

Single Kerr-Schild metric for Taub-NUT instanton

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It is shown that a complex coordinate transformation maps the Taub-Newman-Unti-Tamburino instanton metric to a Kerr-Schild metric. This metric involves a semi-infinite line defect as the gravitational analog of the Dirac string, much like the original metric. Moreover, it facilitates three versions of classical double copy correspondence with the self-dual dyon in electromagnetism, one of which involves a nonlocal operator. The relevance to the Newman-Janis algorithm is briefly noted.

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Introduction. Recently, an exciting program dubbed classical double copy [1–17] has refined analogies between gravity and electromagnetism into concrete mappings between classical solutions while being rooted in the study of scattering amplitudes [17–21]. The prime example is the Kerr-Schild (KS) classical double copy, established in the seminal work [1]. Namely, it is shown that a stationary KS solution $g = \eta + \Phi \ell \otimes \ell$ in general relativity maps to a stationary solution in Maxwell theory described by the gauge potential $A = \Phi \ell$, by taking a “single copy” of the null one-form ℓ . Conversely, one envisions a gravitational counterpart $g = \eta + \Phi \ell \otimes \ell$ of a stationary solution $A = \Phi \ell$ in electromagnetism, by taking a “double copy” of ℓ . Well-known instances of this correspondence are Schwarzschild to point charge and Kerr to a rotating solution dubbed $\sqrt{\text{Kerr}}$ [1,22–24].

From this angle, it is natural to expect that the Taub-Newman-Unti-Tamburino (NUT) solution [25,26] will correspond to a dyon, given the interpretation of the NUT charge as the gravitomagnetic monopole [16,27–33]. Unfortunately, only a *double* KS metric $g = \eta + \Phi_1 \ell_1 \otimes \ell_1 + \Phi_2 \ell_2 \otimes \ell_2$ has been explicitly known for Taub-NUT [34,35], so a mapping to electromagnetic dyons had to be achieved by a *variant* of the KS double copy [6,16].

Remarkably, in the present note we point out that the straightforward application of the *original* KS double copy is, in fact, possible for the *self-dual* Taub-NUT (SDTN) solution, which is an extremal case of the Taub-NUT solution that has gained a revived interest [36–40] because of its intriguing physical properties [41–45]. The crucial

fact is that the SDTN solution remarkably admits a KS metric in the *ordinary* sense: $g = \eta + \Phi \ell \otimes \ell$.

First, a KS metric is obtained by KS double copying the gauge potential of the self-dual dyon (SDD) in electromagnetism. We then show that it describes the SDTN solution by explicitly constructing a coordinate transformation to the well-known Gibbons-Hawking [41,42] instanton metric.

Second, the physical interpretation of the metric as a gravitational dyon is established within the KS description. Most importantly, we confirm the characteristic *Misner string* geometry: the gravitational analog [27,28] of the Dirac string [46,47]. Notably, finding its distributional source [28,29] becomes a simple problem in KS coordinates, thanks to a linearization [48–50].

Third, the classical double copy is established in two more ways. We first show how our KS description concretely validates the Weyl double copy proposed by Refs. [2–4], which has been stated without referencing an explicit metric. We then propose an exotic form of classical double copy employing a *nonlocal* operator, which arises from propagators between dyonic matter [51–55].

Lastly, we briefly remark on how we originally arrived at this KS metric, which was, amusingly, from the Newman-Janis shift property [24] of the Kerr solution.

Derivation of SDTN metric from double copy. The gauge potential of a dyon carrying electric and magnetic charges $(q, -iq)$ is given by

$$A = \frac{q}{4\pi r} (-dt - ir(1 - \cos\theta)d\phi), \quad (1)$$

where we have set up spherical coordinates and oriented the Dirac string [46,47] along the negative z axis (south pole). This describes a complexified solution where the field strength $F = dA$ is self-dual: $*F = +iF$.

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Notably, adding a total derivative $(q/4\pi)d\log(r+z)$ gauge transforms this to a KS form:

$$A = \frac{q}{4\pi r} \left(-dt + \frac{x+iy}{r+z} dx + \frac{y-ix}{r+z} dy + dz \right), \quad (2)$$

where $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$ are the Cartesian coordinates. Importantly, the terms inside the bracket comprise a null one-form.

