$SU(2)$ Chern-Simons theory coupled to competing scalarsJ. M. Pérez Ipiña,¹ F. A. Schaposnik,² and G. Tallarita³¹*Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Buenos Aires 1428, Argentina*²*Departamento de Física, Universidad Nacional de La Plata, Instituto de Física La Plata-CONICET, C.C. 67, 1900 La Plata, Argentina*³*Departamento de Ciencias, Facultad de Artes Liberales, Universidad Adolfo Ibáñez, Santiago 7941169, Chile*

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We study a spontaneously broken $SU(2)$ Chern-Simons-Higgs model coupled through a Higgs portal to an uncharged triplet scalar with a vacuum state competing with the Higgs one. We find vortexlike solutions to the field equations in different parameter space regions. Depending on the scalar coupling constants, we find a parameter region in which the competing order creates a halo about the Chern-Simons-Higgs vortex core, together with two other regions, one where no vortex solutions exist and the other where ordinary Chern-Simons-Higgs vortices can be found. We derive the low-energy theory for the moduli fields on the vortex world sheet and also discuss the connection of our results with those found in studies of competing orders in high-temperature superconductors.

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Among the many applications of vortexlike solutions in gauge theories, there has been a renewed interest in the study of models in which there are two scalars with different ground states, a subject which is relevant in both high-energy and condensed matter physics.

In high-energy physics, the existence of a charged scalar with a nontrivial vacuum expectation value in the core of a vortex can lead to striking effects originally signaled in Ref. [1] in the context of superconducting strings. Also, (center) vortices are relevant to the study of confinement (see [2,3], and references therein for more recent results) and in the effective descriptions in the case of $2 + 1$ space-time where the Chern-Simons term plays a central role [4]. There have also been recent and interesting applications in the construction of low-energy Lagrangians in the world sheet of the Abrikosov-Nielsen-Olesen string by the addition of an extra non-Abelian moduli field [5,6].

Concerning condensed matter, the presence of the two scalars allows for the existence of competing vacuum states in systems in which the superconductivity order is suppressed in the vortex core where a “stripe order” formation takes place [7–9]. These ideas are at present the basis of

many investigations on phase transitions, as, in particular, those referred to competing order in mixed states of high-temperature superconductors [10,11].

Gauge theories with Maxwell and Chern-Simons terms coupled to scalars in $d = 3$ space-time dimensions have vortex solutions, both for Abelian [12] and also for non-Abelian gauge groups [13], provided one includes as many scalars as required to have maximum symmetry breaking. In particular, in the case of an $SU(N)$ gauge group, one needs N scalars in the adjoint representation in order to have topologically nontrivial (vortex) solutions of the field equations. In that case, the surviving symmetry is $\mathbf{Z}_N$, and the vortex solutions belong to $(N - 1)$ homotopy classes. In the particular case of pure Chern-Simons-Higgs theories, first-order field equations exist, and a Bogomol’nyi bound for vortex solutions in both Abelian [14,15] and non-Abelian [16] cases was found.

In this work, we shall consider an $SU(2)$ Chern-Simons gauge theory coupled to two scalar triplets $\vec{\phi}$ and $\vec{\psi}$ with an appropriate potential leading to a symmetry-breaking ground state. This model will be coupled to an $O(3)$ triplet $\vec{\chi}$, uncharged under the gauge group, with a potential $U(|\vec{\phi}|, |\vec{\chi}|)$ such that in certain regions of the parameter space the theory exhibits a competing nontrivial χ vacuum state and vortex solutions. We shall also discuss the moduli fields associated to the $\vec{\chi}$ and the resulting low-energy dynamics of the vortices.

The paper is organized as follows. In Sec. II, we introduce the pure Chern-Simons-Higgs (CSH) model and discuss the pattern of symmetry breaking and the

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topology of CSH vortex configurations. We present in Sec. III the action governing the dynamics of the competing triplet χ field coupled to the CSH model via a Higgs portal mixing and analyze the conditions leading to nontrivial vacuum states for ϕ and χ . We briefly discuss the non-Abelian moduli fields associated to χ localized on the vortex world sheet in Sec. IV and then present in Sec. V the solutions to the coupled field equations, discussing in detail the vortex solutions in different parameter space regions. Finally, in Sec. VI, we summarize and discuss our results, in particular, stressing its connection with superconductivity with competing orders.

