

Alternative to purification in conformal field theory

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In conformal field theories, in contrast to *adding* some auxiliary states into the bipartite mixed state ρ_{AB} as the usual purifications do, we show a pure entangled state ψ_{AB} can be constructed by *subtracting* the undetectable regions. In this pure state ψ_{AB} , the von Neumann entropy $S_{\text{vN}}(A:B)$ naturally captures quantum entanglement between A and B . We verify that $S_{\text{vN}}(A:B)$ is equal to the entanglement wedge cross section E_W in anti-de Sitter spacetime, which is conjectured to be the holographic dual of the entanglement of purification.

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Introduction. Entanglement has fundamental importance for quantum information and quantum gravity. In conformal field theories (CFTs), the entanglement entropy classifies quantum entanglement between complementary parts A and B in a pure entangled state ψ_{AB} , defined by the von Neumann entropy,

$$S_{\text{vN}}(A) = -\text{Tr} \rho_A \log \rho_A, \quad (1)$$

for a reduced density matrix $\rho_A = \text{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|$. However, for disjoint subsystems A and B in a mixed state $\rho_{AB} \neq |\psi_{AB}\rangle\langle\psi_{AB}|$, the entanglement entropy fails to characterize entanglement between A and B , as it yields the same values for entangled and unentangled states.

The typical resolution involves purifying AB by adding auxiliary systems $\bar{A}\bar{B}$, known as purification. The entanglement of purification (EoP or E_P) [1] is defined by minimizing the entanglement entropy $S_{\text{vN}}(A\bar{A}:B\bar{B})$ over all possible purifications. Inspired by the AdS/CFT correspondence [2–4] and the holographic derivation of entanglement entropy, the entanglement of purification E_P is conjectured to be equal to the entanglement wedge cross section (EWCS or E_W) in the anti-de Sitter (AdS) bulk [5]. However, verifying this conjecture, $E_P = E_W$, poses significant challenges due to the complexity involved in minimizing the EoP, which is difficult to compute. As an alternative approach, canonical purification [6] is

suggested as $|\sqrt{\rho_{AB}}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}} \otimes \mathcal{H}_B \otimes \mathcal{H}_{\bar{B}}$, where $\bar{A}$ ($\bar{B}$) is the reflection of A (B). The reflected entropy $S_R(A:B)$ is then defined as the von Neumann entropy $S(\rho_{A\bar{A}})$. Another operationally defined and generally computable candidate is the logarithmic negativity [7] whose explicit form is unknown [8], since it depends on the conformal block that is computationally expensive. Moreover, various measures of mixed states are defined in different ways [9–11], but they are all challenging to compute.

In this Letter, for a bipartite mixed state ρ_{AB} , in contrast to *adding* extra auxiliary parts into the mixed state as the usual purifications do, we show by *subtracting* the undetectable regions in CFT₂, a pure entangled state ψ_{AB} can be constructed on the doubly connected plane (see Fig. 1). In this resulted pure state ψ_{AB} , it is natural to identify the well-defined von Neumann entropy $S_{\text{vN}}(A:B)$ as the entanglement measure between A and B . Our proposal is independent of purification or four-point conformal block data, distinguishing it from other bipartite entanglement measures. We further show that the ordinary entanglement entropies of critical and noncritical quantum field theories are simply limits of the two phases in our proposal. Within the AdS/CFT correspondence [2–4], the gravity dual of a reduced density matrix is called the entanglement wedge, and the EoP is conjectured to be equal to the minimal cross section of entanglement wedge (EWCS) [5], which bipartitions the entanglement wedge region. We verify our proposal holographically and find agreement with the EWCS in the AdS₃ bulk:

$$S_{\text{vN}}(A:B) = E_W(A:B). \quad (2)$$

Entanglement entropy in doubly connected region. In a CFT₂, consider two disjoint subsystems $A = (a_2, b_1)$ and $B = (-\infty, a_1) \cup (b_2, \infty)$ at time $\tau = 0$, such that

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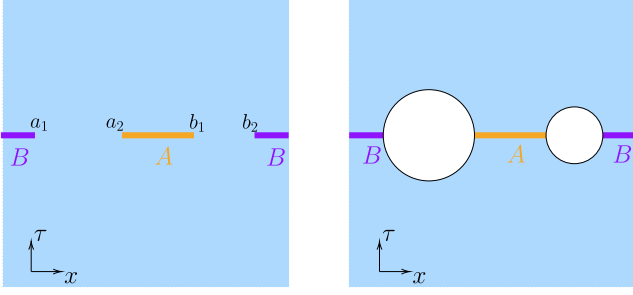


FIG. 1. Left panel: at $\tau = 0$, two subsystems A and B on the x axis are in a mixed state ρ_{AB} . Right panel: by removing the undetectable parts with two disks having different radii, we obtain a two-holed plane, in which two subsystems A and B are now in a pure entangled state ψ_{AB} . The blue shaded region represents the Euclidean path integral.