With this understanding, the KS classical double copy [1] maps Eq. (2) to the following line element, by literally taking “two copies” of the null one-form:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \frac{m}{4\pi r} \left(-dt + \frac{x+iy}{r+z} dx + \frac{y-ix}{r+z} dy + dz \right)^2. \quad (3)$$

Here we have adopted $8\pi G = 1$ units and replaced the charge q with the mass m . Remarkably, Eq. (3) is a metric of the SDTN solution, whose mass and NUT charge are m and $-im$.

The diffeomorphism. To prove this, we simply construct an explicit coordinate transformation to the previously known metric. It is helpful to employ complex coordinates $\zeta = (x+iy)/(r+z)$, $\tilde{\zeta} = (x-iy)/(r+z)$ that arise from stereographically projecting the two-sphere with respect to the south pole, so Eq. (3) translates to

$$ds^2 = -dt^2 + dr^2 + r^2 \frac{4d\tilde{\zeta}d\zeta}{(1+\tilde{\zeta}\zeta)^2} + \frac{m}{4\pi r} \left(-dt + dr - r \frac{2\tilde{\zeta}d\zeta}{1+\tilde{\zeta}\zeta} \right)^2. \quad (4)$$

Amusingly, applying a diffeomorphism given by

$$t \mapsto t, \quad r \mapsto r + \frac{m}{4\pi}, \quad \zeta \mapsto \zeta, \quad \tilde{\zeta} \mapsto \frac{1}{1 + (m/4\pi r)(1 + \tilde{\zeta}\zeta)} \tilde{\zeta}, \quad (5)$$

we obtain

$$ds^2 = - \left(1 + \frac{m}{4\pi r} \right)^{-1} \left(dt + \frac{m}{4\pi r} \left(dr - r \frac{2\tilde{\zeta}d\zeta}{1+\tilde{\zeta}\zeta} \right) \right)^2 + \left(1 + \frac{m}{4\pi r} \right) \left(dr^2 + r^2 \frac{4d\tilde{\zeta}d\zeta}{(1+\tilde{\zeta}\zeta)^2} \right), \quad (6)$$

which is the Gibbons-Hawking metric [41,42] for the SDTN solution with mass m .

This diffeomorphism is surely invertible. In fact, as is explicated in the Appendix, it turns out that Eq. (5) traces back to the flow along the null geodesic congruence

associated with the KS metric. Thus, the inverse map is simply Eq. (5) with m replaced with $-m$.

This completes our *constructive* proof that the KS and Gibbons-Hawking metrics in Eqs. (3) and (6) are diffeomorphic. Consequently, we establish that the SDTN solution admits a construction from the KS classical double copy. Previously, these facts were only recently *conjectured* by Ref. [56] in split signature.

Meanwhile, the astute reader may have noticed an unusual feature of the diffeomorphism in Eq. (5): it describes a *complexified diffeomorphism* since ζ is frozen while $\tilde{\zeta}$ gets scrambled, hence the notation $\tilde{\zeta}$ instead of $\bar{\zeta}$. Yet crucially, this is a legitimate operation since the Lorentzian-signature SDTN solution is inherently a complex saddle as a self-dual solution [36]. Just as the SDD gauge potential described in Eq. (2) is a solution to complexified Maxwell’s equations due to the imaginary magnetic charge $-iq$, the SDTN metric in Eq. (3) solves complexified Einstein’s equations governing holomorphic metrics [36,57–60]. As a consequence of this interpretation, ζ and $\tilde{\zeta}$ can be transformed independently as *holomorphic* coordinates.

Physical interpretation. We shall now examine the physical validity of the KS metric in Eq. (3). To begin with, we point out that Eq. (3) *itself* is quite unusual as a KS metric, in fact. While the KS potential is simply $\Phi = m/4\pi r$, the null one-form ℓ involves a *semi-infinite line defect* along the negative z axis, which is exactly how the gravitomagnetic charge is encoded in this metric:

$$\ell = -dt + \frac{x+iy}{r+z} dx + \frac{y-ix}{r+z} dy + dz. \quad (7)$$

This is intriguing, as the null one-form is usually regular wherever the KS potential is regular—which is especially true for the double KS construction of Taub-NUT [6].

