

Unitarity bounds with subthreshold and anomalous cuts for b -hadron decays

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We derive a generalization of the Boyd-Grinstein-Lebed (BGL) parametrization. Most form factors (FFs) in b -hadron decays exhibit additional branch cuts—namely subthreshold and anomalous branch cuts—beyond the “standard” unitarity cut. These additional cuts cannot be adequately accounted for by the BGL parametrization. For instance, these cuts arise in the FFs for $B \rightarrow D^{(*)}$, $B \rightarrow K^{(*)}$, and $\Lambda_b \rightarrow \Lambda$ processes, which are particularly relevant from a phenomenological standpoint. We demonstrate how to parametrize such FFs and derive unitarity bounds in the presence of subthreshold and/or anomalous branch cuts. Our work paves the way for a wide range of new FF analyses based solely on first principles, thereby minimizing systematic uncertainties.

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State of the art. The thorough study of semileptonic meson decays over the past few decades has led to strong constraints on physics beyond the Standard Model (SM). These constraints are obtained by combining high-precision measurements and theoretical predictions. To enhance the indirect searches for new physics, it is therefore essential to further reduce the theoretical uncertainties. This is particularly important in view of the LHC Run 3 and Belle II programs, which will collect an unprecedented amount of data in the coming years, significantly improving the experimental precision of many observables.

For definiteness, we consider semileptonic B meson decays: $B \rightarrow M\ell_1\ell_2$, where M is a meson and $\ell_{1,2}$ are leptons. Nevertheless, most of the results derived in this letter rely on analyticity and unitarity, making them applicable to other hadron decays.

The primary challenge in obtaining accurate predictions for semileptonic decays lies in calculating the hadronic form factors (FFs), which are scalar functions of the momentum transfer squared q^2 . We denote a specific FF as f_λ , where λ is a label used to distinguish between different FFs. FFs are extremely difficult to calculate because they incorporate nonperturbative quantum chromodynamics (QCD) effects. As a matter of fact, most FFs are only known at a few isolated q^2 points, calculated using lattice QCD [1] (or light-cone sum rules [2–5]). It is

therefore necessary to extrapolate or interpolate between the known q^2 points using a parametrization to obtain predictions of FFs and hence observables in the whole semileptonic region—i.e. for $q^2 \in [(m_{\ell_1} + m_{\ell_2})^2, s_-]$ with $s_- \equiv (m_B - m_M)^2$. Parametrizations are normally formulated in terms of power series of q^2 (or a related variable). Since in practice only a finite number of parameters can be determined, a method of estimating the truncation error of such series is essential to correctly assess the theoretical uncertainties.

It has been shown that the Boyd-Grinstein-Lebed (BGL) parametrization [6,7]

$$f_\lambda(q^2) = \frac{1}{\mathcal{B}_\lambda(q^2)\phi_\lambda(q^2)} \sum_{n=0}^{\infty} a_{\lambda,n} z^n(q^2, s_+), \quad (1)$$

satisfies a unitarity bound (see below). Here, we have introduced the conformal mapping

$$z(q^2, s_+) \equiv z(q^2, s_+, s_0) = \frac{\sqrt{s_+ - q^2} - \sqrt{s_+ - s_0}}{\sqrt{s_+ - q^2} + \sqrt{s_+ - s_0}}, \quad (2)$$

which maps the complex domain $\mathbb{C} \setminus [s_+, \infty)$, with $s_+ \equiv (m_B + m_M)^2$, in the complex q^2 -plane onto the open unit disk $|z| < 1$ in the complex z -plane (see Fig. 1). The parameter s_0 can be freely chosen within the interval $(-\infty, s_+)$, determining the point on the q^2 -plane that maps to the origin of the z -plane. In Eq. (1), the Blaschke product $\mathcal{B}_\lambda$ has zeros at the isolated poles of f_λ , while the outer function ϕ_λ is chosen to be holomorphic with no zeros in $|z| < 1$, ensuring that the bound simplifies to the form