$a_1 < a_2 < b_1 < b_2$. The subsystems A and B are non-complementary parts typically in a mixed state ρ_{AB} (see the left panel of Fig. 1). In order to construct a pure state ψ_{AB} from the mixed state ρ_{AB} , we remove two disks in the Euclidean path integral region between A and B , obtaining a two-holed plane, topologically a doubly connected region, as depicted in Fig. 1. We then impose conformal invariant boundary conditions on two edges of the two-holed plane, corresponding to two boundary states $|a, b\rangle$ [12]. In this two-holed plane, two subsystems A and B are now complementary parts in a pure entangled state ψ_{AB} , allowing us to define the entanglement entropy $S_{\text{vN}}(A)$ [13].

Given that every doubly connected region can be conformally mapped onto an annulus of the type illustrated in Fig. 2, we first calculate the entanglement entropy $S_{\text{vN}}(A)$ between A and B in the annulus. To this end, it is useful to define the width of the annulus,

$$W = \log \frac{r_2}{r_1}, \quad (3)$$

where r_1 and r_2 are the inner and outer radii, respectively. The entanglement entropy $S_{\text{vN}}(A)$ in the annulus is defined through the Rényi entropy:

$$S^{(n)}(A) = \frac{1}{1-n} \log \text{Tr}_A \rho_A^n, \quad (4)$$

$$S_{\text{vN}}(A) = \lim_{n \rightarrow 1} S^{(n)}(A), \quad (5)$$

where $\text{Tr}_A \rho_A^n$ is given by

$$\text{Tr}_A \rho_A^n = \frac{Z_n}{Z_1^n}. \quad (6)$$

Here, Z_1 is the partition function on a single cover of the annulus, and Z_n is the partition function on the n -sheeted cover $\mathcal{M}_n$ obtained by sewing n copies of $\mathcal{M}$ along A . Using the generator of scale transformations, the annulus

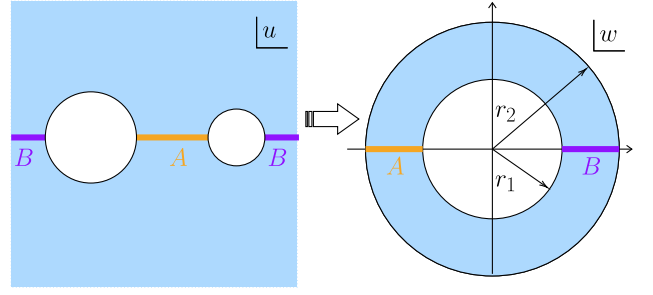


FIG. 2. The two-holed plane can be conformally mapped onto an annulus with the inner radius r_1 and the outer radius r_2 .

partition function can be expressed as [12,15,16]

$$Z_1 = e^{cW/12} \sum_k \langle a|k\rangle \langle k|b\rangle e^{-2\delta_k W}, \quad (7)$$

where c is the central charge, k denotes all allowed scalar operators with dimensions δ_k inside the annulus, and $|a, b\rangle$ are boundary states. Similarly,

$$Z_n = e^{cW/12n} \sum_k \langle a|k\rangle \langle k|b\rangle e^{-2\delta_k W/n}, \quad (8)$$

since the replicated manifold $\mathcal{M}_n$ is conformally equivalent to an annulus with the width $W_n = W/n$. As usual, only the ground state ($k = 0$) concerns us in calculating the entanglement entropy,

$$Z_1 = e^{cW/12} \langle a|0\rangle \langle 0|b\rangle, \quad (9)$$

$$Z_n = e^{cW/12n} \langle a|0\rangle \langle 0|b\rangle, \quad (10)$$

and $\text{Tr}_A \rho_A^n$ read

$$\text{Tr}_A \rho_A^n = e^{\frac{c}{12}(\frac{1}{n}-1)W} (\langle a|0\rangle \langle 0|b\rangle)^{1-n}. \quad (11)$$