II. NON-ABELIAN CHERN-SIMONS VORTICES

The existence of classic nontrivial vortex solutions in non-Abelian gauge theories coupled to scalars requires maximum symmetry breaking [13]. Otherwise, no topologically stable nontrivial solutions can be found, although quantum corrections can stabilize them, for example, if heavy fermions are coupled to the Abelian Higgs model [17].

To this end, in the case of an $SU(2)$ Chern-Simons gauge theory, one has to couple the gauge field A_μ to two scalars in the adjoint representation, $\vec{\phi}$ and $\vec{\psi}$. We shall briefly outline in this section the construction of such topologically nontrivial configurations as derived in Ref. [16].

The dynamics of the model is governed by the following $(2+1)$ -dimensional action

$$S_{\text{CSH}} = \int d^3x \left(\frac{\kappa}{4} \epsilon^{\mu\nu\gamma} \left(\vec{F}_{\mu\nu} \cdot \vec{A}_\gamma - \frac{e}{3} \vec{A}_\mu \cdot (\vec{A}_\nu \wedge \vec{A}_\gamma) \right) + \frac{1}{2} (D_\mu \vec{\phi}) \cdot (D^\mu \vec{\phi}) + \frac{1}{2} (D_\mu \vec{\psi}) \cdot (D^\mu \vec{\psi}) - V(\vec{\phi}, \vec{\psi}) \right). \quad (1)$$

Here the gauge field A_μ ($\mu = 0, 1, 2$) takes value in the algebra of $SU(2)$:

$$A_\mu = A_\mu^a T^a \equiv \vec{A}_\mu \cdot \vec{T} \quad (2)$$

with generators T^a ($a = 1, 2, 3$) satisfying

$$[T^a, T^b] = i\epsilon^{abc} T^c, \quad \{T^a, T^b\} = \frac{1}{2} \delta^{ab}, \quad \text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}. \quad (3)$$

With this, the field strength $\vec{F}_{\mu\nu}$ and covariant derivatives can be written as

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + e \vec{A}_\mu \wedge \vec{A}_\nu, \quad (4)$$

$$D_\mu \vec{\phi} = \partial_\mu \vec{\phi} + e \vec{A}_\mu \wedge \vec{\phi}, \quad (5)$$

and an analogous formula for $D_\mu \vec{\psi}$.

Concerning the symmetry-breaking potential, it takes the form

$$V(\vec{\phi}, \vec{\psi}) = V_1(\vec{\phi}) + V_2(\vec{\psi}) + g''(\vec{\phi} \cdot \vec{\psi})^2, \quad (6)$$

where V_1 and V_2 have to be chosen so that asymptotically gauge symmetry is maximally broken:

$$\lim_{\rho \rightarrow \infty} |\vec{\phi}| = \phi_0, \quad \lim_{\rho \rightarrow \infty} |\vec{\psi}| = \psi_0, \quad \lim_{\rho \rightarrow \infty} \vec{\phi} \cdot \vec{\psi} = 0. \quad (7)$$

Note that the third term in Eq. (6) is the one that forces orthogonality between $\vec{\phi}$ and $\vec{\psi}$, thus ensuring maximal symmetry breaking and the existence of topologically nontrivial vortex solutions. As a result, the invariant group of the vacuum is $\mathbf{Z}_2$, and the relevant homotopy group is $\Pi_1(SU(2)/\mathbf{Z}_2) = \mathbf{Z}_2$.

Concerning the symmetry-breaking potentials, we shall choose for $V_1(|\vec{\phi}|)$ the sixth-order one in order to compare our results with those of a pure CS-Higgs theory at the Bogomol'nyi point [16]:

$$V_1(|\vec{\phi}|) = \frac{\lambda^2}{2} |\vec{\phi}|^2 (|\vec{\phi}|^2 - \phi_0^2)^2. \quad (8)$$

As we shall see below, in view of the minimal energy ansatz that we shall make, it will not be necessary to explicitly choose $V_2(|\vec{\psi}|)$.

The topological charge associated to the non-Abelian vortex configurations can be calculated as usual via a Wilson loop:

$$Q_T = \frac{1}{2} \text{Tr} \mathcal{P} \exp \left(ie \oint_{C_\infty} A_\mu dx^\mu \right) \quad (9)$$

with Tr the $SU(2)$ trace, $\mathcal{P}$ the path-ordering operator, and C_∞ a closed curve at infinity. We shall explicitly see that the suitable ansatz leads to $Q_T = (-1)^n$ so that there are two topologically inequivalent configurations: the “trivial” one with $Q_T = 1$ ($n = 2l$, $l \in \mathbf{Z}$) and those with $Q = -1$ the “nontrivial” one ($n = 2l + 1$).

