

Novel topological classes in black hole thermodynamics

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By viewing black hole solutions as topological defects in thermodynamic parameter space, we unveil a novel topological class and two new topological subclasses, respectively, denoted as $W^{0\leftrightarrow 1+}$, $\bar{W}^{1+}$, and $\hat{W}^{1+}$, that extend beyond the four established categories proposed by Wei *et al.* [*Phys. Rev. D* **110**, L081501 (2024)]. Within the newly identified class and these two novel subclasses, the innermost small black hole states exhibit a distinct sequence of unstable, stable, and stable behaviors, while the outermost large black hole states display a uniform pattern of stable behaviors. These classifications indicate thermodynamic properties both in the low and high Hawking temperature regimes that are strikingly different from the previously known four topological classes. In particular, we demonstrate that the static charged anti-de Sitter black holes in gauged supergravity exhibit an intricate thermodynamic evolution that is notably distinct from that of the Reissner-Nordström anti-de Sitter black hole. From a topological perspective, we emphasize the advantages and potential of investigating thermodynamic phase transitions in these black hole spacetimes, an area that has been rarely explored in the previous research. Our findings not only enrich and sharpen the framework of topological classifications in black hole thermodynamics but also represent a significant stride toward unraveling the fundamental nature of black holes and gravity.

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Introduction. Black holes serve as a crucial testing ground for quantum gravity theories, stemming from the discovery that they exhibit thermal properties, with entropy proportional to the area of their event horizons [1,2]. Over the past two decades, our understanding of black hole mechanics has expanded to include concepts of pressure and volume [3–7]. This area, commonly referred to as black hole chemistry [8], has provided new insights into gravitational phase transitions [9–12] and entropy bounds [13–15], with a holographic interpretation now becoming established that includes heat engines [16], complexity [17], the significance of the central charge [18–24], and its origins in higher dimensions [25].

Although there has been significant progress in recent years, exploring the universal properties of black hole

thermodynamics continues to be a complex undertaking. Recent developments suggest that topology can shed new light on addressing this issue [26–28] by considering black hole solutions as topological defects in the thermodynamic parameter space. Initially, such defects were classified into three categories based on their different topological numbers [26]. This approach was later refined in Ref. [27], categorizing black hole solutions into four more general classes based on their thermodynamic asymptotic behaviors. For more examples of the latest representative developments, see Refs. [29–38]. Furthermore, the topology of thermodynamics and phase transition for certain anti-de Sitter (AdS) black holes have been studied in Refs. [39–42] in the context of bulk, mixed bulk/boundary, and conformal field theory (CFT) thermodynamics by using different holographic dictionaries for the dual CFT. In particular, a residue method was adopted in [39] to study the topological properties of the phase transitions of charged AdS₄ black holes and showed that the bulk and boundary thermodynamics are topologically equivalent for both criticality and first-order phase transition in the canonical ensembles. On the other hand, Ref. [41] showed that a Born-Infeld-AdS black hole and its dual CFT share the identical topology, so there is a parallel transition in the

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dual CFT corresponding to the topological transition that occurs in the bulk.

However the thermodynamic topological classification of black hole solutions into four distinct classes, as proposed in Ref. [27], does not cover all black hole solutions. For example, because of the novel temperature-dependent thermodynamic topological phase transition occurring in the four-dimensional static charged AdS black hole of the Einstein-Maxwell-dilaton-axion (EMDA) gauged supergravity theory and the four-dimensional static two-charge AdS black hole, as well as the five-dimensional static charged AdS black hole within the Kaluza-Klein (K-K) gauged supergravity theory [43], these three black hole solutions cannot be included into any of the existing four topological classes. In addition, the dyonic AdS black hole [44] exhibits certain differences from the existing four topological classes, making it unsuitable for inclusion into these categories.

Therefore, a natural question arises as to whether new topological classes (or subclasses) exist to accommodate these situations. In this Letter, our findings indicate the existence of one novel topological class and two new topological subclasses beyond the four known classes. In addition, we aim to show that static charged AdS black holes in gauged supergravity, compared to the Reissner-Nordström (RN)-AdS black hole, display more distinct and intricate thermodynamic evolution. We also highlight the considerable advantages and potential of investigating thermodynamic phase transitions in static charged AdS black holes within gauged supergravity theories from a topological viewpoint, an area that remains largely unexplored in current research.