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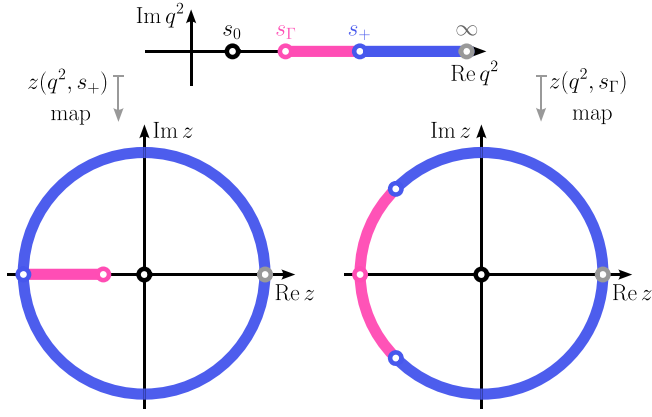


FIG. 1. Illustration of the $z(q^2, s_+)$ and $z(q^2, s_\Gamma)$ mappings. The magenta line represents the subthreshold cut, while the blue line represents the standard branch cut starting at s_+ .

given below in Eq. (4). The analytic expressions of $\mathcal{B}_\lambda$ and ϕ_λ for specific processes can be found in, e.g., Refs. [7,8].

The bound comes from the fact that, using analyticity and unitarity, it is possible to calculate an upper value for the following integral [7]:

$$\frac{1}{\pi} \int_{s_+}^{\infty} dq^2 \left| \frac{dz(q^2, s_+)}{dq^2} \right| |\phi_\lambda f_\lambda|^2 = \frac{1}{2i\pi} \oint_{|z|=1} \frac{dz}{z} |\phi_\lambda f_\lambda|^2 < 1. \quad (3)$$

Here, we have omitted the arguments of ϕ_λ and f_λ for brevity and used the fact that $|\mathcal{B}_\lambda| = 1$ on the unit circle. Since the z^n form a complete set of orthonormal polynomials on the unit circle, it follows directly from Eqs. (1) and (3) that

$$\sum_{n=0}^{\infty} |a_{\lambda,n}|^2 < 1. \quad (4)$$

This inequality is called the *unitarity bound*.

The bound (4) provides a systematic method for quantifying the truncation error of the series in Eq. (1), as it is evident that $|z^n| \rightarrow 0: |z| < 1$ for $n \rightarrow \infty$. However, the BGL parametrization is only valid for FFs with a branch cut along the real axis from the branch point s_+ to infinity. This is e.g. the case for the FFs in $B \rightarrow \pi \ell \bar{\nu}$. When *subthreshold branch cuts* appear—i.e., cuts on the real axis starting at $q^2 = s_\Gamma < s_+$ —the BGL parametrization no longer applies. This is, e.g., the case for the FFs in $B \rightarrow D^{(*)} \ell^+ \ell^-$ and the (local and nonlocal) FFs in $B \rightarrow K^{(*)} \ell^+ \ell^-$. Here, the mapping (2) introduces a branch cut inside the unit disk between $z(s_\Gamma, s_+)$ and $z(s_+, s_+)$ as shown in Fig. 1. Consequently, Eq. (3) [and thus the bound (4)] cannot be used, since the radius of convergence of the series in Eq. (1) is less than one.

Recently, an alternative parametrization has been introduced in Ref. [9], offering a partial solution to this issue. This parametrization uses the mapping $z(q^2, s_\Gamma)$ instead of $z(q^2, s_+)$, with the FFs expanded in terms of polynomials p_n in z that are orthonormal on an arc of the unit circle¹:

$$f_\lambda(q^2) = \frac{1}{\mathcal{B}_\lambda(q^2) \phi_\lambda(q^2)} \sum_{n=0}^{\infty} b_{\lambda,n} p_n(z(q^2, s_\Gamma)). \quad (5)$$