It should be stressed that, as already signaled in the pioneering work of Deser, Jackiw, and Templeton [18], the non-Abelian Chern-Simons action $S_{\text{CS}}[A]$ is not invariant under large gauge transformations $g \in SU(2)$:

$$S_{\text{CS}}[A^g] = S_{\text{CS}}[A] + 8\pi^2 \kappa \omega[g] \quad (10)$$

with $w[g]$ the winding number associated to the gauge transformation:

$$\omega[g] = \frac{1}{24\pi^2} \text{Tr} \epsilon^{\mu\nu\gamma} \int g^{-1} \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\gamma g = n \in \mathbf{Z}. \quad (11)$$

Then, in order to have a gauge-invariant partition function, κ should be chosen as

$$\kappa = \frac{k}{4\pi}, \quad k \in \mathbb{Z}. \quad (12)$$

As discussed in Ref. [16], an axially symmetric ansatz allows one to find vortex solutions with $Q_T = -1$ to the field equations arising from action (1). Moreover, first-order self-dual equations exist for a choice of the symmetry-breaking potential as defined in Eq. (8) for one of the two scalars, say, $\vec{\phi}$. The lowest-energy vortex solution will then correspond to the case in which the second scalar ($\vec{\psi}$) is chosen everywhere in its vacuum expectation value so that its role is just to achieve complete symmetry breaking [19].

Because of the nature of the Gauss law in Chern-Simons theories, vortex solutions should have not only magnetic flux but also electric charge and angular momentum [13], all of them quantized. Finally, it should be noticed that, in the case of non-Abelian gauge theories for both Yang-Mills and CS cases, the bound of the energy is not of topological character. Indeed, as shown in Ref. [16] for the non-Abelian Chern-Simons case, the Bogomol'nyi bound for the energy in $2 + 1$ dimensions is proportional not just to the vortex topological charge Q_T but to the sum of $(Q_T + 2k)$. Here $k \in \mathbb{Z}$ is an integer related to the Cartan subgroup of the gauge group and depends on the gauge element one chooses to use in the asymptotic gauge field behavior. The same happens for the energy per unit length in the $(3 + 1)$ -dimensional Yang-Mills theory as first shown in Ref. [20].

III. ADDING A COMPETING SCALAR

Let us now couple action $S_{\text{CSH}}[A, \phi, \psi]$ with an $O(3)$ global action S_χ for a triplet scalar χ coupled to the Higgs field ϕ in such a way that its vacuum is in competition with the one of the dynamical scalar ϕ .

$$S_\chi = \int d^3x \left(\frac{1}{2} \partial_\mu \vec{\chi} \partial^\mu \vec{\chi} - U(\vec{\chi}, \vec{\phi}) \right) \quad (13)$$

with the potential [5]

$$U(\vec{\chi}, \vec{\phi}) = \gamma [(-\mu^2 + |\vec{\phi}|^2)|\vec{\chi}|^2 + \beta(|\vec{\chi}|^2)^2]. \quad (14)$$

Before a detailed analysis of the potential U , let us stress that, when one couples action (13) to the Chern-Simons-Higgs action (1) with its potential $V(|\vec{\phi}|)$ chosen so as to have BPS equations, the resulting action cannot have first-order (self-dual) BPS equations. A simple way to see this is to recall that self-duality is intimately related to the possibility of extending the original bosonic action to a supersymmetric one [21]. Now the supersymmetric extension of Higgs-portal term $|\vec{\phi}|^2|\vec{\chi}|^2$ in (13) requires the introduction of a second gauge field coupled to χ and the

addition of a gauge-mixing coupling [22,23], which would then completely change the character of the model.

Potential parameters γ and μ are real and positive constants. We are interested in finding vortexlike solutions in which there is a competition between the vacuum states of scalars ϕ and χ . Vorticity implies that $|\phi| \rightarrow \phi_0$ at infinity, and it should vanish at the origin, $\phi(0) = 0$. Concerning χ , we shall impose that $\lim |\chi| \rightarrow 0$ asymptotically and that it takes a nonzero value at the origin fixing $|\vec{\chi}|$ but not its direction so that at short distances the $O(3)$ symmetry is spontaneously broken to $O(2)$. Moreover, we want the length scale of variation of the ϕ field to be larger than or of the order of the χ length scale, thus implying

