***a*-theorem for Horndeski gravity at the critical point**Yue-Zhou Li[†] and Hong Lü^{*}*Department of Physics, Tianjin University, Tianjin 300072, China*

(Received 13 April 2018; revised manuscript received 13 May 2018; published 13 June 2018)

We study holographic conformal anomalies and the corresponding *a*-theorem for Einstein gravity extended with Horndeski terms that involve up to and including linear curvature tensors. We focus on our discussion in $D = 5$ bulk dimensions. For the generic Horndeski coupling, the *a*-charge is the same as that in Einstein gravity, but the inclusion of the Horndeski term violates the *a*-theorem. However, there exists a critical point of the Horndeski coupling, for which the theory admits nearly anti-de Sitter (AdS) spacetimes with nonvanishing Horndeski scalar. The full AdS isometry is broken down by the logarithmic scalar hair to the Poincaré group plus the scale invariance. We find that in this case the *a*-charge depends on the AdS radius ℓ and the integration constant χ_s of the Horndeski scalar. In addition, we find that two new central charges emerge, that are absent in gravities with minimally coupled matter. We call them *b*-charges. These *b*-charges also depend on ℓ and χ_s . We construct an *a*-function for fixed ℓ but with the running Horndeski scalar χ replacing the constant χ_s , and establish the holographic *a*-theorem using the null energy condition in the bulk. Furthermore, we find that there exist analogous monotonous *b*-functions as well. We also obtain the *a*-charge and the *a*-theorem in general odd bulk dimensions.

DOI: [10.1103/PhysRevD.97.126008](https://doi.org/10.1103/PhysRevD.97.126008)**I. INTRODUCTION**

The AdS/CFT correspondence [1] serves as a powerful tool to investigate a certain strongly coupled conformal field theory (CFT) in d dimensions by studying its weakly coupled dual classical anti-de Sitter (AdS) gravity in $D = d + 1$ dimensions.¹ In particular, one can employ the gauge/gravity duality to determine the conformal anomaly of the CFT by means of the Fefferman-Graham (FG) expansion in bulk gravity in $D = 2n + 1$ dimensions [2–5]. The resulting holographic conformal anomaly may arise as the trace of the boundary energy-momentum tensor that would be nonvanishing even at the vacuum. It turns out that the holographic conformal anomaly in the gravitational sector is generally given by [4]

$$\langle T_\mu^\mu \rangle \sim -aE^{(2n)} + \sum_i c_i I_i^{(2n)}, \quad (1.1)$$

where $E^{(2n)}$ is the Euler density and the $I_i^{(2n)}$'s are all the Weyl invariants in $d = 2n$ dimensions. Equation (1.1) is the most general formula for the holographic conformal anomaly arising from a pure gravity action. Any gravity in odd dimensions constructed from arbitrary higher-order curvature invariants exhibits the holographic conformal anomaly as in Eq. (1.1). The details of the theory are encoded in the specific constant coefficients a and c_i , which represent two different types of central charges. These charges are typically expressed in terms of various coupling constants of higher-order terms [6–17]. The form of the holographic conformal anomaly (1.1) coincides precisely with that obtained from the explicit CFT calculation [18–20].

The Wilsonian description of quantum field theory states that higher energy modes are integrated out along the renormalization group (RG) flow from higher energy to lower energy and the degrees of freedom decrease irreversibly. The quantitative statement is typically referred to as the *a*-theorem, namely, there exist *a*-charges measuring the massless degrees of freedom of the CFT at the RG fixed points, and the charge at the ultraviolet (UV) fixed point is always larger than or equal to that at the infrared (IR) fixed point:

$$a_{\text{UV}} \geq a_{\text{IR}}. \quad (1.2)$$

An even stronger statement is that there exists an *a*-function of energy scale μ that becomes a_{UV} and a_{IR} at the respective fixed points, and the function $a(\mu)$ is monotonically increasing, namely

^{*}mrhonglu@gmail.com

[†]liyuezhou@tju.edu.cn

¹In this paper, we shall always use D to refer to the dimensions of bulk gravity, and d to refer to the dimensions of the boundary field theory.

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

$$\frac{d}{d\mu} a(\mu) \geq 0. \quad (1.3)$$

The first important example of an a -theorem is the c -theorem of $d = 2$ CFT established by Zamolodchikov [21]. Further works in higher dimensions can be found in Refs. [22–24].

The a -theorem can also be investigated in the context of the AdS/CFT correspondence [14,25,26], in which the energy scale corresponds to the bulk radius r

$$\frac{d}{dr} a(r) \geq 0, \quad a(r)|_{\text{AdS}} = a, \quad (1.4)$$

for a certain $a(r)$ function, where $r \rightarrow \infty$ is the boundary of the asymptotic AdS geometry, which corresponds to the UV region of the dual CFT. The constant a above is the coefficient in front of the Euler density evaluated at the AdS fixed point. In the holographic framework, one could readily generalize the holographic a -theorem to arbitrary dimensions [15,17], and for a gravity theory involving higher-order curvature invariants, the general dimension version of the holographic a -theorem, in turn, constrains the coupling constants [17]. More properties of the RG flow other than the a -theorem can also be studied in the framework of holography (e.g., Refs. [27–29]).

Matter fields may also contribute to the holographic conformal anomalies. The contributions from minimally coupled scalar fields were studied in Refs. [30–32]. However, the new anomalies do not alter either the form or explicitly the a and c charges in Eq. (1.1) in the gravitational sector, but rather they involve boundary matter fields, and we call them anomalies in the matter sector. The situation becomes different when we consider nonminimally coupled matter and a novel holographic feature may arise. This is because nonminimal couplings mix gravity and matter, which may alter the asymptotic behavior of the matter fields, and hence the FG expansions, yielding novel contributions to the holographic anomaly.

In this paper, we consider Einstein-Horndeski gravity. Horndeski invariants were constructed from curvature tensors and axionic scalars [33]. They were later rediscovered [34] as some Galilean theories, and were deeply investigated in its application in cosmology; see e.g., Refs. [35,36]. For our purpose, we focus on Einstein gravity with a negative cosmological constant, together with the kinetic term for the axion $g_{\mu\nu} \partial^\mu \chi \partial^\nu \chi$ and one Horndeski term $G_{\mu\nu} \partial^\mu \partial^\nu \chi$. The vacuum is the AdS spacetime where the scalar vanishes. It turns out that there exists a critical point of the Horndeski coupling, for which a nearly AdS spacetime can emerge where the scalar is nonvanishing. Properties and applications beyond cosmology have also been actively pursued. Locally asymptotically AdS black hole solutions were constructed in Refs. [37–42], stability and causality were discussed in Refs. [43–45], black hole thermodynamics was analyzed

in Refs. [46,47], the AdS/condensed matter theory (AdS/CMT) properties were given in Refs. [41,48,49], and further properties and applications were studied in Refs. [42,50,51].

In this paper we examine the holographic a -theorem for Einstein-Horndeski gravity. In $D = 5$ dimensions, we derive the holographic conformal anomaly using the FG expansions and find that the result is $\mathcal{A}_{\text{gr}} + \mathcal{A}_m$. The quantity $\mathcal{A}_m$ is given in terms of $\chi^{(0)}$, the boundary field of the Horndeski axion χ . We thus refer to $\mathcal{A}_m$ as the anomaly in the matter sector. The anomalous term $\mathcal{A}_{\text{gr}}$ is expressed in terms of the boundary curvature invariants and hence we refer to it as the anomaly in the gravitational sector. Intriguingly there is no mixing between the two sectors, as if the scalar is minimally coupled, even though the Horndeski scalar is definitely not minimally coupled in the bulk.

For the generic couplings of Einstein-Horndeski gravity we consider in this paper, we find that the anomalous term $\mathcal{A}_{\text{gr}}$ is identical to that of Einstein gravity. In other words, the Horndeski scalar gives contributions to $\mathcal{A}_m$, but not to $\mathcal{A}_{\text{gr}}$. Furthermore, we find that there does not exist an a -function that can lead to the a -theorem (1.4).

