

Universal critical dynamics near the chiral phase transition and the QCD critical point

Johannes V. Roth¹, Yunxin Ye^{2,*}, Sören Schlichting², and Lorenz von Smekal^{1,3}

¹*Institut für Theoretische Physik, Justus-Liebig-Universität, 35392 Giessen, Germany*

²*Fakultät für Physik, Universität Bielefeld, D-33615 Bielefeld, Germany*

³*Helmholtz Research Academy Hesse for FAIR (HFHF), Campus Giessen, 35392 Giessen, Germany*



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We use a novel real-time formulation of the functional renormalization group (FRG) for dynamical systems with reversible mode couplings to study model H, the conjectured dynamic universality class of the quantum chromodynamics (QCD) critical point. We emphasize the structural similarities with model G, conjectured to be the dynamic universality class of the chiral phase transition in the limit of two massless quark flavors. Importantly, our formulation of the real-time FRG preserves all relevant symmetries throughout the FRG flow which, e.g., guarantees the nonrenormalization of the reversible mode couplings, as but one exact result. We derive nonperturbative RG flow equations for the kinetic coefficients of both, model G and H, in parallel, and discuss commonalities and differences in the resulting fixed-point structure and dynamic critical exponents, such as weak-scaling relations which hold in either case versus the characteristic strong scaling of model G which is absent in model H.

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Introduction. Relativistic heavy-ion collision experiments at various beam energies allow to probe the phase structure of quantum chromodynamics (QCD) at finite temperature and baryon density. One of the conjectured features of the QCD phase diagram thereby is the existence of a critical point at the end of a first-order transition line [1]. To identify such a QCD critical point experimentally one searches for signatures of criticality, for example, in cumulants of conserved charges [2]. However, since the fireball created in a heavy-ion collision is a rapidly evolving system, one needs to understand the time evolution of these critical fluctuations [3,4]. Similar to equilibrium/static critical phenomena, dynamic critical phenomena fall into universality classes [5]. The basic idea is that one may be able to understand the dynamics of critical fluctuations in a heavy-ion collision by studying a simpler system from the same dynamic universality class.

The universality classes for dynamic critical phenomena were classified by Hohenberg and Halperin in Ref. [5]. The ones relevant for critical dynamics in QCD, by their classification, are model G of a four-component Heisenberg antiferromagnet, the dynamic universality class of the

chiral transition in the two-flavor chiral limit [6], and model H of the liquid-gas transition in a pure fluid as well as the QCD critical point [7].

To study critical dynamics naturally requires real-time methods. Classical statistical simulations have been used to investigate the dynamic critical behavior of models A, B, and C [8–14], the $O(4)$ model G [15–17], as well as recently also model H [18]. Functional methods such as real-time formulations of the functional renormalization group (FRG) can also be used to study dynamic critical behavior. Previous studies on critical dynamics using the real-time FRG mainly focused on the relaxational models A, B, and C [19–22]. Models G and H, on the other hand, comprise the additional complication of nonlinear reversible mode couplings which are protected from renormalization by Ward identities [23]. Ignoring these subtleties, a first real-time FRG study of model H led to reasonable values for the dynamic critical exponent of the order-parameter fluctuations [24]. Because the truncation scheme was not symmetry preserving, however, additional *ad hoc* assumptions on the static FRG flow and on the reversible mode couplings had to be made to achieve these results. In such a situation it is particularly important to have a formulation of the real-time FRG which manifestly preserves the nontrivial symmetries such as an extended temporal-gauge symmetry of the associated Martin-Siggia-Rose (MSR) action. In Ref. [23] we have demonstrated explicitly how this can be consistently achieved for model G, by introducing the external classical sources as well as the FRG regulators not directly in the MSR action

*Present address: Institute for Theoretical Physics, Friedrich-Schiller-Universität Jena, Fröbelstieg 1, 07743 Jena, Germany.

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but one level lower in the Landau-Ginzburg-Wilson (LGW) free energy of the system. The same methodology applies to model H.

In this paper we therefore review this real-time FRG formulation for model G and, in parallel, derive that for model H, emphasizing the structural similarities and differences between the two. In particular, our earlier work [23] thus allows us to consistently formulate a symmetry-preserving real-time FRG flow for the first time here for model H as well. As one of our main results, we derive nonperturbative RG flow equations for the kinetic coefficients of models G and H. After introducing dimensionless combinations of the dynamic couplings, we obtain flow diagrams and study fixed-point structure, dynamic scaling relations, and the dependence of the resulting dynamic critical exponents on the number of spatial dimensions. Finally, we comment on possible future applications.

