

Entanglement asymmetry in the Hayden-Preskill protocol

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In this paper, we consider the time evolution of entanglement asymmetry of the black hole radiation in the Hayden-Preskill thought experiment. We assume the black hole is initially in a mixed state since it is entangled with the early radiation. Alice throws a diary maximally entangled with a reference system into the black hole. After the black hole has absorbed the diary, Bob tries to recover the information that Alice thought should be destroyed by the black hole. In this protocol, we found that a $U(1)$ symmetry of the radiation emerges before a certain transition time (the time when the vanishing entanglement asymmetry begins to grow). This emergent symmetry is exact in the thermodynamic limit and can be characterized by the vanishing entanglement asymmetry of the radiation. The transition time depends on the initial entropy and the size of the diary. What is more, when the initial state of the black hole is maximally mixed, this emergent symmetry survives during the whole procedure of the black hole radiation. We successfully explained this novel phenomenon using the decoupling inequality.

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I. INTRODUCTION

The black hole information paradox, first raised by Hawking [1,2], poses a significant challenge to our understanding of fundamental physics. Hawking's work suggests that black holes emit radiation (known as Hawking radiation) and eventually evaporate, leading to the loss of information about the matter that initially fell into the black hole. This appears to contradict the principles of quantum mechanics, which assert that information should be conserved. For recent reviews, see [3,4].

Page first suggested that the time evolution of a black hole over a sufficiently long period (longer than the scrambling time) can be described by a Haar-random unitary [5,6]. Page's calculations showed that the entropy of the Hawking radiation initially increases, reaches a maximum at the so-called Page time, and then decreases as the black hole continues to evaporate. This behavior is crucial because it implies that information is not lost but is instead encoded in the Hawking radiation.

One of the most notable contributions by Page is the concept of Page time. Page time refers to the point during the black hole evaporation process when the black hole has

emitted half of its initial entropy in the form of Hawking radiation. Before this point, the radiation is highly mixed and contains little information about the initial state of the black hole. After this point, the radiation starts to carry more information about the initial state, making it possible to reconstruct the original information.

Hayden and Preskill consider a more interesting setup [7]: Alice initially prepares the diary maximally entangled with a reference system R and waits until after the Page time to throw the diary A into the black hole C . In this scenario, the black hole C is maximally entangled with its early radiation E (as shown in Fig. 1). The quantum evolution of the black hole can be modeled by applying a Haar-random unitary operator U to the joint system AC . Subsequently, the system can be reinterpreted as a tensor product of Hawking radiation B and the remaining black hole D . Now, Bob wants to recover the information that Alice has thrown into the black hole. Hayden and Preskill demonstrated that this is possible, and when Bob has access to the early radiation, the information essentially comes out as fast as it possibly could. This is why they refer to “old” black holes as information mirrors.

The Hayden-Preskill protocol highlights the concept of quantum scrambling, where information is rapidly mixed and distributed throughout a quantum system. This is a phenomenon of interest not only in black hole physics but also in many-body quantum systems and quantum computing [8–13]. The ability to scramble and unscramble information efficiently is crucial for developing robust quantum error correction codes and quantum algorithms.

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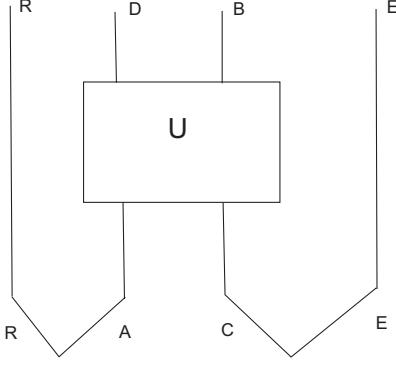


FIG. 1. The Hayden-Preskill protocol: Alice throws N_A qubits diary A , which is maximally entangled with R , into the initial black hole C . The black hole C is entangled with its early radiation E . We only assume that the black hole is in a general mixed state with Rényi-two entropy equal to s . While in the original paper of Hayden and Preskill, they assume that the black hole was just past the Page time and thus is maximally mixed.

Recently, entanglement asymmetry was proposed as a quantity to characterize the breaking and restoration of symmetries [14,15]. On one hand, it has been used as a tool to study the quantum version of the Mpemba effect [16], whose classical counterpart states that hot water can freeze faster than cold water. Since then, entanglement asymmetry and the quantum Mpemba effect have been extensively studied in Hamiltonian dynamics [16–20], quantum field theories [21–24], and random circuits [25–28]. More recently, the quantum Mpemba effects have been observed on quantum simulation platforms [29]. On the other hand, the interplay between entanglement and symmetries in random systems and black hole problems has also attracted much attention in recent years, including studies in [30–35]. In these studies, symmetry-resolved entanglement and entanglement asymmetry play pivotal roles.