The explicit expression of $p_{0,1,2}$ is given in Ref. [9]. The use of these polynomials is imposed by the fact that in this case the dz integral in Eq. (3) does not extend over the entire circle, but only over the arc that goes from $e^{-i \arg(z(s_+, s_\Gamma))}$ to $e^{+i \arg(z(s_+, s_\Gamma))}$ due to the different mapping (blue arc in the right panel of Fig. 1). Although the coefficients $b_{\lambda,n}$ satisfy a unitarity bound analogous to that in Eq. (4), the absolute value of p_n diverges exponentially as $n \rightarrow \infty$ for certain values of z within the unit disk. Therefore, the parametrization (5) is *not* unitarity-bounded, as even a very small coefficient $b_{\lambda,n}$ can produce a large contribution when n is large. In addition, since FFs have a branch point at s_+ , the coefficients $b_{\lambda,n}$ will only decay algebraically with respect to n [i.e., there exists $C > 0$ such that $|b_{\lambda,n}| = o(n^{-C})$ as $n \rightarrow \infty$]. Thus, Eq. (5) will not in general converge for q^2 such that $q^2 \notin [s_\Gamma, s_+]$. For a detailed treatment of the convergence of orthogonal polynomials, we refer the reader to Ref. [11].

Another challenge in FF parametrizations comes from *anomalous branch cuts*. These cuts do not correspond to any physical particle production channel and appear in certain (nonlocal) FFs in rare semileptonic decays—see Ref. [12] for a recent discussion. Notably, these cuts can extend into the complex plane rather than being confined to the real axis. To date, there is no parametrization that both satisfies a unitarity bound and accounts for anomalous branch cuts.

In the remainder of this letter, we derive for the first time bounded parametrizations that account for subthreshold and anomalous branch cuts.

Subthreshold branch cuts. To derive a bounded parametrization for FFs with a subthreshold branch cut, we construct a correlator of the form

$$\Pi^J(q^2) \equiv i \mathcal{P}^J_{\mu\nu} \int d^4x e^{iq \cdot x} \langle 0 | T \{ \mathcal{O}^\mu(x) \mathcal{O}^{\nu,\dagger}(0) \} | 0 \rangle. \quad (6)$$

Here, $\mathcal{O}^\mu$ denotes an operator that can be either local, such as a quark current, or nonlocal, such as the time-ordered

¹An alternative formulation of this parametrization is given in Ref. [10]. While it takes a simpler form by using monomials instead of polynomials, the unitarity bound cannot be expressed in a diagonal form, which prevents an estimation of the truncation error.

product of an effective four-quark operator and the electromagnetic current, as in Ref. [9]. The projectors are defined as

$$\mathcal{P}_{\mu\nu}^0 \equiv \frac{q_\mu q_\nu}{q^2}, \quad \mathcal{P}_{\mu\nu}^1 \equiv \frac{1}{d-1} \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} \right), \quad (7)$$

where d is the number of spacetime dimensions. It is well known that a correlator of the form of Eq. (7) satisfies a subtracted dispersion relation [2,13]

$$\begin{aligned} \chi^J(Q^2, l) &\equiv \frac{1}{l!} \left[\frac{\partial}{\partial q^2} \right]^l \Pi^J(q^2) \Big|_{q^2=Q^2} \\ &= \frac{1}{\pi} \int_0^\infty dq^2 \frac{\text{Im}\Pi^J(q^2)}{(q^2 - Q^2)^{l+1}}, \end{aligned} \quad (8)$$

since Π^J has singularities exclusively along the positive real axis. Here Q^2 denotes the subtraction point, and l represents the number of subtractions, chosen such that χ^J remains finite. For real Q^2 values significantly below the first threshold of Π^J , the function χ^J can be systematically calculated using an operator product expansion (OPE) and perturbation theory. χ_{OPE}^J is known with percent-level accuracy for all phenomenologically relevant cases.

