

Spinning fields on S^d and dS_d , unitary irreducible representations, and ladder operators

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We construct, for spin 0,1,2 tensor fields on S^d , a set of ladder operators that connect the distinct unitary irreducible representations (UIRs) of $SO(d+1)$. This is achieved by relying on the conformal Killing vectors of S^d . For the case of spinning fields, the ladder operators generalize previous expressions with a compensating transformation necessary to preserve the transversality condition. We then extend the results to the exceptional/discrete UIRs of $SO(d,1)$, again relying on the conformal Killing vectors of de Sitter space. Our construction recovers the conventional conformal primary transformations for the scalar fields when the mass term leads to conformal coupling. A similar approach for the spin-2 field leads to the conformal-like operators found recently.

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I. INTRODUCTION

The relation between the unitary irreducible representations (UIRs) of the $ISO(d-1,1)$ flat space isometry group and field equations was elucidated long ago by Wigner [1]. Later, this relation was extended to maximally symmetric spacetimes in [2] (see [3–9] for related work). In this work, we will relate different field realizations of $SO(d+1)$ UIRs on the d -dimensional sphere S^d through differential ladder operators. In addition, we will extend the construction to exceptional and discrete UIRs field realizations on de Sitter space (see [10–12] for nice summaries of $SO(d,1)$ UIRs). As an outcome, we will give a fresh perspective on the conformal-like symmetry operators found in [13] for partially massless gravitons in 4-dimensional de Sitter space. Our setup shows how to generalize these conformal-like symmetry operators to other spacetime dimensions, spins and values of the mass parameter. In particular, for spin-1, we will find a particular tuning of the mass parameter for which the spin-1 field enjoys conformal-like symmetries.

Ladder operators have a long history in physics, their everlasting relevance can be seen for example in recent works where they show up in disparate fields such as black hole Love numbers [14] and spinning conformal

blocks [15]. In the present paper, we will generalize previous works [16,17] which built first-order differential operators that related solutions of the Klein + Gordon scalar field equation with different masses. We will refer to these differential operators as ladder operators. The main ingredient of the construction required the spacetime to possess a closed conformal Killing vector (CCKV), which is also an eigenvector of the Ricci tensor with constant eigenvalue. Here we will focus on *positively curved* maximally symmetric spaces where the eigenvalue condition for CCKV follows naturally from the Einstein character of the space. We will re-derive previous ladder constructions and extend them to spinning fields.

The paper is structured as follows. In Sec. II we start by setting the notation and definitions for conformal primaries in conformally flat spaces. In Sec. III we discuss and enumerate several properties of conformal Killing vectors (CKV) in maximally symmetric spacetimes relevant for our construction. In Sec. IV we construct ladder operators for spin 0,1,2 fields and show how they relate eigenfunctions of the Laplace + Beltrami operator with different eigenvalues. In Sec. V we summarize the results and make some proposals for future work. Then, a number of Appendices follow. In Appendix A we spell out the explicit form of Killing and conformal Killing vectors of S^3 and dS_4 . In Appendix B we summarize the notation for spherical harmonics on S^d . In Appendix C we spell out the explicit computations for the ladder operators acting on tachyon scalar modes corresponding to type I exceptional UIR of $SO(d,1)$. Finally, in Appendix D we work out the action of ladder operators for vector harmonics on S^3 .

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II. PRIMARY FIELDS IN CONFORMALLY FLAT SPACES

In this section, to fix notation, we summarize well known formulas for primary fields in flat space and define primary fields in conformally flat spaces.

A. Primary fields in flat space

Consider x^μ to be the Cartesian coordinates of d -dimensional flat space

$$dx^2 = \eta_{\mu\nu} dx^\mu dx^\nu.$$

Conformal transformations (CT) are diffeomorphisms $x'^\mu(x)$ satisfying

$$\text{Conformal transformation:} \quad dx'^2 = \Omega^2(x) dx^2. \quad (2.1)$$

Any CT of flat space defines a local scale factor $\Omega(x)$ and a local Lorentz transformation $\Lambda^\mu{}_\nu(x)$ which can be read off from

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x) \Lambda^\mu{}_\nu(x). \quad (2.2)$$

Continuous CT at the infinitesimal level take the form

$$x'^\mu(x) \approx x^\mu + \zeta^\mu(x) + \dots, \quad |\zeta| \ll 1$$

with ζ^μ satisfying¹

$$\text{Conformal Killing vector:} \quad \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu = \frac{2}{d} (\partial_\rho \zeta^\rho) \eta_{\mu\nu}. \quad (2.3)$$

Taking the determinant of (2.2) one finds

$$(\Omega(x))^d = \det \left| \frac{\partial x'^\mu}{\partial x^\nu} \right| \approx \left| \frac{\partial(x + \zeta)}{\partial x} \right| \approx |\delta^\mu_\nu + \partial_\nu \zeta^\mu| = 1 + \partial_\mu \zeta^\mu.$$

Thus,

$$\Omega(x) \approx 1 + \frac{1}{d} \partial_\mu \zeta^\mu. \quad (2.4)$$

The ζ -dependence of the local Lorentz transformation $\Lambda^\mu{}_\nu(x)$ can be found from

¹A word on terminology: if the factor $\partial \cdot \zeta$ on the rhs of (2.3) is zero, we say ζ is a Killing vector (KV). If the factor is nonzero, we say ζ is a proper conformal Killing vector. Occasionally, when the factor is constant, ζ is called a homothety.

$$\begin{aligned} \Lambda^\mu{}_\nu(x) &= \frac{1}{\Omega(x)} \frac{\partial x'^\mu}{\partial x^\nu} \approx \left(1 - \frac{1}{d} \partial_\rho \zeta^\rho \right) (\delta^\mu_\nu + \partial_\nu \zeta^\mu) \\ &\approx \delta^\mu_\nu + \partial_\nu \zeta^\mu - \delta^\mu_\nu \frac{1}{d} \partial_\rho \zeta^\rho. \end{aligned} \quad (2.5)$$

It is useful to define infinitesimal generators $\sigma_\zeta(x), \omega_\zeta(x)$ associated to ζ^μ as

$$\sigma_\zeta(x) := \frac{1}{d} \partial_\rho \zeta^\rho, \quad (\omega_\zeta)_{\mu\nu}(x) := \frac{1}{2} (\partial_\nu \zeta_\mu - \partial_\mu \zeta_\nu). \quad (2.6)$$

Then²

$$\Lambda(x) = e^{\omega_\zeta(x)} \quad \text{and} \quad \Omega(x) = e^{\sigma_\zeta(x)}.$$

Spin zero: a scalar primary field with scale dimension Δ is a field Φ which under a CT $x^\mu \rightarrow x'^\mu(x)$ transforms as

$$\text{Primary scalar field:} \quad \Phi(x) \rightarrow \Phi'(x') = \frac{1}{\Omega(x)^\Delta} \Phi(x). \quad (2.7)$$

To first order in ζ we have

$$\begin{aligned} \delta\Phi(x) &:= \Phi'(x) - \Phi(x) \\ &= \left(1 - \frac{\Delta}{d} \partial_\mu \zeta^\mu \right) \underbrace{(\Phi - \zeta^\mu \partial_\mu \Phi)}_{\Phi(x-\zeta)} - \Phi \\ &= - \left(\zeta^\mu \partial_\mu + \frac{\Delta}{d} \partial_\mu \zeta^\mu \right) \Phi \\ &= -(\mathcal{L}_\zeta + \Delta \sigma_\zeta) \Phi \end{aligned} \quad (2.8)$$

The operator inside the parentheses will be extensively used in sections below to construct ladder operators.

Spin one: under conformal transformations (2.2) a primary vector field A_μ with scale dimension Δ transforms as

$$\begin{aligned} \text{Primary vector field:} \quad A_\mu(x) &\rightarrow A'_\mu(x') \\ &= \frac{1}{\Omega(x)^\Delta} \Lambda_\mu{}^\nu(x) A_\nu(x). \end{aligned} \quad (2.9)$$

The $\Lambda_\mu{}^\nu$ can be found from (2.5) to be

$$\Lambda_\mu{}^\nu \approx \delta_\mu^\nu + \partial^\nu \zeta_\mu - \frac{1}{d} (\partial_\rho \zeta^\rho) \delta_\mu^\nu. \quad (2.10)$$

Hence, to first order in ζ we have

²Using (2.3) we can give the alternative expression

$$\omega_{\mu\nu}(x) = \frac{1}{d} \eta_{\mu\nu} (\partial_\rho \zeta^\rho) - \partial_\mu \zeta_\nu.$$

$$\begin{aligned}
\delta A_\mu(x) &= A'_\mu(x) - A_\mu(x) \\
&= -\zeta^\rho \partial_\rho A_\mu + \partial^\nu \zeta_\mu A_\nu - \frac{1}{d} \partial_\rho \zeta^\rho A_\mu - \frac{\Delta}{d} \partial_\rho \zeta^\rho A_\mu \\
&= -(\mathcal{L}_\zeta + (\Delta - 1)\sigma_\zeta) A_\mu,
\end{aligned} \tag{2.11}$$

where $\mathcal{L}_\zeta$ is the Lie derivative on 1-forms

$$\mathcal{L}_\zeta A_\mu = \zeta^\rho \partial_\rho A_\mu + \partial_\mu \zeta^\rho A_\rho.$$

To go from the second to the third line in (2.11) we used (2.3). Equation (2.11) can also be written using (2.6) as [18]

$$\delta A_\mu = -\zeta^\rho \partial_\rho A_\mu + (\omega_\zeta)_\mu{}^\nu A_\nu - \Delta \sigma_\zeta A_\mu \tag{2.12}$$

Each term in this expression respectively manifest the tensorial and scale character of the $A_\mu(x)$ field.