The topological properties of black hole thermodynamics fundamentally depend on the generalized off-shell free energy [45]. In this approach, a black hole with mass M and entropy S is enclosed in a cavity at a fixed temperature $1/\tau$, leading to a generalized free energy $\mathcal{F} = M - S/\tau$, which can also be derived from the gravitational path integral [46]. The free energy reduces to an on-shell quantity only when $\tau = \beta = 1/T$, where T is the Hawking temperature of the black hole. By introducing an additional parameter $\Theta \in (0, \pi)$, a two-component vector field can be defined via

$$\phi = (\phi^{r_h}, \phi^\Theta) = \left(\frac{\partial \mathcal{F}}{\partial r_h}, -\frac{\cos \Theta}{\sin^2 \Theta} \right). \quad (1)$$

The condition $\phi^{r_h} = 0$ identifies black hole states as zero points (or defects) of the vector field. Utilizing Duan's ϕ -mapping topological current theory [47], each zero point or black hole state is then assigned with a topological charge, known as the winding number w [26].

Within this framework, from a local perspective, black hole states that are locally stable have a positive local topological number $w = +1$, while locally unstable states are characterized by a negative local topological number

$w = -1$. Furthermore, for a given class of black holes, the sum of all local topological numbers w_i yields the global topological number W . Therefore, from a global perspective, black holes can be topologically classified according to their global topological number W .

Four known topological classes. Let us start with a brief review of the four known thermodynamic topological classes previously specified in [27]

$$W^{1-}, \quad W^{0+}, \quad W^{0-}, \quad W^{1+}, \quad (2)$$

which correspond to the following asymptotic behaviors of the inverse temperature $\beta(r_h)$:

$$W^{1-}: \beta(r_m) = 0, \quad \beta(\infty) = \infty, \quad (3)$$

$$W^{0+}: \beta(r_m) = \infty, \quad \beta(\infty) = \infty, \quad (4)$$

$$W^{0-}: \beta(r_m) = 0, \quad \beta(\infty) = 0, \quad (5)$$

$$W^{1+}: \beta(r_m) = \infty, \quad \beta(\infty) = 0, \quad (6)$$

where r_m is the minimal radius of the black hole event horizon, which may or may not vanish. For instance, a RN black hole with a fixed charge Q satisfies $r_m = M = Q = r_e$ in the extremal case, while for a Schwarzschild black hole, r_m vanishes. Table II in Ref. [27] and Fig. 1 below provide a comprehensive summary of the properties of the four known

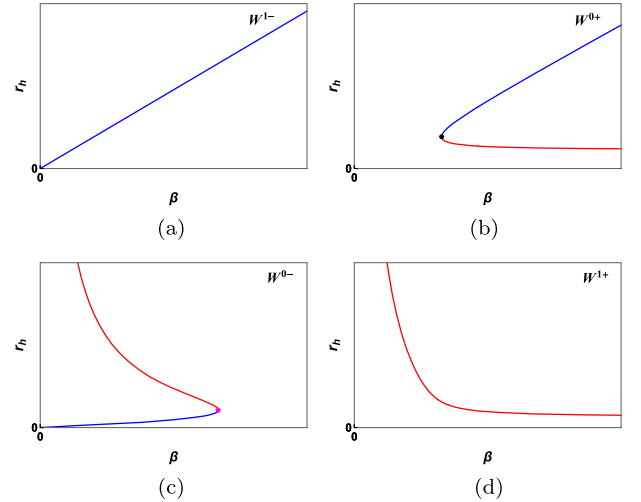


FIG. 1. The zero points of ϕ^{r_h} are shown in the $r_h - \beta$ plane for W^{1-} , W^{0+} , W^{0-} , W^{1+} classes, respectively. The blue line corresponds to a thermodynamically unstable black hole branch with $w = -1$, while the red line corresponds to a thermodynamically stable black hole branch with $w = 1$. The black dot represents the generation point (GP) within the degenerate point (DP), and the pink dot represents the annihilation point (AP) within the DP. (a) Typical case: Schwarzschild black hole. (b) Typical case: RN black hole. (c) Typical case: Schwarzschild-AdS black hole. (d) Typical case: RN-AdS black hole.

topological classes, facilitating a better understanding of their characteristics.