However, there exists a critical point of the Horndeski coupling for which the theory admits a nearly AdS spacetime where the Horndeski scalar is nonvanishing. The FG expansion for the scalar χ at the critical point admits an additional logarithmic mode. This mode can give a nontrivial contribution to the a and c charges. Furthermore, we find that the structure of the anomalous term $\mathcal{A}_{\text{gr}}$ in $D = 2n + 1$ in critical Einstein-Horndeski gravity is augmented from Eq. (1.1) to become

$$\mathcal{A}_{\text{gr}} \sim -aE^{(2n)} + \sum_i c_i I_i^{(2n)} + \sum_k b_k H_k^{(2n)}. \quad (1.5)$$

In other words, new central charges emerge, and we shall call them b -charges. In $D = 5$ dimensions, we perform explicit calculations and find that there are two $H^{(4)}$'s, given by

$$H_1^{(4)} = \square R, \quad H_2^{(4)} = R^{ij} R_{ij} - \frac{1}{4} R^2 - \frac{1}{4} \square R. \quad (1.6)$$

Thus in five dimensions, the inclusion of the Horndeski term at the critical coupling gives rise to a total of four holographic conformal charges in the gravitational sector, associated with all four of the possible boundary curvature terms at the fourth order. With appropriate recombination, they can be grouped as

$$\begin{aligned}
\mathcal{A}_{\text{gr}} &\sim -aE^{(4)} + cI^{(4)} + b_1 H_1^{(4)} + b_2 H_2^{(4)}, \\
E^{(4)} &= R^2 - 4R^{ij}R_{ij} + R^{ijkl}R_{ijkl}, \\
I^{(4)} &= \frac{1}{3}R^2 - 2R_{ij}R^{ij} + R_{ijkl}R^{ijkl}.
\end{aligned} \tag{1.7}$$

By contrast, for gravity with only minimally coupled matter, there are only a and c charges, while the new b -charges are absent. Another distinguishing feature of the central charges for Einstein-Horndeski gravity at the critical point is that they depend not only on the coupling constants of the theory, but also on the integration constant χ_s associated with the scalar hair. In conformal field theory, the anomalous trace term of the form $\square R$, i.e., $H_1^{(4)}$ with a constant coefficient can be removed by some appropriate local counterterm proportional to R^2 . Interestingly, in the case of the holographic conformal anomaly for Einstein gravity (or higher-order curvature gravity) with minimally coupled matter, such a term does not even arise. However, as we shall discuss in the next section, the appearance of the holographic anomalous term $H_1^{(4)}$ in Einstein-Horndeski gravity is not removable by a local counterterm in the boundary field theory owing to the fact that the charge depends on the scalar hair χ_s .

The paper is organized as follows. In Sec. II, we briefly review the Einstein-Horndeski theory. We perform the FG expansion explicitly in $D = 5$ and derive the conformal anomalies for both generic and critical couplings. In Sec. III, we establish that there is no a -theorem for the case with generic couplings. For the critical theory, we find an a -function and establish the a -theorem using the null energy condition. In Sec. IV, we study the conformal

anomaly in general $D = 2n + 1$ dimensions. Owing to the complexity of the calculations, we deduce only the a -charge using the technique developed in Ref. [17]. We then establish an a -theorem using the null energy condition. We conclude the paper in Sec. V.

II. HOLOGRAPHIC CONFORMAL ANOMALY IN $D=5$

A. Horndeski gravity and the FG expansions in general dimensions

We begin by reviewing Einstein-Horndeski gravity involving up to only linear curvature terms. The action in general dimensions D is given by

$$\begin{aligned}
S &= \frac{1}{16\pi} \int d^D x \sqrt{-g} \mathcal{L}, \\
\mathcal{L} &= R - 2\Lambda_0 - \frac{1}{2}(\alpha g_{\mu\nu} - \gamma G_{\mu\nu})\partial^\mu \chi \partial^\nu \chi,
\end{aligned} \tag{2.1}$$

where the bare cosmological constant Λ_0 is taken to be negative in general. The parameters α and γ are coupling constants, and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor. Since χ appears in the action only through a derivative, there is a constant shift symmetry associated with χ , implying that χ is axionic. Note that it is reasonable to expect that $\alpha > 0$, in which case, we can set $\alpha = +1$ without loss of generality. In this paper, we shall let α be arbitrary.

The equations of motions associated with the variations of the metric $g^{\mu\nu}$ and χ are respectively given by

$$\begin{aligned}
E_{\mu\nu} &= G_{\mu\nu} + \Lambda_0 g_{\mu\nu} - \frac{1}{2}\alpha \left(\partial_\mu \chi \partial_\nu \chi - \frac{1}{2} 12 g_{\mu\nu} (\partial\chi)^2 \right) - \frac{1}{2}\gamma \left(\frac{1}{2} \partial_\mu \chi \partial_\nu \chi R - 2 \partial_\rho \chi \partial_{(\mu} \chi R_{\nu)}{}^\rho - \partial_\rho \chi \partial_\sigma \chi R_\mu{}^\rho{}_\nu{}^\sigma \right. \\
&\quad \left. - (\nabla_\mu \nabla^\rho \chi)(\nabla_\nu \nabla_\rho \chi) + (\nabla_\mu \nabla_\nu \chi) \square \chi + \frac{1}{2} G_{\mu\nu} (\partial\chi)^2 - g_{\mu\nu} \left[-\frac{1}{2} (\nabla^\rho \nabla^\sigma \chi)(\nabla_\rho \nabla_\sigma \chi) + \frac{1}{2} (\square \chi)^2 - \partial_\rho \chi \partial_\sigma \chi R^{\rho\sigma} \right] \right) = 0, \\
E_\chi &= \nabla^\mu (\alpha g_{\mu\nu} - \gamma G_{\mu\nu}) \nabla^\nu \chi = 0.
\end{aligned} \tag{2.2}$$

For vanishing χ , these equations admit a maximally symmetric AdS vacuum with $G_{\mu\nu} = -\Lambda_0 g_{\mu\nu}$. As we shall see later, at a certain critical point of the Horndeski coupling γ , there exists a nearly AdS spacetime for which χ is nonvanishing and its integration constant contributes to the effective cosmological constant Λ_{eff} . We shall use Λ_{eff} for the general case and it becomes the bare Λ_0 when $\chi = 0$. We now parametrize Λ_{eff} by

$$\Lambda_{\text{eff}} = -\frac{d(d-1)}{2\ell^2}, \tag{2.3}$$

where ℓ is the AdS radius. We emphasize again here that we shall always use D to denote the total bulk dimensions

and d to denote the boundary dimensions. In this vacuum, the effective kinetic term for the Horndeski scalar χ is

$$\mathcal{L}_\chi = -\frac{1}{2}(\alpha + \gamma \Lambda_{\text{eff}})(\partial\chi)^2. \tag{2.4}$$

The absence of a ghost excitation requires that $\alpha + \gamma \Lambda_{\text{eff}} \geq 0$, with the equality corresponding to the critical point. It follows from Eq. (2.3) that the critical point of the Horndeski coupling is given by [41,42]

$$\gamma = \frac{2\ell^2}{d(d-1)} \alpha. \tag{2.5}$$

Analogous critical points were also found in Einstein-Gauss-Bonnet gravity [52].

In order to investigate the holographic RG flow in Einstein-Horndeski gravity, we analyze the asymptotic behavior using the FG expansions. For the metric ansatz of the form

$$ds^2 = \frac{\ell^2}{4\rho^2} d\rho^2 + \frac{g_{ij}}{\rho} dx^i dx^j, \quad i, j = 1, \dots, d, \quad (2.6)$$

the FG expansion for g_{ij} is

$$\begin{aligned} g_{ij} &= g_{ij}^{(0)} + \rho g_{ij}^{(2)} + \rho^2 g_{ij}^{(4)} + \dots, \\ g^{ij} &= g^{(0)ij} - \rho g^{(2)ij} - \rho^2 \left(g^{(4)ij} - \frac{1}{4} \text{Tr}(g^{(2)2}) g^{(0)ij} \right) + \dots \end{aligned} \quad (2.7)$$

In this expansion, $\rho = 0$ represents the boundary of the AdS spacetime. Since the Einstein-Horndeski theory we study in this paper involves only Ricci tensors, we do not need to compute the FG expansions for the Riemann tensors. The relevant components for the Ricci tensors are

$$\begin{aligned} R_{ij} &= \hat{R}_{ij} - \frac{2\rho}{\ell^2} g'_{ij} + \frac{1}{\ell^2} g^{lk} g'_{lk} g_{ij} + \frac{2\rho}{\ell^2} g^{kl} g'_{lj} g_{ki} \\ &\quad - \frac{\rho}{\ell^2} g^{lk} g'_{lk} g'_{ij} + \frac{d-2}{\ell^2} g'_{ij} - \frac{d}{\ell^2} \frac{1}{\rho} g_{ij}, \\ R_{\rho\rho} &= -\frac{d}{4\rho^2} - \frac{1}{2} g^{ij} g'_{ij} + \frac{1}{4} g^{ik} g^{jl} g'_{ij} g'_{kl}, \end{aligned} \quad (2.8)$$

where a prime here denotes a derivative with respect to ρ and $\hat{R}_{ij}$ is the Ricci tensor of the boundary metric g_{ij} . Expanding $\hat{R}_{ij}$ in terms of ρ , we have

$$\hat{R}_{ij} = \hat{R}_{ij}^{(0)} + \rho \hat{R}_{ij}^{(2)} + \dots, \quad (2.9)$$

where $\hat{R}_{ij}^{(2)}$ is given in terms of $g_{ij}^{(2)}$

$$\begin{aligned} \hat{R}_{ij}^{(2)} &= -\frac{1}{2} \nabla_i^{(0)} \nabla_j^{(0)} (g^{(0)kl} g_{kl}^{(2)}) - \frac{1}{2} \nabla_k^{(0)} \nabla^{(0)k} g_{ij}^{(2)} \\ &\quad + g^{(0)kl} \nabla_k^{(0)} \nabla_l^{(0)} g_{ij}^{(2)}. \end{aligned} \quad (2.10)$$

Its trace by $g_{ij}^{(0)}$ is

$$\hat{R}^{(2)} = -\square^{(0)} \text{Tr} g^{(2)} + \nabla^{(0)i} \nabla^{(0)j} g_{ij}^{(2)}, \quad (2.11)$$

and hence $\sqrt{g^{(0)}} \hat{R}^{(2)}$ is a total derivative. Here $\text{Tr} g^{(2)} = g^{(0)kl} g_{kl}^{(2)}$.