Spin two: a primary symmetric spin-2 field $h_{\mu\nu}$ with scale dimension Δ transforms as

$$h_{\mu\nu} \rightarrow h'_{\mu\nu}(x') = \frac{1}{\Omega(x)^\Delta} \Lambda_\mu{}^\alpha(x) \Lambda_\nu{}^\beta(x) h_{\alpha\beta}(x). \tag{2.13}$$

Proceeding as above one finds

$$\delta h_{\mu\nu} = -(\mathcal{L}_\zeta + (\Delta - 2)\sigma_\zeta) h_{\mu\nu}. \tag{2.14}$$

For completeness, we quote the Lie derivative on a covariant 2-tensor

$$\mathcal{L}_\zeta h_{\mu\nu} = \zeta^\rho \partial_\rho h_{\mu\nu} + \partial_\mu \zeta^\rho h_{\rho\nu} + \partial_\nu \zeta^\rho h_{\rho\mu}. \tag{2.15}$$

Fermions: a spin- $\frac{1}{2}$ conformal primary Ψ_a with scale dimension Δ transforms as

$$\Psi_a \rightarrow \Psi'_a(x') = \frac{1}{\Omega(x)^\Delta} \left(e^{-\frac{1}{2}(\omega_\zeta)^{\mu\nu} \frac{[\gamma_\mu \gamma_\nu]}{4}} \right)_a{}^b \Psi_b(x) \tag{2.16}$$

with $(\omega_\zeta)^{\mu\nu}$ defined in (2.6) and γ_μ the Dirac gamma matrices. At the infinitesimal level one obtains

$$\begin{aligned}
\delta \Psi_a &= -\left(\zeta^\rho \partial_\rho + \frac{1}{4} (\omega_\zeta)^{\mu\nu} \gamma_{\mu\nu} + \Delta \sigma_\zeta \right) \Psi_a \\
&= -(\mathcal{L}_\zeta + \Delta \sigma_\zeta) \Psi_a,
\end{aligned} \tag{2.17}$$

where $\gamma_{\mu\nu} := \frac{1}{2} [\gamma_\mu, \gamma_\nu]$ and $\mathcal{L}_\zeta$ is the Lie derivative acting on spinors

$$\mathcal{L}_\zeta \Psi_a = \zeta^\rho \partial_\rho \Psi_a - \frac{1}{4} \partial^\mu \zeta^\nu (\gamma_{\mu\nu})_a{}^b \Psi_b.$$

Gravitini: a spin- $\frac{3}{2}$ conformal primary $\psi_{a\mu}$ with scale dimension Δ transforms under CT as

$$\psi_{a\mu} \rightarrow \psi'_{a\mu}(x') = \frac{1}{\Omega(x)^\Delta} (e^{-\frac{1}{4}(\omega_\zeta)^{\mu\nu} \gamma_{\mu\nu}})_a{}^b \Lambda_\mu{}^\nu(x) \psi_{b\nu}(x). \tag{2.18}$$

As expected, the infinitesimal transformation reads

$$\delta \psi_{a\mu} = -(\mathcal{L}_\zeta + (\Delta - 1)\sigma_\zeta) \psi_{a\mu} \tag{2.19}$$

with the Lie derivative on gravitini taking the form

$$\begin{aligned}
\mathcal{L}_\zeta \psi_{a\mu} &= \zeta^\rho \partial_\rho \psi_{a\mu} + \partial_\mu \zeta^\nu \psi_{a\nu} + \frac{1}{4} \omega_\zeta^{\mu\nu} (\gamma_{\mu\nu})_a{}^b \psi_{b\mu} \\
&= \zeta^\rho \partial_\rho \psi_{a\mu} + \partial_\mu \zeta^\nu \psi_{a\nu} - \frac{1}{4} \partial^\mu \zeta^\nu (\gamma_{\mu\nu})_a{}^b \psi_{b\mu}.
\end{aligned} \tag{2.20}$$

B. Primary fields in conformally flat spaces

Conformally flat (CF) spaces have line element of the form

$$ds^2 = a^2(x) dx^2. \tag{2.21}$$

It is immediate to realize that flat space conformal transformations $x'(x)$ given by (2.3) are also CT for conformally flat metrics (2.21). The resulting conformal factor can be found simply by noting that under (2.2) we have

$$a^2(x') dx'^2 = a^2(x) \Omega^2(x) dx^2 = \Upsilon^2(x) a^2(x) dx^2, \tag{2.22}$$

where the conformal factor $\Upsilon(x)$ associated to CT in CF spaces is

$$\Upsilon(x) := \frac{a(x') \Omega(x)}{a(x)}. \tag{2.23}$$

Notice that in CF spaces the conformal factor $\Upsilon(x)$ combines $\Omega(x)$ and $a(x)$.

At the infinitesimal level $\Upsilon(x)$ takes the form

$$\begin{aligned}
\Upsilon(x) &\approx \frac{(a(x) + \zeta \cdot \partial a(x) + \dots)(1 + \frac{1}{d} \partial \cdot \zeta + \dots)}{a(x)} \\
&\approx 1 + \frac{1}{d} \partial \cdot \zeta + \zeta \cdot \partial \log a + \dots \\
&\approx 1 + \frac{1}{d} \nabla \cdot \zeta + \dots
\end{aligned} \tag{2.24}$$

In the last line we used that for conformally flat metrics one has³

$$\nabla \cdot \zeta = \partial_\mu \zeta^\mu + d \zeta^\rho \partial_\rho (\log a(x)).$$

³The Christoffel symbols for the conformally flat metric (2.21) are

$$\Gamma_{\alpha\beta}^\mu = (\delta_\alpha^\mu \delta_\beta^\nu + \delta_\beta^\mu \delta_\alpha^\nu - \eta^{\mu\nu} \eta_{\alpha\beta}) \partial_\nu (\log a(x)).$$

Scalars: a scalar conformal primary in a CF space is defined as a field $\Phi(x)$ that, under conformal transformations (2.2), transforms as

$$\begin{aligned} \text{Conformally flat scalar primary: } \Phi(x) &\rightarrow \Phi'(x') \\ &= \frac{1}{\Upsilon(x)^\Delta} \Phi(x). \end{aligned} \quad (2.25)$$

At the infinitesimal level, one gets

$$\begin{aligned} \delta\Phi(x) &= \Phi'(x) - \Phi(x) \\ &= -\left(\zeta^\mu \partial_\mu + \frac{\Delta}{d} (\nabla_\mu \zeta^\mu)\right) \Phi. \end{aligned} \quad (2.26)$$

Tensor fields: a conformally flat spinning primary transforms as⁴

$$\text{CF spinning primary: } \Phi_I(x) \rightarrow \Phi'_I(x') = \frac{1}{\Upsilon(x)^\Delta} \exp\left(-\frac{1}{2}(\omega_\zeta)^{\mu\nu} D(S_{\mu\nu})\right)_I^J \Phi_J(x).$$

Here $D(S_{\mu\nu})_I^J$ denotes the particular representation of the Lorentz generators $S_{\mu\nu}$ carried by Φ_I .

In particular, the transformation law for 1-forms is

$$\text{Conformally flat primary vector: } A_\mu(x) \rightarrow A'_\mu(x') = \frac{1}{\Upsilon(x)^\Delta} \Lambda_\mu{}^\nu(x) A_\nu(x) \quad (2.27)$$

with $\Lambda^\mu{}_\nu(x)$ is defined in (2.2). At the infinitesimal level it reads [cf. (2.12)]

$$\begin{aligned} \delta A_\mu &= -\zeta^\rho \nabla_\rho A_\mu - (\nabla_\mu \zeta^\nu) A_\nu - (\Delta - 1) \frac{1}{d} (\nabla \cdot \zeta) A_\mu \\ &= -\left(\mathcal{L}_\zeta + (\Delta - 1) \frac{1}{d} \nabla \cdot \zeta\right) A_\mu. \end{aligned} \quad (2.28)$$

To obtain the second line we used that the flat CKV fields (2.3) are also solutions to (3.4) for the CF metric (2.21).

Comment on Δ and Weyl weight: in the present context we work with *conformal isometries* which lead to a rescaling of the metric by a conformal factor Υ , see (2.22). Occasionally, CT are presented as a combination of conformal isometries plus Weyl rescalings that leave the metric invariant [20]. In this viewpoint, the factors $1/\Upsilon^{\Delta_{\text{Weyl}}}$ in the field transformation are understood as arising from the Weyl rescaling. If we adopt this perspective the second term in Eq. (2.28) corresponds to the Weyl rescaling, hence $\Delta_{\text{Weyl}} = \Delta - 1$. This result is in agreement with the fact that no Weyl factor is required for gauge fields in $d = 4$.

III. CONFORMAL KILLING VECTOR IDENTITIES IN MAXIMALLY SYMMETRIC SPACES

In this section, we introduce the notation to describe the curvature of maximally symmetric spaces and quote several identities satisfied by Killing and conformal Killing vectors.

d -dimensional maximally symmetric spacetimes (MSS): satisfy

$$\begin{aligned} R_{\mu\nu\alpha\beta}[g] &= \frac{1}{\ell^2} (g_{\mu\alpha} g_{\nu\beta} - g_{\nu\alpha} g_{\mu\beta}) \\ R_{\mu\nu}[g] &= \frac{1}{\ell^2} (d-1) g_{\mu\nu} \quad \text{and} \quad R[g] = \frac{d(d-1)}{\ell^2}. \end{aligned} \quad (3.1)$$

Here $\ell^2 \neq 0$ characterizes the curvature of the space. Positive $\ell^2 > 0$ correspond to spheres and de Sitter spacetimes in Euclidean and Lorentzian signature respectively, while $\ell^2 < 0$ give rise to Hyperbolic and anti-de Sitter (AdS) spacetimes.

Killing vectors: will be denoted by k_μ

$$\text{Killing Vector: } \nabla_\mu k_\nu + \nabla_\nu k_\mu = 0 \rightsquigarrow \nabla_\mu k^\mu = 0. \quad (3.2)$$

Contracting this equation with ∇^μ , commuting covariant derivatives and using Ricci identity one finds

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0 \rightsquigarrow \begin{cases} \nabla^2 k_\nu = -\frac{d-1}{\ell^2} k_\nu \\ \nabla^\mu k_\mu = 0. \end{cases} \quad (3.3)$$

Thus, Killing vectors are transverse eigenvectors of the vector Laplace + Beltrami operator. It is a well-known fact that in positively curved Euclidean MSS (spheres) the set of Killing vectors have the lowest possible eigenvalues of the vector Laplacian [21,22] (see Appendix D).

Conformal Killing vectors (CKV): they are denoted by ζ_μ and satisfy

⁴On a general Lorentz tensor Φ_I , the Lie derivative along a CKV is defined as [19]

$$\mathcal{L}_\zeta \Phi_I = \zeta^\rho \partial_\rho \Phi_I + \frac{1}{2} (\omega_\zeta)^{\mu\nu} D(S_{\mu\nu})_I^J \Phi_J.$$

$$\text{CKV: } \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu = \frac{2}{d} (\nabla \cdot \zeta) g_{\mu\nu}. \quad (3.4)$$

Proper CKV: we denote them by c^μ . These are CKV with $\nabla \cdot c \neq 0$. The set of CKV-solutions will be denoted as $\zeta^\mu = \{k^\mu, c^\mu\}$.

