Relativistic corrections to vacuum polarization contributions in muonic hydrogen

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The method proposed by Kinoshita and Nio to compute higher order vacuum polarization contributions to the Coulomb potential in muonic hydrogen is generalized to obtain relativistic corrections to their results.

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I. INTRODUCTION

The usual calculations of the e^+e^- vacuum polarization corrections to the Coulomb potential in muonic hydrogen are made using nonrelativistic Schrödinger wave functions. These corrections have been calculated to orders α^3 [1,2], α^4 [1–3] and α^5 [4]. The experimental measurements [5,6] are sensitive to relativistic corrections to these results. This effect on the order α^3 contribution has been estimated [3,7] using Dirac wave functions with the Bohr radius defined by the muon-proton reduced mass and by others [8,9] using extensions of nonrelativistic techniques. It is of some interest to obtain a similar estimate for the α^4 correction. It appears that the most convenient way to do this is to use the method introduced by Kinoshita and Nio [4], the authors who derived the order α^5 nonrelativistic vacuum polarization corrections.

In the next section, the nonrelativistic treatment is reviewed and the relativistic generalization is presented. This is followed by some concluding remarks. In what follows, the parameters used are [10] $m_e = 0.510998928(11)$ MeV, $m_\mu = 105.6583715(35)$ MeV, $\mu = 94.9645$ MeV, and $\alpha = 1/137.035999074(44)$.

II. RELATIVISTIC CORRECTIONS

A. Nonrelativistic formulation

In Ref. [4], the evaluation of the α^5 vacuum polarization correction is formulated in momentum space. This is convenient because the various order irreducible vacuum polarization contributions $\Pi_f^{(n)}(\vec{k})$ are usually obtained

from diagrammatic calculations. In the nonrelativistic formulation, the vacuum polarization correction ΔE to the Coulomb potential is given by expansion

$$\begin{aligned} \Delta E = & -\frac{e^2}{(2\pi)^3} \int d^3k \frac{1}{k^2} \left(\Pi_f^{(2)}(\vec{k}^2) + \Pi_f^{(2)}(\vec{k}^2) \Pi_f^{(2)}(\vec{k}^2) \right. \\ & + \Pi_f^{(4)}(\vec{k}^2) + \Pi_f^{(2)}(\vec{k}^2) \Pi_f^{(2)}(\vec{k}^2) \Pi_f^{(2)}(\vec{k}^2) \\ & \left. + 2\Pi_f^{(2)}(\vec{k}^2) \Pi_f^{(4)}(\vec{k}^2) + \Pi_f^{(6)}(\vec{k}^2) + \dots \right) \rho(\vec{k}^2 a^2), \end{aligned} \quad (1)$$

where a denotes the Bohr radius $(\mu\alpha)^{-1}$ and $\rho(\vec{k}^2 a^2)$ is the Fourier transform of the nonrelativistic probability density

$$\rho(\vec{k}^2 a^2) = \int d^3r \langle |\psi(\vec{r})|^2 \rangle e^{-i\vec{k}\cdot\vec{r}}. \quad (2)$$

The angular brackets indicate an average over the degenerate p states and for the $n = 2$ states [4]

$$\begin{aligned} \rho_{2S}(\vec{k}^2 a^2) &= \frac{1 - 3\vec{k}^2 a^2 + 2(\vec{k}^2 a^2)^2}{(1 + (\vec{k}^2 a^2)^2)^4}, \\ \rho_{2P}(\vec{k}^2 a^2) &= \frac{1 - (\vec{k}^2 a^2)^2}{(1 + (\vec{k}^2 a^2)^2)^4}. \end{aligned} \quad (3)$$

Note that Eq. (1) is an order by order expansion in powers of α that contains both reducible e.g., $\Pi_f^{(2)}(\vec{k}^2) \Pi_f^{(2)}(\vec{k}^2)$ and irreducible e.g., $\Pi_f^{(4)}(\vec{k}^2)$ contributions. The sum of these contributions can be obtained if one uses the dispersion representation of Källen [11] and Lehmann [12]. The vacuum polarization corrections to the Coulomb potential take the form

$$D(\vec{k}^2) = -\frac{e^2}{k^2} - e^2 \int_{4m_e^2}^{\infty} \frac{d\lambda}{\lambda} \frac{\Delta(\lambda)}{\lambda + k^2}, \quad (4)$$

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where

$$\Delta(q^2) = -\frac{(2\pi)^3}{3q^2} \sum_n \delta^{(4)}(q - q_n) \langle 0 | j_\mu(0) | n \rangle \langle n | j^\mu(0) | 0 \rangle. \quad (5)$$

For any order in e^2 , the integral in Eq. (4) results in

$$-e^2 \int_{4m_e^2}^{\infty} \frac{d\lambda}{\lambda} \frac{\Delta^{(n)}(\lambda)}{\lambda + \vec{k}^2} = -\frac{e^2}{\vec{k}^2} \Pi^{(n)}(\vec{k}^2), \quad (6)$$

where $\Pi^{(n)}(\vec{k}^2)$ is the sum of all $n/2$ loop vacuum polarization corrections, both reducible and irreducible.