For asymptotically AdS spacetimes, the FG expansion for the Horndeski scalar χ takes the form

$$\chi = \chi^{(0)} + \rho \chi^{(2)} + \rho^2 \chi^{(4)} + \dots \quad (2.12)$$

In the usual context of the AdS/CFT correspondence, this massless scalar is dual to some scalar operator with conformal dimension $\Delta = 4$. To be specific, $\chi^{(0)}$ plays the role of the source of the corresponding scalar operator, and its response appears starting at ρ^2 order. As we shall see in the next subsection, the above expansion is consistent with the equations of motion for general coupling constants. However, at the critical point (2.5), there exist asymptotically nearly AdS spacetimes where there is a new logarithmic term for χ . Thus at the critical coupling, the FG expansion for the scalar becomes

$$\chi = \chi_s \log \rho + \chi^{(0)} + \rho \chi^{(2)} + \rho^2 \chi^{(4)} + \dots, \quad (2.13)$$

where χ_s is set to be a constant. Thus at the critical point, the additional scalar hair χ_s appears. Although the metric is asymptotically AdS, the isometry $SO(2, 4)$ is broken by this mode, and we call the asymptotic geometry nearly AdS, while the boundary field theory is called nearly conformal. To be specific, the surviving symmetry is the Poincaré group together with a scaling invariance, which is a subgroup of $SO(2, 4)$. The corresponding dual theory becomes the scale-invariant but not conformally invariant quantum field theory. This subject has been reviewed in Ref. [53]. As we shall see in Sec. II C, this new logarithmic term in the critical theory gives a nontrivial contribution to the holographic conformal anomaly. In what follows, we shall focus on $D = 5$ bulk dimensions, corresponding to $d = 4$ boundary dimensions.

B. Holographic conformal anomaly for generic couplings

In this subsection, we consider the FG expansions (2.7) for g_{ij} and (2.12) for χ respectively. Substituting all these into Eq. (2.1) and taking $D = 5$, we end up with

$$\begin{aligned} S &= \frac{1}{16\pi} \int d^4x \int_\epsilon d\rho \sqrt{-g} \mathcal{L} \\ &= \frac{1}{16\pi} \int d^4x \sqrt{g^{(0)}} \int_\epsilon d\rho \left(\dots + \frac{\mathcal{A}}{\rho} + \dots \right), \end{aligned} \quad (2.14)$$

where the action has been expanded in a Laurent series around $\rho = 0$, the dots to the left and right of $\mathcal{A}/\rho$ represent negative and positive (and zero) powers of ρ respectively, and the coefficient $\mathcal{A}$ will be given soon. The nearly AdS boundary is located at $\rho = \epsilon \sim 0$ in the FG expansion. It is therefore obvious that the right-dot terms are convergent at the AdS boundary. The left-dot terms are divergent, but can be removed by the appropriate Gibbons-Hawking-type surface terms and the holographic counterterms. The $\mathcal{A}/\rho$ term can only appear in *odd* bulk dimensions, and it gives rise to a logarithmic divergence at the AdS boundary after

integrating over ρ . This divergence cannot be canceled by either the Gibbons-Hawking or holographic counterterms since these surface terms have only the power-law divergence owing to the specific ansatz of the FG expansions.

This term is then interpreted as the holographic conformal anomaly.

After some involved algebra, we find that the anomalous term $\mathcal{A}$ is given by

$$\begin{aligned} \ell^{-1}\mathcal{A} = & \beta_1 \text{Tr}g^{(4)} + A_1 \text{Tr}(g^{(2)2}) + A_2 (\text{Tr}g^{(2)})^2 - \frac{1}{2}(\hat{R}_{ij}^{(0)}g^{(2)ij} - \hat{R}^{(2)}) + \text{Tr}(g^{(2)})\left(\frac{1}{4}\hat{R}^{(0)} + A_3(\nabla^{(0)}\chi^{(0)})^2\right) \\ & + A_4 g^{(2)ij}\nabla_i^{(0)}\chi^{(0)}\nabla_j^{(0)}\chi^{(0)} - \ell^{-2}\beta_2\chi^{(2)2} - \frac{1}{2}\beta_2 g^{(0)ij}\nabla_i^{(0)}\chi^{(2)}\nabla_j^{(0)}\chi^{(0)} + \frac{1}{4}\gamma\hat{G}^{(0)ij}\nabla_i^{(0)}\chi^{(0)}\nabla_j^{(0)}\chi^{(0)}. \end{aligned} \quad (2.15)$$

The various coefficients above are given by

$$\begin{aligned} \beta_1 &= -\frac{3}{\ell^2} - \frac{1}{2}\Lambda_0, & \beta_2 &= \alpha - \frac{6\gamma}{\ell^2}, \\ A_1 &= \frac{1}{\ell^2} + \frac{1}{4}\Lambda_0, & A_2 &= -\frac{1}{4\ell^2} - \frac{1}{8}\Lambda_0, \\ A_3 &= -\frac{1}{8}\alpha + \frac{\gamma}{4\ell^2}, & A_4 &= \frac{1}{4}\alpha - \frac{\gamma}{\ell^2}. \end{aligned} \quad (2.16)$$

Note that $\sqrt{g^{(0)}}\hat{R}^{(2)}$ is a total derivative and hence does not contribute to the equations of motion, but it may contribute to the total anomaly.

The equations of motion of the boundary fields in the FG expansions can be obtained by substituting the ansatz into the full equations of motion (2.2) and solving them order by order in powers of ρ . This can be cumbersome. Since the FG ansatz is self-consistent, we can treat

$$S_{\text{eff}} = \int d^4x \sqrt{g^{(0)}} \mathcal{A} \quad (2.17)$$

as an effective action and derive the equations of motion of the boundary fields. We shall adopt this much simpler approach.

The equation of motion associated with the variation of $g_{ij}^{(4)}$ is $\beta_1 = 0$, which relates the bare cosmological constant with the AdS radius, namely

$$\Lambda_0 = -\frac{6}{\ell^2}. \quad (2.18)$$

Thus in this case, we have $\Lambda_{\text{eff}} = \Lambda_0$. The equation of motion associated with the variation of $\chi^{(2)}$ is then

$$\frac{\beta_2}{\ell^2} \left(\chi^{(2)} - \frac{1}{4}\ell^2 \square^{(0)}\chi^{(0)} \right) = 0. \quad (2.19)$$

One way to solve the above equation is to set $\beta_2 = 0$, corresponding precisely to the critical condition (2.5) in $d = 4$. In this subsection, we shall consider generic couplings with $\beta_2 \neq 0$, for which case the above equation is solved with

$$\chi^{(2)} = \frac{1}{4}\ell^2 \square^{(0)}\chi^{(0)}. \quad (2.20)$$

The variation of the remaining unknown field $g_{ij}^{(2)}$ gives rise to

$$\begin{aligned} \text{Tr}g^{(2)} &= -\frac{1}{12}\ell^2(2\hat{R}^{(0)} - \alpha(\nabla^{(0)}\chi^{(0)})^2), \\ g_{ij}^{(2)} &= \frac{1}{12}\ell^2(\hat{R}^{(0)}g_{ij}^{(0)} - 6\hat{R}_{ij}^{(0)}) \\ &\quad - \frac{1}{24}\ell^2\left(\beta_2(\nabla^{(0)}\chi^{(0)})^2\hat{g}_{ij}^{(0)}\right. \\ &\quad \left.- 6\left(\alpha - \frac{4\gamma}{\ell^2}\right)\nabla_i^{(0)}\chi^{(0)}\nabla_j^{(0)}\chi^{(0)}\right). \end{aligned} \quad (2.21)$$

Substituting these back into Eq. (2.15), we find that the on-shell action contains two parts:

$$\mathcal{A} = \mathcal{A}_{\text{gr}} + \mathcal{A}_m, \quad (2.22)$$

where $\mathcal{A}_{\text{gr}}$ is the anomalous term in the gravitational sector, namely

$$\mathcal{A}_{\text{gr}} = \frac{1}{8}\ell^3\left(\hat{R}_{ij}^{(0)}\hat{R}^{(0)ij} - \frac{1}{3}\hat{R}^{(0)2}\right). \quad (2.23)$$

The quantity $\mathcal{A}_m$ denotes the anomalous term associated with the Horndeski scalar, given by

$$\begin{aligned} \mathcal{A}_m = & -\beta_2\left(\frac{1}{8}\ell^3(\hat{R}_{ij}^{(0)}\nabla^{(0)i}\chi^{(0)}\nabla^{(0)j}\chi^{(0)} - \frac{1}{3}\hat{R}^{(0)}(\partial^{(0)}\chi^{(0)})^2\right) \\ & - \frac{1}{16}\ell^3\chi^{(0)}\square^{(0)2}\chi^{(0)} - \frac{1}{48}\ell(\ell^2\alpha - 3\gamma)(\nabla^{(0)}\chi^{(0)})^4 \\ & + \frac{1}{2}\hat{R}^{(2)}. \end{aligned} \quad (2.24)$$