$$\sqrt{\gamma\mu^2} \gtrsim \lambda^2 \phi_0^2. \quad (15)$$

This condition is usually required in the case of Maxwell gauge dynamics [5,6] so that χ is localized in the center of the vortex core, where the magnetic field has its maximum. In the present case, the Chern-Simons gauge dynamics defined in $2 + 1$ dimensions forces the magnetic field to vanish at the origin (see Refs. [14,15]), so in the spatial plane the magnetic field is not concentrated in the vortex core (a disk), as with the Maxwell theory, but in the annulus centered at the origin, bounded by concentric circles of radii $R_{\min}$, $R_{\max}$. This implies that there are two regions where the competing field χ can form a “halo”: inside the smaller radius circle of the annulus or outside the larger one. We shall discuss in detail in Sec. V the reasons behind condition (15) and the nature of the observed halo.

The action S for the model with two competing scalars that we will investigate is then defined as

$$S = S_{\text{CSH}}[A, \phi, \psi] + S_\chi[\chi, \phi]. \quad (16)$$

The dimensions of fields and parameters are $[A_\mu] = m$; $[\phi] = [\psi] = [\chi] = m^{1/2}$; $[\mu] = m^{1/2}$; $[\gamma] = m$; and $[\kappa] = [e] = [\beta] = [\lambda] = 0$.

Since we choose the field $\vec{\psi}$ to be constant everywhere, the field equations reduce to

$$D_\mu D^\mu \vec{\phi} = -\frac{\partial V_1(\vec{\phi})}{\partial \vec{\phi}} - \frac{\partial U(\vec{\chi}, \vec{\phi})}{\partial \vec{\phi}}, \quad (17)$$

$$\partial_\mu \partial^\mu \vec{\chi} = -\frac{\partial U(\vec{\chi}, \vec{\phi})}{\partial \vec{\chi}}, \quad (18)$$

$$\frac{\kappa}{2} \epsilon^{\mu\nu\gamma} \vec{F}_{\nu\gamma} = \vec{j}^\mu \quad (19)$$

with

$$\vec{j}^\mu = e(D^\mu \vec{\phi}) \wedge \vec{\phi}. \quad (20)$$

The scalar and gauge excitation masses are

$$m_\phi = 2\lambda\phi_0^2, \quad m_A = \frac{e^2\phi_0^2}{\kappa}. \quad (21)$$

Now, using the Gauss law to eliminate $\vec{A}_0$ in terms of the magnetic field,

$$(\vec{A}_0 \wedge \vec{\phi}) \wedge \vec{\phi} = \frac{\kappa}{e^2} \vec{B}, \quad (22)$$

we get for the energy

$$\mathcal{E}_{\text{CSH}\chi} = \int d^2x \left[\frac{\kappa}{2e^2|\vec{\phi}|^2} |\vec{B}|^2 + \frac{1}{2} |D_i \vec{\phi}|^2 + V_1(\vec{\phi}) + \frac{1}{2} |\partial_i \vec{\chi}|^2 + U(\vec{\chi}, \vec{\phi}) \right]. \quad (23)$$

In order to find vortexlike solutions, we propose the following static axially symmetric ansatz:

$$\begin{aligned} \vec{A}_\varphi &= -\frac{1}{e} \frac{a(\rho)}{\rho} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, & \vec{A}_0 &= -\frac{1}{e} a_0(\rho) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ \vec{\phi} &= \phi_0 f(\rho) \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{pmatrix}, & \vec{\chi} &= \chi_0 h(\rho) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \quad (24)$$

with $\chi_0 = \mu/\sqrt{2\beta}$, which corresponds to the minimum of the U potential at the origin. Complete $SU(2)$ symmetry breaking [except for the $SU(2)$ center] implies that

$$\vec{\psi} = \psi_0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (25)$$

with ψ_0 the minimum of the $V_2(|\psi|)$ potential. Finite energy solutions imply the following conditions:

$$\begin{aligned} a(\infty) &= f(\infty) = 1, & a_0(\infty) &= h(\infty) = 0, \\ a(0) &= f(0) = 0, & a_0(0) &= b. \end{aligned} \quad (26)$$

Here b is a constant that is fixed through the Gauss law in terms of κ and e and the ratio of the magnetic and Higgs field squared modulus at the origin. Concerning the χ field, we shall impose $h'(0) = 0$. One can see this by making a Frobenius analysis of the field equations close to the origin. All odd powers of ρ in the h expansion vanish, which in turn implies the derivative there must also vanish (see below).