One novel topological class. Recently we observed [43] a novel temperature-dependent thermodynamic topological phase transition occurring in the four-dimensional static charged AdS black hole within the EMDA gauged supergravity theory, and the four-dimensional static two-charge AdS black hole, as well as the five-dimensional static charged AdS black hole within the K-K gauged supergravity theory.

We will take the four-dimensional static two-charge AdS black hole as an example to construct a new classification that includes black hole solutions exhibiting temperature-dependent thermodynamic topological phase transitions, representing the three similar cases. The metric, Abelian gauge potentials, scalar fields, and thermodynamic quantities of the four-dimensional static two-charge AdS black hole are given in Eqs. (A1) and (A2) in Appendix A.

Consider first the case in which the four electric charge parameters are set to $q_1 \neq q_2 \neq 0$. For a fixed electric charge parameter q_2 and a fixed pressure P , when $q_1 < q_{1c} = 3/(8\pi P q_2)$, the topological number of the four-dimensional static two-charge AdS black hole is either $W = 0$ (at cold temperatures) or $W = 1$ (at high temperatures), exhibited in Fig. 2. It is straightforward to obtain

$$\beta(r_m) = \text{fixed temperature}, \quad \beta(\infty) = 0 \quad (7)$$

for the asymptotic behavior of the inverse temperature $\beta(r_h)$ of this new topological class.

Next, we analyze the systematic ordering for the new topological class. There is at least one black hole state with negative heat capacity and a winding number with -1 , and one black hole state with positive heat capacity and a winding number with $+1$. If additional black hole states are present, then they must emerge in pairs. As the signs of the heat capacities alternate with increasing r_h , the smallest state corresponds to a thermodynamically unstable black hole, while the largest state corresponds to a thermodynamically stable one. The winding numbers associated with the zero points follow the sequence $[-, (+, -), \dots, +]$, where the ellipsis represents pairs of $(-, +)$ winding numbers. For simplicity, this novel topological class can be labeled as $[-, +]$ based upon the signs of the innermost and outermost winding numbers.

Turning to the asymptotic thermodynamic behavior of this new topological class, in the low-temperature limit, $\beta \rightarrow \infty$, no black hole state is present. At the high-temperature limit, $\beta \rightarrow 0$, the system features a stable large black hole. On the base of its unique thermodynamic behavior and following the consistent naming convention from small black hole states to large black hole states, we will name this novel topological classification:

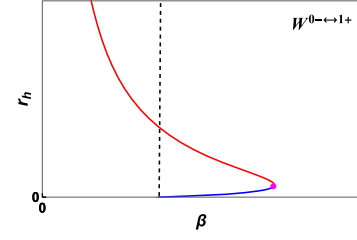


FIG. 2. The zero points of ϕ^{r_h} are shown in the $r_h - \beta$ plane for the novel topological class W^{0-+-+} . A typical example in this category is the four-dimensional static two-charge AdS black hole when $q_1 < q_{1c} = 3/(8\pi P q_2)$, where temperature-dependent thermodynamic topological phase transitions occur, and an annihilation point (pink dot) is displayed. Red (blue) curves denote stable (unstable) branches.

$$W^{0-+-+} \quad (8)$$

indicating that black hole solutions belonging to this classification display thermodynamic topological phase transitions.

Last but not least, let us revisit the unstable black hole branch shown in Fig. 2 (depicted by the blue line) to examine its distinctive thermodynamic behavior, which cannot be observed in the RN-AdS black hole. Thermal instability typically implies that a black hole has a negative heat capacity, implying that as the black hole loses energy, its temperature increases. This negative heat capacity effect renders the black hole unstable under energy absorption or emission. A static charged AdS black hole in gauged supergravity theory exhibiting thermal instability at higher temperatures may evolve along this potential path: at elevated temperatures, the black hole radiation intensifies, potentially leading to a rapid loss of mass and charge, accelerating the evaporation process. If both mass and charge deplete swiftly, then the system may evolve into a horizonless, charged AdS spacetime. During this process, as the black hole mass decreases, its temperature rises further due to the negative heat capacity, enhancing the rate of charge and energy emission. This positive feedback mechanism accelerates the evaporation of the black hole, leading to the disappearance of its event horizon. Note that in the horizonless, charged AdS spacetime, the electric charge typically is manifest as a far-field effect, predominantly concentrated at the boundary of the spacetime or distributed via the electric field to large distances.