Note that $\hat{R}^{(2)}$ can be obtained explicitly in terms of the scalar field:

$$\hat{R}^{(2)} = -\frac{1}{24}\ell^2 \left(3 \left(\alpha - \frac{2\gamma}{\ell^2} \right) \square^{(0)} (\nabla^{(0)} \chi^{(0)})^2 - 6 \left(\alpha - \frac{4\gamma}{\ell^2} \right) \nabla^{(0)i} \nabla^{(0)j} (\nabla_i^{(0)} \chi^{(0)} \nabla_j^{(0)} \chi^{(0)}) \right). \quad (2.25)$$

Thus we see that $\mathcal{A}_{\text{gr}}$ depends only on the boundary curvature invariants, while $\mathcal{A}_m$ depends only on the boundary scalar invariants. Intriguingly there is no mixture of the two types, as if $\chi^{(0)}$ is minimally coupled, even though χ is nonminimally coupled in the bulk. There is a point in parameter space $\alpha = \frac{3\gamma}{\ell^2}$ which suppresses the $(\nabla^{(0)} \chi^{(0)})^4$ term. As we have remarked earlier, $\chi^{(0)}$ is the source of the corresponding scalar operator in the dual field theory. Consequently if we turn off the source, $\mathcal{A}_m$ vanishes while $\mathcal{A}_{\text{gr}}$ remains unchanged.

The anomaly in the gravitational sector $\mathcal{A}_{\text{gr}}$ is associated with the central charges, and it follows from Eqs. (2.23) and (1.7) that we have

$$a = c = \frac{\ell^3}{16}. \quad (2.26)$$

These are precisely the same central charges as in Einstein gravity. While the Horndeski scalar does give an anomalous contribution to $\mathcal{A}$, it does not change $\mathcal{A}_{\text{gr}}$ in the gravitational sector.

C. Holographic conformal anomaly at the critical point

The critical point in $D = 5$ Einstein-Horndeski gravity is

$$\gamma = \frac{1}{6}\ell^2 \alpha, \quad (2.27)$$

corresponding to $\beta_2 = 0$. As we saw in the previous subsection, the equations become degenerate, indicating that there exist additional modes in the FG expansions. In fact, as we mentioned in the Introduction, the theory admits the nearly AdS spacetime, which, in the FG coordinates, is given by

$$ds^2 = \frac{\ell^2}{4\rho^2} d\rho^2 + \frac{1}{\rho} dx^i dx^i, \quad \chi(\rho) = \chi_s \log \rho + \chi_0, \quad \Lambda_0 = -\frac{6}{\ell^2} \left(1 + \frac{2\chi_s^2 \gamma}{\ell^2} \right). \quad (2.28)$$

Although the metric is exactly the AdS spacetime, the full AdS isometry is broken by the scalar χ .

Thus for asymptotically nearly AdS geometries, the correct FG expansion for the Horndeski scalar contains an additional logarithmic mode, and hence the FG expansion is given by Eq. (2.13). Note that the logarithmic mode was indeed present in the black hole constructed in Ref. [38]. In the notation of Ref. [46], it is given by $|\chi_s| = \frac{1}{2}\ell\sqrt{\beta}$, where β is the parameter used in Ref. [46].

Including the logarithmic mode, we find that for generic couplings the coefficient $\mathcal{A}$ in Eq. (2.14) is now given by

$$\begin{aligned} \ell^{-1} \mathcal{A} = & \beta_1 \text{Tr} g^{(4)} + A_1 \text{Tr}(g^{(2)2}) + A_2 (\text{Tr} g^{(2)})^2 + A_5 (\hat{R}_{ij}^{(0)} g^{(2)ij} - \hat{R}^{(2)}) + \text{Tr}(g^{(2)}) \left(-\frac{1}{2} A_5 \hat{R}^{(0)} + A_3 (\nabla^{(0)} \chi^{(0)})^2 - \frac{\chi_s \alpha}{\ell^2} \chi^{(2)} \right) \\ & + A_4 g^{(2)ij} \nabla_i^{(0)} \chi^{(0)} \nabla_j^{(0)} \chi^{(0)} - \frac{4\chi_s \beta_2}{\ell^2} \chi^{(4)} - \frac{\beta_2}{\ell^2} \chi^{(2)2} - \frac{\chi_s \gamma}{\ell^2} \hat{R}^{(0)} \chi^{(2)} - \frac{1}{2} \beta_2 g^{(0)ij} \nabla_i^{(0)} \chi^{(2)} \nabla_j^{(0)} \chi^{(0)} + \frac{1}{4} \gamma \hat{G}^{(0)ij} \nabla_i^{(0)} \chi^{(0)} \nabla_j^{(0)} \chi^{(0)}. \end{aligned} \quad (2.29)$$

The coefficients β_2 , A_3 and A_4 are the same as those given in Eq. (2.16). The remaining coefficients are modified by χ_s and given by

$$\begin{aligned} \beta_1 &= -\frac{3}{\ell^2} - \frac{\chi_s^2}{2\ell^2} \left(\alpha + \frac{6\gamma}{\ell^2} \right) - \frac{1}{2} \Lambda_0, \\ A_1 &= \frac{1}{\ell^2} + \frac{\chi_s^2}{4\ell^2} \left(\alpha + \frac{4\gamma}{\ell^2} \right) + \frac{1}{4} \Lambda_0, \\ A_2 &= -\frac{1}{4\ell^2} - \frac{\chi_s^2}{8\ell^2} \left(\alpha + \frac{2\gamma}{\ell^2} \right) - \frac{1}{8} \Lambda_0, \\ A_5 &= -\frac{1}{2} \left(1 - \frac{\chi_s^2 \gamma}{\ell^2} \right). \end{aligned} \quad (2.30)$$

The equation of motion associated with the variation of $\chi^{(4)}$ yields

$$\beta_2 \chi_s = 0. \quad (2.31)$$

Thus for generic couplings with $\beta_2 \neq 0$, we must have $\chi_s = 0$, leading to the asymptotic FG expansion (2.12) for the Horndeski scalar. At the critical point $\beta_2 = 0$, on the other hand, the coefficient χ_s for the logarithmic term does not have to vanish.

The equation of motion associated with the variation of $g_{ij}^{(4)}$ relates the bare cosmological constant with the effective cosmological constant $\Lambda_{\text{eff}} = -6/\ell^2$, given by Eq. (2.28). The variation of the action with respect to $\chi^{(2)}$ leads to

$$-\frac{6\chi_s}{\ell^4} \left(\text{Tr} g^{(2)} + \frac{1}{6} \ell^2 \hat{R}^{(0)} \right) = 0. \quad (2.32)$$

This equation can be solved by requiring $\chi_s = 0$, in which case we obtain the results in the previous

subsection for generic β_2 . Instead we solve this equation by requiring

$$\text{Tr}g^{(2)} + \frac{1}{6}\ell^2\hat{R}^{(0)} = 0. \quad (2.33)$$

It is interesting to note that with this, the equation of motion associated with the variation of χ_s is automatically satisfied. Finally, the variation of $g_{ij}^{(2)}$ yields $g_{ij}^{(2)}$. Taking the trace with respect to $g_{ij}^{(0)}$ and making use of Eq. (2.33), we can also solve for $\chi^{(2)}$. The results are given by

$$\begin{aligned} \chi^{(2)} &= -\frac{\ell^2}{48\chi_s}(2\chi_s^2\hat{R}^{(0)} + 3(\nabla^{(0)}\chi^{(0)})^2), \\ g_{ij}^{(2)} &= \frac{\ell^2}{12(\ell^2 + \chi_s^2\gamma)}((\ell^2 - 2\chi_s^2\gamma)\hat{R}^{(0)}g_{ij}^{(0)} - 6(\ell^2 - \chi_s^2\gamma)\hat{R}_{ij}^{(0)}) \\ &\quad - \frac{\ell^2\gamma}{8(\ell^2 + \chi_s^2\gamma)}((\nabla^{(0)}\chi^{(0)})^2g_{ij}^{(0)} - 4\nabla_i^{(0)}\chi^{(0)}\nabla_j^{(0)}\chi^{(0)}). \end{aligned} \quad (2.34)$$

Therefore we have

$$\begin{aligned} \hat{R}^{(2)} &= \frac{\ell^2\chi_s^2\gamma}{4(\ell^2 + \chi_s^2\gamma)}\square^{(0)}\hat{R}^{(0)} - \frac{\ell^2\gamma}{8(\ell^2 + \chi_s^2\gamma)}(\square^{(0)}(\nabla^{(0)}\chi^{(0)})^2 \\ &\quad - 4\nabla^{(0)i}\nabla^{(0)j}\nabla_i^{(0)}(\chi^{(0)}\nabla_j^{(0)}\chi^{(0)})). \end{aligned} \quad (2.35)$$