Substituting this into the Rényi entropy expression, we obtain

$$S^{(n)}(A) = \frac{c}{12} \left(1 + \frac{1}{n}\right) W + g_a + g_b,$$

where $g_{a,b} = \log \langle a, b|0\rangle$ are the Affleck-Ludwig boundary entropies [17], which encode the information of the undetectable regions and are irrelevant. Thus, the entanglement between subsystems A and B is quantified by the universal term as

$$S_{\text{vN}}(A:B) := \lim_{n \rightarrow 1} S^{(n)}(A) - g_a - g_b = \frac{c}{6} W. \quad (12)$$

Now, we can map the u plane with two holes onto the annulus in the w plane via the conformal transformations:

$$v(u) = 2 \frac{u - a_1}{a_2 - a_1} - 1, \quad w(v) = \frac{v - \gamma}{\gamma v - 1}, \quad (13)$$

with

$$\gamma = \frac{1 + \alpha\beta + \sqrt{(1 - \alpha^2)(1 - \beta^2)}}{\alpha + \beta},$$

$$\alpha = v(b_1), \quad \beta = v(b_2). \quad (14)$$

The width of the annulus is

$$W = \log \frac{\alpha - \beta}{1 - \alpha\beta + \sqrt{(1 - \alpha^2)(1 - \beta^2)}}. \quad (15)$$

In terms of the cross ratio

$$z = \frac{(a_2 - a_1)(b_2 - b_1)}{(b_1 - a_2)(b_2 - a_1)}, \quad (16)$$

plugging (15) into (12), we get

$$S_{\text{vN}}(A:B) = \frac{c}{6} \log \left[1 + \frac{2}{z} + 2\sqrt{\frac{1}{z} \left(\frac{1}{z} + 1 \right)} \right]. \quad (17)$$

In the limit $b_1 - a_2 = \ell$ and $a_2 - a_1 = b_2 - b_1 = \epsilon \rightarrow 0$ as a UV regulator, this entanglement entropy simplifies to the famous single interval result,

$$S_{\text{vN}}(A:B) = \frac{c}{3} \log \frac{\ell}{\epsilon}. \quad (18)$$

On the other hand, for the same system, it is illuminating to switch the roles of A, B and those of the undetectable regions C, D , as depicted in Fig. 3. So, we are now concerned with the entanglement between $C \in (a_1, a_2)$ and $D = (b_1, b_2)$, while A and B are undetectable regions. Following the similar procedure, we subtract A and B with two disks. The resulted manifold is an asymmetric annulus, where C and D become complementary parts in a pure entangled state ψ_{CD} .

We can map the asymmetric annular region in the u plane onto an annulus in the w plane using the following conformal transformations:

$$\tilde{v}(u) = 2 \frac{u - a_1}{b_2 - a_1} - 1, \quad w(\tilde{v}) = \frac{\tilde{v} - \tilde{\gamma}}{\tilde{\gamma} \tilde{v} - 1}, \quad (19)$$

with

$$\tilde{\gamma} = \frac{1 + \tilde{\alpha}\tilde{\beta} + \sqrt{(1 - \tilde{\alpha}^2)(1 - \tilde{\beta}^2)}}{\tilde{\alpha} + \tilde{\beta}}, \quad (20)$$

$$\tilde{\alpha} = \tilde{v}(b_1), \quad \tilde{\beta} = \tilde{v}(a_2). \quad (21)$$

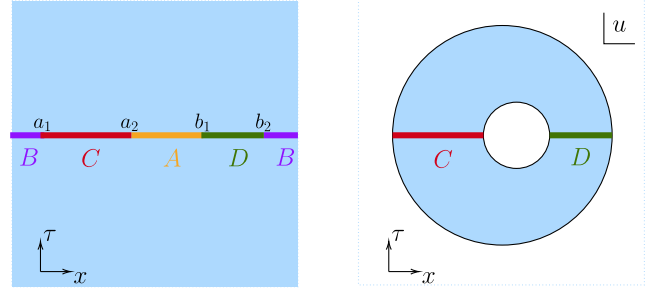


FIG. 3. Left panel: four subsystems $ABCD$ in the u plane. Right panel: by subtracting the undetectable parts A and B with two disks, we obtain an asymmetric annular region in which C and D are in a pure entangled state ψ_{CD} .