For $n = 2$, $\Delta^{(2)}(\lambda)$ is

$$\Delta^{(2)}(\lambda) = \frac{\alpha}{3\pi} (1 + 2m_e^2/\lambda) \sqrt{1 - 4m_e^2/\lambda} \theta(\lambda - 4m_e^2), \quad (7)$$

and, after completing the λ integral, $\Pi^{(2)}(\vec{k}^2)$ is the one loop correction $\Pi_f^{(2)}(\vec{k}^2)$. Using the dispersion representation for $\Pi^{(2)}(\vec{k}^2)/\vec{k}^2$, the expression for the leading vacuum

polarization correction to the Coulomb potential can be expressed as

$$\Delta E^{(2)} = -\frac{2\mu\alpha^3}{3\pi^2} \int_4^\infty \frac{dx}{x} (1 + 2/x) \sqrt{1 - 4/x} \times \int_0^\infty \frac{dy y^2 (\rho_{2P}(y) - \rho_{2S}(y))}{x(\beta^2 x + y^2)}, \quad (8)$$

where $\beta = m_e a$. The remaining integrals can be evaluated using *Mathematica*'s `NINTEGRATE` and give the familiar result [1,2] 205.0074 meV.

The correction of order α^4 can be obtained using the expression for $\Delta^{(4)}(\lambda)$ derived by Källen and Sabry [13]. Explicitly,

$$\Delta E^{(4)} = \frac{\mu\alpha^4}{\pi^3} \int_4^\infty dx \int_0^\infty dy \frac{\Delta^{(4)}(\sqrt{x}/2) y^2 (\rho_{2P}(y) - \rho_{2S}(y))}{x(\beta^2 x + y^2)}, \quad (9)$$

where

$$\begin{aligned} \Delta^{(4)}(x) = & \left(\frac{13}{54x} + \frac{7}{108x^3} + \frac{2}{9x^5} \right) \sqrt{x^2 - 1} + \left(\frac{4}{3x} + \frac{2}{3x^3} \right) \sqrt{x^2 - 1} \log(8x(x^2 - 1)) \\ & + \left(-\frac{44}{9} + \frac{2}{3x^2} + \frac{5}{4x^4} + \frac{2}{9x^6} \right) \text{arccosh}(x) + \left(-\frac{8}{3} + \frac{2}{3x^4} \right) \left[\frac{2\pi^2}{3} - \text{arccosh}^2(x) - \log(8x(x^2 - 1)) \text{arccosh}(x) \right. \\ & \left. - 2\text{Re} \left[\text{Li}_2 \left(\left(x + \sqrt{x^2 - 1} \right)^2 \right) \right] + \text{Li}_2 \left(-\left(x - \sqrt{x^2 - 1} \right)^2 \right) \right]. \end{aligned} \quad (10)$$

Using `NINTEGRATE` to evaluate the integrals gives the usual result [1,3] $\Delta E^{(4)} = 1.5080$ meV. The advantage here is that the bubble and the irreducible contributions need not be computed separately.

B. Relativistic corrections

To obtain the relativistic corrections to the α^3 and α^4 vacuum polarization corrections in the spirit of the muonic hydrogen analysis, all that is necessary is to replace $\langle |\psi(\vec{r})|^2 \rangle$ in Eq. (2) with the Dirac wave functions $\langle (\psi_\kappa^\mu(\vec{r}))^\dagger \psi_\kappa^\mu(\vec{r}) \rangle$. The general form of $\psi_\kappa^\mu(\vec{r})$ is

$$\psi_\kappa^\mu(\vec{r}) = \begin{pmatrix} g_\kappa(r) \chi_\kappa^\mu \\ -i f_\kappa(r) \sigma \cdot \hat{r} \chi_\kappa^\mu \end{pmatrix}, \quad (11)$$

so the expression for $\rho_\kappa(\vec{k})$ is

$$\rho_\kappa(\vec{k}) = \int d^3r (g_\kappa^2(r) + f_\kappa^2(r)) \langle (\chi_\kappa^\mu)^\dagger \chi_\kappa^\mu \rangle e^{-i\vec{k} \cdot \vec{r}}. \quad (12)$$

The angular integral, when averaged over μ_κ , gives $\sin(kr)/(kr)$ for all of the $n = 2$ states and

$$\rho_\kappa(k) = \frac{1}{k} \int_0^\infty dr r (g_\kappa^2(r) + f_\kappa^2(r)) \sin(kr). \quad (13)$$