Crucially, as in the original metric in Eq. (6), the interpretation of this line defect should be the *Misner string* [27,28]: the gravitational analog of the Dirac string, inputting a gravitomagnetic flux of $-im$ at the origin. To show this, we recall that there are at least three senses in which the Taub-NUT metric [27,28] describes a gravitational analog of a dyon: (a) the Misner string is invisible as a coordinate artifact if the gravitational analog of the Dirac quantization condition holds [27], (b) the Misner string implements a flux tube of time monodromy [16,30,31], and (c) the distributional stress-energy of the string describes an energyless thin solenoid of mass current [28,29]. Our goal now is to show that all of these properties hold in the KS description as well.

First, we derive the Dirac quantization condition. For the SDD gauge potential in Eq. (2), adding a term $(q/2\pi)d\log \zeta$ repositions its Dirac string along the positive z axis (north pole). This encodes the complexified group-valued transition function $\zeta^{ieq/2\pi}$, where e is the

electromagnetic coupling. Demanding its single-valuedness on the “equator” [61] implies $ieq \in 2\pi\mathbb{Z}$.

The gravitational Dirac quantization can be derived in a similar fashion, i.e., by hiding the strings with overlapping coordinate patches [16,27,30,31]. In the KS description, we find that the diffeomorphism

$$t \mapsto t + \frac{m}{2\pi} \log \zeta, \quad \tilde{\zeta} \mapsto \frac{1 - (m/4\pi r)(1 + \tilde{\zeta}\zeta)/\tilde{\zeta}\zeta}{1 + (m/4\pi r)(1 + \tilde{\zeta}\zeta)} \tilde{\zeta} \quad (8)$$

with r and ζ fixed transforms the KS metric in Eq. (4) to another KS metric:

$$ds^2 = -dt^2 + dr^2 + r^2 \frac{4d\tilde{\zeta}d\zeta}{(1 + \tilde{\zeta}\zeta)^2} + \frac{m}{4\pi r} \left(-dt + dr + r \frac{2d\zeta}{\zeta(1 + \tilde{\zeta}\zeta)} \right)^2, \quad (9)$$

which develops a string defect along the positive z axis (north pole). For this diffeomorphism to be single-valued within the overlapping region, the time coordinate should be periodic as $t \sim t + im\mathbb{Z}$, given the $2\pi i$ periodicity of $\log \zeta$. Hence the energy e of a test particle is quantized as $iem \in 2\pi\mathbb{Z}$. This is precisely the gravitational Dirac quantization condition due to the NUT charge $-im$.

Second, we study the near-string geometries in an explicit manner. Let $\xi := x + iy$, $\tilde{\xi} := x - iy$ be complex coordinates for the (complexified) $x - y$ plane and suppose a positive infinitesimal ϵ . Applying the replacement $z \mapsto -1/\epsilon + z$ to the gauge potential in Eq. (2) and then taking the limit $\epsilon \rightarrow 0$, we obtain $A = (q/2\pi)d \log \tilde{\xi}$. This precisely describes an infinite Dirac string generated by a large gauge transformation [62,63].

For the line element in Eq. (3), the same limiting procedure leaves out a divergent part that can be compensated by a shift $\xi \mapsto \xi + ((m/\pi)(1/\epsilon - z) - 3m^2/4\pi^2)/\tilde{\xi}$. Amusingly, the resulting line element is simply the Minkowski line element, but with a multivalued time:

$$t' = t + \frac{m}{2\pi} \log \tilde{\xi}. \quad (10)$$

Namely, in the vicinity of the string, the geometry approaches to the image of empty spacetime under a large diffeomorphism. Here, a Sagnac interferometer measures a monodromy in time as the gravitational Aharanov-Bohm phase [31,32,64,65], localized along the z axis (as $\tilde{\xi} = 0$), see Fig. 1. This precisely describes an infinite Misner string as a thin tube of gravitomagnetic flux $-im$ [66].

Third, we derive the distributional stress-energy tensor of this string. Among various approaches [70–72], we particularly consider the “harmonic-gauge linearization” of Ref. [50], which stationary KS metrics enjoy as a luxury. Provided the congruence of $\ell^\mu \partial_\mu$ is geodesic and shear-free

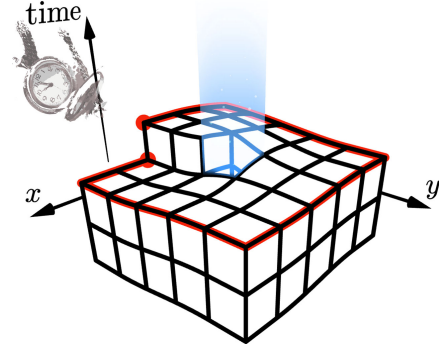


FIG. 1. The Misner string is a “screw dislocation” in spacetime, inducing time monodromy as an Aharanov-Bohm phase.