In Refs. [32,33], the authors examine the case where a Haar-random pure state preserves a $U(1)$ symmetry, and in this situation, the symmetry-resolved entanglement entropy can be defined and the corresponding symmetry-resolved Page curves are derived. In Ref. [34], the authors investigate when a black hole and the radiation can have emergent symmetries. They use the same qubit model of black hole as in [32,33] but adopt the opposite assumption, i.e., they start with a Haar-random pure state that lacks any symmetry. By calculating the average entanglement asymmetry of the emitted radiation, they find that a $U(1)$ symmetry emerges before the Page time. The mixedness of the black hole’s initial state, a key feature of the Hayden-Preskill protocol, along with the findings in [34], motivates us to investigate whether emergent symmetries of black holes and radiation can arise when evolving from a mixed state.

In this paper, we focus on the entanglement asymmetry in the Hayden-Preskill protocol for a general initial mixed

state (not necessarily being maximally mixed as in the original paper of Hayden and Preskill [7]). In the black hole problem, this corresponds to the time evolution of the entanglement asymmetry of radiation. In Ref. [31], the authors extend the original model of Hayden-Preskill to the case where the system has symmetries. They investigate the role of symmetries in the process of information recovery. However, in this work, we assume that no symmetries exist initially: neither the initial state of the black hole nor the Haar-random evolution operator preserves any symmetries. In [36], the authors consider a similar setup to ours, but focus on the scrambling and decoding problem in the Hayden-Preskill protocol at finite temperatures.

Entanglement asymmetry is defined as

$$\Delta S(\rho_A) = S(\rho_A) - S(\rho_{A,Q}). \quad (1.1)$$

Some explanations about this definition are needed here. $\rho_A = \text{tr}_{\bar{A}} \rho$ is the reduced density matrix of subsystem A and $S(\rho_A) = -\text{tr} \rho_A \log \rho_A$ is the von Neumann entropy of ρ_A . While $\rho_{A,Q}$ is obtained by removing the off-diagonal elements of ρ_A ,

$$\rho_{A,Q} = \sum_q \Pi_q \rho_A \Pi_q, \quad (1.2)$$

where Π_q is the projector onto the q th eigenspace of the corresponding symmetry operator. One can define the Rényi entanglement asymmetry as

$$\begin{aligned} \Delta S^{(n)}(\rho_A) &= S^{(n)}(\rho_{A,Q}) - S^{(n)}(\rho_A) \\ &= \frac{1}{1-n} [\log \text{Tr}(\rho_{A,Q}^n) - \log \text{Tr}(\rho_A^n)]. \end{aligned} \quad (1.3)$$

Then $\Delta S(\rho_A)$ can be accessed from $\Delta S^{(n)}(\rho_A)$ by taking the limit $\Delta S(\rho_A) = \lim_{n \rightarrow 1} \Delta S^{(n)}(\rho_A)$.

II. THE SETUP

We model the merged system (diary A plus initially mixed black hole C) with an N -qubit system S whose dynamics is governed by a random unitary U . The local basis of each qubit is denoted by $|0\rangle_i, |1\rangle_i$ with $i = 1, 2, \dots, N$. The global vacuum $|0\rangle$ is defined as $|0\rangle = \bigotimes_{i=1}^N |0\rangle_i$. The dimension of the Hilbert space of S is $d = 2^N$ and U is a $2^N \times 2^N$ random unitary matrix. We also introduce the $U(1)$ symmetry group generator Q , defined by $Q = \sum_{i=1}^N |1\rangle_i \langle 1|$. Clearly, Q counts the number of excitations above the vacuum.

We initially prepare $d_A = 2^{N_A}$ orthonormal states that only differ on the first N_A qubits,

$$\rho_i = |i_A\rangle \langle i_A| \otimes \rho \quad i = 1, 2, \dots, d_A, \quad (2.1)$$

where $|i_A\rangle$ is the orthonormal basis of the Hilbert space of subsystem A and ρ is the state of the remaining $N - N_A$ qubits. We make no assumptions on the explicit form of ρ . Therefore, in general, $[Q, \rho_i] \neq 0$. We also introduce the canonical purification of ρ as $|\sqrt{\rho}\rangle$, which means $\rho = \text{tr}_E |\sqrt{\rho}\rangle\langle\sqrt{\rho}|$. The pure state $|\sqrt{\rho}\rangle$ lives in the Hilbert space of the system C and the early radiation E . Before evolving the system by applying a random unitary, we maximally entangle the first N_A qubits with a reference system R with the same size.