Two new topological subclasses. We now describe two new topological subclasses using two examples. The first is the four-dimensional static two-charge AdS black hole when $q_1 \geq q_{1c} = 3/(8\pi P q_2)$. Its topological number is $W = 1$, and its thermodynamic behavior is shown in Fig. 3. Similar thermodynamic behavior can also be found in the four-dimensional static charged AdS black hole in the EMDA gauged supergravity for $q \geq \sqrt{6}/(4\sqrt{\pi P})$ and the five-dimensional static charged AdS black hole in the K-K gauged supergravity for $q \geq 3/(4\pi P)$ [43]. By comparing

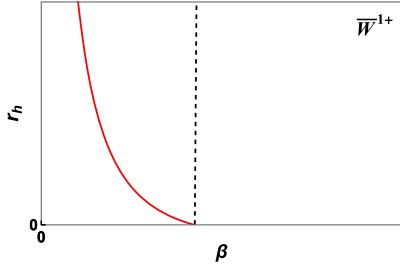


FIG. 3. The zero points of ϕ^{r_h} are shown in the $r_h - \beta$ plane for the first new topological subclass $\bar{W}^{1+}$, which has the topological number $W = 1$. A typical example in this category is the four-dimensional static two-charge AdS black hole when $q_1 \geq q_{1c} = 3/(8\pi P q_2)$.

Figs. 3 and 1(d), it is evident that their thermodynamic behaviors are significantly different. However, since both share the same topological number of one, this should represent a new topological subclass. The asymptotic behavior of the inverse temperature $\beta(r_h)$ for this new topological subclass is the same one as that shown in Eq. (7), but the value of the fixed temperature may be different. This novel topological subclass can be labeled as $[+, +]$ due to the signs of the innermost and outermost winding numbers. In the high-temperature limit, $\beta \rightarrow 0$, the system exhibits a stable large black hole state, which is consistent with that of the W^{1+} class. However, in the low-temperature limit, $\beta \rightarrow \infty$, no black hole state exists, distinguishing it from the W^{1+} class. We thus label this new topological subclass as $\bar{W}^{1+}$.

Our second example is the four-dimensional dyonic AdS black hole in Refs. [44,48]. Its metric and thermodynamic quantities are displayed by Eqs. (B2) and (B3) in Appendix B. Its topological number is $W = 1$, and its thermodynamic behavior is illustrated in Fig. 4. A comparison of Fig. 4 with Figs. 3 and 1(d) reveals that, despite that all of them have a topological number of 1, their thermodynamic behaviors are distinct, indicating a second new topological subclass, which we will designate as $\hat{W}^{1+}$.

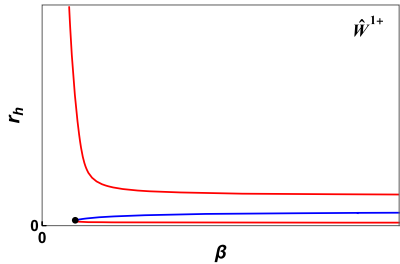


FIG. 4. The zero points of ϕ^{r_h} are shown in the $r_h - \beta$ plane for the second new topological subclass $\hat{W}^{1+}$, which has the topological number $W = 1$. A typical example in this category is the four-dimensional dyonic AdS black hole. The black dot denotes a generation point. Red (blue) curves denote stable (unstable) branches.