Closed CKVs and MSS: an important result to be extensively used below is that proper CKVs, in *curved* maximally symmetric spaces ($\ell^2 \neq 0$), can be chosen to be closed⁵

$$\text{CCKV: } \forall c^\mu \text{ in MSS} \rightsquigarrow c_\mu = \partial_\mu \Psi. \quad (3.5)$$

Moreover the potential Ψ is given by its associated scale factor

$$c_\mu = -\ell^2 \nabla_\mu \sigma_c \quad \text{where } \sigma_c := \frac{1}{d} \nabla \cdot c. \quad (3.6)$$

Notice the need of ℓ^2 required by dimensional analysis [see (A7) and (A10) for the explicit potentials in S^3 and dS_4]. *Proof.* Inserting (3.5) in (3.4) implies

$$\nabla_\mu c_\nu = \nabla_\nu c_\mu = \sigma_c g_{\mu\nu}. \quad (3.7)$$

Contracting this equation with ∇^ν we get

$$\nabla^\nu \cdot (\nabla_\mu c_\nu = \sigma_c g_{\mu\nu}) \rightsquigarrow \nabla^\nu \nabla_\mu c_\nu = \nabla_\mu \sigma_c. \quad (3.8)$$

Then,

$$\begin{aligned} \nabla_\mu \sigma_c &= \nabla^\nu \nabla_\mu c_\nu = \nabla_\mu \nabla^\nu c_\nu + R^\nu_{\mu\nu\alpha} c^\alpha \\ &= d \nabla_\mu \sigma_c + \frac{1}{\ell^2} (dg_{\mu\alpha} - \delta^\nu_\alpha g_{\mu\nu}) c^\alpha \\ &= d \nabla_\mu \sigma_c + \frac{1}{\ell^2} (d-1) c_\mu. \end{aligned} \quad (3.9)$$

From which we obtain (3.6).

Substituting $\nabla_\mu \sigma_c \rightarrow c_\mu$ in (3.8) and taking into account that Eq. (3.5) implies that $\nabla^\nu \nabla_\mu c_\nu = \nabla^\nu \nabla_\nu c_\mu$ we find

$$\begin{cases} \nabla^2 c_\mu = -\frac{1}{\ell^2} c_\mu \\ \nabla^\mu c_\mu \neq 0. \end{cases} \quad (3.10)$$

Thus, CCKVs are longitudinal eigenvectors of the Laplace + Beltrami operator. In Euclidean MSS (S^d), the set of independent CCKVs $\{c_\mu^{(i)}\}$ comprises $(d+1)$ solutions which are in one to one correspondence with the lowest longitudinal eigenvector space of the vector Laplacian [cf. (3.6) and (3.11)] (see [21]).

⁵This fails to be true in flat space as special conformal transformations are not obtained from closed CKVs.

For completeness, taking ∇^μ in (3.6) one obtains

$$\nabla^2 \sigma_c = -\frac{d}{\ell^2} \sigma_c. \quad (3.11)$$

This implies that the scale factors σ_c associated to CCKVs are eigenfunctions of the scalar Laplace + Beltrami operator. In Euclidean signature, the set of scale factors associated to the independent set of CCKV span the lowest nontrivial eigenspace of the scalar Laplacian (see also Appendix B in [22]).⁶

In the following section, we show that the full spectrum of the scalar Laplacian on spheres S^d can be found from the scale factor of the CCKVs using ladder operators (cf. [16,17]). Our aim in the present paper is to generalize these known results to general tensors on spheres.

Comment: it is important to notice, when comparing to the ladder operators to be constructed below, that CCKVs give no local Lorentz rotation

$$\text{Cosed CKV: } \partial_{[\mu} \zeta_{\nu]} = 0 \rightsquigarrow (\omega_\zeta)^{\mu\nu} = 0$$

Hence, under a CCKV c^μ , a spinning primary transforms effectively as a scalar

$$\Phi_I \rightarrow \Phi'_I(x') = \frac{1}{\Omega(x)^\Delta} \Phi_I(x).$$

Alternatively, at the infinitesimal level it transforms as

$$\delta \Phi_I = \mathcal{L}_c \Phi_I + (\Delta - s) \sigma_c \Phi_I.$$

IV. LAPLACIAN EIGENFUNCTIONS ON POSITIVE CURVATURE MSS AND LADDER OPERATORS

We now present a systematic procedure to construct all the $\text{SO}(d+1)$ and $\text{SO}(d,1)$ UIRs labeled by discrete labels by studying wave equations in positively curved MSS.⁷ Specifically, we will build ladder operators $\mathcal{D}, \mathcal{D}^s$ which, by acting on eigenfunctions of the Laplace + Beltrami operator with a given eigenvalue, raise or lower the eigenvalue. These operators will be constructed out of the CCKVs of the space.

A. Scalars

1. Ladder operators

We start by assuming that we have an eigenfunction Φ of the scalar Laplacian

⁶Equations (3.10) and (3.11) are precisely those used in Sec. 5.2 of [23] to argue that the β_n -modes with $n = 0, \pm 1$ correspond to CKV potentials.

⁷For Euclidean signature our procedures leads to all UIRs of $\text{SO}(d+1)$. For the Lorentzian case, we obtain the exceptional/discrete series UIRs of $\text{SO}(d,1)$.

$$\nabla^2 = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu)$$

in a d -dimensional maximally symmetric space

$$\nabla^2 \Phi = \frac{1}{\ell^2} \lambda \Phi, \quad \lambda = \Delta(d-1-\Delta). \quad (4.1)$$

We have parametrized the eigenvalue λ with the *scale dimension* Δ . The reason for coining this name will become clear below (see the comment at the end of this subsection).⁸ It is useful to define the shadow dimension Δ^s as

$$\text{Shadow dimension: } \Delta^s := d-1-\Delta \rightsquigarrow \lambda = \Delta \Delta^s. \quad (4.2)$$

Since $(\Delta^s)^s = \Delta$, any given Laplace + Beltrami eigenfunction Φ can be characterized as having either Δ or Δ^s scale dimension.

To make contact with known expressions, for positive curvature maximally symmetric spaces ($\ell^2 > 0$) we have:

- (i) Euclidean signature: scalar spherical harmonics Φ_k in S^d correspond to UIRs of $\text{SO}(d+1)$.⁹ From

$$\nabla^2 \Phi_k = -\frac{k(k+d-1)}{\ell^2} \Phi_k \quad k = 0, 1, 2, \dots, \quad (4.3)$$

we see they map to the $\Delta = -k$ solutions in (4.1).

- (ii) Lorentzian signature: tachyonic scalars in dS_d with

$$\ell^2 m^2 = -n(n+d-1), \quad n = 0, 1, 2, \dots \quad (4.4)$$

correspond to type I exceptional UIRs of $\text{SO}(1, d)$ [6,10,23,24]. From

$$(\nabla^2 - m^2) \Phi_n = 0, \quad (4.5)$$

we conclude that they correspond to $\Delta = -n$ in (4.1).

Intriguingly (4.5) possesses a (shift) invariance under [25]¹⁰

$$\phi \mapsto \phi + \lambda_n, \quad \lambda_n = s_{I_1 \dots I_n} X^{I_1} \dots X^{I_n}, \quad (4.6)$$

where $s_{I_1 \dots I_n}$ is a real traceless symmetric constant tensor and X^I are the coordinates of the standard ambient space realization of de Sitter space [26,27]. As stressed in [28] and discussed in Appendix C, this invariance should be understood as gauged. For

⁸See also the comment on Δ below.

⁹For ease of notation in this paragraph we denoted with k the first of the quantum numbers in $I = (k, l_{d-1}, \dots, l_1)$. This is the only one relevant for the discussion. See Appendix B for notation.

¹⁰In Euclidean signature, the λ_n -modes correspond to the zero modes of $\mathcal{O} = \ell^2 \nabla^2 + n(n+d-1)$ on S^d .

$n = 0$, (4.6) becomes the familiar constant shift symmetry of the free, massless scalar (see also [29]). The gauging of the shift symmetry for massless scalars in dS_2 was performed recently in [23] (see also [30] for recent work).

In the following sections, as well as in Appendix C and Sec. IV A 3, we discuss the consequences of the set of zero mode solutions arising from the Wick rotation of the *irregular* solutions on the sphere (Legendre Q -function).

Since two scale dimensions (Δ, Δ^s) are associated to each Φ satisfying (4.1), we define operators $\mathcal{D}, \mathcal{D}^s$, built out of CCKVs c^μ , as

$$\begin{aligned} \mathcal{D}\Phi &:= c^\mu \nabla_\mu \Phi + \Delta \sigma_c \Phi, \\ \mathcal{D}^s \Phi &:= c^\mu \nabla_\mu \Phi + \Delta^s \sigma_c \Phi. \end{aligned} \quad (4.7)$$

Notice these expressions resemble the infinitesimal form of conformal scalar primary transformations (2.26).