(GSF), this approach identifies $T^{\mu\nu} = u^{(\mu} J^{\nu)}$ as the source for the double copy solution [50]. Here J^μ is the single copy source, while $u^\mu = \delta^\mu_0$ is the unit timelike Killing vector. For our case, the GSF property of the congruence associated with the null one-form in Eq. (7) can be established from an identity

$$\partial_\mu \ell_\nu = \frac{1}{2r} (\eta_{\mu\nu} + \ell_{(\mu} (2u - \ell)_{\nu)} - i\epsilon_{\mu\nu\rho\sigma} u^\rho \ell^\sigma), \quad (11)$$

while the single copy source J^μ easily follows by acting on the Laplacian $\square := -\eta^{\mu\nu} \partial_\mu \partial_\nu$ to Eq. (1). Applying the formula $T^{\mu\nu} = u^{(\mu} J^{\nu)}$ then immediately reproduces Bonnor’s stress-energy tensor [28,29], describing a point mass m attached to a massless semi-infinite rotating rod.

To sum up, we have convinced ourselves from multiple angles that the KS metric in Eq. (3) indeed exhibits the geometrical and physical characteristics of a Taub-NUT spacetime arising from the Misner string. This analysis demonstrates that the KS description of the solution is not merely a formal construct but physical. Moreover, the quick derivation of the distributional source demonstrates its usefulness as well.

Weyl double copy in position space. Finally, let us unlock more classical double copy relations for the SDTN solution. First, we explicate how our KS metric solidly verifies the *Weyl double copy* correspondence [3,4].

The Weyl double copy directly constructs the Weyl curvatures of algebraically special spacetimes by “squaring” electromagnetic field strengths [2–5] while not referencing a metric nor a gauge potential. For example, take the Coulomb and Schwarzschild solutions. In the spinor language, their field strength and Weyl tensor are

$$F_{\dot{\alpha}\dot{\beta}} = \frac{2q}{4\pi r^2} \tilde{o}_{(\dot{\alpha}} \tilde{l}_{\dot{\beta})}, \quad C_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = -\frac{6m}{4\pi r^3} \tilde{o}_{(\dot{\alpha}} \tilde{o}_{\dot{\beta}} \tilde{l}_{\dot{\gamma}} \tilde{l}_{\dot{\delta})}, \quad (12)$$

where we have omitted the anti-self-dual counterparts. The Weyl double copy is the squaring relation $C_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = F_{(\dot{\alpha}\dot{\beta}} F_{\dot{\gamma}\dot{\delta})}/S$ [2], which holds with $S \propto 1/r$.

Crucially, this relation is *chiral*: there is no mixing between the self-dual and anti-self-dual parts. Thus the Weyl double copy for SDTN seems to simply follow from that of Schwarzschild, if one posits that the SDTN solution is the “self-dual part” of the Schwarzschild solution in the sense that its self-dual Weyl curvature and the principal spinors exactly equal those of the Schwarzschild solution [3,4]. This physical anticipation, however, may appear a bit subtle due to the nonlinearity of gravity. However luckily, straightforward calculations with the explicit KS metric in Eq. (3) concretely verifies this claim, so the Weyl double copy construction of the SDTN solution is completely validated at the fully nonlinear order.

Specifically, the principal spinors for the KS metric in Eq. (3) are found as $\tilde{o}^{\dot{\alpha}} = (1, \zeta)$ and $\tilde{t}^{\dot{\alpha}} = (-\tilde{\zeta}, 1)/(1 + \tilde{\zeta}\zeta)$, one of which being encoded in the null one-form:

$$\ell^\mu = (\sigma^\mu)_{\alpha\dot{\alpha}} \tilde{o}^{\dot{\alpha}} o^\alpha, \quad \tilde{o}^{\dot{\alpha}} = \begin{pmatrix} 1 \\ \zeta \end{pmatrix}, \quad o^\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (13)$$

Crucially, the *real* combinations $[\tilde{o}_\alpha]^* \tilde{o}_\beta$, $[\tilde{t}_\alpha]^* \tilde{t}_\beta$ of these spinors exactly reproduce the null one-forms $-dt \pm dr$ associated with the KS metrics of the Schwarzschild solution as stated in Refs. [3,4,73].

