

Holomorphic quasimodular bootstrap

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In this paper, we propose a generalization to the holomorphic modular bootstrap approach featuring flavored modular differential equations arising from null states of the algebra. The approach can be applied to rational and the more general quasimodular chiral algebras that arise in the study of four-dimensional superconformal field theories. The transformation property of the flavored equations reveals hidden and universal effective action of the modular group and spectral flow on the null states. As shown in examples, the equations compute the full ordinary and nonordinary spectra both in the untwisted and the twisted sector.

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Introduction. The classification of two-dimensional unitary rational conformal field theories (RCFTs) has been reinvigorated in recent years using the holomorphic modular bootstrap approach [1–15], which has close relation with the active study of unitary modular tensor category (MTC) [16]. A rational CFT has a factorized symmetry algebra $\mathbb{V} \otimes \bar{\mathbb{V}}$, where the chiral algebra $\mathbb{V}$ has finitely many irreducible representations $M_{i=1,\dots,N}$. The corresponding characters ch_i transform under the modular group $\Gamma \subset \text{SL}(2, \mathbb{Z})$ as a vector-valued modular function. The modularity is condensed into an unflavored modular linear differential equation (MLDE), an ordinary differential equation that constrains the spectrum $\{\text{ch}_i\}$ of the chiral algebra,

$$\left(D_q^{(N)} + \sum_{r=0}^{N-1} \phi_r(\tau) D_q^{(N-r)} \right) \text{ch}_i = 0. \quad (1)$$

Here, the derivatives $D_q^{(n)} := \partial_{(2n-2)} \cdots \partial_{(2)} \partial_{(0)}$, $\partial_{(k)} := q \partial_q + k E_2(\tau)$, and $\phi_r(\tau)$ are modular forms of modular-weight $2k$ with respect to the modular group Γ . The unflavored MLDE is homogeneous in modular weight, transforming covariantly under Γ . This bootstrap approach starts by postulating a generic form of MLDE and analyzes its solutions to classify possible unitary RCFTs.

Nonunitary and/or nonrational CFTs/chiral algebras have long been playing significant roles in describing

critical phenomena in non-Hermitian physics and two-dimensional percolation [17], a classic example being the Lee-Yang model [18] and, more recently, [19]. They are also important objects of interest in studying topological quantum field theories (TQFTs) and supersymmetric field theories. In [20], it is shown that any 4D $\mathcal{N} = 2$ unitary SCFT $\mathcal{T}$ is associated to a nonunitary chiral algebra $\mathbb{V}[\mathcal{T}]$ that serves as an algebra-valued superconformal invariant that classifies 4D $\mathcal{N} = 2$ SCFTs. More recently, 3D nonunitary TQFTs are found to describe the infrared fixed points of a special renormalization group flow triggered from deforming unitary ultraviolet 3D theories, which supports nonunitary chiral algebras at the boundary [21]. Additionally, the spectrum of nonunitary chiral algebras are given geometric interpretation in terms of fixed points of the Hitchin moduli space through 4D mirror symmetry [22,23]. With all these classic results and recent development, it is therefore an interesting and important task to develop tools to analyze the spectrum of nonunitary CFTs and their chiral algebras.

In this paper, we focus on the (non)unitary quasimodular chiral algebras [24], which are the simplest generalization of the rational ones. Examples of quasimodular algebras include the $\hat{\mathfrak{g}}_k$ Kac-Moody algebras at integral level and admissible levels, which give rise to nonunitary minimal models via coset construction. A vast source of such algebras also comes from $\mathbb{V}[\mathcal{T}]$ [25].

The spectrum of a quasimodular algebra is often more complicated than a rational one. In any irreducible representation of a rational chiral algebra, the weight space of a given conformal weight h is finite dimensional: Such a representation is called an “ordinary” representation, while in a quasimodular algebra, there may be nonordinary representations. The ordinary spectrum of a quasimodular algebra can be studied using unflavored MLDE just as in the holomorphic modular bootstrap [25,26]. This unflavored MLDE is related to the “nilpotent property” of the stress tensor [24,25,27,28], namely, $L_{-2}^N|0\rangle$ equals a null state