In the following, we show that the action of $\mathcal{D}, \mathcal{D}^s$ on modes Φ satisfying (4.1), i.e. with scale dimension Δ , generates eigenmodes Φ' and Φ'' ,

$$\Phi' := \mathcal{D}\Phi, \quad \Phi'' := \mathcal{D}^s \Phi,$$

which are eigenfunctions of the Laplace + Beltrami operator with shifted scale dimension. Explicitly,

$$\begin{aligned} \nabla^2 \Phi' &= \frac{1}{\ell^2} \Delta' (d-1-\Delta') \Phi' \quad \text{with } \Delta' = \Delta + 1, \\ \nabla^2 \Phi'' &= \frac{1}{\ell^2} \Delta'' (d-1-\Delta'') \Phi'' \quad \text{with } \Delta'' = \Delta - 1. \end{aligned} \quad (4.8)$$

Alternatively, $\mathcal{D}, \mathcal{D}^s$ can be viewed as shifting the shadow dimension as $\Delta^s \mapsto \Delta^s \mp 1$

$$\begin{aligned} \nabla^2 \Phi &= \frac{1}{\ell^2} \Delta \Delta^s \Phi \\ \Rightarrow \begin{cases} \nabla^2 \Phi' = \frac{1}{\ell^2} \Delta' \Delta'^s \Phi', & \Delta'^s = \Delta^s - 1, \\ \nabla^2 \Phi'' = \frac{1}{\ell^2} \Delta'' \Delta''^s \Phi'', & \Delta''^s = \Delta^s + 1. \end{cases} \end{aligned}$$

This means that $\mathcal{D}, \mathcal{D}^s$ raise and lower the associated scale dimensions of the Laplace eigenfunction Φ . The commutation relations with the Laplace + Beltrami operator are

$$\begin{aligned} [\ell^2 \nabla^2, \mathcal{D}] \Phi &= (d-2-2\Delta) \mathcal{D}\Phi, \\ [\ell^2 \nabla^2, \mathcal{D}^s] \Phi &= (d-2-2\Delta^s) \mathcal{D}^s \Phi. \end{aligned} \quad (4.9)$$

Proof. We assume we are in a MSS, then

$$\begin{aligned} [\mathcal{D}, \nabla^2] \Phi &= -(\nabla^2 c^\mu) \nabla_\mu \Phi - 2\sigma_c \nabla^2 \Phi - \frac{d-1}{\ell^2} c^\mu \nabla_\mu \Phi \\ &\quad - \Delta (\nabla^2 \sigma_c) \Phi - 2\Delta (\nabla^\rho \sigma_c) \nabla_\rho \Phi. \end{aligned} \quad (4.10)$$

Using (3.10) and (3.11) we get

$$\begin{aligned}
 [\mathcal{D}, \ell^2 \nabla^2] \Phi &= c^\mu \nabla_\mu \Phi - 2\sigma_c \Delta \Delta^s \Phi - (d-1) c^\mu \nabla_\mu \Phi \\
 &\quad + d\sigma_c \Delta \Phi + 2\Delta c^\mu \nabla_\mu \Phi \\
 &= (2\Delta - d + 2) c^\mu \nabla_\mu \Phi + \sigma_c \Delta (-2\Delta^s + d) \Phi \\
 &= (d - 2\Delta^s) [c^\mu \nabla_\mu \Phi + \Delta \sigma_c \Phi] \\
 &= (d - 2\Delta^s) \mathcal{D} \Phi.
 \end{aligned} \tag{4.11}$$

We conclude that

$$[\mathcal{D}, \nabla^2] = \frac{1}{\ell^2} (d - 2\Delta^s) \mathcal{D}, \quad [\mathcal{D}^s, \nabla^2] = \frac{1}{\ell^2} (d - 2\Delta) \mathcal{D}^s. \tag{4.12}$$

The second identity follows from $(\Delta^s)^s = \Delta$. Therefore, it now follows from (4.12) that

$$\begin{aligned}
 \nabla^2 \Phi' &= \nabla^2 (\mathcal{D} \Phi) = \mathcal{D} \nabla^2 \Phi - \frac{1}{\ell^2} (d - 2\Delta^s) \mathcal{D} \Phi \\
 &= \frac{1}{\ell^2} (\Delta \Delta^s - d + 2\Delta^s) \mathcal{D} \Phi \\
 &= \frac{1}{\ell^2} (\Delta + 1)(d - 1 - (\Delta + 1)) \Phi' \\
 &= \frac{1}{\ell^2} (\Delta + 1)(\Delta + 1)^s \Phi'.
 \end{aligned} \tag{4.13}$$

Similarly, we have

$$\begin{aligned}
 \nabla^2 \Phi'' &= \frac{1}{\ell^2} (\Delta^s + 1)(\Delta^s + 1)^s \Phi'' \\
 &= \frac{1}{\ell^2} (\Delta - 1)(d - \Delta) \Phi''.
 \end{aligned} \tag{4.14}$$

Comment on conformal coupling: the right-hand side (rhs) in (4.11) vanishes for $\Delta^s = \frac{d}{2}$ which is equivalent to $\Delta = \frac{d}{2} - 1$. This means that the $\mathcal{D}$ -operators preserve the $\lambda = (\frac{d}{2} - 1) \frac{d}{2}$ eigenspace. Notice that the value $\Delta = \frac{d}{2} - 1$ coincides with the *classical scale dimension* of a scalar field in d -dimensions. This situation has a neat interpretation as $\lambda = \ell^2 m^2 = (\frac{d}{2} - 1) \frac{d}{2}$ is precisely the “mass” value equivalent to conformal coupling to the scalar curvature that guarantees Weyl invariance of the field equation [31] and therefore invariance under the CKVs of the MSS (cf. right column top in Fig. 2).

2. Building discretely labeled UIRs on $\ell^2 > 0$ MSS

Sphere: the constant function on the sphere $\Phi_0(x) = \text{const}$ has vanishing Laplace eigenvalue, $\lambda = 0$. The scale dimensions associated to Φ_0 are $\Delta = 0$ and $\Delta^s = d - 1$. From (4.7) we learn that $\mathcal{D} \Phi_0 = 0$ vanishes identically. However, the action of $\mathcal{D}^s$ for any of the $(d + 1)$ CCKVs c^μ

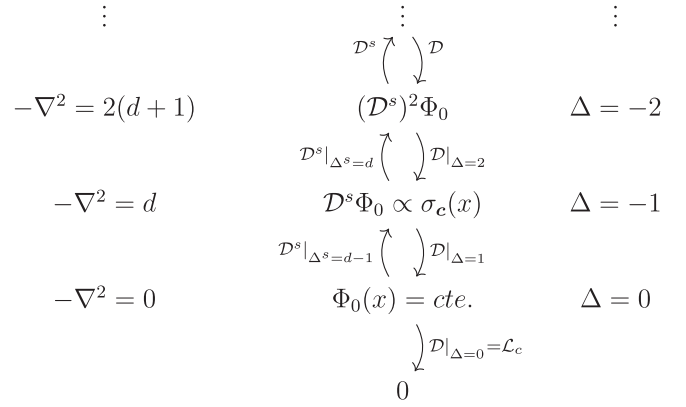


FIG. 1. Construction of the full tower of scalar spherical harmonics on spheres out of $\Phi_0(x) = \text{const}$. The action of ladder operators $\mathcal{D}^s$, built out of CCKVs c^μ , takes us up indefinitely along the tower. At each level, denoted by $\Delta = -k$, the remaining states of the UIR multiplet are obtained by acting with Lie derivatives along the Killing vectors of the sphere.

gives rise $(d + 1)$ nontrivial eigenfunctions¹¹

$$\mathcal{D}^s \Phi_0 \propto \sigma_c.$$

Comparing with (3.11) and in accordance with (4.14), we recognize that the set $\{\sigma_c\}$ built from each of the $(d + 1)$ linearly independent c^μ 's, coincides with the first nontrivial harmonics on S^d , i.e. those corresponding to $\lambda = -d$ or $k = 1$ in (4.3). Naturally, they mix under the action of the Killing vectors. If we continue recursively, acting with $(\mathcal{D}^s)^k$ on Φ_0 , we obtain scalar spherical harmonics with all possible eigenvalues of the Laplacian (see Fig. 1). The remaining states on the multiplet can then be found by Lie derivative actions along the Killing vectors on the sphere. The standard inner product on S^d gives rise to the $\mathbf{l} = (k, 0, \dots, 0)$ finite dimensional UIR of $\text{SO}(d + 1)$.

de Sitter space and tachyonic shift symmetries: the Lorentzian signature case presents additional features. We start by reminding the reader that from

$$ds^2 = \ell^2 (d\theta^2 + \sin^2 \theta d\Omega_{d-1}^2),$$

with $d\Omega^2$ the round sphere metric, we obtain the de Sitter metric by making the Wick rotation

$$\theta \mapsto \frac{\pi}{2} - it. \tag{4.15}$$

The result is

$$ds^2 = \ell^2 (-dt^2 + \cosh^2 t d\Omega_{d-1}^2). \tag{4.16}$$

¹¹The $(d + 1)$ CCKVs are customarily denoted as $\mathbf{D}, \mathbf{K}_i$ with $i = 1, \dots, d$.

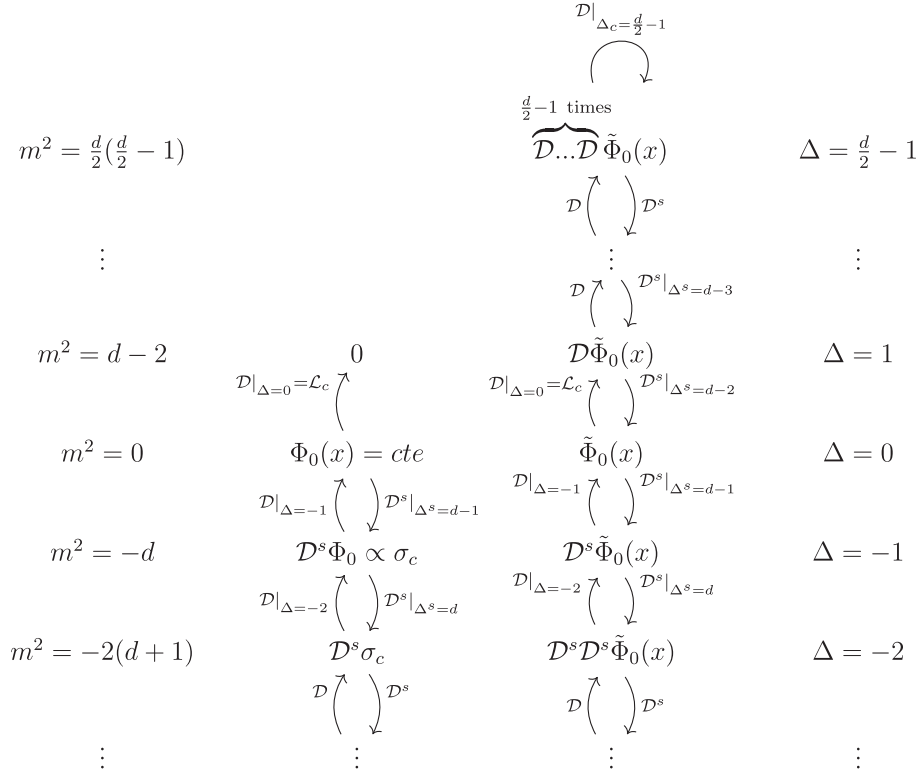


FIG. 2. On the left column we show that all regular gauge modes (4.6) at different levels are connected by the $\mathcal{D}$, $\mathcal{D}^s$ ladder operators acting on the constant function Φ_0 . On the right column we display that the irregular modes are also connected through $\mathcal{D}$, $\mathcal{D}^s$ ladders. Amusingly in even dimensions the action of $\mathcal{D}$ -ladder stops at level $\Delta = \frac{d}{2} - 1$ (cf. top right column).