The initial state of the whole system (including the reference qubits and the early radiation) can be written as

$$|\Psi\rangle = \left(\frac{1}{\sqrt{d_A}} \sum_{i=1}^{d_A} |i_R\rangle |i_A\rangle \right) \otimes |\sqrt{\rho}\rangle. \quad (2.2)$$

As mentioned before, we describe the evolution of the merged system by a Haar-random unitary U .¹ Note that the operator U only acts on the N -qubit system and does not have effect on the early radiation. Then the time-evolved state is

$$|\Psi(t)\rangle = \frac{1}{\sqrt{d_A}} \sum_{i=1}^{d_A} |i_R\rangle U_S \otimes I_E |i_A, \sqrt{\rho}\rangle, \quad (2.3)$$

where we denote the N -qubit system as S and $S = A \cup C = B \cup D$.

III. AVERAGED PURITY OF REDUCED STATE

Tracing out the early radiation E and the reference qubits R , we obtain the reduced density matrix ρ_S of the N -qubit system as

$$\rho_S = \frac{1}{d_A} \sum_{i=1}^{d_A} U \rho_i U^\dagger, \quad (3.1)$$

where ρ_i is defined in Eq. (2.1) and we have omitted the subscript in U_S for simplicity. Here, again, no symmetry restrictions on U are imposed. We have $[\rho_S, Q] \neq 0$ and then $[Q_B, \rho_B] \neq 0$, which allows one to define the entanglement asymmetry of ρ_B .

To compute the averaged purity of ρ_B , i.e., $\mathbb{E}_U[\text{tr}(\rho_B^2)]$, we can first compute $\mathbb{E}_U[\rho_S^{\otimes 2}]$,

$$\mathbb{E}_U[\rho_S^{\otimes 2}] = \frac{1}{d_A^2} \sum_{i,j=1}^{d_A} \mathbb{E}_U[U \rho_i U^\dagger \otimes U \rho_j U^\dagger]. \quad (3.2)$$

For evaluating the expectation value involving four Haar-random unitary matrices U and two operators $\mathcal{O}, \mathcal{O}'$, the following formula is useful:

¹We assume that the black hole absorbs the diary instantly.

$$\mathbb{E}_U[U \mathcal{O} U^\dagger \otimes U \mathcal{O}' U^\dagger] = \sum_{\sigma=\pm} \frac{\text{tr} \mathcal{O} \text{tr} \mathcal{O}' + \sigma \text{tr}(\mathcal{O} \mathcal{O}')}{d(d+\sigma)} \frac{I + \sigma \mathcal{F}}{2}, \quad (3.3)$$

where I and $\mathcal{F}$ are the identity operator and the swap operator on the two “replicas” of the system S , respectively. Denoting $|\alpha\rangle, |\beta\rangle$ as the basis of the Hilbert space of S , one can write $I = \sum_{\alpha, \beta \in \mathcal{H}_S} |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|$ and $\mathcal{F} = \sum_{\alpha, \beta \in \mathcal{H}_S} |\alpha\rangle\langle\beta| \otimes |\beta\rangle\langle\alpha|$.

Using the above formula, it is convenient to analyze the diagonal terms ($i = j$ terms) and cross terms ($i \neq j$ terms) in Eq. (3.2) separately. All the diagonal terms give the same contribution,

$$\begin{aligned} \mathbb{E}_U[U^{\otimes 2} \rho_i^{\otimes 2} U^{\dagger \otimes 2}] &= \frac{(\text{tr} \rho)^2 I + \text{tr}(\rho^2) \mathcal{F}}{d^2 - 1} - \frac{(\text{tr} \rho)^2 \mathcal{F} + \text{tr}(\rho^2) I}{d(d^2 - 1)} \\ &= \frac{I + 2^{-s} \mathcal{F}}{d^2 - 1} - \frac{\mathcal{F} + 2^{-s} I}{d(d^2 - 1)}. \end{aligned} \quad (3.4)$$

Here we have defined $s \equiv -\log_2 \text{tr}(\rho^2)$ as the initial Rényi-two entropy of the black hole. Since all the ρ_i are density matrices, we have $\text{tr} \rho_i = \text{tr} \rho = 1$. We have also assumed they are mixed states, so $\text{tr} \rho_i^2 = \text{tr}(\rho^2) \neq 1$. From the definition of ρ_i , it is obvious that $\text{tr}(\rho_i \rho_j) = 0$ for $i \neq j$.