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$|\mathcal{N}_T\rangle$ up to a state $|c_2\rangle$ given by linear combinations of states of the form $a_{-h_a-1}|b\rangle$ [29]

$$L_{-2}^N|0\rangle = |c_2\rangle + |\mathcal{N}_T\rangle. \quad (2)$$

However, to analyze the nonordinary spectra, the infinite degeneracy must be lifted. When $\mathcal{T}$ enjoys continuous flavor symmetries, the degeneracy often can be lifted by introducing flavor refinement, which turns on the flavor fugacities b_i when defining the characters $\text{ch}(q, b, y) = y^k \text{tr} q^{L_0-c/24} b_i^{f_i}$. Here, $q = e^{2\pi i \tau}$, and we also denote $b_i := e^{2\pi i b_i}$, $y = e^{2\pi i \eta}$ [30].

In this paper, we propose a generalization to the holomorphic modular bootstrap, simply called *holomorphic quasi-modular bootstrap*, to encompass the associated chiral algebras $\mathbb{V}[\mathcal{T}]$, and more generally, quasimodular algebras. Our method introduces flavor refinement and exploits a hidden quasimodularity structure. We demonstrate our approach by looking at examples of Kac-Moody algebras $\hat{\mathfrak{g}}_k$ associated to simple Lie algebras $\mathfrak{g}$.

The starting point of our approach is a postulation of the generic form of the null state $|\mathcal{N}_T\rangle$. By Zhu's flavored recursion relation [25,27,28,31–34], the state $|\mathcal{N}_T\rangle$ gives rise to one flavored MLDE, a partial differential equation with respect to q and the flavor fugacities b_i . Under the modular transformation and spectral flow of flavor fugacities, the $|\mathcal{N}_T\rangle$ equation transforms inhomogeneously and generates a set of additional flavored modular differential equations of varying modular weights. From this set of equations, we can derive the full spectra in both the untwisted and twisted sectors of the chiral algebra.

To simplify the initial postulation, we propose an alternative characterization of $|\mathcal{N}_T\rangle$ for Kac-Moody algebras, instead of directly referencing T , $|c_2\rangle$, or it being null. Moreover, we construct a closed-form expression for $|\mathcal{N}_T\rangle$ when possible.

We apply our approach to a series of Kac-Moody examples. There we further demonstrate that all the equations in the modular orbit are associated with a set of null states $\{\mathcal{N}\}$. We reveal a hidden modular and spectral-flow structure among the null states $\{\mathcal{N}\}$ by analyzing the transformation property of the flavored MLDEs. For the example we studied, the modular and spectral-flow structure among $\{\mathcal{N}\}$ share elegant universal forms.

Quasimodularity. Let us focus on Kac-Moody algebras $\hat{\mathfrak{g}}_k$, where the level k and algebra $\mathfrak{g}$ of rank r are to be determined. The generators of the algebra are J_m^a with commutation relations

$$[J_m^a, J_n^b] = f^{ab}{}_c J_{m+n}^c + km K^{ab} \delta_{m+n,0}, \quad (3)$$

where the symmetric bilinear form $K^{ab} := K(J^a, J^b)$ denotes the Killing form of $\mathfrak{g}$. We often use the

Chevalley basis $h^i, i = 1, \dots, r$ for the Cartan generators, and $K^{ij} := K(h^i, h^j)$.

We propose that the null state $|\mathcal{N}_T\rangle$ can be alternatively characterized by the following $|\mathcal{N}_T\rangle$ conditions [35]:

$$J_0^a |\mathcal{N}_T\rangle = h_{n \geq 2}^i |\mathcal{N}_T\rangle = L_2 |\mathcal{N}_T\rangle = 0. \quad (4)$$

The first condition simply requires $|\mathcal{N}_T\rangle$ to be $\mathfrak{g}$ invariant, as it is part of the $\mathfrak{g}$ -invariant nilpotency condition. The conditions (4) also implies $|\mathcal{N}_T\rangle$ to be a Virasoro primary, $L_{n \geq 1} |\mathcal{N}_T\rangle = 0$. In addition, we may also consider a trivial null state $|\mathcal{N}_{\text{Sug}}\rangle = L_{-2}|0\rangle - \frac{1}{2(k+h^\vee)} K_{ab} J_{-1}^a J_{-1}^b |0\rangle$, which implements the standard Sugawara construction of the stress tensor $T = T_{\text{Sug}}$.