The width of the annulus is

$$\tilde{W} = \log \frac{1 - \tilde{\alpha}\tilde{\beta} + \sqrt{(1 - \tilde{\alpha}^2)(1 - \tilde{\beta}^2)}}{\tilde{\alpha} - \tilde{\beta}}. \quad (22)$$

In terms of the cross ratio (16), the entanglement entropy between C and D is

$$S_{\text{vN}}(C:D) = \frac{c}{6} \tilde{W}$$

$$= \frac{c}{6} \log [1 + 2z + 2\sqrt{z(z+1)}]. \quad (23)$$

In the limit $b_1 = -a_2 = \epsilon \rightarrow 0$ and $b_2 = -a_1 = \xi \rightarrow \infty$, we reproduce the half-space entanglement entropy

$$S_{\text{vN}}(C:D) = \frac{c}{6} \log \frac{\xi}{\epsilon}. \quad (24)$$

Holographic interpretation. Notably, $S_{\text{vN}}(C:D)$ is related to $S_{\text{vN}}(A:B)$ by

$$z \rightarrow \frac{1}{z}, \quad (25)$$

and satisfies

$$\sinh \left[\frac{3}{c} S_{\text{vN}}(A:B) \right] \sinh \left[\frac{3}{c} S_{\text{vN}}(C:D) \right] = 1. \quad (26)$$

This equation is an exclusive constraint in hyperbolic geometry [18] for the only two perpendicular geodesics inside a quadrilateral, as depicted in Fig. 4. This observation is reminiscent of the AdS/CFT correspondence, where it has been conjectured that the minimal cross section of the entanglement wedge E_W in the AdS bulk is a holographic dual of the EoP in a given bipartite mixed state [5]. With the same setup, the EWCS could be computed in the AdS_3 bulk, and we immediately find that the EWCS precisely

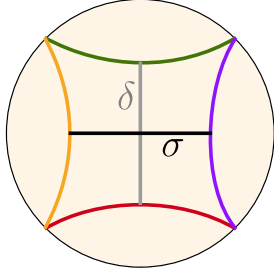


FIG. 4. In the hyperbolic disk, a quadrilateral is bounded by colored geodesics, and the only two perpendicular geodesics (black and gray lines) satisfy the equation $\sinh \frac{\delta}{2} \sinh \frac{\sigma}{2} = 1$, where σ and δ denote their lengths.

agrees with the entanglement entropy S_{vN} in the doubly connected region:

$$E_W(A:B) = S_{\text{vN}}(A:B), \quad (27)$$

$$E_W(C:D) = S_{\text{vN}}(C:D). \quad (28)$$

Thus, the entanglement entropy S_{vN} in the doubly connected region should be regarded as an alternative to EoP.

In addition, our proposal provides a novel interpretation of the entanglement phase transition for mixed states. From Eq. (26), we identify a critical point:

$$S_{\text{vN}}(A:B) = S_{\text{vN}}(C:D) = S_{\text{vN}}^*, \quad (29)$$

where the critical value

$$S_{\text{vN}}^* = \frac{c}{3} \operatorname{arcsinh} 1 = \frac{c}{6} \log(3 + 2\sqrt{2}) \quad (30)$$

matches the phase transition point of the EWCS [5]. This entanglement phase transition is usually interpreted as the mutual information $I(A:B) = \frac{c}{6} \log z$ between two subsystems, vanishing when $z \leq 1$. We now see from our results this is precisely a transition of the dominant contribution between s channel and t channel [19]. It indicates that the entanglement transition near $z = 1$ signifies a transition from entanglement between A and B to entanglement between C and D , or vice versa.

Conclusion and discussion. In this Letter, we proposed an alternative to purification for bipartite mixed states in CFT_2 . Our proposal is completely determined by the definition of von Neumann entropy and the replica trick, independent of any purification or four-point conformal block data. We confirmed the proposed S_{vN} is the dual of EWCS and equal to EoP.

Our choice of regulators, by removing disks, is *canonical*, although other choices with different shapes may exist. The width W is a *conformal invariant*; if two annuli have different widths, they cannot be conformally mapped onto

each other. The set of all doubly connected regions falls into classes of conformally equivalent regions, with each class characterized by the width of that class. Therefore, any other possible regulators with different shapes must be conformally equivalent to our choice.

Our proposal suggests that the von Neumann entropy in the multiconnected regions may correspond to an entanglement measure for multipartite mixed states. Having successfully extracted the EWCS from the annulus CFT, it is of interest to further study the connections between the entanglement wedge and the CFT in multiconnected regions.

In [20], the EWCSs for two parallel strips with equal widths l in AdS_{d+1} are given. It turns out our proposal can be used to find the dual entanglement entropy in CFT_4 .