Equations of motion for model G/H. The static critical behavior of an N -component order parameter ϕ is described by the Landau-Ginzburg-Wilson free energy,

$$F = \int d^d x \left\{ \frac{1}{2} (\partial^i \phi_a) (\partial^i \phi_a) + \frac{m^2}{2} \phi_a \phi_a + \frac{\lambda}{4!N} (\phi_a \phi_a)^2 \right\} \quad (1)$$

in d spatial dimensions, with $m^2 < 0$ to spontaneously break the $O(N)$ symmetry, and with quartic coupling λ . To study *dynamic* critical phenomena, one also needs to specify the equations of motion which reflect the dynamic universality class of the system. According to the classification by Halperin and Hohenberg [5], this depends on whether the order parameter is conserved or not, whether there are other conserved quantities present in the system, and whether these exhibit nontrivial couplings with the order parameter, as expressed through nonvanishing Poisson-bracket relations.

In case of $O(N)$ model G [25] the N -component order parameter is not conserved, but couples reversibly to a set of $N(N-1)/2$ conserved-charge densities, which can be summarized in an antisymmetric tensor n_{ab} . Statically, the fluctuations of the charge densities are noncritical and as such appropriately described by a quadratic term with susceptibility χ in the free energy, replacing

$$F \rightarrow F + \int d^d x \frac{n_{ab} n_{ab}}{4\chi}. \quad (2)$$

The charge densities and the order parameter satisfy the Poisson-bracket algebra

$$\{\phi_a(\vec{x}), n_{bc}(\vec{x}')\} = (\delta_{ac} \phi_b - \delta_{ab} \phi_c) \delta(\vec{x} - \vec{x}'), \quad (3a)$$

$$\{n_{ab}(\vec{x}), n_{cd}(\vec{x}')\} = (\delta_{ac} n_{bd} + \delta_{bd} n_{ac} - \delta_{ad} n_{bc} - \delta_{bc} n_{ad}) \delta(\vec{x} - \vec{x}'), \quad (3b)$$

reflecting the fact that in model G the charge densities are the generators of $O(N)$ transformations. With these Poisson brackets, the equations of motion for model G can be written as [26,27]

$$\frac{\partial \phi_a}{\partial t} = -\Gamma \phi \frac{\delta F}{\delta \phi_a} + \frac{g}{2} \{\phi_a, n_{bc}\} \frac{\delta F}{\delta n_{bc}} + \theta_a, \quad (4a)$$

$$\begin{aligned} \frac{\partial n_{ab}}{\partial t} = & \gamma \vec{\nabla}^2 \frac{\delta F}{\delta n_{ab}} + g \{n_{ab}, \phi_c\} \frac{\delta F}{\delta \phi_c} \\ & + \frac{g}{2} \{n_{ab}, n_{cd}\} \frac{\delta F}{\delta n_{cd}} + \zeta_{ab}, \end{aligned} \quad (4b)$$

where the Gaussian white noises θ and ζ have variances set by the fluctuation-dissipation relation (FDR),

$$\begin{aligned} \langle \theta_a(t, \vec{x}) \theta_b(t', \vec{x}') \rangle &= 2\Gamma_0 T \delta_{ab} \delta(\vec{x} - \vec{x}') \delta(t - t'), \\ \langle \zeta_{ab}(t, \vec{x}) \zeta_{cd}(t', \vec{x}') \rangle &= -2\gamma T (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) \vec{\nabla}^2 \delta(\vec{x} - \vec{x}') \delta(t - t'), \end{aligned} \quad (5)$$

and drive the system toward the equilibrium Boltzmann distribution. For $N = 4$, the $O(4)$ model G represents the dynamic universality class of the $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$ chiral phase transition in QCD with two massless quark flavors and anomalously broken axial $U_A(1)$ [6], where the six charge densities belong to the conserved isovector and isoaxial-vector currents.