The radial wave functions for the $n = 2$, $2s_{1/2}$, $2p_{1/2}$ and $2p_{3/2}$, have $\kappa = -1$, $\kappa = 1$ and $\kappa = -2$, respectively, and can be found Rose [14]. For example, $\rho_{-1}(k)$ is

$$\begin{aligned} \rho_{-1}(k^2) = & 0.99999_2 F_1 \left[1.49997, 1.99997, \frac{3}{2}, -2.08229k^2 \right] \\ & - 2.99996_2 F_1 \left[1.99997, 2.49997, \frac{3}{2}, -2.08229k^2 \right] \\ & + 2.99997_2 F_1 \left[2.49997, 2.99997, \frac{3}{2}, -2.08229k^2 \right], \end{aligned} \quad (14)$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ is the hypergeometric function. Its parameters are related to those occurring in any particular

relativistic wave function and the numbers in Eq. (14) are generated by *Mathematica* when calculating $\langle(\psi_k^{\mu_k}(\vec{r}))^\dagger \psi_k^{\mu_k}(\vec{r})\rangle$. The role of the Bohr radius a is played by the square root of the coefficient of k^2 in Eq. (14). This coefficient depends on the total angular momentum j of the state, resulting in two slightly different values of a , $a_{1/2} = 1.44301 \text{ MeV}^{-1}$ and $a_{3/2} = 1.44302 \text{ MeV}^{-1}$. For the $n=2$ states, $a_{1/2} = W/\mu\alpha$, where $W = \sqrt{(1 + \sqrt{1 - \alpha^2})}/2$ and $a_{3/2} = 1/\mu\alpha$. The corresponding values of the parameter $\beta_j = m_e a_j$ are $\beta_{1/2} = 0.737379$ and $\beta_{3/2} = 0.737384$. With this in mind, the expressions for the relativistic corrections $\Delta E_k^{(2)}$ and $\Delta E_k^{(4)}$ are

$$\Delta E_k^{(2)} = -\frac{2\alpha^2}{3\pi^2 a_j} \int_4^\infty \frac{dx}{x} (1 + 2/x) \sqrt{1 - 4/x} \times \int_0^\infty \frac{dy y^2 (\rho_k(y) - \rho_{-1}(y))}{x(\beta_j^2 x + y^2)} - \Delta E^{(2)} \quad (15)$$

$$\Delta E_k^{(4)} = \frac{\alpha^3}{\pi^3 a_j} \int_4^\infty dx \times \int_0^\infty dy \frac{\Delta^{(4)}(\sqrt{x}/2) y^2 (\rho_k(y) - \rho_{-1}(y))}{x(\beta_j^2 x + y^2)} - \Delta E^{(4)}, \quad (16)$$

with

$$\rho_{-1}(x) = 0.99999 {}_2F_1 \left[1.49997, 1.99997, \frac{3}{2}, -x^2 \right] - 2.99996 {}_2F_1 \left[1.99997, 2.49997, \frac{3}{2}, -x^2 \right] + 2.99997 {}_2F_1 \left[2.49997, 2.99997, \frac{3}{2}, -x^2 \right], \quad (17)$$

$$\rho_1(x) = 9.98481 \times 10^{-6} {}_2F_1 \left[1.49997, 1.99997, \frac{3}{2}, -x^2 \right] + 9.98468 \times 10^{-6} {}_2F_1 \left[1.99997, 2.49997, \frac{3}{2}, -x^2 \right] + 0.99998 {}_2F_1 \left[2.49997, 2.99997, \frac{3}{2}, -x^2 \right], \quad (18)$$

$$\rho_{-2}(x) = {}_2F_1 \left[2.49999, 2.99999, \frac{3}{2}, -x^2 \right]. \quad (19)$$

The numerical integrations are evaluated using NINTEGRATE.

The leading order relativistic corrections from Eq. (15) are

$$\Delta E_1^{(2)} = 0.02080 \text{ meV}, \quad \Delta E_{-2}^{(2)} = 0.0258 \text{ meV}, \quad (20)$$

which confirms the known results [3]. The order α^4 corrections from Eq. (16) are

$$\Delta E_1^{(4)} = 0.0001 \text{ meV}, \quad \Delta E_{-2}^{(4)} = 0.0002 \text{ meV}. \quad (21)$$

These are obviously small but of the order that tends to be kept in calculations of contributions to the $n = 2$ levels of muonic hydrogen.

III. CONCLUSIONS

The calculation of the relativistic corrections in momentum space is quite straightforward given the tools available to handle numerical integration and hypergeometric functions. In this calculation, the dispersion approach is particularly convenient because the explicit forms of the leading order $\Pi_f^{(2)}(\vec{k}^2)$ or the next-to-leading order $\Pi_f^{(4)}(\vec{k}^2)$ need not be known. The next-to-next-to-leading order is irrelevant when it comes to relativistic corrections. However, it is unfortunate that $\Delta^{(6)}(\lambda)$ is not available because it would simplify the calculation of the vacuum polarization correction at order α^5 [4].

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