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Appendix. In this appendix, we present a simple example to illustrate the correspondence established in this Letter:

$$S_{\text{vN}}(A:B) = E_W(A:B). \quad (\text{A1})$$

In 2D CFT, as a pure state can be viewed as an adjacent limit of a mixed state, the traditional entanglement entropy for a pure state is similarly a limit of the von Neumann entropy $S_{\text{vN}}(A:B)$ for a mixed state. To demonstrate this, consider a time slice of CFT_2 , with two setups for intervals A and B .

$A = (a_2, b_1)$ and $B = (-\infty, a_1) \cup (b_2, \infty)$ with $a_1 < a_2 < b_1 < b_2$: Using our method, the von Neumann entropy $S_{\text{vN}}(A:B)$ can be computed without the need for purification, measuring the entanglement entropy between the disjoint subsystems A and B in a mixed state. The result is given by

$$S_{\text{vN}}(A:B) = \frac{c}{6} \log \left[1 + \frac{2}{z} + 2\sqrt{\frac{1}{z} \left(\frac{1}{z} + 1 \right)} \right], \quad (\text{A2})$$

where $z = \frac{(a_2 - a_1)(b_2 - b_1)}{(b_1 - a_2)(b_2 - a_1)}$. This result could not be obtained by previous methods and has not been directly related to the entanglement wedge cross section (EWCS) until now. In our framework, using the $\text{AdS}_3/\text{CFT}_2$ correspondence, the entropy $S_{\text{vN}}(A:B)$ is computed on a time slice of the AdS_3 bulk, using Poincaré coordinate $ds^2 = z^{-2}(dx^2 + dz^2)$ (with the AdS radius set to 1). The intervals are taken as $A = (a_2, b_1)$ and $B = (-\infty, a_1) \cup (b_2, \infty)$ at the boundary

$z = 0$. The entanglement wedge is enclosed by geodesics connecting the boundaries of A and B . The EWCS measures the minimal cross section of the entanglement wedge, resulting in

$$E_W(A:B) = \frac{1}{4G_N^{(3)}} \log \left[1 + \frac{2}{z} + 2\sqrt{\frac{1}{z} \left(\frac{1}{z} + 1 \right)} \right], \quad (\text{A3})$$

which equals the von Neumann entropy $S_{\text{vN}}(A:B)$ through the identification $c = 3/2G_N^{(3)}$. This equivalence is depicted in the left-hand side of Fig. 5.

Furthermore, the von Neumann entropy $S_{\text{vN}}(A:B)$ for the mixed state can be reduced to the traditional entanglement entropy $S_{EE}(A)$ for a pure state by setting $a_2 - a_1 = b_2 - b_1 = 2\epsilon$ and $b_1 - a_2 = l$. This yields the result

$$S_{\text{vN}}(A:B) \rightarrow S_{EE}(A) = \frac{c}{3} \log \frac{l}{\epsilon}, \quad (\text{A4})$$

where l is the size of the entangling region A , and ϵ is a UV cutoff. This expression corresponds to the well-known entanglement entropy for a quantum system composed

of two parts, A and B . This reduction is depicted in the right-hand side of Fig. 5.

$A = [-b, -a]$ and $B = [a, b]$ with $0 < a < b$: In this setup, we can also apply our method to compute the von Neumann entropy $S_{\text{vN}}(A:B)$ for the disjoint subsystems A and B in a mixed state. The result is

$$S_{\text{vN}}(A:B) = \frac{c}{6} \log \frac{b}{a}. \quad (\text{A5})$$

Similarly, we can calculate the EWCS in the Poincaré metric $ds^2 = z^{-2}(dx^2 + dz^2)$ for the same regions $A = [-b, -a]$ and $B = [a, b]$ at the boundary $z = 0$. The EWCS is computed as the minimal cross section of the entanglement wedge in the bulk geometry:

$$E_W(A:B) = \frac{1}{4G_N^{(3)}} \log \frac{b}{a}. \quad (\text{A6})$$

We can see the equivalence using the relation $c = 3/2G_N^{(3)}$. This correspondence is depicted in the left-hand side of Fig. 6.