The imaginary part of Π^J can be determined using unitarity, i.e. by inserting a complete set of states in Eq. (7) [7,14]. For the sake of concreteness, we consider the operator $\mathcal{O}^\mu = \bar{s}\gamma^\mu b$ and $J = 1$ in the following derivation, although it can easily be generalized to other operators and to $J = 0$. In this case, the two-particle contribution of the $\bar{B}K$ states to $\text{Im}\Pi^J$ reads²

$$\text{Im}\Pi_{\text{had}}^{1,bs}(q^2) = \frac{|\lambda_{\text{kin}}|^{\frac{3}{2}}}{24\pi(q^2)^2} |f_+^{BK}(q^2)|^2 \theta(q^2 - s_+) + \dots, \quad (9)$$

where $\lambda_{\text{kin}} \equiv \lambda(q^2, m_B^2, m_K^2)$ is the Källén function and the isospin limit has been used. The ellipsis denotes the contribution of all the other states. The contribution of the one-particle state, (i.e., $\bar{B}_s^*$) or other two-particle states [e.g., $\bar{B}K^{(*)}$, $\bar{B}_s\phi$, and $\Lambda_b\Lambda$] can easily be added to the right-hand side (rhs) of Eq. (9) [8,16,17]. The FF f_+^{BK} is defined as [4]

$$\mathcal{P}_{\mu\nu}^1 \langle \bar{K}(k) | \bar{s}\gamma^\mu b | \bar{B}(q+k) \rangle = \frac{2}{3} \left(k_\nu - \frac{q \cdot k}{q^2} q_\nu \right) f_+^{BK}(q^2). \quad (10)$$

By equating the OPE result and the hadronic representation (9) of $\chi^{1,bs}$ we obtain

²The mathematical steps to derive this equation are standard and are omitted here for brevity. Detailed derivations can be found, for example, in Refs. [3,7,15].

$$\chi_{\text{OPE}}^{1,bs}(0, l) > \int_{s_+}^\infty dq^2 \frac{|\lambda_{\text{kin}}|^{\frac{3}{2}}}{24\pi^2(q^2)^{l+3}} |f_+^{BK}(q^2)|^2, \quad (11)$$

where $s_+ \equiv (m_B + m_K)^2$. For simplicity, we have set $Q^2 = 0$, which is the most common choice in the literature [7,14]. Hereafter we omit the first argument of χ^J , namely $\chi^J(0, l) \equiv \chi^J(l)$. It has been shown that the minimum number of subtractions required to obtain a finite $\chi^{1,bs}$ is $l = 2$ [7]. Note that neglecting the additional contributions denoted by the ellipsis in Eq. (9) and using an inequality sign in Eq. (11) is justified, as these contributions are positive definite [7,17].

The domain of analyticity for f_+^{BK} is $\mathbb{C} \setminus \{m_{B_s^*}^2\} \setminus [s_\Gamma, \infty)$. In this case $s_\Gamma \equiv (m_{B_s} + m_\pi)^2$, since $\bar{B}_s\pi_0$ is the lightest multiparticle state that goes on-shell [17]. In addition there is a simple pole at $q^2 = m_{B_s^*}^2$. In the standard approach, the branch cut between s_+ and s_Γ is typically ignored, allowing the direct application of the procedure outlined in the previous section, as Eq. (11) can be easily recast in the form of Eq. (3) to yield Eq. (4) [7,8]. Although it can be argued that these subthreshold cuts give only a minor contribution in certain cases, neglecting them is generally unjustified. Indeed, expanding a FF using a power series in a nonanalytic region introduces unknown systematic uncertainties.

To overcome these limitations we add the same (positive) term on both sides of Eq. (11) and set $l = 2$:

$$\Delta\chi^{1,bs}(2) \equiv \int_{s_\Gamma}^{s_+} dq^2 \frac{|\lambda_{\text{kin}}|^{\frac{3}{2}}}{24\pi^2(q^2)^5} |f_+^{BK}(q^2)|^2. \quad (12)$$