Weyl double copy in twistor space. Meanwhile, the Weyl double copy has been also discussed in *twistor space* [3,4]. This is essentially a statement about the *linearized* Weyl tensor as a massless spin-2 field on flat background constructed from Penrose transform [74,76,77]. The Penrose transform for a massless spin- h field is given by [78]

$$\phi_{\dot{\alpha}_1 \dots \dot{\alpha}_{2h}}(x) = \oint \frac{\tilde{\pi}_\gamma d\tilde{\pi}^{\dot{\gamma}}}{2\pi i} \frac{\tilde{\pi}_{\dot{\alpha}_1} \dots \tilde{\pi}_{\dot{\alpha}_{2h}}}{(f(\tilde{\pi}, \tilde{\pi}x))^{h+1}}, \quad (14)$$

and the self-dual linearized Weyl tensor of the Schwarzschild and SDTN solutions arises from Eq. (14) for $h = 2$ by taking the twistor function [4,80–83]

$$f(\tilde{Z}) = -\tilde{\omega}^\alpha u_{\alpha\dot{\alpha}} \tilde{\pi}^{\dot{\alpha}}, \quad u_{\alpha\dot{\alpha}} := u_\mu (\sigma^\mu)_{\alpha\dot{\alpha}}, \quad (15)$$

whose zeros encode the principal spinors $\tilde{o}_\alpha, \tilde{t}_\alpha$ as per the Kerr theorem [74,84,85]. The twistor-space Weyl double copy implies that taking $h = 1$ with the same function derives the single copy field strength in Eq. (12) [3,4].

Here, the crucial clarification, which the explicit KS metric provides, is that this twistor-space construction from Penrose transform has, in fact, described the *exact* Weyl tensor of the SDTN solution at the fully nonlinear level. Namely, the Cartesian coordinates of the flat background employed in the Penrose transform secretly described KS coordinates for the curved SDTN geometry.

Note that the “position vector” $\vec{x}$, which one identifies from the Killing spinor as $\tilde{\chi}^{\dot{\alpha}}_{\dot{\beta}} = \frac{1}{2}(\vec{x} \cdot \vec{\sigma})^{\dot{\alpha}}_{\dot{\beta}}$ [36], also describes the (spatial part of) our KS coordinates.

Self-dual double copy. Moreover, our KS coordinates are also the Plebański coordinates [57,86] for the second heavenly equation, on which the *self-dual double copy* [1,7,17] is based. Notably, the self-dual double copy would imply fascinating conclusions such as “SDTN is a large- N dyon” in the context of Moyal deformation of the heavenly equation [87–92]. Strangely, however, the SDTN metric is not straightforwardly the self-dual double copy of SDD in terms of the Plebański scalar field [93]: $A_{\alpha\dot{\alpha}} = o_\alpha \partial_{0\dot{\alpha}}(q\zeta/4\pi)$, $h_{\alpha\dot{\alpha}\beta\dot{\beta}} = o_\alpha o_\beta \partial_{0\dot{\alpha}} \partial_{0\dot{\beta}}(m(2r+z)\zeta^2/24\pi)$, if $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ denotes the KS metric perturbation. This puzzle deserves further investigation, in comparison with the Eguchi-Hanson case [7].

Operator KS double copy. Interestingly, we still find that an operator KS double copy is viable, but in a more exotic form. The operator KS double copy promotes the null one-form ℓ_μ in the KS ansatz to a differential operator $\hat{k}_\mu$, which may factorize as $-\frac{1}{2}(\sigma^\mu)_{\alpha\dot{\alpha}} \hat{k}_\mu = o_\alpha \hat{k}_{\dot{\alpha}}$ so that the fields are manifestly put in the light cone gauge [1,7,8]. Intriguingly, we find that Eqs. (2) and (3) precisely arise in this way if the operator $\hat{k}_{\dot{\alpha}}$ is given by

$$\hat{k}_{\dot{\alpha}} = \left(-\frac{\partial_x + i\partial_y}{\partial_z}, 1 \right), \quad (16)$$