The null state $|\mathcal{N}_T\rangle, |\mathcal{N}_{\text{Sug}}\rangle$ gives rise to two flavored MLDEs by applying Zhu's recursion. Under a modular transformation

$$\tau \rightarrow -\frac{1}{\tau}, \quad \mathfrak{b}_i \rightarrow \frac{\mathfrak{b}_i}{\tau}, \quad \eta \rightarrow \eta - \frac{1}{2\tau} \sum_{i,j=1}^r K^{ij} \mathfrak{b}_i \mathfrak{b}_j, \quad (5)$$

each of the two equations transforms into a linear combination of equations $\text{eq}_{i\dots}$ of equal or lower modular weights, which schematically reads

$$\text{eq} \rightarrow \tau^w \text{eq} + \tau^{w-1} \mathfrak{b}_i \text{eq}_i + \tau^{w-2} \mathfrak{b}_i \mathfrak{b}_j \text{eq}_{ij} + \dots \quad (6)$$

One may further perform additional modular transformation on the $\text{eq}_{i\dots}$ to construct a full modular orbit of both the $|\mathcal{N}_T\rangle$ and $|\mathcal{N}_{\text{Sug}}\rangle$ equations. At this point, we impose *quasimodularity*, requiring all the characters of the chiral algebra to be solutions to all the induced equations in the entire modular orbit. This places strong constraint on the spectrum both in the untwisted and twisted sector of the algebra, and we conjecture that it actually is strong enough to fully determine the spectrum. We shall see this in the examples below.

Transformation (6) reveals a hidden structure of null states. In all the examples we studied, the equations in the orbit are associated to a set of null states $\{|\mathcal{N}\rangle, |\mathcal{N}_i\rangle, \dots\}$. Schematically,

$$\begin{array}{ccccc} \text{eq} & \xrightarrow{\gamma} & \text{eq}_i & \oplus & \text{eq}_{ij} & \oplus \dots \\ \downarrow & & \downarrow & & \downarrow & \\ |\mathcal{N}\rangle & \xrightarrow{\gamma} & |\mathcal{N}_i\rangle & & |\mathcal{N}_{ij}\rangle & \dots \end{array} \quad (7)$$

by working out $|\mathcal{N}_{i\dots}\rangle$, we show that (6) implies an effective action of the modular group Γ on formal sums of null states with $\tau, \mathfrak{b}_i$ as the coefficients. Explicitly, the action has a universal form; for instance, when $\Gamma = \text{SL}(2, \mathbb{Z})$,

$$S|\mathcal{N}\rangle = \sum_{\ell \geq 0} \sum_{i_1, \dots, i_\ell=1}^r \frac{1}{\ell!} \tau^{h[\mathcal{N}] - \ell} \mathfrak{b}_{i_1} \dots \mathfrak{b}_{i_\ell} h_{i_1}^{i_1} \dots h_{i_\ell}^{i_\ell} |\mathcal{N}\rangle, \quad (8)$$

and T acts only on the τ in a formal sum. Using an identity $\sum_{\ell=0}^{h[\mathcal{N}]} \frac{(-1)^\ell}{\ell!(L-\ell)!} = \delta_{L,0}$ for $L \in \mathbb{N}$, we verify $S^2 = 1$. Similarly, $(ST)^3 = 1$ as expected.

There are additional translations σ of $\mathfrak{b}_i$ that can be performed on the equations,

$$\mathfrak{b}_i \xrightarrow{\sigma} \mathfrak{b}_i + n_i \tau, \quad (9)$$

$$\mathfrak{y} \xrightarrow{\sigma} \mathfrak{y} + \sum_{i,j=1}^r K^{ij} n_j \mathfrak{b}_i + \sum_{i,j=1}^r \frac{1}{2} K^{ij} n_i n_j \tau. \quad (10)$$

The translations are generalized spectral-flow automorphisms [36–39], which we continue to refer to as spectral-flow operations. Spectral flow with suitable integrality condition on n_i maps the flavored MLDEs and the spectrum within or between the untwisted and twisted sectors. At the level of the null states, σ takes the universal form

$$|\mathcal{N}\rangle \xrightarrow{\sigma} \sum_{\ell \geq 0} \sum_{i_1 \leq \dots \leq i_\ell} \frac{(-1)^\ell}{\ell!} n_{i_1} \dots n_{i_\ell} h_{i_1}^{i_1} \dots h_{i_\ell}^{i_\ell} |\mathcal{N}\rangle, \quad (11)$$

provided $(\sum_{i=1}^r n_i \alpha_i^\vee, \alpha) \in \frac{1}{2}\mathbb{Z}$ for all roots α of $\mathfrak{g}$.