The asymptotic behavior of the inverse temperature $\beta(r_h)$ aligns with the pattern presented in Eq. (6), and we can label this subclass $[+, +]$ according to the signs of its innermost and outermost winding numbers. In the high-temperature limit, $\beta \rightarrow 0$, the system exhibits a stable large black hole state, paralleling the behavior of the W^{1+} and $\bar{W}^{1+}$ classes. However, in the low-temperature limit, $\beta \rightarrow \infty$, it features two stable small black hole states and one unstable small black hole state, which sets it apart from the W^{1+} and $\bar{W}^{1+}$ classes.

Holographic dual CFT side. Through the holographic dictionary [22]: $E = M/\omega$, $\tilde{T} = T/\omega$, and $\tilde{S} = S$, one can define the free energy for a canonical ensemble in the dual boundary CFT: $\tilde{\mathcal{F}} = E - \tilde{S}/\tilde{\tau} = \mathcal{F}/\omega$, where the Euclidean period time in the dual boundary theory is related to that in the bulk via $\tilde{\tau} = \omega\tau$ by virtue of the above dictionary, and $\omega = \mathcal{R}/l$ is an arbitrary conformal factor of the boundary AdS metric, in which $\mathcal{R}$ is the boundary curvature radius, which is different from the bulk curvature radius l . Clearly, one can reasonably assume that the boundary conformal factor should be independent of the horizon radius, that is, $\partial\omega/\partial r_h = 0$. Then similarly to the bulk case, one can define a two-component vector field $\tilde{\phi}$ in the boundary CFT with its radial component being $\tilde{\phi}^{r_h} = \phi^{r_h}/\omega$ that only reduces or enlarges the size of ϕ^{r_h} . This implies that the orientations of the vector fields both in the bulk and in the boundary theories are coincident, which leads us to draw a *universal* conclusion that the topology of boundary CFT thermodynamics shares the same one of the bulk thermodynamics in the sense of canonical ensembles. This would be an very expected and promising result in that it reflects the gravity-gauge duality in terms of topology, which depicts the global character of the geometry. Otherwise, it would be rather odd and unappreciated.

Conclusions and discussions. We have introduced a new thermodynamic topological class $W^{0 \leftrightarrow 1+}$ and two novel subclasses $\bar{W}^{1+}$, and $\hat{W}^{1+}$ that extend beyond the four previously established categories [27]. For each new topological (sub)class, we analyzed its thermodynamic behavior in both the small and large black hole limits, as well as in the low and high temperature limits. A concise overview of these results is presented in Table I. Our results emphasize the significance and potential of investigating the thermodynamic phase transition behavior of static charged AdS black holes in gauged supergravity theories, whose thermodynamic behavior is far richer than the corresponding RN-AdS case.

In summary, our findings highlight both the necessity and great potential of studying the thermodynamic phase transition behavior of static charged AdS black holes in gauged supergravity theories, though we have just grasped a fraction of these insights from the perspective of thermodynamic topology. More generally, we have enhanced and

TABLE I. Thermodynamical properties of the black hole states for the one novel topological class and two new topological subclasses of $W^{0 \leftrightarrow 1+}$, $\bar{W}^{1+}$, and $\hat{W}^{1+}$, respectively.

Classes	Innermost	Outermost	Low T ($\beta \rightarrow \infty$)	High T ($\beta \rightarrow 0$)	DP	W
$W^{0 \leftrightarrow 1+}$	Unstable	Stable	No	Stable large	One More AP	0 or +1
$\bar{W}^{1+}$	Stable	Stable	No	Stable large	In pairs	+1
$\hat{W}^{1+}$	Stable	Stable	Unstable small + two stable small	Stable large	One More GP	+1

refined the framework of topological classifications in black hole thermodynamics, while also marking a significant step toward uncovering the fundamental nature of black holes and gravity.