In model H, the order parameter ϕ has one component, is conserved, and couples reversibly to the transverse part of the conserved momentum density $\vec{j}$ [5]. The static fluctuations of the latter are again noncritical. In the free energy (1) the momentum density appears in the form of a kinetic energy with mass density ρ ,

$$F \rightarrow F + \int d^d x \frac{\vec{j}^2}{2\rho}. \quad (6)$$

The components of the momentum density are the generators of spatial translations, so their Poisson-bracket algebra with the order parameter is given by [28]

$$\begin{aligned} \{\phi(\vec{x}), j_l(\vec{x}')\} &= \phi(\vec{x}') \frac{\partial}{\partial x'_l} \delta(\vec{x} - \vec{x}'), \\ \{j_l(\vec{x}), j_m(\vec{x}')\} &= \left[j_l(\vec{x}') \frac{\partial}{\partial x'_m} - j_m(\vec{x}) \frac{\partial}{\partial x_l} \right] \delta(\vec{x} - \vec{x}'). \end{aligned} \quad (7)$$

With these Poisson brackets, we can express the equations of motion of model H [5,29] as

$$\frac{\partial \phi}{\partial t} = \sigma \vec{\nabla}^2 \frac{\delta F}{\delta \phi} + g \{\phi, \vec{j}\} \cdot \frac{\delta F}{\delta \vec{j}} + \theta, \quad (8a)$$

$$\frac{\partial j_l}{\partial t} = \mathcal{T}_{lm} \left[\eta \vec{\nabla}^2 \frac{\delta F}{\delta j_m} + g \{j_m, \phi\} \frac{\delta F}{\delta \phi} + g \{j_m, j_n\} \frac{\delta F}{\delta j_n} \right] + \xi_l, \quad (8b)$$

where $\mathcal{T}$ is a transverse projector, which in momentum space acts as $\mathcal{T}_{lm}(\vec{p}) = \delta_{lm} - p_l p_m / \vec{p}^2$. The Gaussian noise terms θ and $\vec{\xi}$ have variances

$$\begin{aligned} \langle \theta(t, \vec{x}) \theta(t', \vec{x}') \rangle &= -2\sigma T \vec{\nabla}^2 \delta(\vec{x} - \vec{x}') \delta(t - t'), \\ \langle \xi_l(t, \vec{x}) \xi_m(t', \vec{x}') \rangle &= -2\eta T \mathcal{T}_{lm} \vec{\nabla}^2 \delta(\vec{x} - \vec{x}') \delta(t - t'). \end{aligned} \quad (9)$$

Comparing the equations of motion of model G (4) and model H (8), one immediately notices that the two merely differ by: (a) the number of components of the order parameter field, (b) its kinetic coefficient reflecting whether it is conserved (model H) or not (model G), and (c) the detailed expressions for the reversible mode couplings. In model G, the term in (4a) describes Larmor precession of the order parameter around the magnetic field generated by the fluctuating charge densities n_{ab} , while in model H, the corresponding term in (8a) describes advection of the order parameter with the momentum density $\vec{j}$. Other than that, the structural similarities ensure that the real-time FRG methods developed in [23] can also be directly applied to model H.

Real-time FRG approach to model G/H. Nonperturbative field-theoretic methods are needed to describe the critical behavior in the vicinity of a second-order thermal phase transition in $d = 3$ spatial dimensions. One such method which is particularly well approved for describing static critical phenomena is the Euclidean FRG, e.g., see [30,31]. To describe *dynamic* critical phenomena, one evidently needs real-time formulations, however. Since these phenomena are essentially classical, effective descriptions are usually based on the Martin-Siggia-Rose (MSR) formalism [32–34], where one introduces an auxiliary response field $\tilde{\psi}$ for every classical field ψ , which allows us to express multitime correlation functions as a path integral over an associated MSR action.

By adding source terms of the form $J\tilde{\psi}$ and $\tilde{J}\psi$ directly to the MSR action, one can obtain a generating functional $Z[J, \tilde{J}]$ for multitime correlation functions, following standard field theory techniques. In presence of reversible mode couplings, however, in order to obtain physical correlation functions [35], it was suggested in Ref. [23] to insert classical source terms of the form $J\psi$ in the free energy (1) instead of adding $J\tilde{\psi}$ in the MSR action. Due to the nontrivial Poisson-brackets in the equations of motion (4) and (8), this has the consequence that—on the level of the MSR action—these classical sources J then couple to composite fields $\tilde{\Psi}$, which are related to the standard response fields $\tilde{\psi}$ by a linear but field dependent transformation. We will therefore refer to those $\tilde{\Psi}$ as the composite response fields.

By coupling the sources at the level of the free-energy, i.e., $F \rightarrow F - \int J\psi$, it is ensured that (i) the classical symmetry of thermal equilibrium [36] as well as (ii) an

extended temporal gauge symmetry related to the reversible mode couplings, are maintained exactly in the presence of classical external sources. The extended symmetry (ii) expresses an invariance under time-gauged $O(N)$ transformations in model G [23], and the invariance under time-gauged Galilean boosts in model H [37].