Inserting the vortex ansatz in Eq. (23), we get for the vortex energy

$$\begin{aligned} \mathcal{E} &= 2\pi\phi_0^2 \int \xi d\xi \left[\frac{a'(\xi)^2}{2\xi^2 f^2(\xi)} + \frac{1}{2} \left(f'(\xi)^2 + \frac{f^2(\xi)}{\xi^2} (a(\xi) - 1)^2 \right) \right. \\ &\quad + \frac{c_1}{8} f^2(\xi) (f^2(\xi) - 1)^2 + \frac{c_2}{2c_3} h'(\xi)^2 \\ &\quad \left. + c_3 h^2(\xi) \left(\frac{1}{2} \left(\frac{\mu}{\phi_0} \right)^2 (h^2(\xi) - 2) + f^2(\xi) \right) \right] \\ &\equiv \int \epsilon(\xi) d\xi, \end{aligned} \quad (27)$$

where we have introduced the dimensionless radial variable $\xi = m_A \rho$, $f' = d/d\xi$, and the dimensionless parameters

$$c_1 = \frac{m_\phi^2}{m_A^2}, \quad c_2 = \frac{\gamma\chi_0^2}{m_A^2}, \quad c_3 = \frac{\gamma\phi_0^2}{m_A^2}. \quad (28)$$

Finally, $\epsilon(\xi)$ is the energy density.

Concerning the field equations, after inserting the ansatz (24) and (25) in Eqs. (17)–(19), we get the following system of nonlinear coupled equations:

$$\begin{aligned} f''(\xi) + \xi^{-1} f'(\xi) - \xi^{-2} f(\xi) (1 - a(\xi))^2 + (a'(\xi))^2 / (\xi^2 f^3(\xi)) \\ = \frac{c_1}{4} f(\xi) (f^2(\xi) - 1) (3f^2(\xi) - 1) + 2c_2 f(\xi) h^2(\xi), \end{aligned} \quad (29)$$

$$h''(\xi) + \xi^{-1} h'(\xi) = 2\beta c_2 h(\xi) (h^2(\xi) - 1) + 2c_3 h(\xi) f^2(\xi), \quad (30)$$

$$a''(\xi) - \left[\frac{1}{\xi} + 2 \frac{f'(\xi)}{f(\xi)} \right] a'(\xi) + f^4(\xi) [1 - a(\xi)] = 0. \quad (31)$$

Here we have used the Gauss law to write $\vec{A}_0$ in terms of the magnetic and Higgs fields.

As mentioned above, following a Frobenius analysis of the field equations close to the origin, the Neumann condition imposed on χ can be justified. We set $a(0) = a'(0) = 0$ in Eqs. (29) and (30), and we expand the scalar fields $f(\xi)$ in odd powers of ξ and the $h(\xi)$ field in all powers of ξ :

$$f = \xi f_1 + \xi^3 f_2 + \dots,$$

$$h = h_0 + \xi h_1 + \xi^2 h_2 + \dots,$$

where ... denote higher powers in ξ . Then, inserting the expansions in the field equations, a ξ power by power analysis of the equations reveals that all odd powers of ξ in the χ expansion vanish, which in turn implies its derivatives must vanish.

Let us end this section stressing that, due to the lack of a second gauge field with a suitable coupling to $\vec{A}_\mu$ and to $\vec{\chi}$, one cannot reduce the second-order field equations to first-order BPS ones.

IV. THE χ FIELD ORIENTATIONAL MODULI

Before finding the axially symmetric solutions to the field equations, we shall discuss the orientational moduli associated to the presence of the χ field [5]. To this end, instead of considering the specific choice proposed for χ in ansatz (24), we allow for a more general parameterization which implies a time dependence:

$$\chi^i = \sqrt{\frac{\mu^2}{2\beta}} h(\rho) n^i(t), \quad (32)$$

where $n^i(t)$ is satisfying

$$n^i(t) n^i(t) = 1. \quad (33)$$

Inserting (32) in the action (16), we obtain the low-energy action for the orientational moduli:

$$S_O = \frac{I_O}{2} \int dt n^i(t) n^i(t), \quad (34)$$

where

$$I_O = \frac{4\pi\mu^2}{\beta} \int d\rho \rho^2 h(\rho)^2. \quad (35)$$

This is the action for the $CP(1)$ nonlinear sigma model, as expected from the pattern of global symmetry breaking in the χ sector: $SU(2)/U(1) \rightarrow CP(1)$. We can proceed to quantize it by parameterizing the unit vector n^i in terms of polar and azimuthal angles (α, ϑ) :

$$\vec{n} = \begin{pmatrix} \cos \alpha \sin \vartheta \\ \sin \alpha \sin \vartheta \\ \cos \vartheta \end{pmatrix}. \quad (36)$$