It is straightforward to show that a constant function on de Sitter spacetime, which we denote $\Phi_0(x) = \text{const.}$, is a seed for all the gauge modes λ_n displayed in (4.6) (see left column in Fig. 2). As well known, the gauge modes can be written as homogeneous polynomials in embedding space coordinates. Their explicit form in de Sitter global coordinates (4.16) is obtained by performing the Wick rotation (4.15) on the standard spherical harmonics $Y_l(\theta, \Omega)$ (see Appendix C).¹²

However, in Lorentzian signature at level $\Delta = 0$, a second (nontrivial) solution exists which we denote as $\tilde{\Phi}_0(x)$. Explicitly, Φ_0 and $\tilde{\Phi}_0$ are solutions to the massless KG equation independent of the (spatial) coordinates on the sphere in (4.16). They are obtained as

$$\partial_t((\cosh t)^{d-1} \partial_t \Phi) = 0 \rightsquigarrow \Phi = c_1 + c_2 \int^t \frac{1}{(\cosh t')^{d-1}} dt'. \quad (4.17)$$

The term involving the integral in (4.17) is what we call $\tilde{\Phi}_0$. It can be understood as the Lorentzian version of the Legendre Q -function (see Appendix C). Interestingly, both

¹²We succinctly denote by $\mathbf{l}$ the full set of quantum numbers labeling the harmonic (see Appendix B).

$\mathcal{D}$ and $\mathcal{D}^s$ act nontrivially on it.¹³ Amusingly, in even dimensions, the action of $\mathcal{D}$ stops at level $\Delta = \frac{d}{2} - 1$. At this particular value the mass in the KG equation coincides with conformal coupling and $\mathcal{D}$ commutes with the Laplace operator [cf. (4.12)]. Equivalently, for $\Delta = \frac{d}{2} - 1$ the ladder $\mathcal{D}$ preserves the space of solutions of the Klein + Gordon equation. $\mathcal{D}$ is nothing but the infinitesimal form of a combined Weyl and conformal diffeomorphism transformation of the scalar field [see comment below (2.28)].

3. Action of $\mathcal{D}, \mathcal{D}^s$ on dS_d tachyonic mode functions

Although the construction works for any CCKV in positive curvature MSS, simple expressions are found for

Sphere: $\mathbf{c} = \sin \theta \partial_\theta$ in $\frac{ds^2}{\ell^2} = d\theta^2 + \sin^2 \theta d\Omega_{d-1}^2$

de Sitter: $\mathbf{c} = \cosh t \partial_t$ in $\frac{ds^2}{\ell^2} = -dt^2 + \cosh^2 t d\Omega_{d-1}^2$.

Our construction shows that the set of states $\{\mathcal{D}^s \dots \mathcal{D} \Phi_0\}$ on the left column in Fig. 2 correspond to the regular gauge

¹³The t -dependent *zero mode* for the massless scalar in $4d$ de Sitter was originally discussed in [32] (see also [33–36]).

modes discussed in the context of scalar tachyons in de Sitter (see [10,23,37]).

In this section we spell the details of action of $\mathcal{D}$, $\mathcal{D}^s$ on the mode functions for scalar tachyons satisfying

$$(-\nabla^2 + m^2)\Psi = 0$$

The relevant feature of the following discussion will be on the shift of the mass m^2 , parametrized by $\Delta = -n$. Therefore, in the present section, we suppress the angular quantum numbers $\mathbf{l}$ and simply denote the tachyon field with a subscript n . We write

$$\Phi_n(t, \Omega) = (\cosh t)^{-(d-2)/2} P_{n+\frac{d-2}{2}}^{-(l+\frac{d-2}{2})}(i \sinh t) Y_l(\Omega) \quad (4.18)$$

$$\tilde{\Phi}_n(t, \Omega) = (\cosh t)^{-(d-2)/2} Q_{n+\frac{d-2}{2}}^{-(l+\frac{d-2}{2})}(i \sinh t) Y_l(\Omega). \quad (4.19)$$

It is sufficient to compute the action of the ladder $\mathcal{D}^s \Phi_n$ with respect to only one (out of d) CKV c^μ . The ladders with respect to the remaining CKV can be found using the de Sitter isometries k_μ as

$$\begin{aligned} [\mathcal{L}_k, \mathcal{D}_c^s] \Phi_n &= (\mathcal{L}_{[k,c]} + (n+d-1)\sigma_{[k,c]}) \Phi_n \\ &= \mathcal{D}_{[k,c]}^s \Phi_n. \end{aligned} \quad (4.20)$$

In this equation, we have explicitly denoted the CKV dependence of the ladder operator by a subscript, $\mathcal{D}_c^s$.

We start by choosing, in global coordinates, the CKV given by

$$c_\mu = -\partial_\mu \sinh t \rightsquigarrow \sigma_c = \sinh t. \quad (4.21)$$

This choice is particularly convenient since it does not involve the spatial coordinates Ω .

Rising ladder: our aim is to compute

$$\mathcal{D}^s \Phi_n = (\mathcal{L}_c + (n+d-1)\sigma_c) \Phi_n. \quad (4.22)$$

More explicitly, one has

$$\mathcal{D}^s \Phi_n = (\cosh t \partial_t + (n+d-1) \sinh t) \Phi_n. \quad (4.23)$$

Inserting (4.18) into this equation, and introducing $z = i \sinh t$, we straightforwardly find

with $m^2 = -n(n+d-1)$. The mode solutions to the Klein + Gordon equation in global coordinates take the form (see Appendix C for notation and details)

$$\Psi_l(t, \Omega) = f_l(t) Y_l(\Omega)$$

where

$$f_l(t) \in \left\{ (\cosh t)^{-(d-2)/2} P_{n+\frac{d-2}{2}}^{-(l+\frac{d-2}{2})}(i \sinh t), (\cosh t)^{-(d-2)/2} Q_{n+\frac{d-2}{2}}^{l+\frac{d-2}{2}}(i \sinh t) \right\}.$$

$$\begin{aligned} \mathcal{D}^s \Phi_n &= -i Y_l(\Omega) (\cosh t)^{-\frac{d-2}{2}} \\ &\times \left((z^2 - 1) \frac{\partial}{\partial z} + \left(n + \frac{d}{2} \right) z \right) P_{n+\frac{d-2}{2}}^{-(l+\frac{d-2}{2})}(z). \end{aligned} \quad (4.24)$$

Using [Eq. (14.10.4) in [38]]

$$\left((x^2 - 1) \frac{\partial}{\partial x} + (\nu + 1)x \right) P_\nu^\mu(x) = (\nu + \mu + 1) P_{\nu+1}^\mu(x)$$

one obtains

$$\mathcal{D}^s \Phi_n = -i(n+l+d-1) \Phi_{n+1}. \quad (4.25)$$

Similarly, for the Q -modes (4.19) we obtain

$$\mathcal{D}^s \tilde{\Phi}_n = -i(1+n-l) \tilde{\Phi}_{n+1}. \quad (4.26)$$

Some comments are in order:

- (i) Gauge modes $l \leq n$ of $\Delta = -n$ are mapped onto gauge modes of the $\Delta = -(n+1)$ theory.
- (ii) The lowest physical modes, i.e., $l = n+1$, of the $\Delta = -n$ theory are mapped onto gauge modes of the $\Delta = -(n+1)$ theory.
- (iii) All higher physical modes of the $\Delta = -n$ theory (i.e., modes with $l \geq n+2$) are mapped onto physical modes of the $\Delta = -(n+1)$ theory.
- (iv) Q -modes with $l = n+1$ (physical regime) are zero-modes of $\mathcal{D}^s$.

Lowering ladder: we now wish to compute

$$\mathcal{D} \Phi_n = (\mathcal{L}_c - n \sigma_c) \Phi_n. \quad (4.27)$$

Proceeding as above we find

$$\begin{aligned} \mathcal{D} \Phi_n &= i Y_l(\Omega) (\cosh t)^{-\frac{d-2}{2}} \\ &\times \left(-(z^2 - 1) \frac{\partial}{\partial z} + \left(n + \frac{d-2}{2} \right) z \right) P_{n+\frac{d-2}{2}}^{-(l+\frac{d-2}{2})}(z). \end{aligned} \quad (4.28)$$

We now use [Eq. (14.10.5) in [38]]

$$\left(- (x^2 - 1) \frac{\partial}{\partial x} + \nu x\right) P_\nu^{-\mu}(x) = (\nu - \mu) P_{\nu-1}^{-\mu}(x),$$

and arrive at

$$\mathcal{D}\Phi_n = i(n - l)\Phi_{n-1}. \quad (4.29)$$

For the Q -modes, the result is

$$\mathcal{D}\tilde{\Phi}_n = i(n + l + d - 2)\tilde{\Phi}_{n-1}. \quad (4.30)$$

We conclude this section with the following comments:

- (i) Physical modes of the $\Delta = -n$ theory ($l \geq n + 1$) are mapped onto physical modes of the $\Delta = -(n - 1)$ theory.
- (ii) Gauge modes with $l = n$ of the $\Delta = -n$ theory are zero-modes of $\mathcal{D}$. Hence they cannot transform into physical modes under the action of $\mathcal{D}$.

B. Vectors

1. Longitudinal modes

Longitudinal eigenmodes of the vector Laplacian are well known to be given by derivatives of the scalar harmonics. We do not elaborate on them since they follow easily from the discussion of the previous section [21,39].

2. Transverse modes

We now proceed to construct ladder operators for transverse vector harmonics A_μ satisfying

$$\nabla^2 A_\mu = \frac{1}{\ell^2} \lambda A_\mu, \quad \nabla^\mu A_\mu = 0, \quad \lambda = 1 + \Delta \Delta^s \quad (4.31)$$

with Δ^s defined in (4.2) (see Appendix D for the allowed eigenvalues on spheres).

We define two Ladder operators

$$\mathcal{D}A_\mu := \mathcal{L}_c A_\mu + (\Delta - 1)\sigma_c A_\mu - \frac{1}{\Delta} \nabla_\mu (c^\rho A_\rho), \quad (4.32)$$

and

$$\mathcal{D}^s A_\mu := \mathcal{L}_c A_\mu + (\Delta^s - 1)\sigma_c A_\mu - \frac{1}{\Delta^s} \nabla_\mu (c^\rho A_\rho). \quad (4.33)$$

They depend on a CCKV c^μ and the scale dimension of A_μ , while they also include a compensating *gauge-looking* transformation necessary to preserve the transversality condition. The reader should keep in mind that the gauge-looking transformation becomes an actual invariance only if λ is tuned to the strictly massless value [corresponding to $m^2 = 0$ in (4.42)]. Notice again the similarity of (4.32) and (4.33) with the infinitesimal form of a conformal vector primary transformation in (2.28).