which is *nonlocal*. Specifically, it follows from $\hat{k}_{\dot{\alpha}}(1/r) = (1/r)\tilde{o}_{\dot{\alpha}}$ and $\hat{k}_{\dot{\alpha}}\hat{k}_{\dot{\beta}}(1/r) = (1/r)\tilde{o}_{\dot{\alpha}}\tilde{o}_{\dot{\beta}}$ that a double copy correspondence between SDD and SDTN is realized as

$$A_{\alpha\dot{\alpha}} = o_\alpha \hat{k}_{\dot{\alpha}} \frac{q}{4\pi r}, \quad h_{\alpha\dot{\alpha}\beta\dot{\beta}} = o_\alpha o_\beta \hat{k}_{\dot{\alpha}} \hat{k}_{\dot{\beta}} \frac{m}{4\pi r}. \quad (17)$$

Notably, this new form of classical double copy admits a nice physical explanation from an electric-magnetic dual perturbation theory. The key fact is that the SDD can be viewed as an object in the “extended” Maxwell theory where the violation of Bianchi identity by *magnetic matter* $J^\star$ is allowed by the introduction of the dual gauge potential $A^\star$ [47]: $F = dA + *dA^\star$. In this picture, the distributional source for the SDD is

$$J^\mu = \partial_\nu F^{\mu\nu} = qu^\mu \delta^{(3)}(\vec{x}), \\ J^{\star\mu} = -\partial_\nu *F^{\mu\nu} = -iqu^\mu \delta^{(3)}(\vec{x}). \quad (18)$$

We may redescribe these in the “helicity basis” as $J^\mu_+ = J^\mu - iJ^{\star\mu} = 0$ and $J^\mu_- = J^\mu + iJ^{\star\mu} = 2qu^\mu \delta^{(3)}(\vec{x})$, while having $A^\pm_\mu := \frac{1}{2}(A_\mu \pm iA^\star_\mu)$ for the potentials.

As a dynamical field theory, a formulation of this extended Maxwell theory has been given by Zwanziger [51–55]. We note that the propagators of this theory can be summarized into a single formula in the spinor notation:

$$\hat{\Delta}_{\alpha\dot{\alpha}\beta\dot{\beta}}^{+-} = \frac{n_{\alpha\dot{\beta}}\partial_{\beta\dot{\alpha}}}{-n \cdot \partial} \frac{1}{\square}, \quad (19)$$

where n is an auxiliary four-vector that traces back to the Dirac string. Equation (19) describes the position-space propagator from $A_{\beta\dot{\beta}}^-$ to $A_{\alpha\dot{\alpha}}^+$, while the $\langle A^- A^- \rangle$ and $\langle A^+ A^+ \rangle$ propagators vanish. A simple calculation then shows that the SDD gauge potential in Eq. (2) is precisely reproduced by this propagator, if one takes $n_{\alpha\dot{\alpha}} = -o_{\alpha}\bar{o}_{\dot{\alpha}}$ for the auxiliary vector so that $\hat{k}_{\alpha} = -\partial_{1\dot{\alpha}}/\partial_{00}$.

Amusingly, the graviton field arises in the same fashion by using the linearized gravity equivalent of Zwanziger propagator, which is given by doubling Eq. (19) as is advocated in Ref. [55]. In this case, the sources in the two-potential theory are $T_+^{\mu\nu} = 0$ and $T_-^{\mu\nu} = 2mu^{\mu}u^{\nu}\delta^{(3)}(\vec{x})$. The resulting linearized metric perturbation happens to be also the full metric perturbation by taking the KS form [48–50].

In summary, the double copy structure proposed in Eq. (17) can be traced back to propagators between dyonic matter in electrodynamics and linearized gravity:

$$A_{\alpha\dot{\alpha}}^+ = \hat{\Delta}_{\alpha\dot{\alpha}\beta\dot{\beta}}^{+-} J_{\beta\dot{\beta}}^+, \quad h_{\alpha\dot{\alpha}\beta\dot{\beta}}^+ = \hat{\Delta}_{\alpha\dot{\alpha}\gamma\dot{\gamma}}^{+-} \hat{\Delta}_{\beta\dot{\beta}\delta\dot{\delta}}^{+-} T_{\gamma\dot{\gamma}\delta\dot{\delta}}^+. \quad (20)$$