Action (34) is the action for a symmetric top with moment of inertia I_O so that upon canonical quantization the Hamiltonian operator reads

$$\hat{H} = -\frac{1}{2I_O} \left(\frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial}{\partial \vartheta} \right) + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \alpha^2} \right), \quad (37)$$

which of course has spherical harmonic eigenfunctions with eigenvalues

$$E_{\mathcal{L}} = \frac{1}{2I_O} S(S+1), \quad (38)$$

where S is an integer. Therefore, the vortices found below for which $\chi \neq 0$ correspond to vortices with “isospin.” Note that these orientational gapless excitations can be lifted by the introduction of a “spin-orbit” coupling of the form $\epsilon(\partial_\mu \chi^i)^2$ to the Lagrangian [24]. As shown in Ref. [6], this

is the starting point for an interesting correspondence between this system with a new kind of superconducting liquid crystal phase, with the cholesteric formed by the introduction of a $\eta \epsilon_{ijk} \chi_i \partial_j \chi_k$ term to the original Lagrangian. We will leave the detailed analysis of this connection (with regards to this specific model) to a future publication.

V. SOLUTIONS

Action (16) has the trivial extremum $\vec{\phi} = \vec{\chi} = 0$. Of course, for $\chi = 0$ one has the non-Abelian vortex CS-Higgs solution discussed in Ref. [16]. There exists also a spatially uniform solution in which both ϕ and χ are constant (with $A_\mu = 0$):

$$|\vec{\phi}| = \phi_0, \quad |\vec{\chi}| = \sqrt{\frac{\mu^2 - \phi_0^2}{2\beta}}. \quad (39)$$

Note that this solution is valid only provided $\phi_0^2/\mu^2 < 1$ and was also found in Ref. [10], where it corresponds to the region of coexisting order in the schematic zero-temperature diagram that the authors present.

Concerning $(\vec{A}, \vec{\phi})$ -vortex configurations and fully non-trivial χ , we have found, numerically, solutions of Eqs. (29)–(31) using a second-order central finite difference procedure with an accuracy of $\mathcal{O}(10^{-4})$. We tested the solver for the particular case $\chi = 0$, and we accurately reproduced the exact result of the Bogomol’nyi lowest bound for the lowest-energy $n = 1$ vortex, $E = \phi_0^2 \pi$ [16].

Typical field profiles of the scalar and electromagnetic fields are shown in Figs. 1 and 2 for the case in which the gauge coupling constant e , the Chern-Simons coefficient κ , and the Higgs potential coupling constant λ are chosen to the values in which the CSH model is defined at the Bogomol’nyi point ($m_\phi = m_A$).

FIG. 1. Higgs (solid line) and χ (dashed line) profiles [$f(\xi)$, $h(\xi)$] for $e = 2$, $\lambda = 1$, $\kappa = 2$ ($c_1 = 1$), $\beta = 160$, $\gamma = 200$, and $\mu/\phi_0 = 0.8$ (dimensionful parameters have been normalized by the gauge field mass).

FIG. 2. Magnetic field (solid line) and electric field (dashed line) for $e = 2$, $\lambda = 1$, $\kappa = 2$ ($c_1 = 1$), $\beta = 160$, $\gamma = 200$, and $\mu/\phi_0 = 0.8$ (dimensionful parameters have been normalized by the gauge field mass).

The competing orders are clearly shown in the $\vec{\phi}$, $\vec{\chi}$ profiles in Fig. 1, where one can see that the coherence lengths of both scalar fields are of the same order. This is a relevant property for condensed matter models that describe the quantum phase transition between a pure superconducting phase from one in which a competing order coexists with superconductivity. The result is a “halo” about the vortex core whose existence has been inferred from the charge-stripe order that has been found experimentally (see [10] and references therein).

The electromagnetic field profiles in Fig. 2 are qualitatively similar to those in the pure CSH model. As happens when a CS term is present, in addition to the magnetic field, vortices are electrically charged and both fields should vanish at the origin where the Higgs field has to vanish to ensure regularity of the solution.