Next, to formulate the FRG flow equations [38], one furthermore adds an RG-scale k dependent (spatial) regulator term also on the level of the free energy, i.e. $F \rightarrow F + \frac{1}{2} \int \psi R_k \psi$, analogous to the way the (classical) external sources are introduced. The central object in the real-time generalization [39] of the standard FRG approach by Wetterich [38] is the effective *average* MSR action Γ_k , which is defined as the modified Legendre transform of the scale-dependent Schwinger functional $-i \log Z_k$, see Eq. (S.5) in Supplemental Material [40]. The RG-scale dependence of Γ_k is described by an (exact) real-time flow equation [39]

$$\partial_k \Gamma_k = -\frac{i}{2} \text{Tr} \left\{ \frac{\partial_k R_k}{\Gamma_k^{(2)} - R_k} \right\}. \quad (10)$$

Based on the above symmetries (i) and (ii), it was shown in Ref. [23] that (a) the mode-coupling constant g is protected from renormalization, and (b) the static FRG flow for the free energy (1) is independent of the dynamics and simply given by the standard d -dimensional Euclidean flow equation [41]. In practice, these two exact results can largely simplify the derivation of flow equations.

Introducing the source and regulator terms on the level of the MSR action in the standard way (as if there were no reversible mode couplings), and as done in Ref. [24] also for model H, the symmetries (i) and (ii) are generally not preserved. The violation of symmetries during the FRG flow, on the other hand, typically introduces RG relevant operators that would otherwise be excluded by symmetry. These operators then need to be carefully fine-tuned to vanish in the IR again in order to reach the correct fixed point. To avoid such an additional fine tuning, and to guarantee the exact results that we discuss below, it is therefore necessary to set up an FRG flow that manifestly maintains all the relevant symmetries.

For illustration, we first truncate the effective average MSR action by simply promoting all couplings (m^2 , λ , Γ^ϕ , and γ for model G, and m^2 , λ , σ , and η for model H) to depend on the FRG scale k . For the static FRG flow this amounts to a simple “ ϕ^4 -truncation” which in combination with the flow of the kinetic coefficients here has the advantage to provide an entirely analytical description of the main qualitative features of dynamic scaling. We employ the optimized regulator $R_k^\phi(\vec{p}) = (k^2 - \vec{p}^2)\theta(k^2 - \vec{p}^2)$ [42] for the order parameter ϕ and set the regulators for the charge densities n_{ab} and $\vec{j}$ to zero, as their static fluctuations are anyway noncritical, and corresponding spatial regulator terms thus RG irrelevant.

The flow equations for the static quantities m_k^2 and λ_k are given for arbitrary N in Refs. [23,41,43]. In addition, we use the FDR to project the flow onto the kinetic coefficients, as discussed for the order-parameter damping rate Γ_k^ϕ and charge mobility γ_k of model G in Ref. [23]. For model H, the scale-dependent order-parameter mobility σ_k and shear viscosity η_k can similarly be obtained from second functional derivatives of the effective average MSR action Γ_k with respect to composite response fields $\tilde{\Phi}$ and $\tilde{J}$ [cf. Eq. (S.4) in [40]] which are appropriate generalizations of the ones introduced for model G in Ref. [23],

$$\frac{1}{\sigma_k} = \frac{1}{2iT} \lim_{\vec{p} \rightarrow 0} \vec{p}^2 \lim_{\omega \rightarrow 0} \frac{\delta^2 \Gamma_k}{\delta \tilde{\Phi}(-p) \delta \tilde{\Phi}(p)} \Big|_0, \quad (11)$$

$$\frac{1}{\eta_k} = \frac{1}{2iT} \lim_{\vec{p} \rightarrow 0} \vec{p}^2 \lim_{\omega \rightarrow 0} \frac{\mathcal{T}_{lm}(\vec{p})}{d-1} \frac{\delta^2 \Gamma_k}{\delta \tilde{J}_l(-p) \delta \tilde{J}_m(p)} \Big|_0, \quad (12)$$

with four momenta $p \equiv (\omega, \vec{p})$ and evaluated at vanishing field expectation values as indicated by $(\cdots)|_0$.