In the following, we will demonstrate the above proposal by studying a few examples explicitly.

Deligne-cvitanović series. Let us assume $|\mathcal{N}_T\rangle$ to be at conformal weight $h = 4$. The most general $\mathfrak{g}$ -invariant state can be written as

$$\begin{aligned} |\mathcal{N}_T\rangle = & (L_{-2}^2 + \beta K_{AB} J_{-3}^A J_{-1}^B + \gamma K_{AB} J_{-2}^A J_{-2}^B) |0\rangle \\ & + \delta_1 d_{ABC} J_{-2}^A J_{-1}^B J_{-1}^C |0\rangle \\ & + \delta_2 d_{ABCD} J_{-1}^A J_{-1}^B J_{-1}^C J_{-1}^D |0\rangle. \end{aligned} \quad (12)$$

First, we apply the $|\mathcal{N}_T\rangle$ conditions, which fixes $\delta_{1,2} = 0$, and

$$\begin{aligned} \beta = & \frac{-3(2k + h^\vee)}{(k + h^\vee)(6k + h^\vee)}, \quad \gamma = \frac{3h^\vee}{2(k + h^\vee)(6k + h^\vee)}, \\ \dim \mathfrak{g} = & \frac{-2(6k - 5h^\vee)(k + h^\vee)}{k(6k + h^\vee)}. \end{aligned} \quad (13)$$

Next we apply Zhu's recursion to the state $|\mathcal{N}_{\text{Sug}}\rangle$ and $|\mathcal{N}_T\rangle$, giving the $|\mathcal{N}_{\text{Sug}}\rangle$ equation

$$\begin{aligned} 0 = & 2(k + h^\vee) D_q^{(1)} \text{ch} - K_{ij} D_{b_i} D_{b_j} \text{ch} \\ & - \sum_{\alpha} K_{\alpha, -\alpha} f^{\alpha, -\alpha} E_1 \left[\frac{1}{b^\alpha} \right] D_{b_i} \text{ch} \\ & - kr E_2(\tau) \text{ch} - k \sum_{\alpha} E_2 \left[\frac{1}{b^\alpha} \right] \text{ch}, \end{aligned} \quad (14)$$

and the $|\mathcal{N}_T\rangle$ equation,

$$\begin{aligned} 0 = & D_q^{(2)} \text{ch} + \left(\frac{c}{2} + 3kr(\beta - 2\gamma) \right) E_4(\tau) \text{ch} \\ & + (\beta - 2\gamma) \sum_{\alpha \in \Delta} \sum_{i=1}^r K_{\alpha, -\alpha} f^{\alpha, -\alpha} E_3 \left[\frac{1}{b^\alpha} \right] D_{b_i} \text{ch} \\ & + 3k(\beta - 2\gamma) \sum_{\alpha \in \Delta} E_4 \left[\frac{1}{b^\alpha} \right] \text{ch}. \end{aligned} \quad (15)$$

We perform modular transformation on the two equations, which produces $r(r+1)/2$ additional flavored MLDEs at modular-weight two and r at modular-weight three. Quasimodularity requires the roots of $\mathfrak{g}$ to satisfy an algebraic constraint for any weight $\mu \in \mathfrak{h}^*$

$$(\mu, \mu)^2 = \lambda_{\mathfrak{g}} \sum_{\alpha \in \Delta} (\mu, \alpha)^4, \quad \lambda_{\mathfrak{g}} \in \mathbb{R}, \quad (16)$$

for some constant $\lambda_{\mathfrak{g}}$. Among all the simple Lie algebras, there are only those in the Deligne-Cvitanović (DC) exceptional series [2,24,25,40–42] (also known as the Mathur-Mukhi-Sen series [2]) that satisfy the constraint [43],

$$\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{g}_2, \mathfrak{d}_4, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8. \quad (17)$$