Using all these facts mentioned in the above paragraph and the formula (3.3), we find that the contribution from the cross terms in Eq. (3.2) is the same,

$$\mathbb{E}_U[U \rho_i U^\dagger \otimes U \rho_j U^\dagger] = \frac{I + \mathcal{F}}{2d(d+1)} + \frac{I - \mathcal{F}}{2d(d-1)}. \quad (3.5)$$

There are d_A diagonal terms and $d_A^2 - d_A$ cross terms. Adding all the terms together, we find

$$\begin{aligned} \mathbb{E}_U[\rho_S^{\otimes 2}] &= \frac{I + 2^{-s} \mathcal{F}}{d_A(d^2 - 1)} - \frac{\mathcal{F} + 2^{-s} I}{d_A d(d^2 - 1)} \\ &\quad + \left(1 - \frac{1}{d_A}\right) \left[\frac{I + \mathcal{F}}{2d(d+1)} + \frac{I - \mathcal{F}}{2d(d-1)} \right] \\ &= \frac{(d_A d - 2^{-s}) I}{d_A d(d^2 - 1)} + \frac{(2^{-s} d - d_A) \mathcal{F}}{d_A d(d^2 - 1)}. \end{aligned} \quad (3.6)$$

The averaged purity of ρ_B , can be obtained from $\mathbb{E}_U[\rho_S^{\otimes 2}]$ as

$$\begin{aligned} \mathbb{E}_U[\text{tr} \rho_B^2] &= \mathbb{E}_U[\text{tr}(\rho_S^{\otimes 2} \mathcal{F}_B \otimes I_D)] \\ &= \frac{(d_A d - 2^{-s}) d_B d_D^2}{d_A d(d^2 - 1)} + \frac{(2^{-s} d - d_A) d_B^2 d_D}{d_A d(d^2 - 1)} \\ &= \frac{(2^N - 2^{-s-N_A}) 2^{N-N_B}}{2^{2N} - 1} + \frac{(2^{N-s-N_A} - 1) 2^{N_B}}{2^{2N} - 1}, \end{aligned} \quad (3.7)$$

where we have used the fact $d_B d_D = d_A d_C = d$ and denote the number of qubits in subsystem B as N_B .

IV. AVERAGED PURITY OF “PRUNED” REDUCED STATE

Now, we consider calculating the moments of $\rho_{B,Q}$, i.e., $\text{tr}(\rho_{B,Q}^n)$. The standard strategy to compute this quantity is to use the integral representation of the projector Π_q , converting it to the charged moments of $\rho_{B,Q}$ and then evaluating the integral. This method has the advantage that in some cases it is possible to derive a formula for $\text{tr}(\rho_{B,Q}^n)$ for general n . However, in our case, we focus only on the Rényi-two entanglement asymmetry, as it captures the main features of entanglement asymmetry. Instead, we adopt a different method. This method was previously used in Ref. [26], where the authors studied the quantum Mpemba effect in random circuit dynamics.

Recall that the pruned reduced state is defined as

$$\rho_{B,Q} = \sum_q \Pi_{q,B} \rho_B \Pi_{q,B}, \quad (4.1)$$

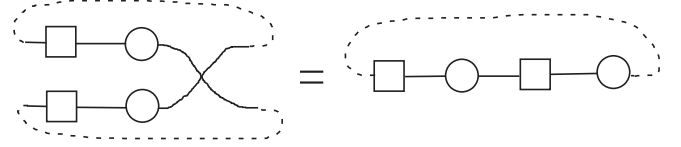


FIG. 2. Diagram representation of identity (4.3).

where $\Pi_{q,B}$ is the projector onto the q -charge sector restricted on subsystem B . Then

$$\begin{aligned} \text{tr}(\rho_{B,Q}^2) &= \sum_{q=0}^{N_B} \text{tr}(\rho_B \Pi_{q,B} \rho_B \Pi_{q,B}) \\ &= \sum_{q=0}^{N_B} \text{tr}(\rho_B^{\otimes 2} (\Pi_{q,B}^{\otimes 2} \mathcal{F}_B) \otimes I_D). \end{aligned} \quad (4.2)$$

In the above equation, we have used the identity

$$\text{tr}(ABAB) = \text{tr}(A^{\otimes 2} (B^{\otimes 2} \mathcal{F})), \quad (4.3)$$

which can be easily proved using the tensor network diagram. See Fig. 2 for an illustration.