The energy density of the vortex solution as a function of the dimensionless radial coordinate ξ is shown in Fig. 3. It is important to notice that at short distances the energy density is negative due to the fact that the potential $U[\vec{\chi}, \vec{\phi}]$ is negative there and all other field contributions tend to

FIG. 3. Energy density $\epsilon(\xi)/m_A$ for $e = 2$, $\lambda = 1$, $\kappa = 2$ ($c_1 = 1$), $\beta = 160$, $\gamma = 200$, and $\mu/\phi_0 = 0.8$ (dimensionful parameters have been normalized by the gauge field mass).

zero when one approaches the origin. Because of this fact, when both orders are in competition, the total energy $\mathcal{E}_{\text{CSH}\chi}$ is lower than the energy $\mathcal{E}_{\text{CSH}}$ of the pure CS-Higgs vortices. Indeed, in the latter case, the BPS bound leads, for the minimal energy solution with topological charge $n = 1$ [16] so that for the gauge coupling constant $e = 2$ chosen in Fig. 3 the result is

$$\mathcal{E}_{\text{CSH}} = \frac{e\phi_0^2}{2}\pi = \pi\phi_0^2, \quad (40)$$

while for the solution in which the competing χ field is present we get

$$\mathcal{E}_{\text{CSH}\chi} = 2.66623\phi_0^2. \quad (41)$$

Hence, the presence of the competing field χ lowers the energy by $\Delta\mathcal{E} = -0.778\phi_0^2$. This phenomenon has been reported in many studies of the so-called intertwined orders in high-temperature superconductors (see [25] and references there) and more recently in a variational study of a phenomenological nonlinear sigma model with two competing orders [26].

We have explored a wide range of parameter values. Figure 4 shows the effect of changing the Higgs mass $m_\phi = 2\lambda\phi_0^2$ by changing λ with all other parameters fixed.

As the Higgs mass grows, the size of the vortex core decreases, as can be seen in Fig. 4, and also the χ field profile shrinks in both height and extension. We find that the same happens when one takes smaller β and γ values in such a way that condition (15) and stability requirements hold.

In Fig. 5, we have plotted the magnetic field and the χ field as a function of (x, y) . Note that, in contrast with what happens for the Maxwell gauge dynamics, here the magnetic field does vanish at the origin so that the core of the vortex is not a disk but an annulus. The χ halo therefore was found to be inside the smaller radius circle, as was mentioned before.

The values in Figs. 1–5 were chosen because they lead to an enhanced value of χ . But, in fact, the region of competing orders was found for a wide range of parameter values.

Based on our numerical analysis of the solutions for different parameter ranges, we present in Fig. 6 the three regions in which the competing scalar field solutions are trivial or purely CS vortexlike or they show a competition between two order parameters. We have chosen for the horizontal axis $\alpha = \phi_0^2/\mu^2$ and studied the fate of the solutions when one changes the value of the $\vec{A}_\mu - \vec{\phi}$ coupling constant e . We chose $1/e^2$ for the vertical axis because, as it grows, the magnetic field contribution to the energy density also grows [see Eq. (23)] so that the figure can be identified with the usual magnetic field versus control parameter diagrams.

FIG. 4. Field profiles at distinct values of m_ϕ . The figure on the left shows Higgs profiles $f(\xi)$, while in the right one it is the competing χ field profiles $h(\xi)$ that are shown. From right to left, the values of λ grow according to $\lambda = 1, 3, 5, 7, 9, 11, 13$ so that the Higgs mass also grows consistently as $m_\phi = 2\lambda\phi_0^2$. The other parameters remain fixed at $e = 2, \kappa = 2, \beta = 160, \gamma = 200$, and $\mu/\phi_0 = 0.8$ (dimensionful parameters have been normalized with respect to the gauge field mass). As was to be expected, as the mass grows larger, the Higgs field reaches earlier its vacuum value.

In the region to the right of the dashed curve, vortices are those of a CS-Higgs model as discussed in Ref. [16]: No competing order with the χ field takes place. To the left of the solid line, no nontrivial (ϕ, χ) vortex solutions exist. Solely in the region between the solid and dashed lines, there is a competing order where the ϕ and χ fields are nontrivial and behave as in Fig. 1. As α grows, the halo produced by the χ field about the vortex core reduces in size until no competing order is left. This is consistent with the definition of α as ϕ_0^2/μ^2 , since a big value of α would imply the Higgs field dominating over the competing χ field.

Figure 6 can be interpreted at the light of the schematic zero-temperature phase diagram of a layered superconductor in the presence of two order parameters proposed in

Ref. [10]. To this end, one can identify the two axes in their diagram, the magnetic induction B and a control parameter α , with $1/e^2$ and our α parameter (note that when $1/e^2$ grows the contribution of the magnetic field to the energy density also grows).