Further more, quasimodularity also requires $k = 1$ or the nonunitary value $k = -\frac{h^\vee}{6} - 1$, and recovers one dimension formula [46] for the DC exceptional series,

$$\dim \mathfrak{g} = \frac{2(h^\vee + 1)(5h^\vee - 6)}{h^\vee + 6}. \quad (18)$$

With k and $\mathfrak{g}$ determined, the exact form of $|\mathcal{N}_T\rangle$ is also fixed. Note that the nonunitary DC-series with simply-laced algebra $\mathfrak{g}$ are chiral algebras of 4D $\mathcal{N} = 2$ SCFTs [20,25,38,47–49].

Using all the equations that appear in the modular orbit of $|\mathcal{N}_T\rangle$ and $|\mathcal{N}_{\text{Sug}}\rangle$, one can solve the spectrum [33,50]. This is achieved with the ansatz

$$\text{ch}(q, b) = q^\sigma \sum_{n=0}^{+\infty} c_n(b) q^n. \quad (19)$$

The $|\mathcal{N}_T\rangle$ equation determines the allowed values of σ and solves algebraically $c_{n>0}$ in terms of c_0 . The remaining

equations determine the leading function $c_0(b_i)$. In particular, expanding

$$c_0(b_i) = a_1^{n_1} \dots a_r^{n_r} \sum_{\ell_i \geq 0} c_{0;\ell_1, \dots, \ell_r} a_1^{\ell_1} \dots a_r^{\ell_r}, \quad (20)$$

the n_i are completely determined by the equations

$$\begin{aligned} 0 &= 2(k + h^\vee)\sigma - \sum_{i,j} (\mathbf{n}, \mathbf{n}) + \sum_{\alpha > 0} (\alpha, \mathbf{n}) + \frac{k}{12} \dim \mathfrak{g}, \\ 0 &= k \left[\sigma - \frac{1}{12} - \frac{h^\vee}{12} (\beta - 2\gamma) \right] (\mathbf{b}, \mathbf{b}) \\ &\quad - \frac{1}{2} (\beta - 2\gamma) \sum_{\alpha > 0} (\mathbf{n}, \alpha) (\alpha, \mathbf{b})^2 + (\mathbf{b}, \mathbf{n})^2, \end{aligned} \quad (21)$$

where $\mathbf{n} := \sum_{i=1}^r n_i \alpha_i$. The solutions of n_i give the integrable highest weights (in the simple root basis) when $k = 1$, the admissible weights for $\mathfrak{a}_1$, $\mathfrak{a}_2$, and the highest weights for $\mathfrak{g}_2$, $\mathfrak{d}_4$, $\mathfrak{f}_4$, $\mathfrak{e}_{6,7,8}$ studied in [51–54] when $k = -\frac{h^\vee}{6} - 1$. The function $c_0(b_i)$ is simply the $\mathfrak{g}$ character of the corresponding irreducible representation.

SU(2) Kac-Moody algebra. It is straightforward to explore $|\mathcal{N}_T\rangle$ of higher conformal weight, though it is a computational heavy task. For simplicity, in the following discussions, we fix $\mathfrak{g}$ to be the simplest Lie algebra $\mathfrak{a}_1$, with the level k to be determined.

We start with our initial postulation by solving (4) at conformal weight $h[\mathcal{N}_T] = 4, 6, \dots, 12$, which turns out to simultaneously determine the exact form of $|\mathcal{N}_T\rangle$ and the allowed values of level k . Here, we list the results in the following table.