The averaged purity of $\rho_{B,Q}$ can be calculated as

$$\begin{aligned} \mathbb{E}_U[\text{tr}(\rho_{B,Q}^2)] &= \sum_{q=0}^{N_B} \text{tr}[\mathbb{E}_U[\rho_B^{\otimes 2}] (\Pi_{q,B}^{\otimes 2} \mathcal{F}_B) \otimes I_D] \\ &= \sum_{q=0}^{N_B} \text{tr} \left[\frac{(d_A d - 2^{-s})I + (2^{-s}d - d_A)\mathcal{F}}{d_A d (d^2 - 1)} (\Pi_{q,B}^{\otimes 2} \mathcal{F}_B) \otimes I_D \right] \\ &= \sum_{q=0}^{N_B} \frac{(d_A d - 2^{-s})d_D^2}{d_A d (d^2 - 1)} \text{tr}(\Pi_{q,B}^2) + \frac{(2^{-s}d - d_A)d_D}{d_A d (d^2 - 1)} [\text{tr}(\Pi_{q,B})]^2 \\ &= \sum_{q=0}^{N_B} \frac{(d_A d - 2^{-s})d_D^2}{d_A d (d^2 - 1)} \binom{N_B}{q} + \frac{(2^{-s}d - d_A)d_D}{d_A d (d^2 - 1)} \binom{N_B}{q}^2 \\ &= \frac{(d_A d - 2^{-s})d_D^2 d_B}{d_A d (d^2 - 1)} + \frac{(2^{-s}d - d_A)d_D}{d_A d (d^2 - 1)} \binom{2N_B}{N_B} \\ &= \frac{(2^N - 2^{-s-N_A})2^{N-N_B}}{2^{2N} - 1} + \frac{(2^{N-s-N_A} - 1)2^{N_B}}{2^{2N} - 1} \frac{(2N_B)!}{(2^{N_B} N_B!)^2}. \end{aligned} \quad (4.4)$$

It is important to note that all the irreducible representations of the $U(1)$ group are one-dimensional and can be labeled by an integer q . $\Pi_{q,B}$ is identity on the q -charge subspace. The dimension of the q -charge subspace equals the number of occupation configurations of q charges distributed on N_B sites. Thus, we have $\text{tr}(\Pi_{q,B}) = \text{tr}(\Pi_{q,B}^2) = \binom{N_B}{q}$. In the above calculations, we also have used the identity

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}. \quad (4.5)$$

V. AVERAGED RÉNYI-TWO ENTANGLEMENT ASYMMETRY

The averaged Rényi-two entanglement asymmetry is defined as

$$\mathbb{E}_U[\Delta S^{(2)}(\rho_B)] = \mathbb{E}_U[\log \text{tr}(\rho_B^2)] - \mathbb{E}_U[\log \text{tr}(\rho_{B,Q}^2)]. \quad (5.1)$$

To proceed, we assume that $\mathbb{E}_U[\log \text{tr}(\rho_B^2)]$ and $\mathbb{E}_U[\log \text{tr}(\rho_{B,Q}^2)]$ can be approximated by $\log \mathbb{E}_U[\text{tr}(\rho_B^2)]$ and $\log \mathbb{E}_U[\text{tr}(\rho_{B,Q}^2)]$, respectively.² From the expression of $\mathbb{E}_U[\text{tr}(\rho_B^2)]$ and $\mathbb{E}_U[\text{tr}(\rho_{B,Q}^2)]$ in Eqs. (3.7) and (4.4), respectively, we find that if $s = N - N_A$, this means that the black hole is initially maximally mixed. Then $\mathbb{E}_U[\text{tr}(\rho_B^2)] = \mathbb{E}_U[\text{tr}(\rho_{B,Q}^2)]$, which leads to $\mathbb{E}[\Delta S^{(2)}(\rho_B)] = 0$. For $s \neq N - N_A$, the averaged Rényi-two entanglement asymmetry is

$$\begin{aligned} \mathbb{E}[\Delta S^{(2)}(\rho_B)] &\approx -\log \frac{\mathbb{E}_U[\text{tr}(\rho_{B,Q}^2)]}{\mathbb{E}_U[\text{tr}(\rho_B^2)]} \\ &= -\log \left[1 + \frac{(2N_B)!(2^{N_B}N_B!)^{-2} - 1}{\frac{2^{N+N_A}-2^{-s}}{2^{N-s}-2^{N_A}} 2^{N-2N_B} + 1} \right]. \end{aligned} \quad (5.2)$$

In the thermodynamic limit $N \rightarrow \infty, N_B \rightarrow \infty$ with N_B/N kept fixed, we can use the Stirling formula $n! \approx n^n e^{-n} \sqrt{2\pi n}$ for large n to simplify the factor involving the factorial in the equation above as

$$\frac{(2N_B)!}{(2^{N_B}N_B!)^2} \approx \frac{1}{\sqrt{\pi N_B}} \quad (5.3)$$

and

$$\frac{2^{N+N_A}-2^{-s}}{2^{N-s}-2^{N_A}} 2^{N-2N_B} = \frac{2^{N_A}-2^{-s-N}}{2^{-s}-2^{N_A-N}} 2^{N-2N_B} \approx 2^{N+N_A+s-2N_B}. \quad (5.4)$$