Using the definitions in (11) and (12), or the corresponding ones for model G [23], one can project the flow equation (10) onto the various kinetic coefficients, see Eqs. (S.13)–(S.16) in the SM [40] for the explicit expressions. The physically relevant information is then encoded in the following dimensionless combinations of the kinetic coefficients [27], first defined for model G,

$$w_G \equiv \chi \frac{\Gamma_k^\phi}{\gamma_k}, \quad f_G \equiv \frac{d\Omega_d g^2 T k^{d-4}}{(2\pi)^d \Gamma_k^\phi \gamma_k}, \quad (13)$$

and second for model H,

$$w_H \equiv \rho \frac{\sigma_k k^2}{\eta_k}, \quad f_H \equiv \frac{d\Omega_d g^2 T k^{d-4}}{(2\pi)^d \sigma_k \eta_k}. \quad (14)$$

Here, w_G and w_H are just the ratios between the kinetic coefficients of order parameter and conserved charges, while f_G and f_H conveniently represent the prefactors of the loop diagrams in the flow of the kinetic coefficients.

Inserting the usual dimensionless mass parameter $\bar{m}^2 \equiv k^{-2} m^2$ into the flow equations for the kinetic coefficients [cf. Eqs. (S.13)–(S.16) in the SM [40]], one obtains the dimensionless flow equations for f_G and w_G in model G,

$$\begin{aligned} k\partial_k f_G &= f_G(d-4) \\ &+ f_G^2 \left(\frac{2}{d(1+\bar{m}_k^2)^3} - (N-1)I_d(\bar{m}^2, w_G) \right), \\ k\partial_k w_G &= w_G f_G \left[\frac{2}{d(1+\bar{m}_k^2)^3} + (N-1)I_d(\bar{m}^2, w_G) \right], \end{aligned} \quad (15)$$

with the shorthand notation

$$I_d(\bar{m}^2, w_G) \equiv -\frac{1}{(1+\bar{m}^2)^2} \left\{ \frac{1}{1+(1+\bar{m}^2)w_G} + \frac{4-d}{d-2} \left[1 - {}_2F_1 \left(1, \frac{d-2}{2}; \frac{d}{2}; -\frac{1}{(1+\bar{m}^2)w_G} \right) \right] \right\}.$$

And the dimensionless flow equations for f_H and w_H in model H are obtained analogously as

$$\begin{aligned} k\partial_k f_H &= (d-4)f_H + f_H^2 \left[\frac{1}{d(d+2)} \frac{1}{(1+\bar{m}^2)^3} \right. \\ &\quad \left. - \frac{2(d-1)}{d(d-2)} \left(\frac{1}{(1/w_H + (1+\bar{m}^2))^2} - \frac{1}{(1+\bar{m}^2)^2} \right) \right], \\ k\partial_k w_H &= 2w_H + w_H f_H \left[\frac{1}{d(d+2)} \frac{1}{(1+\bar{m}^2)^3} \right. \\ &\quad \left. + \frac{2}{d-2} \left(\frac{(d-1)}{d(1/w_H + (1+\bar{m}^2))^2} - \frac{d-1}{d(1+\bar{m}^2)^2} \right) \right]. \end{aligned} \quad (16)$$

Importantly, Eqs. (15)–(17) no longer depend on the nonuniversal parameters ρ , g , and T , which stresses the universal character of these nonperturbative flow equations.

Dynamic critical behavior of model G/H. Close to a critical point, the dynamic critical exponent z describes the divergence of the correlation time $\xi_t \sim \xi^z$ relative to the correlation length ξ . Sufficiently close to the critical point, on the other hand, the relevant infrared cutoff during the FRG flow is set by the FRG scale $k > 0$, and the correlation length effectively behaves as $\xi \sim 1/k$, while the correlation time ξ_t can be inferred from the position ω_p of the lowest lying pole of the dynamic response function, as $\xi_t \sim 1/\text{Im}(\omega_p)$. Since for both, model G and H, the kinetic coefficients themselves show critical power-law divergences with the FRG scale k , which can be quantified in terms of logarithmic k -derivatives of the kinetic coefficients in the scaling regime, i.e. $x_{\Gamma^\phi} = -k\partial_k \log \Gamma_k^\phi$ and $x_\gamma = -k\partial_k \log \gamma_k$ for model G, or $x_\sigma = -k\partial_k \log \sigma_k$ and $x_\eta = -k\partial_k \log \eta_k$ for model H, these divergences also contribute to the dynamic critical exponents.