This term is visually represented by the magenta arc in the right panel of Fig. 1, whereas the rhs of Eq. (11) corresponds to the blue arc. This yields

$$\tilde{\chi}^{1,bs}(2) > \int_{s_\Gamma}^\infty dq^2 \frac{|\lambda_{\text{kin}}|^{\frac{3}{2}}}{24\pi^2(q^2)^5} |f_+^{BK}(q^2)|^2, \quad (13)$$

where $\tilde{\chi}^{1,bs} \equiv \chi_{\text{OPE}}^{1,bs} + \Delta\chi^{1,bs}$. To estimate $\Delta\chi^{1,bs}$ one can approximate f_+^{BK} using its scaling large q^2 calculated in perturbative QCD as [18,19]

$$|f_+^{BK}(q^2)|^2 \simeq K \left(\frac{s_\Gamma}{q^2} \right)^2. \quad (14)$$

Following Ref. [20], it is possible to show that the constant K is expected to be smaller than one (see also Refs. [21–23]). Even assuming $K \sim \mathcal{O}(10^2)$ the contribution of $\Delta\chi^{1,bs}$ remains less than one per mille of $\chi_{\text{OPE}}^{1,bs}$, which is significantly smaller than the uncertainty of $\chi_{\text{OPE}}^{1,bs}$ itself. The discussion about the appropriate approximation for f_+^{BK} is therefore irrelevant, allowing us to simply set $\tilde{\chi}^{1,bs} = \chi_{\text{OPE}}^{1,bs}$. This is not surprising, as the interval $[s_\Gamma, s_+]$ is relatively minuscule, making its contribution naturally negligible.

Based on the considerations above, we propose to parametrize f_+^{BK} as

$$f_+^{BK}(q^2) = \frac{1}{\mathcal{B}_+(q^2)\phi_+(q^2)} \sum_{n=0}^{\infty} c_{+,n} z^n(q^2, s_\Gamma), \quad (15)$$

where $\mathcal{B}_+(q^2) = z(q^2, s_\Gamma, m_{B_s^*}^2)$ and

$$\begin{aligned} \phi_+(q^2) &= \frac{1}{\sqrt{24\pi\tilde{\chi}^{1,bs}(2)}} \frac{(s_+ - q^2)^{\frac{3}{4}}}{\sqrt{-dz(q^2, s_\Gamma)/dq^2}} \\ &\times \frac{\left(\sqrt{s_\Gamma - q^2} + \sqrt{s_\Gamma - s_-}\right)^{\frac{3}{2}}}{\left(\sqrt{s_\Gamma - q^2} + \sqrt{s_\Gamma}\right)^5}. \end{aligned} \quad (16)$$

Note that this outer function depends on both thresholds: s_Γ and s_+ . Using the mapping $z(q^2, s_\Gamma)$ and Eqs. (16), (13) can be written as

$$\frac{1}{2i\pi} \oint_{|z|=1} \frac{dz}{z} |\phi_+(q^2) f_+^{BK}(q^2)|_{q^2=z(s_\Gamma)}^2 < 1. \quad (17)$$

which has exactly the same form of Eq. (3) and hence, using Eq. (15),

$$\sum_{n=0}^{\infty} |c_{+,n}|^2 < 1. \quad (18)$$

We have therefore derived the first model-independent and unitarity-bounded parametrization for f_+^{BK} . This parametrization can be extended to other FFs and transitions. While its extension is generally straightforward—requiring only the calculation of outer functions and the use of appropriate Blaschke factors—two specific cases require further consideration.

The first is when the term $\Delta\chi^J$ is not negligible. This can potentially occur when $(s_+ - s_\Gamma)/s_\Gamma \sim \mathcal{O}(1)$, a condition that is satisfied by certain nonlocal FFs in rare semileptonic decays.³ These FFs are discussed in the next section.

The second is when the considered FF has a pole between s_Γ and s_+ . This happens for instance in the FF $f_0^{BK}(q^2)$, defined as

$$\mathcal{P}_{\mu\nu}^1 \langle \bar{K}(k) | \bar{s} \gamma^\mu b | \bar{B}(q+k) \rangle = q_\nu \frac{\sqrt{s_- s_+}}{q^2} f_0^{BK}(q^2), \quad (19)$$

which has a simple pole at $q^2 = m_{B_{s0}}^2 = (5.711 \text{ GeV})^2$ [24], and hence $s_\Gamma < m_{B_{s0}}^2 < s_+$. The minimal number of

³The quantity $(s_+ - s_\Gamma)/s_\Gamma$ can be relatively large even for (local) FFs of decays involving excited mesons or baryons. In such cases, a careful estimate of $\Delta\chi^J$ is necessary. Importantly, a technique to suppress the contribution of $\Delta\chi^J$ is discussed at the end of this section.