Substituting these back into $\mathcal{A}$, we find again that $\mathcal{A}$ is the sum of two parts

$$\mathcal{A} = \mathcal{A}_{\text{gr}} + \mathcal{A}_m, \quad (2.36)$$

where

$$\begin{aligned} \mathcal{A}_{\text{gr}} &= \frac{\ell(\ell^2 - 3\chi_s^2\gamma)}{8}\left(\hat{R}_{ij}^{(0)}\hat{R}^{(0)ij} - \frac{1}{3}\hat{R}^{(0)2}\right) + \frac{\ell^3\chi_s^2\gamma}{8(\ell^2 + \chi_s^2\gamma)}\square^{(0)}\hat{R}^{(0)} + \frac{\ell\chi_s^4\gamma^2}{2(\ell^2 + \chi_s^2\gamma)}\left(\hat{R}_{ij}^{(0)}\hat{R}^{(0)ij} - \frac{1}{4}(\hat{R}^{(0)2} + \square^{(0)}\hat{R}^{(0)})\right), \\ \mathcal{A}_m &= \frac{\gamma^2\ell((3(\nabla^{(0)}\chi^{(0)})^2 - 4\chi_s^2\hat{R}^{(0)})(\nabla^{(0)}\chi^{(0)})^2 + 16\chi_s^2\hat{R}_{ij}^{(0)}\nabla^{(0)i}\chi^{(0)}\nabla^{(0)j}\chi^{(0)})}{32(\ell^2 + \chi_s^2\gamma)} \\ &\quad + \frac{1}{16\ell}\frac{\ell^2 - \chi_s^2\gamma}{\ell^2 + \chi_s^2\gamma}(\square^{(0)}(\nabla^{(0)}\chi^{(0)})^2 - 4\nabla^{(0)i}\nabla^{(0)j}\nabla_i^{(0)}(\chi^{(0)}\nabla_j^{(0)}\chi^{(0)})). \end{aligned} \quad (2.37)$$

The holographic conformal anomaly for the gravitation sector is quite surprising. Comparing the results with Eq. (1.7), we find that in addition to the a - and c -charges, namely

$$a = c = \frac{\ell^3}{16}\left(1 - \frac{3\chi_s^2\gamma}{\ell^2}\right), \quad (2.38)$$

we have two new charges, which we call b -charges. Specifically, they are given by

$$b_1 = \frac{\ell^3\chi_s^2\gamma}{8(\ell^2 + \chi_s^2\gamma)}, \quad b_2 = \frac{\ell\chi_s^4\gamma^2}{2(\ell^2 + \chi_s^2\gamma)}. \quad (2.39)$$

It is worth noting that the overall coefficient of the term $\square^{(0)}R^{(0)}$ in $\mathcal{A}_{\text{gr}}$ is

$$b_1 - \frac{1}{4}b_2 = \frac{\ell\chi_s^2\gamma}{8}\frac{\ell^2 - \chi_s^2\gamma}{\ell^2 + \chi_s^2\gamma}. \quad (2.40)$$

Comparing with the shear viscosity/entropy ratio obtained for the black hole in the Einstein-Horndeski theory [46], namely

$$\frac{\eta}{s} = \frac{1}{4\pi}\sqrt{\frac{\ell^2 - \chi_s^2\gamma}{\ell^2 + \chi_s^2\gamma}}, \quad (2.41)$$

we can thus express the above results in terms of purely anomalous charges:

$$\frac{\eta}{s} = \frac{1}{4\pi}\sqrt{\frac{2(4b_1 - b_2)}{3(\ell^3 - 16a)}}. \quad (2.42)$$

It is worth noticing that the η/s combination also appears as a coefficient of the total derivative term of $\mathcal{A}_m$ in Eq. (2.37).

As in the previous case, when the source mode $\chi^{(0)}$ vanishes, the anomalous term $\mathcal{A}_m$ vanishes also, but the quantity $\mathcal{A}_{\text{gr}}$ is completely unchanged. The occurrence of the b_1 -charge requires further comments. In a conformal field theory, the trace anomaly of the type $\square R$ with a constant coefficient can be removed by adding the local counterterm R^2 with an appropriate constant coefficient. Intriguingly in the holographic picture, such a term does not even arise in the bulk gravity calculation, when gravity

and matter are minimally coupled. However, in Einstein-Horndeski gravity at the critical point, whose boundary field theory is not conformal, but nearly conformal, the anomalous $\Box R$ term does emerge. This leads to the question whether this term is removable by adding appropriate counterterms in the field theory.

In the previously studied examples of AdS gravities in the literature where matter is minimally coupled, the holographic central charges depend only on the “fixed” coupling constants of the theory, such as the cosmological constant or the couplings of higher-order curvature invariants. A distinguishing feature that occurs in Einstein-Horndeski gravity at the critical point is that all the central charges depend also on the integration constant χ_s of the axionic scalar χ . It follows that the anomalous $\Box R$ cannot be removed by the local counterterms. In other words, the scalar hair χ_s appearing in the b_1 -charge is not a coupling constant, and it should be viewed as the expectation value of a local field that couples to $\Box R$.

In fact, since the b_2 -charge is forbidden by any conformal field theory, its emergence also indicates that the boundary field theory is not conformal. It is a reflection of the fact that the background is nearly AdS. In other words, although the metric is AdS, the full AdS symmetry is broken spontaneously by the scalar hair χ_s . The surviving symmetry is the Poincaré group together with the scaling invariance, while the special conformal invariance is broken.

III. ESTABLISHING AN a -THEOREM IN $D=5$

In the previous section, we studied the holographic conformal anomalies in $D=5$ Einstein-Horndeski gravity. We found that for general couplings, the holographic anomaly in the gravitational sector is identical to that in pure Einstein gravity, given by Eq. (2.26). The situation changes significantly at the critical point of the couplings. In addition to the usual $a=c$ charge, which differs from that in Einstein gravity, new conformal charges b_1 and b_2 also arise.

In this section, we examine whether there exists an a -theorem for each of the above charges. We follow Ref. [14] and consider the cohomogeneity-one domain wall ansatz

$$ds_5^2 = dr^2 + e^{2A(r)}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2), \quad \chi = \chi(r). \quad (3.1)$$

The AdS vacuum is given by $A(r) = r/\ell$, where ℓ is the AdS radius. The function $A(r)$ describe the flow to the AdS “fixed” point. For asymptotically AdS domain walls where we adopt the convention that the asymptotic region is located at $r \rightarrow \infty$, the coordinate r is related to the FG coordinate ρ (with flat boundary) by

$$r = -\frac{1}{2}\ell \log \rho. \quad (3.2)$$

A. No a -theorem for generic couplings

As we established in Sec. II B, for generic couplings in Einstein-Horndeski gravity, the a -charge is given by Eq. (2.26), depending only on the AdS radius ℓ . This leads to a natural choice for the a -function

$$a(r) = \frac{1}{16A^3}, \quad (3.3)$$

which implies that $a' = -3A''/(16A^4)$. We now assume that there is an additional matter energy-momentum tensor $T_{\mu\nu}^{\text{mat}}$ for some generic minimally coupled matter in Einstein-Horndeski gravity, and the full Einstein equations of motion become

$$E_{\mu\nu} = T_{\mu\nu}^{\text{mat}}, \quad (3.4)$$

where $E_{\mu\nu}$ is given by Eq. (2.2). Thus for our domain wall ansatz, we have

$$\begin{aligned} -E_t^t + E_r^r &= -3A'' - \frac{3}{2}\gamma A'\chi'\chi'' + \frac{1}{4}(-2\alpha + 12\gamma A'^2 - 3\gamma A'')\chi'^2 \\ &= -(T^{\text{mat}})_t^t + (T^{\text{mat}})_r^r. \end{aligned} \quad (3.5)$$

The null energy condition for the matter field, $-(T^{\text{mat}})_t^t + (T^{\text{mat}})_r^r \geq 0$, implies that

$$16a'A'^4 \left(1 + \frac{1}{4}\gamma\chi'\right)^2 - \frac{3}{2}\gamma A'\chi'\chi'' + \frac{1}{2}(6\gamma A'^2 - \alpha)\chi'^2 \geq 0. \quad (3.6)$$

Thus we see that for a nonvanishing coupling constant γ of the Horndeski term, we cannot establish the a -theorem $a' \geq 0$. (Note that $a' \geq 0$ when $\gamma = 0$ for $\alpha \geq 0$, as one would expect.)

B. An a -theorem at the critical point

A novel RG flow property arises when the coupling parameters are at the critical point. In particular, the equations of motion imply that the Horndeski scalar no longer vanishes in the AdS vacuum (with Λ_{eff}) but is given by

$$\chi(r) = -\frac{2}{\ell}\chi_s r + \chi_0. \quad (3.7)$$

It follows from Eq. (3.2) that this linear r term corresponds to the logarithmic term in the FG expansion of the Horndeski scalar. As was mentioned earlier, the full AdS isometry of the vacuum is broken down by the scalar to the Poincaré group plus the scale invariance.