We first focus on the critical dynamics of the order parameter field ϕ . In model G the order parameter is not conserved, and an explicit calculation of its (retarded) propagator in our FRG truncation simply yields $\omega_\phi = -i\Gamma_k^\phi m_k^2 \sim k^{2-x_{\Gamma^\phi}}$, we therefore have $z_\phi = 2 - x_{\Gamma^\phi}$. In contrast, in model H the order parameter, defined from the entropy per particle, is conserved, and one obtains the dispersion relation $\omega_\phi = -i\sigma_k \vec{p}^2 (m_k^2 + \vec{p}^2)$. Setting the external momentum to $|\vec{p}| \sim k$, one thus obtains an additional factor k^2 in the dispersion relation, yielding $\omega_\phi \sim \sigma_k k^4 \sim k^{4-x_\sigma}$, so that $z_\phi = 4 - x_\sigma$ in model H.

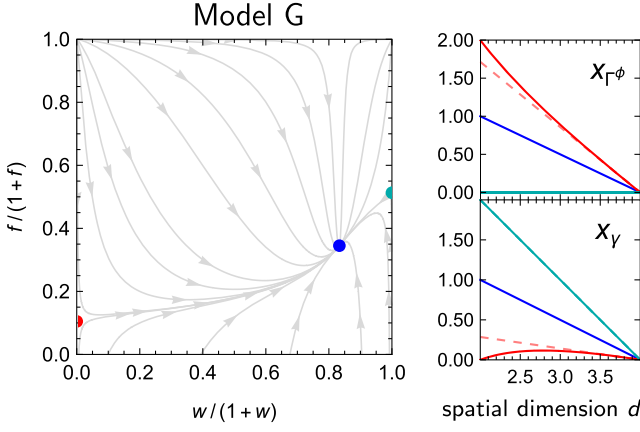


FIG. 1. Left: flow diagram of 3d model G with one purely attractive stable strong-scaling fixed point (blue) at $(f_G^*, w_G^*) \approx (0.527, 4.998)$ in our ϕ^4 -truncation, and two unstable weak-scaling fixed points at $w_G^* = \{0, \infty\}$ (red and green) [27]. Right: the critical exponents $x_{\Gamma\phi}$ and x_γ of order-parameter damping rate and charge mobility at the various fixed points (solid lines in matching colors), compared to the results from the 1st-order ϵ -expansion in Ref. [35] (dashed lines).

Note that we have omitted the anomalous dimension $\eta_\perp$ of the order-parameter field here, which vanishes in our ϕ^4 -truncation of the spatial flow. For $\eta_\perp \neq 0$, these dynamic critical exponents generalize to $z_\phi = 2 - \eta_\perp - x_{\Gamma\phi}$ in model G, and $z_\phi = 4 - \eta_\perp - x_\sigma$ in model H, respectively.

Moreover, in either case, the conserved charge densities n_{ab} and $\vec{j}$ of model G and H, both also experience critical slowing down due to the reversible couplings to the order parameter. Setting the external momentum to $|\vec{p}| \sim k$ again, their dispersion relations, $\omega_n = -i\gamma_k \vec{p}^2/\chi$ and $\omega_j = -i\eta_k \vec{p}^2/\rho$, both exhibit power laws, $\omega_n \sim k^{2-x_\gamma}$ and $\omega_j \sim k^{2-x_\eta}$, as well, from which one can deduce the dynamic critical exponents $z_n = 2 - x_\gamma$ for the charge densities n_{ab} in model G and $z_j = 2 - x_\eta$ for the transverse momentum density $\vec{j}$ in model H.

Since the critical point corresponds to a fixed point of the FRG flow, it is instructive to investigate the flow diagrams for model G and H, which are shown in Figs. 1 and 2, respectively, where the FRG flow lines are plotted in the compactified $(w_{G/H}, f_{G/H})$ plane [44]. One can see that for model G there are two unstable fixed points at $w_G^* = 0$ and $w_G^* = \infty$. In addition, there is the stable fixed point that describes the critical dynamics of model G. In contrast, in the flow diagram of model H, there is only one stable fixed point at $w_H^* = 0$. With this information we can then look for the stable fixed points in the dimensionless flow equations of $f_{G/H}$ analytically:

For model G, with $w_G^* \neq 0$ and $f_G^* \neq 0$ in Eq. (15), the stable fixed point (in our truncation) is located at

$$f_G^* = \frac{(4-d)d(1+\bar{m}^2)^3}{4}. \quad (17)$$

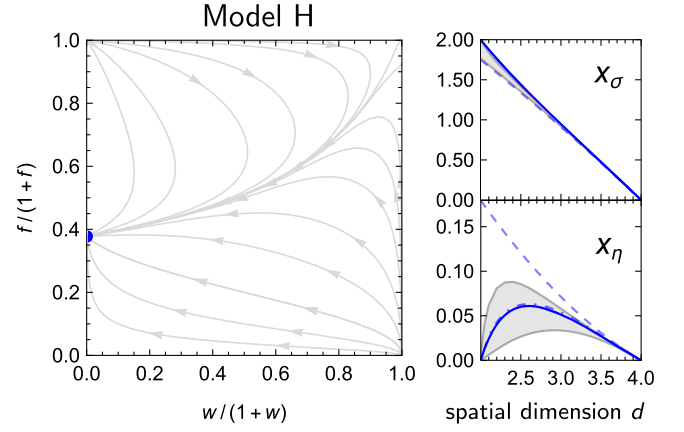


FIG. 2. Left: flow diagram of 3d model H with only a single stable weak-scaling fixed point at $w_H^* = 0$. Right: the associated critical exponents x_σ and x_η of order-parameter mobility and shear viscosity in our ϕ^4 -truncation (solid blue lines) compared to the 2nd-order ϵ -expansion for model H in Refs. [45,46] (dashed blue), and the result for x_η (bottom) by Ohta and Kawasaki [47] from mode-coupling theory (dashed-dotted blue). Gray bands between corresponding real-time FRG results from full LPA' truncations of the free energy, based on the fixed expansion point $\phi_{0,k} = 0$ (upper gray line) and on using the scale-dependent minimum $\phi_{0,k} = \phi_{\min,k}$ (lower gray line), are included to assess the quantitative uncertainty of the ϕ^4 -truncation.

For model H, $w_H^* = 0$ implies that the term with $1/w_H$ in the denominator in (17) will tend to zero at the stable fixed point and thus can be safely neglected, yielding

$$f_H^* = \frac{(d-2)(4-d)(1+\bar{m}^2)^3}{\frac{2(d-1)}{d}(1+\bar{m}^2) + \frac{d-2}{d(d+2)}}. \quad (18)$$

Now, using $k\partial_k f_G = (d-4+\eta_\perp+x_{\Gamma\phi}+x_\gamma)f_G$ for model G and $k\partial_k f_H = (d-4+\eta_\perp+x_\sigma+x_\eta)f_H$ for model H, which follow directly from the definitions in Eqs. (13) and (14) (here including field renormalization), we can deduce that as long as the fixed-point value f^* is finite (i.e., $0 < f^* < \infty$), the following *weak-scaling* relations [27,35] hold at the fixed point in either case,

$$x_{\Gamma\phi} + x_\gamma = x_\sigma + x_\eta = 4 - d - \eta_\perp. \quad (19)$$

If w^* assumes a finite fixed-point value (i.e., $0 < w^* < \infty$) as well, as in the case of the stable fixed point in model G, then, from $k\partial_k w_G = w_G(x_\gamma - x_{\Gamma\phi} - \eta_\perp)$, we also have the *strong-scaling* relation,

$$x_{\Gamma\phi} = x_\gamma - \eta_\perp. \quad (20)$$

With both w_G^* and f_G^* nonvanishing and finite at the stable strong-scaling fixed point of model G (cf., Fig. 1), Eqs. (19) and (20) are sufficient to uniquely fix the scaling exponents of both kinetic coefficients to

$$x_{\Gamma\phi} + \eta_\perp = x_\gamma = 2 - \frac{d}{2}, \quad (21)$$

as represented by the solid blue lines for $x_{\Gamma\phi}$ and x_γ in the right panel of Fig. 1. Moreover, with $z_\phi = 2 - \eta_\perp - x_{\Gamma\phi}$ and $z_n = 2 - x_\gamma$ from above, these dynamic scaling relations imply that one exactly recovers the dynamic critical exponents $z_\phi = z_n = d/2$ at this strong-scaling fixed point in model G.

With $w_H^* = 0$ but $f_H^* \neq 0$ and finite at the model H fixed point (cf., the left panel of Fig. 2), on the other hand, there is no such strong-scaling relation as in (20). Instead, inserting Eq. (18) into the FRG flow equations for σ_k and η_k [cf. Eqs. (S.13) and (S.14) in [40]], we then obtain the following analytical expressions for model H,