subtractions for $\chi^{0,bs}$ is $l = 1$ [7]. Using Eq. (8) and setting $Q^2 = 0$ once again, we construct the following equation:

$$\begin{aligned} \chi_\Sigma^{0,bs} &\equiv m_{B_{s0}}^4 \chi_{\text{OPE}}^{0,bs}(3) - 2m_{B_{s0}}^2 \chi_{\text{OPE}}^{0,bs}(2) + \chi_{\text{OPE}}^{0,bs}(1) \\ &= \frac{1}{\pi} \int_0^\infty dq^2 (q^2 - m_{B_{s0}}^2)^2 \frac{\text{Im}\Pi^{0,bs}(q^2)}{(q^2)^4}. \end{aligned} \quad (20)$$

By substituting $\text{Im}\Pi^{0,bs}$ with the contribution from $\bar{B}K$ states (see, e.g., Refs. [3,7,15]) and following a procedure similar to that used to derive Eq. (13), we obtain

$$\tilde{\chi}^{0,bs} > \int_{s_\Gamma}^\infty dq^2 \frac{s_- s_+ |\lambda_{\text{kin}}|^{\frac{1}{2}}}{8\pi^2 (q^2)^6} (q^2 - m_{B_{s0}}^2)^2 |f_0^{BK}(q^2)|^2. \quad (21)$$

Here $\tilde{\chi}^{0,bs} \equiv \chi_\Sigma^{0,bs} + \Delta\chi^{0,bs}$ and $\Delta\chi^{0,bs}$ is defined as the integral in the above equation, evaluated over the range from s_Γ to s_+ . In this way the pole at $q^2 = m_{B_{s0}}^2$ in the integrals has been manifestly removed. We therefore parametrize f_0^{BK} as

$$f_0^{BK}(q^2) = \frac{1}{\phi_0(q^2)} \sum_{n=0}^{\infty} c_{0,n} z^n(q^2, s_\Gamma), \quad (22)$$

The outer function reads

$$\begin{aligned} \phi_0(q^2) &= \sqrt{\frac{s_- s_+}{8\pi\tilde{\chi}^{0,bs}}} \frac{(s_+ - q^2)^{\frac{1}{4}}}{\sqrt{-dz(q^2, s_\Gamma)/dq^2}} \\ &\times \frac{\left(\sqrt{s_\Gamma - q^2} + \sqrt{s_\Gamma - s_-}\right)^{\frac{1}{2}}}{\left(\sqrt{s_\Gamma - q^2} + \sqrt{s_\Gamma}\right)^6} (m_{B_{s0}}^2 - q^2). \end{aligned} \quad (23)$$

Note that no Blaschke factor is needed in this case, as there are no poles below s_Γ . Similar to the case of f_+^{BK} , we can derive a unitarity bound using Eqs. (21) and (22)

$$\sum_{n=0}^{\infty} |c_{0,n}|^2 < 1. \quad (24)$$

The procedure presented in this section to derive unitarity-bounded parametrizations for FFs with subthreshold cuts is general and can therefore be applied to other hadronic decays. The only step that requires special attention is the evaluation of $\Delta\chi^J$, which involves calculating (or at least estimating an upper bound for) the weighted integral of the squared modulus of the FF along the subthreshold cut. However, the contribution of $\Delta\chi^J$ is typically negligible in most applications, as subthreshold cuts are usually short. It is also possible to perform additional subtractions to further suppress the contribution of $\Delta\chi^J$. This can be accomplished by applying the same procedure we used to remove the pole at $m_{B_{s0}}^2$ in f_0^{BK} , replacing $m_{B_{s0}}^2$ with $(s_\Gamma + s_+)/2$, i.e., the midpoint between s_Γ and s_+ .

We conclude this section by noting that, in addition to the approach of Refs. [9,10] described above, another method for dealing with subthreshold cuts was proposed in Refs. [21,22]. This method uses the mapping $z(q^2, s_+)$ and relies on the cancellation of the subthreshold branch cut within the unit disk. However, the precise functional form of the FFs along the cut is not known, making it impossible to exactly cancel the cuts within the unit disk. As a result, while the method of Refs. [21,22] can provide a good approximation in many cases, it is generally not reliable and model dependent.