$h[\mathcal{N}_T]$	Allowed k
4	1, $-4/3$
6	2, $-8/5$, $-1/2$
8	3, $-12/7$
10	4, $-16/9$, $-5/4$
12	5, $-20/11$, $-7/5$, $-2/3$, $1/2$

In the table, we recover the integrable levels $k \in \mathbb{N}$, and the fractional admissible levels $k = \frac{t}{v}$, where $t \in \mathbb{Z}_{\neq 0}$, $v \in \mathbb{N}_{>0}$ are coprime, $t + 2v \geq 2$. There are the boundary admissible levels $k + 2 = \frac{h^\vee}{v}$, as well as nonboundary admissible levels. Recall that boundary admissible Kac-Moody algebras $\widehat{\mathfrak{su}}(2)_{k=-\frac{4n}{2n+1}}$ are the associated chiral algebras of the Argyres-Douglas theories (A_1, D_{2n+1}) [49,55], while there is no known 4D correspondence for the remaining $\widehat{\mathfrak{su}}(2)_k$. To the best of our knowledge, integrable and admissible $\widehat{\mathfrak{su}}(2)_k$ are the only known quasilisse $\mathfrak{su}(2)$ Kac-Moody algebras.

We observe that $|\mathcal{N}_T\rangle$ in all $\widehat{\mathfrak{su}}(2)_k$ share a universal closed form. Let $|v_{\text{sing}}\rangle$ be the first nontrivial singular vector of the algebra. At an integrable level $k \in \mathbb{N}$, $|v_{\text{sing}}\rangle = (J_{-1}^+)^{k+1}|0\rangle$, which is the highest weight state of an $\mathfrak{su}(2)$ -representation $\mathcal{R}_{\text{sing}}$ of spin $j = (k+1)$. Similarly, at an admissible level with $k+2 = \frac{u}{v}$, the singular vector $|v_{\text{sing}}\rangle$ has conformal weight $h_{\text{sing}} = (u-1)v$ and the finite $\mathfrak{su}(2)$ weight (Dynkin label) $\lambda_{\text{sing}} = 2u-2$, and is also the highest weight state of an $\mathfrak{su}(2)$ -representation $\mathcal{R}_{\text{sing}}$ of spin $j = (u-1)$ [56,57]. The null state $|\mathcal{N}_T\rangle$ is then given by the universal formula (up to a normalization constant)

$$|\mathcal{N}_T\rangle = K_{a_1 b_1} \dots K_{a_j b_j} J_{-1}^{a_1} \dots J_{-1}^{a_j} |v^{b_1 \dots b_j}\rangle, \quad (22)$$

where $|v^{b_1 \dots b_j}\rangle$ denotes a basis in the $\mathcal{R}_{\text{sing}}$, which is totally symmetric and traceless (with respect to the Killing form) in all the indices, such that $|v^{+\dots+}\rangle = |v_{\text{sing}}\rangle$. From (22), we conclude that

$$h[\mathcal{N}_T] = 2j = \begin{cases} 2(k+1), & k \in \mathbb{N}_{>0} \\ (v+1)(u-1), & k+2 = \frac{u}{v}. \end{cases} \quad (23)$$

For boundary admissible cases, $h[\mathcal{N}_T]/2$ recovers the order of unflavored MLDE that annihilates the unflavored Schur index of (A_1, D_{2n+1}) [25]. Although we don't have proof for uniqueness, (22) is a solution to the $|\mathcal{N}_T\rangle$ conditions. For example, at $k = -\frac{4n}{2n+1}$, $u = 2$, $j = 1$, $|\mathcal{N}_T\rangle$ and $|v^a\rangle$ are given by explicitly

$$\begin{aligned} |\mathcal{N}_T\rangle &= K_{ab} J_{-1}^a |v^b\rangle, & |v^+\rangle &= |v_{\text{sing}}\rangle, \\ |v^1\rangle &= -J_0^- |v_{\text{sing}}\rangle, & |v^-\rangle &= -\frac{1}{2} J_0^- J_0^- |v_{\text{sing}}\rangle. \end{aligned} \quad (24)$$

$|\mathcal{N}_T\rangle$ is obviously $\mathfrak{su}(2)$ invariant. Direct computation using the commutation relations and $J_{n \geq 1}^a |v_{\text{sing}}\rangle = 0$ shows that $h_{n \geq 2}^1 |\mathcal{N}_T\rangle = L_2 |\mathcal{N}_T\rangle = 0$.