The action of $\mathcal{D}$, $\mathcal{D}^s$ on an A -mode with scale dimension Δ gives¹⁴

$$A' := \mathcal{D}A \quad \text{and} \quad A'' := \mathcal{D}^s A.$$

The modes A' , A'' have scale dimensions shifted by one unit

$$\Delta' = \Delta + 1 \quad \text{and} \quad \Delta'' = \Delta - 1.$$

These expressions generalize (4.8) to the case of vectors.

Details and proofs: consider the ansatz

$$\mathcal{D}A_\sigma := \mathcal{L}_c A_\sigma + \gamma \sigma_c A_\sigma + \alpha c^\mu \nabla_\sigma A_\mu \quad (4.34)$$

The coefficients γ and α can be fixed by demanding the transformed function to satisfy

$$\text{Eigenmode:} \quad \nabla^2 (\mathcal{D}A_\sigma) = \frac{1}{\ell^2} (\lambda + \delta\lambda) \mathcal{D}A_\sigma \quad (4.35)$$

with $\delta\lambda$ the change in the eigenvalue, and

$$\text{Transversality:} \quad 0 = \nabla^\sigma (\mathcal{D}A_\sigma). \quad (4.36)$$

The need for a compensating “gauge-looking transformation” can be seen by taking the divergence of (4.34). Indeed,

$$\begin{aligned} 0 &= \nabla^\sigma (c^\mu \nabla_\mu A_\sigma + (1 + \gamma)\sigma_c A_\sigma + \alpha c^\mu \nabla_\sigma A_\mu) \\ &= c^\mu \nabla^\sigma \nabla_\mu A_\sigma - \frac{1}{\ell^2} (1 + \gamma - \alpha\lambda) c^\mu A_\mu \\ &= \frac{d-1}{\ell^2} c^\mu A_\mu - \frac{1}{\ell^2} (1 + \gamma - \alpha\lambda) c^\mu A_\mu, \\ &\rightsquigarrow \gamma = d - 2 + \alpha(1 + \Delta \Delta^s). \end{aligned} \quad (4.37)$$

To go from the first to the second line we use (3.6), (3.7), (3.10), and (4.31). To get the third line we used that in MSS $[\nabla_\sigma, \nabla_\mu]A^\sigma = \frac{d-1}{\ell^2} A_\mu$. The eigenmode condition (4.35) implies

$$[\ell^2 \nabla^2, \mathcal{D}]A_\sigma = \delta\lambda \mathcal{D}A_\sigma. \quad (4.38)$$

Alternatively computing the left-hand side (lhs) for a MSS space with c a CCKV one obtains

¹⁴We denote coordinate free 1-forms as $A = A_\mu dx^\mu$.

$$\begin{aligned}
[\ell^2 \nabla^2, \mathcal{D}]A_\sigma &= \underbrace{(d-2-2(\gamma+1)+2\alpha)}_{\delta\lambda} c^\mu \nabla_\mu A_\sigma \\
&\quad + \underbrace{(2\lambda-d(\gamma+1)+2(d-1)\alpha)}_{\delta\lambda(1+\gamma)} \sigma_c A_\sigma \\
&\quad + \underbrace{(2+(d-2)\alpha)}_{\delta\lambda\alpha} c^\mu \nabla_\sigma A_\mu. \quad (4.39)
\end{aligned}$$

Below the underbraces, we have written what each term should equal for (4.38) to be satisfied. Since c is a CCKV, using (3.7), (3.10), and (3.11) we get an overdetermined linear system of equations, i.e., three equations for two unknowns $\delta\lambda, \alpha$ [γ is given by (4.37)]

$$\begin{aligned}
d-2-2(\gamma+1)+2\alpha &= \delta\lambda \\
2\lambda-d(\gamma+1)+2(d-1)\alpha &= (1+\gamma)\delta\lambda \\
2+(d-2)\alpha &= \alpha\delta\lambda. \quad (4.40)
\end{aligned}$$

Amusingly, this system of equations has two solutions corresponding to the rising and lowering ladder operators. The two solutions are

$$(\gamma, \alpha, \delta\lambda) = \begin{cases} \text{I. } \left(\frac{\Delta^2-\Delta-1}{\Delta}, -\frac{1}{\Delta}, d-2-2\Delta \right) \\ \text{II. } \left(\frac{(\Delta^s)^2-\Delta^s-1}{\Delta^s}, -\frac{1}{\Delta^s}, d-2-2\Delta^s \right). \end{cases} \quad (4.41)$$

Inserting these solutions in (4.34) we obtain (4.32) and (4.33).

Comment on conformal symmetry: as for scalar fields, we have $\delta\lambda = 0$ when the scale dimension coincides with the classical value, i.e., $\Delta = \frac{d}{2} - 1$. This means that the action of the ladder operators (4.32) preserves the space of solutions (4.31). It is satisfying to recognize that in $d = 4$ where the classical engineering dimension is $\Delta = 1$ we have

$$\nabla^2 A_\mu = \frac{3}{\ell^2} A_\mu, \quad \nabla_\mu A^\mu = 0.$$

These are Maxwell's equations in Lorenz gauge (see [4,5,40]). Moreover, for $\Delta = 1$ the ladder $\mathcal{D}$ reduces to the Lie action plus a compensating gauge transformation. Hence, the fact that the action of the ladder operators preserves the solution space simply expresses the conformal invariance of Maxwell's equations in $d = 4$.¹⁵

On the other hand, for $d \neq 4$, vector fields with $\Delta = \frac{d}{2} - 1$ are massive but still enjoy a symmetry generated by CCKV. This means that the space of solutions of

$$\nabla^2 A_\mu = \frac{1}{\ell^2} \left(d-1 + \underbrace{\frac{(d-2)(d-4)}{4}}_{m^2} \right) A_\mu, \quad \nabla^\mu A_\mu = 0 \quad (4.42)$$

is preserved under the action of

$$\mathcal{D}|_{\Delta=\frac{d}{2}-1} A_\mu = c^\rho \left(\nabla_\rho A_\mu - \frac{2}{d-2} \nabla_\mu A_\rho \right) + \frac{d(d-4)}{2(d-2)} \sigma_c A_\mu.$$

Moreover, for $d \geq 4$, the spin-1 fields in (4.42) satisfy the Higuchi bound $m^2 \geq 0$ (4.60) and, thus, they are unitary. See the discussions below [around (4.60)] for more details on the Higuchi bound.

For completeness, we quote the commutation relations with the Laplace + Beltrami operator for arbitrary d and Δ

$$\begin{aligned}
[\ell^2 \nabla^2, \mathcal{D}]A_\mu &= (d-2-2\Delta)\mathcal{D}A_\mu, \\
[\ell^2 \nabla^2, \mathcal{D}^s]A_\mu &= (d-2-2\Delta^s)\mathcal{D}^s A_\mu. \quad (4.43)
\end{aligned}$$

Comment on Killing vectors and ladder operators: Killing vectors k_μ are annihilated by $\mathcal{D}$. Comparing (3.3) with (4.31), we learn that Killing vectors have $\Delta = -1$. Inserting this value in (4.34) and using (4.41), we obtain

$$\begin{aligned}
\mathcal{D}k_\mu &= \mathcal{L}_c k_\mu - \sigma_c k_\mu + c^\rho \nabla_\mu k_\rho \\
&= c^\rho (\nabla_\rho k_\mu - \nabla_\mu k_\rho) + \nabla_\mu c^\rho A_\rho - \sigma_c k_\mu = 0. \quad (4.44)
\end{aligned}$$

The last two terms cancel by virtue of (3.7). The result (4.44) is the generalization of $\mathcal{D}|_{\Delta=0}\Phi_0 = 0$, shown in Figs. 1 and 2, to the case of vector fields. In passing we comment that there is a tower of shift-symmetric tachyonic vector fields in de Sitter [25]. The lowest level field has Killing vectors as zero-modes. However, they give non-unitary representations.

C. Gravitons

To conclude we consider the case of a symmetric transverse traceless 2-tensors (STT)

$$\text{Spin-2: } h_{\mu\nu} = h_{\nu\mu}, \quad \nabla_\mu h^{\mu\nu} = 0, \quad h^\mu{}_\mu = 0$$

satisfying

$$\nabla^2 h_{\mu\nu} = \frac{1}{\ell^2} \lambda h_{\mu\nu}, \quad \lambda = 2 + \Delta\Delta^s. \quad (4.45)$$

Ladder operators $\mathcal{D}, \mathcal{D}^s$ depend on a CCKV c^μ of the positive curvature MSS, include a compensating *gauge-looking* transformation and take the form¹⁶

¹⁵Confront the discussion around Eq. (15) in [41].

¹⁶We denote symmetrization by $B_{(\mu\nu)} := \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu})$.

$$\begin{aligned}\mathcal{D}h_{\mu\nu} &:= \mathcal{L}_c h_{\mu\nu} + (\Delta - 2)\sigma_c h_{\mu\nu} - \frac{2}{\Delta + 1} \nabla_{(\mu}(h_{\nu)\rho} c^\rho), \\ \mathcal{D}^s h_{\mu\nu} &:= \mathcal{L}_c h_{\mu\nu} + (\Delta^s - 2)\sigma_c h_{\mu\nu} - \frac{2}{\Delta^s + 1} \nabla_{(\mu}(h_{\nu)\rho} c^\rho).\end{aligned}\quad (4.46)$$

The commutation relations with the Laplace + Beltrami operator are

$$\begin{aligned}[\ell^2 \nabla^2, \mathcal{D}]h_{\mu\nu} &= (d - 2 - 2\Delta)h_{\mu\nu} \\ [\ell^2 \nabla^2, \mathcal{D}^s]h_{\mu\nu} &= (d - 2 - 2\Delta^s)h_{\mu\nu},\end{aligned}\quad (4.47)$$

which imply that $\mathcal{D}$ shifts as $\Delta \rightsquigarrow \Delta + 1$, while $\mathcal{D}^s$ lowers the scale dimension by one unit, $\Delta \rightsquigarrow \Delta - 1$.