Our results also raise an interesting question: Are there any new topological classes or subclasses yet to be identified? By examining the symmetry of functions (r_h, β) , one can conjecture there might still exist another three new thermodynamic topological (sub)classes based upon the three novel (sub)classes we have supplemented, although it remains unclear whether their corresponding black hole solutions actually exist or not, which needs to be verified in future research. These possible new (sub)classes are listed as follows:

$$W^{0 \leftrightarrow 1-}, \quad \bar{W}^{1-}, \quad \hat{W}^{1-}, \quad (9)$$

which are aligned with the asymptotic behaviors of the inverse temperature $\beta(r_h)$:

$$\begin{aligned} W^{0 \leftrightarrow 1-}: \beta(r_m) &= \text{fixed temperature}, & \beta(\infty) &= \infty, \\ \bar{W}^{1-}: \beta(r_m) &= \text{fixed temperature}, & \beta(\infty) &= \infty, \\ \hat{W}^{1-}: \beta(r_m) &= 0, & \beta(\infty) &= \infty. \end{aligned}$$

We have also provided a schematic diagram in Fig. 5 and summarized in Table II the thermodynamical properties of the three potentially existing topological (sub)classes of $W^{0 \leftrightarrow 1-}$, $\bar{W}^{1-}$, and $\hat{W}^{1-}$, respectively. Certainly, further exploration is necessary to verify the potential existence of the corresponding black hole solutions. Perhaps we have only touched the tip of the iceberg: these and many other questions remain intriguing subjects for future investigation.

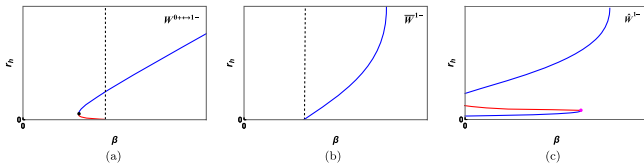


FIG. 5. The schematic diagram of the zero points of ϕ^{r_h} is shown in the $r_h - \beta$ plane for the possible new topological (sub)classes: (a) $W^{0 \leftrightarrow 1-}$, (b) $\bar{W}^{1-}$, and (c) $\hat{W}^{1-}$.

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Data availability. No data were created or analyzed in this study.

Appendix A: $D = 4$ static four-charge AdS black hole in STU gauged supergravity. For the general static four-charge AdS black hole in four-dimensional STU gauged supergravity, the metric, the Abelian gauge potentials, and the scalar fields are given by [49]

$$\begin{aligned} ds^2 &= - \prod_{i=1}^4 H_i^{-1/2} f dt^2 + \prod_{i=1}^4 H_i^{1/2} (f^{-1} dr^2 + r^2 d\Omega_2^2), \\ A^i &= \frac{\sqrt{q_i(q_i + 2m)}}{2(r + q_i)} dt, \quad X_i = H_i^{-1} \prod_{j=1}^4 H_j^{1/4}, \end{aligned} \quad (A1)$$

where

$$f = 1 - \frac{2m}{r} + \frac{r^2}{l^2} \prod_{i=1}^4 H_i, \quad H_i = 1 + \frac{q_i}{r},$$

in which l is the AdS radius, m and q_i are the mass and four independent electric charge parameters, respectively.

For the static charged AdS black hole metric described by Eq. (A1), the most general solution features four independent electric charge parameters. In addition, based upon the classification of black hole solutions shown in Fig. 1 of Ref. [50], there are several notable truncated supergravity solutions. For instance, when $q_1 \neq q_2 \neq 0$, $q_3 = q_4 = 0$, this corresponds to the four-dimensional static two-charge AdS black hole solution [51]. When $q_1 = q_2 \neq 0$ and $q_3 = q_4 = 0$, it describes the static charged

TABLE II. Thermodynamical properties of the black hole states for the possible new topological (sub)classes of $W^{0+\leftrightarrow 1-}$, $\bar{W}^{1-}$, and $\hat{W}^{1-}$, respectively.

Classes	Innermost	Outermost	Low T ($\beta \rightarrow \infty$)	High T ($\beta \rightarrow 0$)	DP	W
$W^{0+\leftrightarrow 1-}$	Stable	Unstable	Unstable large	No	One More GP	0 or -1
$\bar{W}^{1-}$	Unstable	Unstable	Unstable large	No	In pairs	-1
$\hat{W}^{1-}$	Unstable	Unstable	Unstable large	Stable small + two unstable small	One More AP	-1

AdS black hole solution in the EMDA gauged supergravity theory [52,53]. In addition, when $q_1 = q_2 = q_3 = q_4 \neq 0$, the solution reduces to the familiar RN-AdS black hole after a coordinate transformation $\rho = r + q_i$, etc.