The crossover lines in the phase diagram of Ref. [10] can be identified with the lines separating the three regions in Fig. 6 that we found without any approximation from the numerical solution of the coupled system of equations. It is also important to note that $\alpha_c = 1$, the point at which the constant nontrivial solution for the competing scalar vanishes, corresponds in Ref. [10] to the point in which there is a continuous transition at $B = 0$ between a pure superconducting phase and a phase with coexisting orders.

FIG. 5. In the vertical axis, we represent the χ field as a function of (x, y) in a color gradient from yellow to red and around it the magnetic field modulus in a color gradient from white to black for $e = 2, \lambda = 1, \kappa = 2$ ($c_1 = 1$), $\beta = 160, \gamma = 200$, and $\mu/\phi_0 = 0.8$ (dimensionful parameters have been normalized by the gauge field mass).

FIG. 6. Analysis of the solutions in different parameter regions (at $\lambda = 1, \kappa = 2, \beta = 400$, and $\gamma = 200$, normalized with respect to the gauge mass). The horizontal and vertical axes are $\alpha = \phi_0^2/\mu^2$ and $1/e^2$, respectively. To the left of the solid line, no vortex solutions were found. To the right of the dashed line, there is no competing order and ordinary vortex CSH solutions can be obtained. In between these regions, one finds vortex solutions with nontrivial χ order.

In contrast, because our potential $U[\phi, \chi]$ has a larger number of parameters to fix, suitable choices allow us to find solutions with coexisting orders for both $\alpha \lesssim \alpha_c$, provided one is not too far from α_c .

The figure also shows why Eq. (15) needs to be satisfied in order to have vortex solutions with a competing field halo. As α grows, the halo reduces in size until the condition is no longer satisfied, and we are left with ordinary CS vortices to the right of the dashed line. Physically, this means that the Higgs field is dominant with respect to the competing field, and the latter becomes irrelevant. A similar effect occurs when λ grows and the rhs of condition (15) becomes larger, as can be seen from Fig. 4. Finally, we observed the same behavior when changing the value of γ , considering that a large γ implies a strong interaction between the fields and for small γ the competing orders decouple, and, again, no halo can be found.

Moving to smaller α , condition (15) is easily satisfied, and one would expect to still find vortex solutions with a competing halo. However, as mentioned before, there is another solution to the field equations playing a role in this region. This is the constant fields solution from Eq. (39), valid for $\alpha < 1$. This solution has a lower energy than the vortex configuration, so it destabilizes the latter when one works with small values of α . This argument provides a physical explanation to why numerically no vortex solutions were found at small values of the parameter α . Note that topological stability does not, in general, guarantee that the solution cannot decay to a lower-energy one. It happens, for example, in the case of the $O(3)$ sigma model as a size instability. This also takes place in the decay of vortex solutions. Indeed, when the Landau parameter value is $\lambda > 1$, the energy of a vortex with topological charge $n = 2$ is larger than that of two vortices with $n = 1$, as has been shown numerically [27].

VI. SUMMARY AND DISCUSSION

In this work, we have analyzed how the vortexlike solutions of an $SU(2)$ Chern-Simons-Higgs theory are

affected when an uncharged scalar triplet χ is coupled to one of the Higgs scalars with a potential chosen so that χ develops a nontrivial vacuum state in the vortex core. As originally discussed in Ref. [5], this is a simple mechanism through which vortex strings can acquire non-Abelian moduli localized on their world sheets.

Models with two competing vacuum states are also of interest in studies of superconductivity phenomena, and in this respect it is interesting to compare our results with the analysis of a layered superconductor in the vicinity of a quantum critical point separating a pure superconducting phase from a phase in which a competing order coexists with superconductivity [10]. The main results in that paper are summarized in the schematic zero-temperature phase diagram that the authors propose, which can be compared with the diagram presented in our Fig. 6.

Theories where two sectors with different field content are coupled through a potential like the one we employed here are also actively investigated in connection with cosmological problems like that of the fractional cosmic neutrinos [28] and the dark matter problem [29]. Their supersymmetric extension requires, apart from the (scalar-scalar) Higgs portal, the addition of a kinetic gauge mixing [30]. As a result, coupling constants accommodate to the Bogomol'nyi point for which the second-order field equations can be reduced to the first-order ones originally found in the study of the Ginzburg-Landau theory of superconductivity when the Landau parameter takes the value $K = 1/\sqrt{2}$ [31]. We expect in future work to extend the analysis of the present work in some of the directions described above.

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