Details and proofs: we start with

$$\mathcal{D}h_{\mu\nu} = c^\rho \nabla_\rho h_{\mu\nu} + (\gamma + 2)\sigma_c h_{\mu\nu} + \alpha c^\rho \nabla_{(\mu} h_{\nu)\rho}. \quad (4.48)$$

Again by taking the divergence of (4.48), one finds

$$\nabla^\mu h_{\mu\nu} = 0 \rightsquigarrow \gamma = \frac{2\alpha - 2d + 4}{(\Delta + 1)(d - \Delta)}. \quad (4.49)$$

To arrive at (4.47), we start by computing

$$\begin{aligned}[\nabla^2, \nabla_\rho]h_{\mu\nu} &= \nabla^\sigma \nabla_\sigma \nabla_\rho h_{\mu\nu} - \frac{\lambda}{\ell^2} \nabla_\rho h_{\mu\nu} \\ &= -\frac{\lambda}{\ell^2} \nabla_\rho h_{\mu\nu} + \nabla^\sigma (\nabla_\rho \nabla_\sigma h_{\mu\nu} + R_{\mu\alpha\sigma\rho} h^\alpha{}_\nu + R_{\nu\alpha\sigma\rho} h_\mu{}^\alpha) \\ &= -\frac{\lambda}{\ell^2} \nabla_\rho h_{\mu\nu} + \nabla^\sigma \nabla_\rho \nabla_\sigma h_{\mu\nu} + \frac{1}{\ell^2} (\nabla_\mu h_{\rho\nu} + \nabla_\nu h_{\rho\mu}) \\ &= \frac{1}{\ell^2} (\nabla_\mu h_{\rho\nu} + \nabla_\nu h_{\rho\mu}) + R_{\sigma\alpha}{}^\sigma{}_\rho \nabla^\alpha h_{\mu\nu} + R_{\mu\alpha}{}^\sigma{}_\rho \nabla_\sigma h^\alpha{}_\nu + R_{\nu\alpha}{}^\sigma{}_\rho \nabla_\sigma h_\mu{}^\alpha \\ &= \frac{d-1}{\ell^2} \nabla_\rho h_{\mu\nu} + \frac{4}{\ell^2} \nabla_{(\mu} h_{\nu)\rho},\end{aligned}\quad (4.50)$$

where we used the first identity in (3.1) and the transversality condition. Using this expression we obtain

$$\begin{aligned}[\nabla^2, c^\rho \nabla_\rho]h_{\mu\nu} &= (\nabla^2 c^\rho) \nabla_\rho h_{\mu\nu} + 2(\nabla_\sigma c^\rho)(\nabla^\sigma \nabla_\rho h_{\mu\nu}) + c^\rho [\nabla^2, \nabla_\rho]h_{\mu\nu} \\ &= \frac{d-2}{\ell^2} c^\rho \nabla_\rho h_{\mu\nu} + \frac{2\lambda}{\ell^2} \sigma_c h_{\mu\nu} + \frac{4}{\ell^2} c^\rho \nabla_{(\mu} h_{\nu)\rho}.\end{aligned}\quad (4.51)$$

Now, the commutator of the Laplacian with the second term in (4.48) gives

$$\begin{aligned}[\nabla^2, \sigma_c]h_{\mu\nu} &= (\nabla^2 \sigma_c)h_{\mu\nu} + 2(\nabla_\sigma \sigma_c)(\nabla^\sigma h_{\mu\nu}) \\ &= -\frac{d}{\ell^2} \sigma_c h_{\mu\nu} - \frac{2}{\ell^2} c^\sigma \nabla_\sigma h_{\mu\nu}.\end{aligned}\quad (4.52)$$

Finally, to find the commutator with the last term in (4.48) we compute

$$\begin{aligned}[\nabla^2, c^\rho \nabla_\mu]h_{\rho\nu} &= (\nabla^2 c^\rho) \nabla_\mu h_{\rho\nu} + 2(\nabla^\sigma c^\rho)(\nabla_\sigma \nabla_\mu h_{\rho\nu}) + c^\rho [\nabla^2, \nabla_\mu]h_{\rho\nu} \\ &= \frac{d-2}{\ell^2} c^\rho \nabla_\mu h_{\rho\nu} + 2\sigma_c \nabla^\rho \nabla_\mu h_{\rho\nu} + \frac{2}{\ell^2} c^\rho (\nabla_\rho h_{\nu\mu} + \nabla_\nu h_{\rho\mu}) \\ &= \frac{d}{\ell^2} c^\rho \nabla_\mu h_{\rho\nu} + 2\sigma_c (R_{\rho\alpha}{}^\rho{}_\mu h^\alpha{}_\nu + R_{\nu\alpha}{}^\rho{}_\mu h_\rho{}^\alpha) + \frac{2}{\ell^2} c^\rho \nabla_\rho h_{\nu\mu} - \frac{2}{\ell^2} c^\rho \nabla_{[\mu} h_{\nu]\rho} \\ &= \frac{2}{\ell^2} c^\rho \nabla_\rho h_{\nu\mu} + \frac{d}{\ell^2} c^\rho \nabla_\mu h_{\rho\nu} + 2\frac{d}{\ell^2} \sigma_c h_{\mu\nu} - \frac{2}{\ell^2} c^\rho \nabla_{[\mu} h_{\nu]\rho},\end{aligned}\quad (4.53)$$

where we used the transversality and traceless condition in going to the third and fourth lines. The $(\mu\nu)$ -symmetrization results in

$$[\nabla^2, c^\rho \nabla_{(\mu} h_{\nu)\rho}] = \frac{2}{\ell^2} c^\rho \nabla_\rho h_{\mu\nu} + \frac{d}{\ell^2} c^\rho \nabla_{(\mu} h_{\nu)\rho} + \frac{2d}{\ell^2} \sigma_c h_{\mu\nu}. \quad (4.54)$$

Putting everything together and collecting one finds

$$[\ell^2 \nabla^2, \mathcal{D}]h_{\mu\nu} = \underbrace{(d-2-2(\gamma+2)+2\alpha)}_{\delta\lambda} c^\rho \nabla_\rho h_{\mu\nu} + \underbrace{(2\lambda-d(\gamma+2)+2\alpha d)}_{(\gamma+2)\delta\lambda} \sigma_c h_{\mu\nu} + \underbrace{(4+\alpha d)}_{\alpha\delta\lambda} c^\rho \nabla_{(\mu} h_{\nu)\rho}, \quad (4.55)$$

where, in the rhs, we have the desired results below the underbraces. This gives an overconstrained system of three equations for two unknowns $\delta\lambda, \alpha$. As before it has non-trivial solutions, they are

$$(\gamma, \alpha, \delta\lambda) = \begin{cases} \text{I. } \left(\frac{\Delta^2 - \Delta - 4}{\Delta + 1}, -\frac{2}{\Delta + 1}, d - 2 - 2\Delta \right) \\ \text{II. } \left(\frac{(\Delta^s)^2 - \Delta^s - 4}{\Delta^s + 1}, -\frac{2}{\Delta^s + 1}, d - 2 - 2\Delta^s \right). \end{cases} \quad (4.56)$$

Substituting these values of $\gamma, \alpha, \delta\lambda$ into (4.48), and massaging, we find (4.46).

Comment on Killing + Stäckel tensors and ladder operators: Let us show that Killing + Stäckel tensors are annihilated by the $\mathcal{D}$ -operator.

We start from the definition of a (symmetric) Killing tensor [42–45]

$$\text{Killing tensor: } \nabla_\mu k_{\nu\rho} + \nabla_\nu k_{\rho\mu} + \nabla_\rho k_{\mu\nu} = 0, \quad (4.57)$$

which is assumed to be traceless. Contracting with ∇^μ , commuting covariant derivatives and using (3.1), one finds

$$\nabla^2 k_{\mu\nu} = -\frac{2d}{\ell^2} k_{\mu\nu}. \quad (4.58)$$

Hence, the STT Killing tensors are solutions of the Laplace + Beltrami operator with $\Delta = -2$. The action of $\mathcal{D}$ is given by (4.46), and thus

$$\begin{aligned} \mathcal{D}k_{\mu\nu} &= \mathcal{L}_c k_{\mu\nu} - 4\sigma_c k_{\mu\nu} + 2\nabla_{(\mu} (k_{\nu)\rho} c^\rho) \\ &= c^\rho (\nabla_\rho k_{\mu\nu} + \nabla_\mu k_{\nu\rho} + \nabla_\nu k_{\mu\rho}) = 0. \end{aligned}$$

There exists also a tower of tachyonic shift-symmetric spin-2 fields in de Sitter, the lowest level of which have STT Killing tensors as zero-modes. They are also nonunitary (see [25] for further details).

Comments on conformal-like symmetries and partially massless gravitons: as for scalars and vectors, the ladder operator $\mathcal{D}$ preserves the space of solutions of the spin-2 field equation (4.45) if $\Delta = \frac{d}{2} - 1$. We call these STT *conformal-like gravitons*, they satisfy

$$\nabla^2 h_{\mu\nu} = \frac{1}{\ell^2} \left(2 + \underbrace{\frac{d}{4}(d-2)}_{m^2} \right) h_{\mu\nu}. \quad (4.59)$$

We should confront this equation with the conditions for having a unitary spin-2 field in de Sitter space. As found in [21,46], STT fields of spin- s satisfying

$$\begin{aligned} (-\nabla^2 + s + (2-s)(s+d-3) + m^2) h_{\mu_1 \dots \mu_s} &= 0 \\ \nabla^\mu h_{\mu\mu_2 \dots \mu_s} &= 0, \quad h^\mu{}_{\mu\mu_3 \dots \mu_s} = 0, \end{aligned}$$

give rise to (massive) unitary theories for

$$\ell^2 m^2 > (s-1)(s+d-4). \quad (4.60)$$

However, for the special values [5,46–49]

$$\ell^2 m^2 = (\tau-1)(2s+d-4-\tau), \quad \tau = 1, \dots, s, \quad (4.61)$$

the theory is also unitary, by virtue of enjoying a certain gauge invariance. Spin- s fields with masses taking the discrete values given by (4.61) are known as partially massless field of depth τ .¹⁷ For the case of spin-2 particles, partially massless gravitons arise for $\ell^2 m^2 = d-2$.¹⁸ We then conclude that in $d \geq 4$ the condition (4.60) is satisfied and hence the field equation gives rise to a unitary theory. To conclude, we remark that demanding the gravitons to be conformal-like [in the sense of (4.59)] and partially massless at the same time is only possible for $d=4$ [13]. Indeed, for the partially massless gravitons ($\Delta = \frac{d}{2} - 1 = 1$) in $d=4$, the operator (4.46) takes the form

$$\mathcal{D}h_{\mu\nu} = c^\rho (\nabla_\rho h_{\mu\nu} - \nabla_{(\mu} h_{\nu)\rho}), \quad (4.62)$$

which coincides with the unconventional conformal symmetry of partially massless graviton uncovered earlier in [13].