The thermodynamic quantities are given by [7]

$$\begin{aligned}
M &= m + \frac{1}{4} \sum_{i=1}^4 q_i, & Q_i &= \frac{1}{2} \sqrt{q_i(q_i + 2m)}, \\
S &= \pi \prod_{i=1}^4 (r_h + q_i)^{1/2}, & T &= \frac{f'(r_h)}{4\pi} \prod_{i=1}^4 H_i^{-1/2}(r_h), \\
\Phi_i &= \frac{\sqrt{q_i(q_i + 2m)}}{2(r_h + q_i)}, & P &= \frac{3}{8\pi l^2}, \\
V &= \frac{\pi r_h^3}{3} \prod_{i=1}^4 H_i(r_h) \sum_{j=1}^4 \frac{1}{H_j(r_h)}. \quad (A2)
\end{aligned}$$

It is easy to verify that these thermodynamic quantities simultaneously obey the first law and the Bekenstein-Smarr mass formula

$$dM = TdS + \sum_{i=1}^4 \Phi_i dQ_i + VdP, \quad (A3)$$

$$M = 2TS + \sum_{i=1}^4 \Phi_i Q_i - 2VP. \quad (A4)$$

Appendix B: $D = 4$ dyonic AdS black hole. In four dimensions, the Lagrangian of dyonic AdS black holes can be expressed as [48]

$$\mathcal{L} = \sqrt{-g} \{ R - 2\Lambda - F^2 - \alpha[(F^2)^2 - 2F^{(4)}] \}, \quad (B1)$$

where the standard Maxwell Lagrangian and quasitopological electromagnetism are defined, respectively, via $F^2 = -F^\mu_\nu F^\nu_\mu$, $F^{(4)} = F^\mu_\nu F^\nu_\rho F^\rho_\sigma F^\sigma_\mu$, and the Faraday-Maxwell tensor reads $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with A_μ being the Abelian gauge potential. In the above,

Λ denotes the cosmological constant, the coupling constant $\alpha_2 = \alpha$ has the dimension of [length]². (Note that here we have set $\alpha_1 = 1$.)

The black hole solution in the static and spherically symmetric spacetime background has been obtained by [48]

$$\begin{aligned}
ds^2 &= -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \\
f(r) &= 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + \frac{p^2}{r^2} + \frac{q^2}{r^2} {}_2F_1\left[\frac{1}{4}, 1; \frac{5}{4}; -\frac{4\alpha p^2}{r^4}\right], \quad (B2)
\end{aligned}$$

where p and q are the electric charge and magnetic charge parameters, respectively.

The thermodynamic quantities are [44,48]

$$\begin{aligned}
M &= \frac{r_h^2 + p^2}{2r_h} + \frac{4\pi}{3} P r_h^3 + \frac{q^2}{2r_h} {}_2F_1\left[\frac{1}{4}, 1; \frac{5}{4}; -\frac{4\alpha p^2}{r_h^4}\right], \\
Q_e &= q, & Q_m &= p, & S &= \pi r_h^2, & P &= -\frac{\Lambda}{8\pi}, \\
T &= \frac{r_h^2 - p^2}{4\pi r_h^3} + 2P r_h - \frac{q^2 r_h}{4\pi(r_h^4 + 4\alpha p^2)}, \\
\Phi_e &= \frac{q}{r_h} {}_2F_1\left[\frac{1}{4}, 1; \frac{5}{4}; -\frac{4\alpha p^2}{r_h^4}\right], & V &= \frac{4\pi}{3} r_h^3, \\
\Phi_m &= \frac{p}{r_h} + \frac{q^2 r_h^3}{4p(r_h^4 + 4\alpha p^2)} \\
&\quad - \frac{q^2}{4p r_h} {}_2F_1\left[\frac{1}{4}, 1; \frac{5}{4}; -\frac{4\alpha p^2}{r_h^4}\right], \quad (B3)
\end{aligned}$$

and have been shown to satisfy the first law

$$dM = TdS + \Phi_e dQ_e + \Phi_m dQ_m + VdP \quad (B4)$$

for fixed α . It is also possible to verify the Smarr relation by treating α as a thermodynamic variable and finding its conjugate potential.

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