Anomalous branch cuts. Nonlocal FFs, which parametrize the hadronic matrix elements of nonlocal operators [9,25–30], are known to potentially exhibit anomalous branch cuts [12,31,32]. Anomalous branch cuts, unlike the unitarity cuts discussed in the previous section, are not restricted to lie on the real axis.

To date, there is no unitary-bounded parametrization that can account for anomalous branch cuts. Given the critical role of nonlocal FFs in predicting observables for rare semileptonic decays [25–28], the development of such a parametrization is of utmost importance.

For definiteness we consider the nonlocal FF appearing in $B \rightarrow K\ell^+\ell^-$ decays

$$i \int d^4x e^{iq \cdot x} \langle \bar{K}(k) | T \{ \bar{c} \gamma_\mu c(x), \mathcal{Q}(0) \} | \bar{B}(q+k) \rangle$$

$$= 2m_B^2 \left(k_\mu - \frac{q \cdot k}{q^2} q_\mu \right) f_{\text{nl}}^{BK}(q^2), \quad (25)$$

where $\mathcal{Q} \equiv C_1 \mathcal{Q}_1 + C_2 \mathcal{Q}_2$, with $C_{1,2}$ denoting the Wilson coefficients of the four-quark effective operators $\mathcal{Q}_{2(1)} \equiv [\bar{s}_L \gamma^\mu (T^a) q_L][\bar{q}_L \gamma_\mu (T_a) b_L]$ [33]. The FF f_{nl}^{BK} serves as an excellent example not only because it exhibits anomalous cuts in the complex plane but also due to its significant phenomenological relevance, especially in light of the tensions between measurements and predictions in $B \rightarrow K\mu^+\mu^-$ decays [28,34–37].

The rescattering process $B \rightarrow DD_s^* \rightarrow K\ell^+\ell^-$ induces an anomalous branch cut in f_{nl}^{BK} between the points $s_\Gamma = 4m_D^2$ and $q^2 = s_A \equiv 24.1 - 3.5i$, as shown in Ref. [12].⁴ In addition, f_{nl}^{BK} presents a unitarity cut along the real axis starting at s_Γ . This domain is illustrated in the left panel of Fig. 2.

The first step in deriving a parametrization for this case is to identify a mapping $\hat{z}(q^2) \equiv z(q^2, s_A, s_\Gamma, s_0)$ that conformally maps the domain

⁴To simplify the discussion, we omit the additional anomalous cuts generated by other rescattering processes. While their inclusion would require additional effort, it can be done systematically using the method presented here.

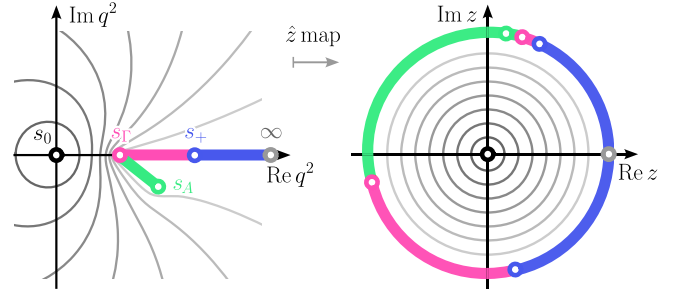


FIG. 2. Illustration of the conformal mapping $\hat{z}$ with $s_0 = 0$. We use the same color scheme as in Fig. 1 with the anomalous branch cut represented in green.