$$x_\sigma = \frac{2(d-1)f_H^*}{d(d-2)(1+\bar{m}^2)^2}, \quad (22a)$$

$$x_\eta = \frac{f_H^*}{d(d+2)(1+\bar{m}^2)^3}, \quad (22b)$$

with f_H^* given by (18), for $\eta_\perp = 0$ (the generalization for $\eta_\perp \neq 0$ is given in [40]). One can explicitly verify that an expansion of (22) in $\epsilon = 4 - d$ around $\epsilon = 0$ reproduces the standard result from 1st-order ϵ -expansion [45] exactly, but deviates at higher orders. Our FRG results (22) are plotted as the solid blue lines in the right panel of Fig. 2 along with results from 2nd-order ϵ -expansion [45,46], which currently is the highest order available in this case. While the critical exponent of the mobility x_σ deviates from the latter only on a quantitative level, that of the shear viscosity shows an entirely different qualitative behavior. In fact, our FRG result for x_η in the ϕ^4 -truncation closely resembles that of a self-consistent mode-coupling calculation with $\eta_\perp = 0$ by Ohta and Kawasaki [47]. Specifically, with $z_\phi = d + x_\eta$, in $d = 3$ spatial dimensions, we obtain the dynamic critical exponent $z_\phi \approx 3.051$ while the result from [47] corresponds to $z_\phi \approx 3.054$; for comparison, the 2nd-order ϵ -expansion yields $z_\phi \approx 3.071$ [46].

To assess the robustness of our results, we also consider the extended local-potential approximation (LPA') truncation for the static flow [48,49], as shown as the gray bands in Fig. 2. For $d = 3$ we obtain the values $z_\phi \approx 3.058$ for fixed expansion point $\rho_{0,k} = 0$ in field space, and $z_\phi \approx 3.034$ for an expansion around the scale-dependent minimum. In particular, the latter includes a non-vanishing $\eta_\perp \approx 0.0443$. We refer to [40] for details.

Summary and outlook. We have used our novel real-time FRG approach to dynamical systems with reversible mode couplings from Ref. [23] to study, in parallel, the critical dynamics of model G and model H, the conjectured dynamic universality classes of the chiral transition in QCD with two flavors of nearly massless quark and of the QCD critical point at finite baryon density, respectively. Carefully adding external sources and regulators at the level of the LGW free energy, this formalism ensures the preservation of all relevant symmetries and, in particular,

provides Ward identities to protect the reversible mode coupling constant g from renormalization.

Within this framework we have derived one-loop exact nonperturbative FRG flow equations for the kinetic coefficients of model G and H. The associated fixed-point structure is qualitatively in line with 1st-order ϵ -expansion [27], comprising one stable strong-scaling and two unstable weak-scaling fixed points in model G, as compared to only a single weak-scaling fixed point in model H. We have obtained the corresponding strong and weak dynamic relations and computed the dynamic critical exponents x_i of the kinetic coefficients in $2 < d < 4$ spatial dimensions. For the $O(4)$ chiral transition in QCD these determine the singular behavior of chiral order-parameter and Goldstone pion damping rate $\Gamma_k^\phi \sim k^{-x_{\Gamma\phi}}$, and the iso-(axial-)vector charge mobilities $\gamma_k \sim k^{-x_\gamma}$ of model G. The mobility $\sigma_k \sim k^{-x_\sigma}$ of the conserved order parameter of model H determines the singular behavior of the entropy per baryon at the QCD critical point and that of transverse momentum densities the shear viscosity $\eta_k \sim k^{-x_\eta}$. For the dynamic critical exponents z_i the strong-scaling relations of model G imply the exact result $z_\phi = z_n = d/2$. In model H, we obtain numerically in $d = 3$ spatial dimensions $x_\sigma \approx 0.949$ and $x_\eta \approx 0.051$ in quite close agreement with the mode-coupling calculations by Ohta and Kawasaki [47], implying $z_\phi \approx 3.051$ for the critical fluctuations of the entropy per baryon.

Evidently, the true power of this formalism is that it can straightforwardly be extended to include more sophisticated truncations of the effective action, e.g. improving the static truncation for the effective LGW free energy, including derivative couplings and anomalous dimensions, for improved precision. Moreover, including self-consistent frequency and/or momentum dependencies of the kinetic coefficients $\sigma_k(\omega, \vec{p})$ and $\eta_k(\omega, \vec{p})$, cf. Refs. [22,23], the framework can be used to compute universal dynamic scaling functions [10,11,23] that go beyond the common Kawasaki approximation [50–52].

The developments presented in this letter also provide the basis for applications of this real-time FRG approach to various other systems with reversible mode couplings. One example is stochastic hydrodynamics, where the nonlinear self-coupling of the momentum density $\vec{j}$ [cf., Eq. (8b)] leads to a nontrivial renormalization of the shear viscosity [53]. By using the present techniques, this effect could be computed nonperturbatively and compared to the one-loop result of Ref. [53].

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