A few comments are in order. First, it is instructive to translate the propagator in Eq. (19) into the vector notation, where one can identify a combination of operators $(\vec{n} \times \vec{\partial} / \vec{n} \cdot \vec{\partial})$ connecting between the Gilbertian and Ampèrian implementations [95] of multipoles. This implies that the nonlocality of the operator in Eq. (16) has encoded the Dirac string structures. Second, our construction could be justified from the on-shell perspective. The “numerator” of the off-shell propagator in Eq. (19) correctly factorizes into the null polarization vectors so that the double copy of sources from $J_-^{\mu} = 2qu^{\mu}\delta^{(3)}(\vec{x})$ to $T_-^{\mu\nu} = 2mu^{\mu}u^{\nu}\delta^{(3)}(\vec{x})$ encodes the squaring relation of the massive on-shell amplitudes [55,96–98]. In this perspective, the nonlocality traces back to that of the x factor. Lastly, it is interesting to apply this approach to construct generic Taub-NUT spacetimes, in which case one naturally obtains a double KS metric perturbation.

Conclusions. In this work, we showed that a complexified diffeomorphism reveals a KS metric for the SDTN solution. The validity of this diffeomorphism and the metric was then verified while highlighting their complex nature. Notably, the KS metric which also exhibits the Misner string defect, supporting its physical interpretation as a gravitational dyon. Further, we established three concrete senses in which the SDTN solution is precisely the gravitational counterpart of the SDD in electromagnetism: KS, Weyl, and nonlocal operator KS classical double copies. Notably, the last one adds to the inventory of classical

double copies as a new entry, whose physical origin is the propagators between dyonic matter.

The most distinctive charm of this work lies in the exciting development along the very first three equations. The usual punchline of classical double copy has been starting with a gravitational solution and then deducing the electromagnetic counterpart. This work, in contrast, takes a reverse path: we constructed an electromagnetic solution and then applied the double copy to yield a new statement in general relativity.

It is also insightful to frame this work in the angle of “classical double copy of gravitational instantons.” The only instance has been the self-dual double copy for the Eguchi-Hanson instanton [99–101] provided by the work [7], which traces back to Refs. [102,103]. This work adds another instance to the discourse, namely the Taub-NUT instanton. It is amusing that the correspondence between self-dual objects in electromagnetism and gravity, which the Gibbons-Hawking ansatz [41,42] had suggested a long time ago, has been strengthened from new perspectives by the advent of the classical double copy program.

The KS metric of the SDTN solution conveys an aesthetic elegance in terms of its minimalist and spinorial nature. To investigate what facilitates this remarkable gem, we could examine its relation to the self-dual limit of the double KS metric in Refs. [6,104].

Lastly, we anticipate that the explicit KS metric we have established here will contribute to future studies on the SDTN solution, both conceptually and practically. In fact, we have already seen that it largely clarifies the subtle aspects of the Weyl double copy. For further consequences or applications, we suggest the following directions: asymptotic symmetries [105–107], integrability of test-particle motion [43,44], black hole perturbation theory [36,37,111,112], or exact graviton propagators [113–115], which will facilitate self-force calculations [116–119] in an idealized setup. Especially, explorations in the scattering context are likely to shed insights on the Kerr black hole scattering for an integrable subsector, on account of the Newman-Janis shift [24]: the “self-dual part” of Kerr is a SDTN solution [36,38,56,120–122].

Amusingly, about a year and a half ago, we deciphered a complete physical picture behind the Newman-Janis algorithm, generalizing the above statement. As is depicted in Fig. 2, we realized that a simple complex coordinate transformation maps the Kerr metric in KS coordinates to another KS metric describing two point masses joined by a Misner string. By zooming to the upper tip of the string, we *by construction* obtained a KS metric, which was precisely Eq. (3). Crucially, this had to be describing a Taub-NUT instanton on account of the Newman-Janis shift. The effort to verify this strong physical conviction gave birth to this work. We will release more details in the future [123].