Starting from $|\mathcal{N}_T\rangle$, we apply Zhu's recursion to obtain the corresponding flavored $|\mathcal{N}_T\rangle$ equation. Modular transformations on the equation generate additional flavored MLDEs with modular weight $h[\mathcal{N}_T] - 1, \dots, 3, 2$. Apart from the integrable $k = 1$ case, the weight-two equation in the orbit of $\mathcal{N}_T$ equation reproduces the Sugawara equation. This set of $h[\mathcal{N}_T] - 1$ partial differential equations uniquely determines the integrable and admissible spectrum and all flavored characters. For example, there are five flavored MLDEs for the (nonboundary) admissible $\widehat{\mathfrak{su}}(2)_{-1/2}$. By assuming an ansatz of the flavored character $\text{ch} = q^\sigma \sum_{n=0}^{+\infty} a_n(b_1) q^n$, we find [58]

$$\begin{cases} \sigma = +1/24 & a_0(b_1) = 1 \\ \sigma = +13/24 & a_0(b_1) = b_1 + b_1^{-1} \\ \sigma = -1/12, & a_0(b_1) = \frac{c_1 b_1^{-3/2} + c_2 b_1^{-1/2}}{1 - b_1^{-2}} \end{cases}.$$

The remaining coefficients $a_{n>0}(b_1)$ can be recursively solved by applying the five flavored MLDEs. The solutions shown above precisely reproduce the admissible representations with finite highest weights $[0], [1], [-\frac{1}{2}]$, and $[-\frac{3}{2}]$ written in Dynkin labels.

Spectral-flow and twisted sector. For a Kac-Moody algebra $\hat{\mathfrak{g}}_k$, one can define the spectral-flow automorphisms τ_i associated to the simple root $\alpha_i^\vee$, which rearrange the affine generators and the Virasoro generators. The spectral flows map between different modules. At the level of the characters, one may generalize the automorphisms to the following translation:

$$\mathfrak{b}_i \xrightarrow{\sigma} \mathfrak{b}_i + n_i \tau, \quad (25)$$

$$\mathfrak{y} \xrightarrow{\sigma} \mathfrak{y} + \sum_{i,j=1}^r K^{ij} n_j \mathfrak{b}_i + \sum_{i,j=1}^r \frac{1}{2} K^{ij} n_i n_j \tau, \quad (26)$$

$$(\mathbf{n}^\vee, \alpha) \in \frac{1}{2}\mathbb{Z}, \quad \mathbf{n}^\vee := \sum_{i=1}^r n_i \alpha_i^\vee, \quad \forall \alpha \in \Delta. \quad (27)$$

The numbers n_i are spectral-flow parameters, satisfying the integrality condition in the last line. When $\mathfrak{g} = \mathfrak{su}(2)$ and $n_1 = \ell/2$ for $\ell \in \mathbb{Z}$, the translations correspond to the $\ell/2$ -unit spectral flow in [39].

In all cases we have studied, under the spectral-flow with $(\mathbf{n}^\vee, \alpha) \in \mathbb{Z}$, the flavored MLDEs that we derived from the null states transform among themselves. This is equivalent to an effective spectral flow on the null states,

$$|\mathcal{N}\rangle \xrightarrow{\sigma} \sum_{\ell \geq 0} \sum_{i_1 \leq \dots \leq i_\ell} \frac{(-1)^\ell}{\ell!} n_{i_1} \dots n_{i_\ell} h_{-1}^{i_1} \dots h_{-1}^{i_\ell} |\mathcal{N}\rangle. \quad (28)$$

As a result, the solutions to the flavored MLDEs form a representation under the corresponding translation group. In particular, the translations map the vacuum character to other nonvacuum characters. For example, for $n_1 = 1, 2, 3$, the spectral flows map the vacuum character of $\widehat{\mathfrak{su}}(2)_{-1/2}$ to highest weight characters corresponding to $-\frac{1}{2}\omega_1, \omega_1, -\frac{3}{2}\omega_1$.