Finally, in the thermodynamic limit, the averaged Rényi-two entanglement asymmetry can be written as

$$\mathbb{E}[\Delta S^{(2)}(\rho_B)] = -\log \left[1 + \frac{1/\sqrt{\pi N_B} - 1}{2^{N+N_A+s-2N_B} + 1} \right], \quad (5.5)$$

which is the main result in this paper.

From the equation above, we can see that, when $N_B < (N + N_A + s)/2$, the averaged Rényi-two entanglement asymmetry is almost zero. When N_B begins to exceed $(N + N_A + s)/2$, the entanglement asymmetry rapidly saturates to the maximum $\frac{1}{2} \log(\pi N_B)$. This behavior of

²In fact, this approximation can be proved to be valid in the thermodynamic limit [37,38], which matches the regime of interest in this paper.

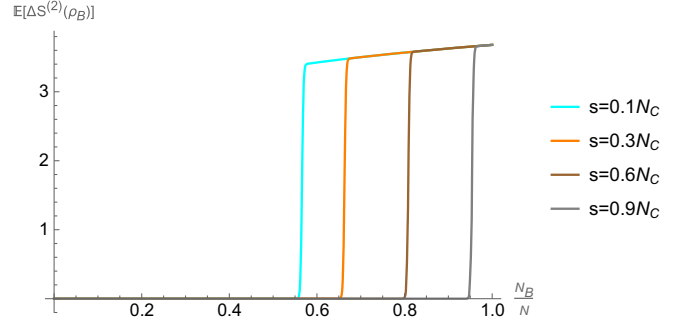


FIG. 3. The averaged Rényi-two entanglement asymmetry as a function of $\frac{N_B}{N}$ [cf. Eq. (5.5) or Eq. (5.2)], where we take $N = 500, N_A = 15$. The maximum allowable value for s is $N_C = N - N_A$.

the averaged Rényi-two entanglement asymmetry as a function of $\frac{N_B}{N}$ is plotted in Fig. 3.

As shown in the figure, the averaged Rényi-two entanglement asymmetry of ρ_B vanishes when N_B is smaller than some number that depends on the initial Rényi entropy s . When N_B becomes slightly greater than that number, the entanglement asymmetry grows to its maximum sharply.

VI. DISCUSSION AND CONCLUSION

Similar to the random pure state case [34], the behavior of entanglement asymmetry can be understood using the decoupling inequality. For the readers' convenience, we give the derivation in the Appendix. In our case, it reads

$$(\mathbb{E}_U[\|\rho_B - \rho_B^{\max}\|_1])^2 \leq \frac{d_B}{d_D} \text{tr} \rho_{AC}^2, \quad (6.1)$$

where $\rho_B^{\max} = \frac{1}{d_B} I_B$ is the maximally mixed state on B . $\|\mathcal{O}\|_1 = \text{tr} \sqrt{\mathcal{O}^\dagger \mathcal{O}}$ is the trace norm for operator $\mathcal{O}$, also called the L_1 norm. ρ_{AC} is the initial density matrix of the N -qubit system,

$$\rho_{AC} = \text{tr}_R \rho_{ACR} = \frac{1}{d_A} I_A \otimes \rho. \quad (6.2)$$

Therefore, we have

$$\text{tr} \rho_{AC}^2 = \frac{1}{d_A^2} \text{tr}(I_A) \text{tr}(\rho^2) = \frac{2^{-s}}{d_A}. \quad (6.3)$$

Then the decoupling inequality (6.1) becomes

$$(\mathbb{E}_U[\|\rho_B - \rho_B^{\max}\|_1])^2 \leq \frac{1}{2^{N+N_A+s-2N_B}}, \quad (6.4)$$

which says that if $N_B < (N + N_A + s)/2$, then ρ_B becomes almost maximally mixed, while the maximally mixed state has vanishing entanglement asymmetry. Therefore, we can conclude that the entanglement asymmetry of ρ_B is almost

vanishing for $N_B < (N + N_A + s)/2$. In particular, when the black hole is initially maximally mixed, i.e., $s = N - N_A$, the entanglement asymmetry is always suppressed. Our result is also applicable in the late-time regime of local unitary dynamics or even Hamiltonian dynamics.

