed in Tables I–III according to each tensorial type.

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