Since the additional matter fields we consider are minimally coupled, the equation of motion for χ remains $E_\chi = 0$ as in Eq. (2.2). Since in the domain wall ansatz, all fields are functions of r , it follows that $E_\chi = 0$ implies that

$$(\alpha g_{rr} - \gamma G_{rr})\partial^r \chi = (\alpha - 6\gamma A'^2)\chi' = C, \quad (3.8)$$

where C is an integration constant. Since as $r \rightarrow \infty$, $(\alpha - 6\gamma A') \rightarrow 0$, it follows that in order for χ to behave as Eq. (3.7) for $r \rightarrow \infty$, we must have $C = 0$. Thus we have

$$\alpha - 6\gamma A'^2 = 0. \quad (3.9)$$

This equation keeps the domain wall at the AdS vacuum $A(r) = r/\ell$ irrespective of the existence of other minimally coupled matters. This seems to suggest that the AdS vacuum remains rigidly “fixed” and there can be no RG flow. This is indeed true for the AdS radius parameter; however, there is another integration constant χ_s that contributes to the a -charge and the scalar χ can run along the RG flow. In fact in this critical case, the scalar χ is not determined by the scalar equation $E_\chi = 0$, but rather it is determined by the Einstein equations where the nonminimally matter energy-momentum tensor contributes.

It is thus natural to replace χ_s by the function $\chi_s = -\ell\chi'/2$, and substitute this into the a -charge (2.38). We thus propose an a -function:

$$a(r) = \frac{1}{16}\ell^3 \left(1 - \frac{3}{4}\chi'^2\gamma\right), \Rightarrow a' = -\frac{3}{32}\ell^3\gamma\chi'\chi''. \quad (3.10)$$

At the critical point, with the help of Eq. (3.9), it is easy to see from Eq. (3.5) that we have

$$0 \leq -(T^{\text{mat}})_t^t + (T^{\text{mat}})_r^r = -E_t^t + E_r^r = -\frac{3}{2\ell}\gamma\chi'\chi'' = \frac{16}{\ell^4}a'. \quad (3.11)$$

Thus the a -theorem holds true indeed, namely

$$a'(r) \geq 0. \quad (3.12)$$

Hence Einstein-Horndeski gravity satisfies the a -theorem at the critical point in $D = 5$.

As we established in Sec. II C, in addition to the $a = c$ charges, Einstein-Horndeski gravity at the critical point has also two b -charges. The physical interpretation of these charges is not clear to us. To understand their properties better, we use the same argument and propose two $b(r)$ -functions

$$b_1(r) = \frac{\ell^3\chi'^2\gamma}{8(4 + \gamma\chi'^2)}, \quad b_2(r) = \frac{\ell^3\gamma^2\chi'^4}{8(4 + \gamma\chi'^2)}. \quad (3.13)$$

The derivatives of these two functions respectively give

$$b'_1 = \frac{\ell^3\gamma(\chi'^2)'}{2(4 + \gamma\chi'^2)^2}, \quad b'_2 = \frac{\ell^3\gamma^2\chi'^2(8 + \chi'^2\gamma)(\chi'^2)'}{8(4 + \gamma\chi'^2)^2}. \quad (3.14)$$

It follows from the null energy condition (3.11) that b_1 is monotonously decreasing, namely

$$b'_1 \leq 0. \quad (3.15)$$

The situation for b'_2 is somewhat more complicated. In the region $\gamma > 0$, it is clear from Eq. (3.11) that $(\chi'^2)' \leq 0$, and therefore $b'_2 \leq 0$. On the other hand, in the region $\gamma < 0$, we must have $(\chi'^2)' \geq 0$, and hence $\chi'^2\gamma < 0$ is monotonously decreasing. The quantity $\chi'^2\gamma$ thus reaches its minimum at the AdS boundary $r \rightarrow \infty$ in this case, i.e., $\chi'^2\gamma|_\infty = 4\gamma\chi_s^2/\ell^2$. Thus as long as even this minimum renders $8 + \chi'^2\gamma \geq 0$, then $b'_2 \geq 0$ is true for the entire r region. We finally conclude that

$$b'_2 \leq 0, \quad \text{when } \gamma > 0, \\ b'_2 \geq 0, \quad \text{when } \gamma < 0 \quad \text{and} \quad \chi_s^2 \leq \frac{2\ell^2}{(-\gamma)}. \quad (3.16)$$

It is worth commenting that the critical coupling for γ is given by Eq. (2.5) with $d = 4$. It is natural to take α for the kinetic term of χ to be positive, in which case, only the first case above arises. For completeness, we also give the second case above for which α is negative.

IV. A HOLOGRAPHIC a -THEOREM IN GENERAL DIMENSIONS

In Sec. III, we obtained the holographic conformal anomaly for $D = 5$ Einstein-Horndeski gravity for general and critical couplings. In particular, new anomalous terms emerge in the gravitational sector at the critical point. In general $D = 2n + 1$ ($d = 2n$) dimensions, we expect that the total anomalous term at the critical point takes the form

$$\mathcal{A}_{\text{gr}} = -aE^{(2n)} + \sum_i c_i I_i^{(2n)} + \sum_k b_k H_k^{(2n)}, \quad (4.1)$$

where $H_k^{(2n)}$ are some specific invariants constructed from the boundary curvature tensors and their covariant derivatives. Using the general FG expansion to determine these (a, b_k, c_i) charges is a formidable task. In fact, we have not obtained the explicit forms for the $H_k^{(2n)}$ terms except for $D = 5$. We shall give a conjecture of their defining property presently. In this section, we shall be content to obtain only the a -charge, in which case, we can adopt the trick discussed in Ref. [17]. First we note that if $g_{ij}^{(0)}$ is conformally flat, then all $I_i^{(2n)}$ vanish. Furthermore, in $D = 5$, there are two b -charges, i.e., the coefficients of the following two terms:

$$H_1^{(4)} = \square^{(0)} \hat{R}^{(0)},$$

$$H_2^{(4)} = \hat{R}_{ij}^{(0)} \hat{R}^{(0)ij} - \frac{1}{4} (\hat{R}^{(0)2} + \square^{(0)} \hat{R}^{(0)}). \quad (4.2)$$

If we choose $g_{ij}^{(0)}$ to be an Einstein metric, then both $H_1^{(4)}$ and $H_2^{(4)}$ vanish identically. Thus if we chose a special class of FG expansions where g_{ij} is a conformally flat and Einstein metric, then we can read off the a -charge only. Thus if our task is to read off the a -charge only, we can choose a subclass of FG expansions, for which the calculation becomes much simpler.

In order to obtain the a -charge in general dimensions, we make a conjecture that all $H_k^{(2n)}$ vanish when g_{ij} is an Einstein metric. This can be viewed as the defining property of $H_k^{(2n)}$.

A. The a -charge in higher dimensions

The simplest Einstein metric that is conformally flat is that for a round sphere. We follow Ref. [17] and consider the reduced FG expansion, with the metric

$$ds^2 = \frac{\ell^2}{4\rho^2} d\rho^2 + f(\rho) d\Omega_d^2, \quad (4.3)$$

where $d\Omega_d^2$ is the metric for the round d -sphere S^d . The function $f(\rho)$ and the Horndeski scalar $\chi(\rho)$ are expanded as

$$f = f_0 + f_2\rho + f_4\rho^2 + f_6\rho^3 + \dots,$$

$$\chi = \chi_s \log \rho + \chi_2\rho + \chi_4\rho^2 + \chi_6\rho^3 + \dots, \quad (4.4)$$

where f_{2i} , χ_{2i} and χ_s are all constant variables. Now in $D = 2k + 1$, the anomaly $\mathcal{A}$ is a functional of $(f_2, f_4, \dots, f_{2k})$, $(\chi_2, \chi_4, \dots, \chi_{2k})$ and χ_s . In particular the terms involving χ_{2k} and f_{2k} are linear in these two variables, given by

$$\mathcal{A} = -\frac{d}{2\ell^3 f_0} \left(\ell^4 \Lambda_0 + \frac{d(d-1)}{2} \gamma \chi_s^2 \right. \\ \left. + \ell^2 \left(\frac{1}{2} d(d-1) + \alpha \chi_s^2 \right) f_{2k} \right. \\ \left. - \frac{d}{\ell^3} \left(\ell^2 \alpha - \frac{1}{2} d(d-1) \gamma \right) \chi_s \chi_{2k} + \dots \right). \quad (4.5)$$

The variation of χ_{2k} implies

$$\left(\alpha \ell^2 - \frac{1}{2} d(d-1) \gamma \right) \chi_s = 0. \quad (4.6)$$

The solution is either that $\chi_s = 0$ or we take the critical condition (2.5). The former leads to an a -charge that is identical to that in Einstein gravity and the Horndeski term

gives no contribution. As in the $D = 5$ case discussed in Sec. III, we can find no a -function for a nonvanishing Horndeski coupling that satisfies the a -theorem.