For n_i satisfying $(\mathbf{n}^\vee, \alpha) \in \mathbb{Z} + \frac{1}{2}$, the flavored MLDEs are mapped to a new set of twisted equations where some of the $E_k[\frac{+1}{b}] \rightarrow E_k[\frac{-1}{b}]$. This set of twisted equations can also be derived using the twisted Zhu's recursion from the null states. In the twisted sector, spectral flow also acts on the nulls, taking the same form as above,

$$|\mathcal{N}\rangle \xrightarrow{\sigma} \sum_{\ell \geq 0} \sum_{i_1 \leq \dots \leq i_\ell} \frac{(-1)^\ell}{\ell!} n_{i_1} \dots n_{i_\ell} h_{-1}^{i_1} \dots h_{-1}^{i_\ell} |\mathcal{N}\rangle. \quad (29)$$

Solutions of this new set of equations also have crucial physical importance: They are simply the characters of the same algebra in the twisted sector. Take $\widehat{\mathfrak{su}}(2)_{-1/2}$ as an example: The five original flavored MLDEs of modular weight $6, 5, \dots, 2$ now turn into five twisted equations of equal weights. There are precisely four half-integral q -series solutions $\text{ch} = q^\sigma \sum_{n=0}^{+\infty} c_n(b_1) q^{n/2}$, corresponding to the characters of the spectral flowed modules $\sigma^{-1/2} M_0$, $\sigma^{-1/2} M_{-\frac{1}{2}\omega_1}$, $\sigma^{-1/2} M_{\omega_1}$, and $\sigma^{-1/2} M_{-\frac{3}{2}\omega_1}$. This is the complete twisted spectrum of $\widehat{\mathfrak{su}}(2)_{-1/2}$. By solving the equations, we show that all four twisted modules are ordinary, although M_0 and M_{ω_1} are the only nontwisted ordinary modules. In fact, from the form of twisted flavored MLDEs, we easily conclude that all twisted modules of $\widehat{\mathfrak{su}}(2)_k$ are ordinary since the coefficients of all twisted equations are holomorphic at $b_1 \rightarrow 1$, which is compatible with the results in [39].

Other algebras. We believe our approach can be generalized to algebras with a specified set of generators beyond $\hat{\mathfrak{g}}_k$. For example, consider the generators $G_\pm, \tilde{G}_\pm, J^a$ of the 2d $\mathcal{N} = 4$ small superconformal algebra with unknown $SU(2)$ level k . A weight-four $|\mathcal{N}_T\rangle$ satisfying the generalized $|\mathcal{N}_T\rangle$ -condition $L_2 |\mathcal{N}_T\rangle = h_{n \geq 2} |\mathcal{N}_T\rangle = G_{5/2} |\mathcal{N}_T\rangle = \tilde{G}_{5/2} |\mathcal{N}_T\rangle = 0$ fixes $k = -3/2$, giving the algebra associated to the 4D $\mathcal{N} = 4$ $SU(2)$ theory. The null state then gives rise to a modular orbit of equations, from which we solve the full spectrum of the algebra [59].

Conclusion. In this paper, we propose a generalization of the holomorphic modular bootstrap by introducing flavor refinement to the modular differential equations. The generalization reveals hidden quasimodular/spectral-flow structure of the flavored MLDEs and the associated null states, which turns out to be a power tool to construct and analyze the spectrum in both the untwisted and twisted sector. We also propose alternative characterization and closed-form expression for the null state $|\mathcal{N}_T\rangle$.

It would be interesting to apply the proposal to more chiral algebras. For example, the bootstrap for the Kac-Moody algebra $\hat{\mathfrak{g}}_k$ may reveal the candidates of quasilisse algebras. For other types of algebras with different set of generators, one needs to modify the $|\mathcal{N}_T\rangle$ condition to accommodate other generators in order to initiate the bootstrap; it is natural to ask what the condition should be in general. Besides the usual chiral algebras, there are “intermediate vertex operator algebras” that are explored recently [44,60], which could be investigated from our bootstrap point of view. For example, the coset construction [61] $(E_{7+1/2})_1 = (E_8)_1 / (LY)_1$ gives rise to flavored characters of the well-known unitary

Lee-Yang model [62], and it would be interesting to verify if such algebras can be constrained by quasimodular bootstrap.

The quasimodularity is closely tied to the modular structure of the untwisted and twisted spectrum of the chiral algebra $\mathbb{V}[\mathcal{T}]$ associated to a 4D theory $\mathcal{T}$. The modular structure is believed to map to the fixed-point geometry of the Coulomb branch moduli space $\mathcal{M}[\mathcal{T}]$ via the so-called 4D mirror symmetry [22,23,63]. Therefore, it is intriguing to ask if there is a corresponding geometric structure or constraint that corresponds to

the flavored MLDEs or null states on the chiral algebra side.

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