We know that, typically, random pure states always have the maximal possible bipartite entanglement entropies. A similar phenomenon happens here, when $N_B > (N + N_A + s)/2$, the averaged Rényi-two entanglement asymmetry quickly saturates the value $\frac{1}{2}\log(\pi N_B)$. Although no rigorous proofs show that this is the maximal possible value of entanglement asymmetry, there is some other evidence that this might be true. In Ref. [14], the authors showed that the maximal Rényi- α entanglement asymmetry for product states is $\frac{1}{2}\log(\pi N_B \alpha^{1/(\alpha-1)}/2)$, which coincides with our result for $\alpha = 2$. In [39], the authors extended this result to matrix product states and to non-Abelian symmetries.

In summary, in this paper, using the entanglement asymmetry as a modern tool to quantify symmetry breaking, we found a $U(1)$ symmetry of the radiation emerges before a certain transition time (the time when the vanishing entanglement asymmetry begins to grow) in the Hayden-Preskill thought experiment. We confirmed this emergent symmetry by calculating the entanglement asymmetry of the radiation. We found that the transition time depends on the initial entropy s and the size of the diary N_A . When the black hole is maximally entangled with the early radiation, this emergent symmetry survives throughout the entire radiation process. Using the decoupling inequality, we can understand our findings to some extent.

It is quite interesting to generalize our results to more realistic models of black hole evaporation [12] and consider the entanglement asymmetry in random unitary dynamics with structures as in [26,40].

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: THE DECOUPLING INEQUALITY

In this section, we give the derivation of Eq. (6.1). This is a simplified version of the so-called decoupling inequality. However, it is enough for our purpose. For the more general case, we encourage the reader to consult Ref. [41]. For the readers' convenience, we restate the inequality here,

$$(\mathbb{E}_U[\|\rho_B - \rho_B^{\max}\|_1])^2 \leq \frac{d_B}{d_D} \text{tr} \rho_{AC}^2. \quad (\text{A1})$$

To prove it, we also need to introduce the L_2 norm of an operator $\mathcal{O}$, which is defined as $\|\mathcal{O}\|_2 = \sqrt{\text{tr}(\mathcal{O}^\dagger \mathcal{O})}$. It is easy to show that the L_1 norm and L_2 norm satisfy

$$\|\mathcal{O}\|_1 \leq \sqrt{d_{\mathcal{O}}} \|\mathcal{O}\|_2, \quad (\text{A2})$$

where $d_{\mathcal{O}}$ is the dimension of the Hilbert space that $\mathcal{O}$ acts on.

By Jensen's inequality, we know that, for any non-negative function f , one has

$$(\mathbb{E}_U[f(U)])^2 \leq \mathbb{E}_U[f(U)^2]. \quad (\text{A3})$$

A combination of Eqs. (A2) and (A3) gives

$$(\mathbb{E}_U[\|\rho_B - \rho_B^{\max}\|_1])^2 \leq \mathbb{E}_U[\|\rho_B - \rho_B^{\max}\|_1^2] \leq d_B \mathbb{E}_U[\|\rho_B - \rho_B^{\max}\|_2^2]. \quad (\text{A4})$$

The right-hand side of Eq. (A4) reminds us to compute

$$\|\rho_B - \rho_B^{\max}\|_2^2 = \text{tr}(\rho_B - \rho_B^{\max})^2 = \text{tr}(\rho_B^2) - \frac{1}{d_B}, \quad (\text{A5})$$

and

$$\mathbb{E}_U[\text{tr} \rho_B^2] = \mathbb{E}_U[\text{tr}(\rho_S^{\otimes 2} \mathcal{F}_B \otimes I_D)]. \quad (\text{A6})$$

Substituting $\rho_S = U \rho_{AC} U^\dagger$ into the equation above, we obtain

$$\mathbb{E}_U[\text{tr} \rho_B^2] = \mathbb{E}_U[\text{tr}(U^{\dagger \otimes 2} (\mathcal{F}_B \otimes I_D) U^{\otimes 2} \rho_{AC}^{\otimes 2})], \quad (\text{A7})$$

where we have used the cyclicity of the trace.

Now use the identity

$$\begin{aligned} & \mathbb{E}_U[U^{\dagger \otimes 2} (\mathcal{F}_B \otimes I_D) U^{\otimes 2}] \\ &= \frac{1}{d_B} \frac{1 - 1/d_D^2}{1 - 1/d^2} I + \frac{1}{d_D} \frac{1 - 1/d_D^2}{1 - 1/d^2} \mathcal{F}. \end{aligned} \quad (\text{A8})$$

Here, as before, we have used the notation that operators without subscripts such as U , I , and $\mathcal{F}$ act on the full system S . This identity is somewhat complicated to prove. In order not to affect the coherence of the main proof, we will give the proof at the end of this section.