The situation becomes more interesting if we solve Eq. (4.6) by requiring the critical condition (2.5). At the critical point, the variation of f_{2k} leads to

$$\Lambda_0 = -\frac{d(d-1)}{2\ell^2} \left(1 + \frac{2\chi_s^2 \gamma}{\ell^2} \right). \quad (4.7)$$

Then the variation of the remaining variables $(f_2, f_4, \dots, f_{2(k-1)})$ and $(\chi_2, \chi_4, \dots, \chi_{2(k-1)})$ implies that

$$\chi_{2n} = (-1)^n \frac{\ell^{2n} \chi_s}{2^{2n-1} n f_0}, \quad 1 \leq n \leq k-1.$$

$$f_2 = -\frac{1}{2} \ell^2, \quad f_4 = \frac{\ell^4}{16 f_0}, \quad f_{2n} = 0, \quad 3 \leq n \leq k-1. \quad (4.8)$$

We can then read off the a -charge, up to an overall numerical constant, as

$$a = \ell^{d-1} \left(1 - (d-1) \frac{\chi_s^2 \gamma}{\ell^2} \right). \quad (4.9)$$

In addition to the round sphere boundary, we verified that many other boundaries can also give Eq. (4.9), as long as they are Einstein metrics. This is because when the bulk theory involves only Ricci tensors, all the c -charges are proportional to the a -charge, giving rise to a net a -charge, up to a pure numerical factor [17]. The fact that we get the same a -charge for various different Einstein metrics as boundaries confirms our original conjecture that the b -charge terms all vanish when the boundary metrics are Einstein.

B. An a -theorem in general dimensions

As discussed in Sec. III for $D = 5$, we consider here the domain wall ansatz in general dimensions:

$$ds_{d+1}^2 = dr^2 + e^{2A(r)} (-dt^2 + dx_i dx^i), \quad \chi = \chi(r). \quad (4.10)$$

At the critical point, by the same argument in $D = 5$, we find that the equation of motion for the Horndeski scalar is reduced to

$$\alpha g_{rr} - \gamma G_{rr} = \alpha - \frac{1}{2} d(d-1) \gamma A'^2 = 0. \quad (4.11)$$

Thus the AdS geometry is fixed and not running. We thus instead substitute $\chi_s = -\ell \chi' / 2$ into the general a -charge (4.9) and define an a -function at the critical point

$$a(r) = \ell^{d-1} \left(1 - \frac{1}{4}(d-1)\chi'^2 \gamma \right), \quad (4.12)$$

which implies that

$$a'(r) = -\frac{1}{2}(d-1)\ell^{d-1}\gamma\chi'\chi''. \quad (4.13)$$

The null energy condition in general dimensions implies that

$$\begin{aligned} -(T^{\text{mat}})_t{}^t + (T^{\text{mat}})_r{}^r &= -E_t^t + E_r^r \\ &= -(d-1)A'' - \frac{1}{2}(d-1)\gamma A'\chi'\chi'' \\ &\quad + \frac{1}{4}(-2\alpha + d(d-1)\gamma A'^2 \\ &\quad - (d-1)\gamma A'')\chi'^2 \geq 0. \end{aligned} \quad (4.14)$$

At the critical point, using Eq. (4.11), Eq. (4.14) becomes

$$-(T^{\text{mat}})_t{}^t + (T^{\text{mat}})_r{}^r = -\mathcal{E}_t^t + \mathcal{E}_r^r = -\frac{d-1}{2\ell}\gamma\chi'\chi'' = \frac{a'}{\ell^d} \geq 0. \quad (4.15)$$

Thus the a -theorem

$$a'(r) \geq 0 \quad (4.16)$$

is established at the critical point with the null energy condition.

C. Holographic conformal anomaly in $D=3$

In Sec. II, we considered the $D=5$ example and used the full FG expansion to construct the holographic conformal anomalies. For general dimensions, the full FG expansion becomes very complicated and we adopted the technique of Ref. [17] and used the reduced FG expansion ansatz for $D \geq 7$ dimensions.

For $D=3$, the situation becomes simpler than that in $D=5$ and we can easily analyze the holographic conformal anomaly using the full FG expansions (2.7) and (2.13). We find

$$\begin{aligned} S &= \frac{1}{16\pi} \int d^2x \int_\epsilon d\rho \sqrt{-g} \mathcal{L} \\ &= \frac{1}{16\pi} \int d^2x \sqrt{g^{(0)}} \int_\epsilon d\rho \left(\cdots + \frac{\mathcal{A}}{\rho} + \cdots \right), \end{aligned} \quad (4.17)$$

where the quantity $\mathcal{A}$ is

$$\ell^{-1}\mathcal{A} = A_1 \hat{R}^{(0)} + A_2 \text{Tr}g^{(2)} + A_3 \chi^{(2)} + A_4 (\partial^{(0)}\chi^{(0)})^2. \quad (4.18)$$

The various coefficients A_i above are

$$\begin{aligned} A_1 &= \frac{\ell^2 - \chi_s^2 \gamma}{2\ell^2}, & A_2 &= -\frac{\ell^2 + \chi_s^2(\gamma + \ell^2 \alpha) + \ell^4 \Lambda_0}{2\ell^4}, \\ A_3 &= \frac{2\chi_s(\gamma - \alpha\ell^2)}{\ell^4}, & A_4 &= \frac{\gamma - \alpha\ell^2}{4\ell^2}. \end{aligned} \quad (4.19)$$

The variations of $\chi^{(2)}$ and $g_{ij}^{(2)}$ provide the critical condition and determine the cosmological constant:

$$\alpha = \frac{\gamma}{\ell^2}, \quad \Lambda_0 = -\frac{1}{\ell^2} \left(1 + \frac{2\chi_s^2 \gamma}{\ell^2} \right). \quad (4.20)$$

Then the holographic conformal anomaly in $D=3$ is simply

$$\mathcal{A} = \frac{1}{2}\ell \left(1 - \frac{\chi_s^2 \gamma}{\ell^2} \right) \hat{R}^{(0)}. \quad (4.21)$$

Interestingly, there is no anomaly in the matter sector at all at the critical point. The a -charge is given by

$$a = \ell \left(1 - \frac{\chi_s^2 \gamma}{\ell^2} \right). \quad (4.22)$$

This gives Eq. (4.9) with $d=2$. The corresponding a -theorem was established in the previous subsection.

V. CONCLUSION

In this paper, we studied Einstein-Horndeski gravity with the Horndeski terms involving only up to and including the linear curvature tensor. We took the cosmological constant to be negative so that the theory admitted the AdS spacetimes. The theory may be expected to be dual to some strongly coupled conformal field theory in the AdS boundary, by virtue of the AdS/CFT correspondence. Applications to the AdS/CMT correspondence were studied in the literature. In this paper, we analyzed the holographic conformal anomaly and studied the corresponding a -theorem.

We focused our discussion on $D=5$ bulk dimensions, corresponding to $d=4$ boundary dimensions. We adopted the full FG expansion to derive the conformal anomalies. We found that the total anomaly splits into two parts. One is the anomaly in the gravitational sector, given by the curvature tensor invariants of the boundary metric. The other is the anomaly in the matter sector, given by the invariants constructed from the Horndeski scalar. There is no mixing between the two sectors as if the theory were minimally coupled. For the generic Horndeski coupling, we found that the anomaly in the gravitational sector was the same as that in Einstein gravity, and the Horndeski scalar gives no contribution to the $a=c$ charges. There exists a critical point of the Horndeski

coupling for which case the theory admits a nearly AdS spacetime with the nonvanishing scalar. The full AdS isometry is broken by the scalar hair to the Poincaré group plus the scale invariance. The corresponding scale-invariant quantum field theory has a much richer anomalous structure. Not only does the Horndeski scalar contribute to the $a = c$ charges, but it also generates two additional b -charges in the gravitational sector.

Intriguingly, the boundary curvature polynomials $H^{(4)}$ associated with the b -charges vanish identically for any Einstein metric. Although we have not computed the $H^{(2n)}$ explicitly except for $D = 5$, we conjectured that this would be true in general $D = 2n + 1$ dimensions. In fact this may be the defining property of $H^{(2n)}$. This conjecture allowed us to consider general odd dimensions and use the reduced FG expansions to deduce the a -charge by taking the boundary metrics in the FG expansion to be Einstein metrics. We obtained the general formula for the a -charge at the critical point in general odd bulk dimensions. We used a variety of different Einstein metrics and obtained the same result for the a -charge, confirming our conjecture.

We then used the domain wall ansatz to study the holographic a -theorem. For generic Horndeski couplings, we failed to find an a -function that would satisfy the a -theorem. On the other hand, for the critical couplings,

we found that the a -theorem could be established for general odd bulk dimensions. In this case, the a -charge depends on two parameters: one is the AdS radius, and the other is an integration constant of the Horndeski scalar. We found that at the critical point, the AdS radius was rigidly fixed and the domain wall was the rigidly AdS metric, but the Horndeski scalar could run with the holographic RG flow. In $D = 5$, we also obtained explicit expressions for the two new b -charges, and we found that there exist monotonous b -functions. The physical meanings of these two b -charges, and their apparent “ b -theorems” remain to be further investigated.

Not all gravity theories yield sensible conformal or nearly conformal field theories under the AdS/CFT correspondence. The dual conformal field theory of Einstein-Horndeski gravity is far from clear. Our results provide encouraging signs for the theory at the critical point.

ACKNOWLEDGMENTS

We are grateful to the referee for raising questions on the holographic b_1 -charge, and to Zhao-Long Wang for useful discussions to clarify the issues. This work is supported in part by National Natural Science Foundation of China Grants No. 11475024 and No. 11235003.