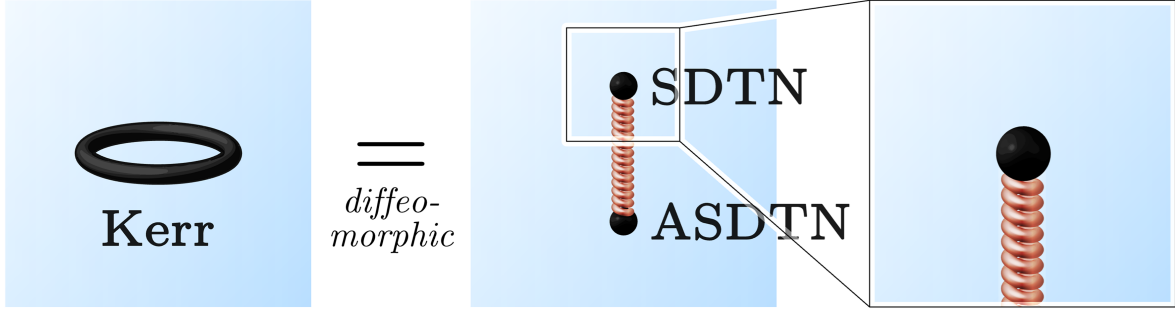


FIG. 2. The very physical origin of the Newman-Janis shift is the fact that the Kerr black hole is secretly a “dumbbell” of Taub-NUT instantons [123].

Note added. The work [124], which appeared recently, provided an *existence* proof for a real diffeomorphism connecting between the KS and Gibbons-Hawking metrics in split signature. However, this paper has already provided a *constructive* proof for all signatures by taking the complexified point of view, e.g., the result and conjecture of Refs. [56,124] are reproduced by taking the real slice $t, x, iy, z \in \mathbb{R}$.

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Appendix: Kerr-Schild to Gibbons-Hawking by null geodesic flow. Suppose a vector field ℓ^μ in complexified Minkowski space that defines a geodesic congruence: $\ell^\mu{}_{;\nu}\ell^\nu = 0$. The time- ϵ flow generated by this vector field is simply $x^\mu \mapsto x^\mu + \epsilon\ell^\mu(x)$, as geodesics are straight lines in flat space. The pullback of the flat metric $\eta = \eta_{\mu\nu}dx^\mu \otimes dx^\nu$ is $\eta + 2\epsilon\ell_{(\mu;\nu)}dx^\mu \otimes dx^\nu + \epsilon^2\eta_{\rho\sigma}\ell^\rho{}_{;\mu}\ell^\sigma{}_{;\nu}dx^\mu \otimes dx^\nu$.

The null vector field associated with the KS metric of the SDTN solution, $\ell^\mu = (1, \zeta, -i\zeta, 1)$, is stationary, geodesic,

and shear free. With the self-duality condition, they together imply that $2\partial_\nu\ell^\mu = \rho\tilde{m}^\mu m_\nu$, when described by a null tetrad $(\ell^\mu, n^\mu = 2u^\mu - \ell^\mu, m^\mu, \tilde{m}^\mu)$ where $u^\mu := \delta^\mu_0$. The complex expansion computes to $\rho = 1/r$.

As a result, it follows that the flat metric transforms to $\eta + \epsilon\rho\tilde{m} \odot m$ if the geodesic flow $x^\mu \mapsto x^\mu + \epsilon\ell^\mu(x)$ is used as a diffeomorphism, where $\odot$ denotes the symmetrized tensor product. In turn, the KS metric $\eta + \phi\ell^2$ transforms to

$$\eta + \epsilon\rho\tilde{m} \odot m + \phi'\ell^2, \quad (\text{A1})$$

where $\phi'(x) := \phi(x + \epsilon\ell(x))$.

Meanwhile, using $\eta = -2u \odot \ell + \ell^2 + \tilde{m} \odot m$, the Gibbons-Hawking metric $-(1 + \phi)^{-1}(u - \phi(\ell - u))^2 + (1 + \phi)(u^2 + \eta)$ boils down to

$$\eta + \phi\tilde{m} \odot m + (\phi^{-1} + 1)^{-1}\ell^2. \quad (\text{A2})$$

Therefore, if we can flow along the geodesic congruence such that $\phi^{-1} + 1 = \phi'^{-1}$ and $\epsilon\rho = \phi$, then the KS metric is mapped to the Gibbons-Hawking metric. Evidently, $\epsilon = m/4\pi$ satisfies both of these conditions. Thus the diffeomorphism from the KS to Gibbons-Hawking metrics is given by $t \mapsto t + m/4\pi$, $x \mapsto x + m\zeta/4\pi$, $y \mapsto y - im\zeta/4\pi$, $z \mapsto z + m/4\pi$, which is equivalent to Eq. (5) up to dropping the constant shift in the time coordinate. It could be interesting if this “trick” also proves to be useful for more general cases such as non-self-dual solutions or multicentered Gibbons-Hawking metrics.

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