Since $d = d_B d_D$, this means $d > d_B$ and $d > d_D$. Additionally, from Eq. (A8), we know that

$$\mathbb{E}_U[U^{\dagger \otimes 2} (\mathcal{F}_B \otimes I_D) U^{\otimes 2}] \leq \frac{1}{d_B} I + \frac{1}{d_D} \mathcal{F}. \quad (\text{A9})$$

Using the inequality above and substituting it into Eq. (A7), we find

$$\begin{aligned}\mathbb{E}_U[\text{tr}(\rho_B^2)] &\leq \text{tr}\left(\left(\frac{1}{d_B}I + \frac{1}{d_D}\mathcal{F}\right)\rho_{AC}^{\otimes 2}\right) \\ &= \frac{1}{d_B}(\text{tr}\rho_{AC})^2 + \frac{1}{d_D}\text{tr}(\rho_{AC}^2) = \frac{1}{d_B} + \frac{1}{d_D}\text{tr}(\rho_{AC}^2).\end{aligned}\quad (\text{A10})$$

Finally, we have

$$d_B\mathbb{E}_U[\|\rho_B - \rho_B^{\max}\|_2^2] \leq \frac{d_B}{d_D}\text{tr}(\rho_{AC}^2). \quad (\text{A11})$$

From Eq. (A4) we know that the decoupling inequality (A1) is indeed valid.

Now we give the proof of the identity (A8). It is useful to introduce the following two types of maximally entangled states:

$$\begin{aligned}|\mathcal{I}^+\rangle &= |\mathcal{I}^+\rangle_B |\mathcal{I}^+\rangle_D = \sum_{\alpha\beta\gamma\delta} |\alpha\alpha\beta\beta\rangle_B |\gamma\gamma\delta\delta\rangle_D, \\ |\mathcal{I}^-\rangle &= |\mathcal{I}^-\rangle_B |\mathcal{I}^-\rangle_D = \sum_{\alpha\beta\gamma\delta} |\alpha\beta\alpha\beta\rangle_B |\gamma\delta\gamma\delta\rangle_D,\end{aligned}\quad (\text{A12})$$

where the summation indices α, β take values $1, 2, \dots, d_B$, γ, δ take values $1, 2, \dots, d_D$. The Choi-Jamolkowski

mapping allows us to view operators on the two replicas as states,

$$|\mathcal{O} \otimes \mathcal{O}\rangle = (\mathbb{I} \otimes \mathcal{O} \otimes \mathbb{I} \otimes \mathcal{O})|\mathcal{I}^+\rangle. \quad (\text{A13})$$

In this way, the identity operator I and swap operator $\mathcal{F}$ are mapped to $|I\rangle = |\mathcal{I}^+\rangle$ and $|\mathcal{F}\rangle = |\mathcal{I}^-\rangle$, respectively. Then by the Choi-Jamolkowski isomorphism, the identity (A8) is equivalent to

$$\begin{aligned}\langle \mathcal{I}_+ |_B \langle \mathcal{I}_- |_D \mathbb{E}_U[U \otimes U^* \otimes U \otimes U^*] \\ = \frac{1}{d_B} \frac{1 - 1/d_D^2}{1 - 1/d^2} \langle \mathcal{I}_+ |_+ + \frac{1}{d_D} \frac{1 - 1/d_B^2}{1 - 1/d^2} \langle \mathcal{I}_- |_+.\end{aligned}\quad (\text{A14})$$

From the Haar identity for the fourth moment of U ,

$$\begin{aligned}\mathbb{E}_U[U \otimes U^* \otimes U \otimes U^*] \\ = \frac{1}{d^2 - 1} \left[(|\mathcal{I}_+\rangle \langle \mathcal{I}_+| + |\mathcal{I}_-\rangle \langle \mathcal{I}_-|) \right. \\ \left. - \frac{1}{d} (|\mathcal{I}_+\rangle \langle \mathcal{I}_-| + |\mathcal{I}_-\rangle \langle \mathcal{I}_+|) \right],\end{aligned}\quad (\text{A15})$$

and the facts that $\langle \mathcal{I}_\pm | \mathcal{I}_\pm \rangle = d^2$ and $\langle \mathcal{I}_\pm | \mathcal{I}_\mp \rangle = d$ (similar expression for subsystem B, D), we can easily check that Eq. (A14) is indeed satisfied. In fact, the identity (3.3) can also be easily proved using this operator-state mapping method.

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