-
- [1] J. M. Maldacena, The large- N limit of superconformal field theories and supergravity, *Int. J. Theor. Phys.* **38**, 1113 (1999); *Adv. Theor. Math. Phys.* **2**, 231 (1998).
 - [2] M. Henningson and K. Skenderis, The holographic Weyl anomaly, *J. High Energy Phys.* **07** (1998) 023.
 - [3] M. Henningson and K. Skenderis, Holography and the Weyl anomaly, *Fortschr. Phys.* **48**, 125 (2000).
 - [4] C. Imbimbo, A. Schwimmer, S. Theisen, and S. Yankielowicz, Diffeomorphisms and holographic anomalies, *Classical Quantum Gravity* **17**, 1129 (2000).
 - [5] A. Schwimmer and S. Theisen, Universal features of holographic anomalies, *J. High Energy Phys.* **10** (2003) 001.
 - [6] S. Nojiri and S. D. Odintsov, On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence, *Int. J. Mod. Phys. A* **15**, 413 (2000).
 - [7] M. Blau, K. S. Narain, and E. Gava, On subleading contributions to the AdS/CFT trace anomaly, *J. High Energy Phys.* **09** (1999) 018.
 - [8] M. Banados, A. Schwimmer, and S. Theisen, Chern-Simons gravity and holographic anomalies, *J. High Energy Phys.* **05** (2004) 039.
 - [9] P. Kraus and F. Larsen, Holographic gravitational anomalies, *J. High Energy Phys.* **01** (2006) 022.
 - [10] M. Banados, R. Olea, and S. Theisen, Counterterms and dual holographic anomalies in CS gravity, *J. High Energy Phys.* **10** (2005) 067.
 - [11] N. Banerjee and S. Dutta, Shear viscosity to entropy density ratio in six derivative gravity, *J. High Energy Phys.* **07** (2009) 024.
 - [12] R. C. Myers and B. Robinson, Black holes in quasi-topological gravity, *J. High Energy Phys.* **08** (2010) 067.
 - [13] R. C. Myers, M. F. Paulos, and A. Sinha, Holographic studies of quasi-topological gravity, *J. High Energy Phys.* **08** (2010) 035.
 - [14] R. C. Myers and A. Sinha, Seeing a c -theorem with holography, *Phys. Rev. D* **82**, 046006 (2010).
 - [15] R. C. Myers and A. Sinha, Holographic c -theorems in arbitrary dimensions, *J. High Energy Phys.* **01** (2011) 125.
 - [16] Y. Z. Li, H. S. Liu, and H. Lü, Quasi-topological Ricci polynomial gravities, *J. High Energy Phys.* **02** (2018) 166.
 - [17] Y. Z. Li, H. Lü, and J. B. Wu, Causality and a -theorem constraints on Ricci polynomial and Riemann cubic gravities, *Phys. Rev. D* **97**, 024023 (2018).
 - [18] M. J. Duff, Observations on conformal anomalies, *Nucl. Phys.* **B125**, 334 (1977).
 - [19] M. J. Duff, Twenty years of the Weyl anomaly, *Classical Quantum Gravity* **11**, 1387 (1994).
 - [20] L. Bonora, P. Pasti, and M. Bregola, Weyl cocycles, *Classical Quantum Gravity* **3**, 635 (1986).
 - [21] A. B. Zamolodchikov, Irreversibility of the flux of the renormalization group in a 2D field theory, *Pis'ma Zh. Eksp. Teor. Fiz.* **43**, 565 (1986) [*JETP Lett.* **43**, 730 (1986)].

- [22] J. L. Cardy, Is there a c theorem in four-dimensions?, *Phys. Lett. B* **215**, 749 (1988).
- [23] Z. Komargodski and A. Schwimmer, On renormalization group flows in four dimensions, *J. High Energy Phys.* **12** (2011) 099.
- [24] Y. T. Huang and C. Wen, Soft theorems from anomalous symmetries, *J. High Energy Phys.* **12** (2015) 143.
- [25] L. Girardello, M. Petrini, M. Porrati, and A. Zaffaroni, Novel local CFT and exact results on perturbations of $N = 4$ superYang Mills from AdS dynamics, *J. High Energy Phys.* **12** (1998) 022.
- [26] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, Renormalization group flows from holography supersymmetry and a c -theorem, *Adv. Theor. Math. Phys.* **3**, 363 (1999).
- [27] J. de Boer, The holographic renormalization group, *Fortschr. Phys.* **49**, 339 (2001).
- [28] K. Skenderis, Lecture notes on holographic renormalization, *Classical Quantum Gravity* **19**, 5849 (2002).
- [29] S. Rajagopal, A. Stergiou, and Y. Zhu, Holographic trace anomaly and local renormalization group, *J. High Energy Phys.* **11** (2015) 216.
- [30] S. Nojiri and S. D. Odintsov, Conformal anomaly for dilaton coupled theories from AdS/CFT correspondence, *Phys. Lett. B* **444**, 92 (1998).
- [31] S. Nojiri, S. D. Odintsov, and S. Ogushi, Finite action in 5D gauged supergravity and dilatonic conformal anomaly for dual quantum field theory, *Phys. Rev. D* **62**, 124002 (2000).
- [32] I. Papadimitriou, Holographic renormalization of general dilaton-axion gravity, *J. High Energy Phys.* **08** (2011) 119.
- [33] G. W. Horndeski, Second-order scalar-tensor field equations in a four-dimensional space, *Int. J. Theor. Phys.* **10**, 363 (1974).
- [34] A. Nicolis, R. Rattazzi, and E. Trincherini, The Galileon as a local modification of gravity, *Phys. Rev. D* **79**, 064036 (2009).
- [35] L. Amendola, Cosmology with nonminimal derivative couplings, *Phys. Lett. B* **301**, 175 (1993).
- [36] C. Germani and A. Kehagias, New Model of Inflation with Non-Minimal Derivative Coupling of Standard Model Higgs Boson to Gravity, *Phys. Rev. Lett.* **105**, 011302 (2010).
- [37] M. Rinaldi, Black holes with non-minimal derivative coupling, *Phys. Rev. D* **86**, 084048 (2012).
- [38] A. Anabalón, A. Cisterna, and J. Oliva, Asymptotically locally AdS and flat black holes in Horndeski theory, *Phys. Rev. D* **89**, 084050 (2014).
- [39] E. Babichev and C. Charmousis, Dressing a black hole with a time-dependent Galileon, *J. High Energy Phys.* **08** (2014) 106.
- [40] A. Cisterna and C. Erices, Asymptotically locally AdS and flat black holes in the presence of an electric field in the Horndeski scenario, *Phys. Rev. D* **89**, 084038 (2014).
- [41] W. J. Jiang, H. S. Liu, H. Lü, and C. N. Pope, DC conductivities with momentum dissipation in Horndeski theories, *J. High Energy Phys.* **07** (2017) 084.
- [42] X. H. Feng, H. S. Liu, W. T. Lu, and H. Lü, Horndeski gravity and the violation of reverse isoperimetric inequality, *Eur. Phys. J. C* **77**, 790 (2017).
- [43] J. B. Jimenez, R. Durrer, L. Heisenberg, and M. Thorsrud, Stability of Horndeski vector-tensor interactions, *J. Cosmol. Astropart. Phys.* **10** (2013) 064.
- [44] T. Kobayashi, H. Motohashi, and T. Suyama, Black hole perturbation in the most general scalar-tensor theory with second-order field equations. II. The even-parity sector, *Phys. Rev. D* **89**, 084042 (2014).
- [45] M. Minamitsuji, Causal structure in the scalar-tensor theory with field derivative coupling to the Einstein tensor, *Phys. Lett. B* **743**, 272 (2015).
- [46] X. H. Feng, H. S. Liu, H. Lü, and C. N. Pope, Black hole entropy and viscosity bound in Horndeski gravity, *J. High Energy Phys.* **11** (2015) 176.
- [47] X. H. Feng, H. S. Liu, H. Lü, and C. N. Pope, Thermodynamics of charged black holes in Einstein-Horndeski-Maxwell theory, *Phys. Rev. D* **93**, 044030 (2016).
- [48] M. Baggioli and W. J. Li, Diffusivities bounds and chaos in holographic Horndeski theories, *J. High Energy Phys.* **07** (2017) 055.
- [49] H. S. Liu, H. Lü, and C. N. Pope, Holographic heat current as Noether current, *J. High Energy Phys.* **09** (2017) 146.
- [50] E. Caceres, R. Mohan, and P. H. Nguyen, On holographic entanglement entropy of Horndeski black holes, *J. High Energy Phys.* **10** (2017) 145.
- [51] W. J. Geng, S. L. Li, H. Lü, and H. Wei, Godel metrics with chronology protection in Horndeski gravities, *Phys. Lett. B* **780**, 196 (2018).
- [52] Z. Y. Fan, B. Chen, and H. Lü, Criticality in Einstein-Gauss-Bonnet gravity: gravity without graviton, *Eur. Phys. J. C* **76**, 542 (2016).
- [53] Y. Nakayama, Scale invariance vs. conformal invariance, *Phys. Rep.* **569**, 1 (2015).