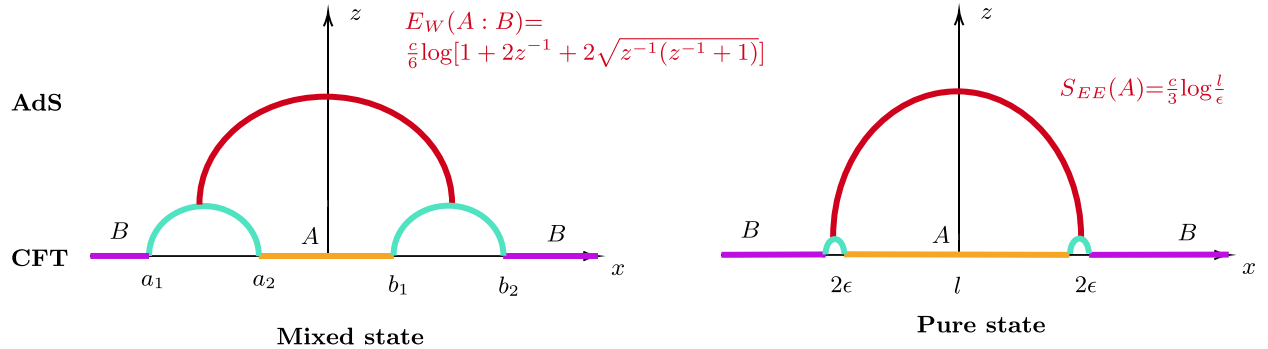


FIG. 5. The left-hand side picture shows the EWCS in the time slice of AdS_3 between regions $A = (a_2, b_1)$ and $B = (-\infty, a_1) \cup (b_2, \infty)$. The geodesic length $E_W(A:B)$ matches the von Neumann entropy $S_{\text{vN}}(A:B)$, as derived in this Letter. The right-hand side picture also shows the adjacent limit. This limit reduces to the well-known entanglement entropy for a pure state.

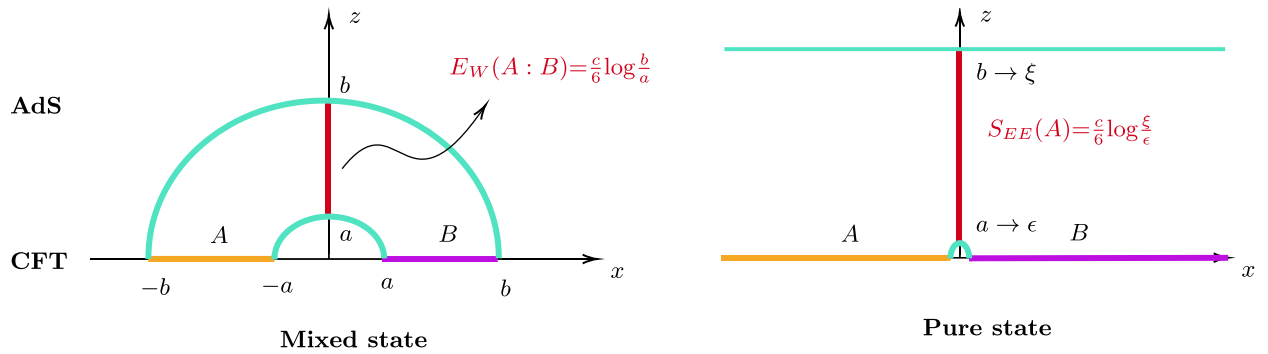


FIG. 6. The left-hand side picture shows the EWCS in the time slice of AdS_3 between regions $A = [-b, -a]$ and $B = [a, b]$. The geodesic length $E_W(A:B)$ matches the von Neumann entropy $S_{\text{vN}}(A:B)$, as derived in this Letter. On the right-hand side, the adjacent limit is depicted, where regions A and B are brought together. This limit reproduces the well-known entanglement entropy for a pure state, demonstrating the consistency of our method.

Moreover, we can observe how the von Neumann entropy $S_{\text{vN}}(A:B)$ for the mixed state reduces to the traditional entanglement entropy $S_{EE}(A)$ for the pure state. This occurs when A and B become adjacent. In our framework, this corresponds to the limits $a = \epsilon \rightarrow 0$ and $b = \xi \rightarrow \infty$. Thus, the von Neumann entropy simplifies to

$$S_{\text{vN}}(A:B) \rightarrow S_{EE}(A) = \frac{c}{6} \log \frac{\xi}{\epsilon}. \quad (\text{A7})$$

This describes a free CFT perturbed by a relevant operator, where ξ represents the correlation length in the perturbed theory and ϵ is a UV cutoff. This limit is depicted in the right-hand side of Fig. 6.

In short, our method is general and applies to any 2D conformal field theory (CFT). It allows us to obtain valid entanglement measures for both mixed and pure states.

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