$$\Omega = \mathbb{C} \setminus ([s_\Gamma, \infty] \cup \{(1-t)s_A + ts_\Gamma : t \in [0, 1]\}) \quad (26)$$

to the unit disk $D = \{z : |z| < 1\}$ such that $\hat{z}(s_0) = 0$. The existence of such a mapping is ensured by the Riemann mapping theorem. To compute it we first compute the conformal mapping g from D to Ω such that $g(-1) = s_A$, $g(-i) = s_\Gamma$, $g(1) = \infty$. We note that since Ω is a polygon with vertices $w_1 = s_\Gamma$, $w_2 = s_A$, $w_3 = s_\Gamma$, $w_4 = \infty$, with corresponding angles $\phi_1 = -\arg(s_A - s_\Gamma)$, $\phi_2 = 2\pi$, $\phi_3 = 2\pi - \phi_1$, $\phi_4 = -2\pi$, g may be computed using the Schwarz–Christoffel formula

$$g(z) = A + C \int_0^z d\zeta \prod_{k=1}^4 \left(1 - \frac{\zeta}{z_k} \right)^{\phi_k/\pi-1}, \quad (27)$$

where $z_2 = -1$, $z_3 = -i$, and $z_4 = 1$; z_1 , A , and C are parameters that must be determined. We proceed with the same approach as the Schwarz–Christoffel Toolbox [38] which is described in detail in [39]. We first determine z_1 with $|z_1| = 1$ such that g satisfies $\int_{z_1}^{z_2} |d\zeta| |g'(\zeta)| = \int_{z_2}^{z_3} |d\zeta| |g'(\zeta)|$ using a quasi-Newton method. We then fix A and C by enforcing that $g(z_1) = s_\Gamma$ and $g(z_2) = s_A$. After g is determined, we can evaluate g^{-1} in Ω using a Newton iteration. We now compute $z_0 = g^{-1}(s_0)$ and note that the Möbius transform $h(z) = (z + z_0 e^{i\theta}) / (e^{i\theta} + \bar{z}_0 z)$ with $\theta = \arg((1 - \bar{z}_0)/(1 - z_0))$ maps D to D , and satisfies $h(0) = z_0$ and $h(1) = 1$. Thus, $\hat{z}^{-1} = g \circ h$ and $\hat{z}$ can be computed using a Newton iteration on $\hat{z}^{-1}$. The Python code for computing the conformal mapping with a single branch cut, as well as its inverse and derivatives, is included as Supplemental Material [40].

Now that the mapping has been obtained, it is possible to follow the procedure presented in the previous section to obtain a parametrization for f_{nl}^{BK} . In fact, from a conceptual point of view, anomalous cuts can be treated as subthreshold cuts. The OPE calculation and the imaginary part in terms of $\bar{B}K$ states of the corresponding correlator can be found in Ref. [9]. The derivation of the outer function is standard and is not shown here. The only point requiring further clarification is the calculation of $\Delta\chi^J$, which is particularly

challenging in this case and beyond the scope of this work. Nevertheless, an upper bound for $\Delta\chi^J$ can be obtained by using the fact that it can be decomposed as $f_{\text{nl}}^{BK} = f_{\text{unit}}^{BK} + f_{\text{an}}^{BK}$. The part containing only the unitarity cut f_{unit}^{BK} can be computed with the local OPE of Refs. [41,42], while the part containing the anomalous cut f_{an}^{BK} can be estimated using, e.g., the approach of Ref. [12].

Conclusion and outlook. We have presented a systematic procedure for deriving hadronic form factor (FF) parametrizations that satisfy unitarity bounds in the presence of subthreshold and/or anomalous branch cuts. While FFs with subthreshold cuts are relatively common in hadron decays, anomalous cuts are a distinctive feature of FFs arising from more complicated (nonlocal) operators.

Our parametrization can be regarded as an extension of the BGL parametrization [6,7], which is not valid when the aforementioned cuts are present. Moreover, our approach supersedes previous attempts in the literature to account for

subthreshold cuts, as it allows a rigorous estimation of the truncation error and does not rely on numerical cancellation of singularities [9,10,21,22].

This work is important not only from a conceptual and theoretical perspective, but also from a phenomenological one. In fact, the method presented here paves the way for new phenomenological analyses that can rely on first-principles constraints, i.e., the unitarity bounds, to estimate the truncation error. These constraints will contribute significantly to solving pressing problems such as the tensions in $B \rightarrow K^{(*)}\mu^+\mu^-$ decays [28,34–37].

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