V. CONCLUSIONS

In the present paper, we have built ladder operators relating distinct discretely labeled UIRs on S^d and dS_d . These ladder operators were constructed relying on CCKV on those spaces and they were explicitly written for spins $s=0, 1, 2$ with generalizations to higher integer spins

¹⁷The case $\tau=1$ corresponds to the strictly massless case. In 4d, a strictly massless field has two propagating degrees of freedom with helicities $\pm s$, while a partially massless field of depth τ has 2τ of them: $(\pm s, \pm(s-1), \dots, \pm(s-\tau+1))$. Strictly massless fields are the closest analogs of Minkowskian massless fields, while partially massless fields of depth $\tau > 1$ have no Minkowskian counterparts.

¹⁸The gauge invariance of partially massless gravitons which eliminates the zero helicity component of the field is $\delta h_{\mu\nu} = (\nabla_{(\mu} \nabla_{\nu)} + \frac{1}{\ell^2} g_{\mu\nu}) a$, where $a(x)$ is a scalar gauge parameter.

being straightforward. Our results generalized previous known formulas for scalar fields [16,17].

The operators $\mathcal{D}, \mathcal{D}^s$ we found connect solutions to free field equations in S^d and dS_d . It is well known that the mass in these equations is related to the quadratic Casimir of the isometry group of the space, therefore, our ladder operators are mass-shifting operators. It is interesting to notice that the form of the ladder operators resembles the spinning conformal primaries transformation laws.

Our setup hopefully sheds light on the appearance of conformal-like symmetries for partially massless gravitons in 4-dimensional de Sitter space [13,50,51]. Moreover, for $d \neq 4$, we found that massive spin-1 and spin-2 fields with special tunings of their mass parameters ($\Delta = d/2 - 1$) enjoy conformal-like symmetries in the sense that the operator $\mathcal{D}$ which preserves their solution space is built from CKV. The case with $d = 4$ is special in the sense that spin-1 and spin-2 fields with $\Delta = 1$ are gauge potentials corresponding to the Discrete Series UIRs of $SO(4,1)$, i.e., the Maxwell field and the partially massless graviton on dS_4 . The former is well known to be $SO(4,2)$ invariant, however, the algebra closure of the partially massless graviton conformal-like symmetry is currently under scrutiny [13].

We leave for future work the computation of the algebra generated by our ladder operators and the possibility of realizing the conformal-like symmetries at the Lagrangian level.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: POSITIVE CURVATURE MSS AND CCKV

de Sitter: in global conformal coordinates the dS_4 metric is given by

$$ds^2 = \frac{1}{\sin^2 t} (-dt^2 + d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)) \quad (A1)$$

here $t \in (-\pi, 0)$. Notice this parametrization maps dS_4 to a portion of Einstein static universe [52].

The Killing vectors on dS_4 can be separated as those acting closely on $t = \text{const}$ (spatial sections), $\{J_i, P_i\}$, which close a $SO(4)$ algebra, and $\{D, K_i\}$ which move us away from the $t = \text{const}$ hypersurface. Their explicit form is

$$\begin{aligned} J_1 &= \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi \\ J_2 &= -\cos \phi \partial_\theta + \cot \theta \sin \phi \partial_\phi \\ J_3 &= -\partial_\phi \end{aligned} \quad (A2)$$

$$\begin{aligned} P_1 &= \sin \theta \cos \phi \partial_\chi + \cot \chi \cos \theta \cos \phi \partial_\theta \\ &\quad - \cot \chi \csc \theta \sin \phi \partial_\phi \\ P_2 &= \sin \theta \sin \phi \partial_\chi + \cot \chi \cos \theta \sin \phi \partial_\theta \\ &\quad + \cot \chi \csc \theta \cos \phi \partial_\phi \\ P_3 &= \cos \theta \partial_\chi - \cot \chi \sin \theta \partial_\theta, \end{aligned} \quad (A3)$$

$$D = \sin t \cos \chi \partial_t + \cos t \sin \chi \partial_\chi$$

$$K_1 = -\sin t \sin \chi \sin \theta \cos \phi \partial_t + \cos t \cos \chi \sin \theta \cos \phi \partial_\chi + \cos t \csc \chi \cos \theta \cos \phi \partial_\theta - \cos t \csc \chi \csc \theta \sin \phi \partial_\phi$$

$$K_2 = -\sin t \sin \chi \sin \theta \sin \phi \partial_t + \cos t \cos \chi \sin \theta \sin \phi \partial_\chi + \cos t \csc \chi \cos \theta \sin \phi \partial_\theta + \cos t \csc \chi \csc \theta \cos \phi \partial_\phi$$

$$K_3 = -\sin t \sin \chi \cos \theta \partial_t + \cos t \cos \chi \cos \theta \partial_\chi - \cos t \csc \chi \sin \theta \partial_\theta. \quad (A4)$$

The set of Killing vectors $\{J_i, P_i, D, K_i\}$ with $i = 1, 2, 3$ close a $SO(4,1)$ algebra

$$\begin{aligned} [J_i, J_j] &= \epsilon_{ijk} J_k, & [J_i, P_j] &= \epsilon_{ijk} P_k, & [J_i, K_j] &= \epsilon_{ijk} K_k \\ [P_i, P_j] &= \epsilon_{ijk} J_k, & [K_i, K_j] &= \epsilon_{ijk} J_k \\ [D, P_i] &= -K_i, & [D, K_i] &= -P_i, & [P_i, K_j] &= -D \delta_{ij}. \end{aligned} \quad (A5)$$

In addition dS_4 possess five proper CKV which we write

$$\begin{aligned}
\tilde{D} &= \cos t \cos \chi \partial_t - \sin t \sin \chi \partial_\chi \\
\tilde{K}_1 &= \sin \theta \cos t \sin \chi \cos \phi \partial_t + \sin \theta \sin t \cos \chi \cos \phi \partial_\chi + \cos \theta \sin t \csc \chi \cos \phi \partial_\theta - \csc \theta \sin t \csc \chi \sin \phi \partial_\phi \\
\tilde{K}_2 &= \sin \theta \cos t \sin \chi \sin \phi \partial_t + \sin \theta \sin t \cos \chi \sin \phi \partial_\chi + \cos \theta \sin t \csc \chi \sin \phi \partial_\theta + \csc \theta \sin t \csc \chi \cos \phi \partial_\phi \\
\tilde{K}_3 &= \cos \theta \cos t \sin \chi \partial_t + \cos \theta \sin t \cos \chi \partial_\chi - \sin \theta \sin t \csc \chi \partial_\theta \\
\tilde{T} &= -\partial_t
\end{aligned} \tag{A6}$$

The conformal symmetry algebra of dS_4 is $SO(4,2)$ which in addition to (A5) reads

$$\begin{aligned}
[\tilde{K}_i, \tilde{K}_j] &= -\epsilon_{ijk} J_k, & [\tilde{K}_i, K_j] &= \tilde{T}, & [P_i, \tilde{K}_j] &= \tilde{D} \delta_{ij}, & [D, \tilde{K}_i] &= 0 \\
[D, \tilde{D}] &= \tilde{T}, & [\tilde{D}, \tilde{K}_i] &= P_i, & [\tilde{D}, P_i] &= \tilde{K}_i, & [J_i, \tilde{K}_j] &= \epsilon_{ijk} \tilde{K}_k \\
[D, \tilde{T}] &= \tilde{D}, & [\tilde{K}_i, \tilde{T}] &= K_i, & [K_i, \tilde{T}] &= -\tilde{K}_i \\
[\tilde{D}, J_i] &= [\tilde{D}, P_i] = [\tilde{T}, J_i] = [\tilde{T}, P_i] = 0.
\end{aligned}$$

The proper CKV (A6) were chosen to be closed, if we pull down the index one can verify that all each of them can be written $c_\mu = \partial_\mu \Psi_c$ [cf. (3.5)]. The potentials are

$$\begin{aligned}
\tilde{D}: \quad \Psi_D &= \csc t \cos \chi \\
\tilde{K}_1: \quad \Psi_{K_1} &= \sin \theta \csc t \sin \chi \cos \phi \\
\tilde{K}_2: \quad \Psi_{K_2} &= \sin \theta \csc t \sin \chi \sin \phi \\
\tilde{K}_3: \quad \Psi_{K_3} &= \cos \theta \csc t \sin \chi \\
\tilde{T}: \quad \Psi_T &= -\cot t.
\end{aligned} \tag{A7}$$

It is amusing to verify that the proper CKV potentials satisfy

$$(\Psi_D)^2 + (\Psi_{K_i})^2 - (\Psi_T)^2 = 1. \tag{A8}$$

This is in agreement with the fact that they coincide with the embedding coordinates $\Psi \sim X^I$ (see [53,54]) when representing de Sitter dS_d as a quadric in Minkowski embedding space $\eta_{IJ} X^I X^J = 1$ with $I, J = 0, \dots, d$.

Sphere: the S^3 metric is written as

$$ds^2 = d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2). \tag{A9}$$

The isometries are generated by $\{J_i, P_i\}$ closing a $SO(4)$ algebra

$$[J_i, J_j] = \epsilon_{ijk} J_k, \quad [J_i, P_j] = \epsilon_{ijk} P_k, \quad [P_i, P_j] = \epsilon_{ijk} J_k.$$

The proper CKV on the sphere can be obtained from the late-time limit behavior of dS_4 Killing vector *boosts* (A4). Their covariant expression are given by

$$\begin{aligned}
D &= d(-\cos \chi) \\
K_1 &= d(\sin \chi \sin \theta \cos \phi) \\
K_2 &= d(\sin \chi \sin \theta \sin \phi) \\
K_3 &= d(\sin \chi \cos \theta)
\end{aligned} \tag{A10}$$

with $d = dx^\mu \partial_\mu$ the exterior differential operator. In analogy with (A8), on the sphere, one can check that the proper CKV potentials Ψ_I ($I = \{D, K_i\}$) satisfy

$$(\Psi_D)^2 + (\Psi_{K_i})^2 = 1$$

APPENDIX B: SCALAR HARMONICS ON S^{d-1}

We write the metric of S^{d-1} in standard fashion

$$d\Omega_{d-1}^2 = d\theta_{d-1}^2 + \sin^2 \theta_{d-1} d\Omega_{d-2}^2. \tag{B1}$$

The full set of coordinates is $\Omega = (\theta_{d-1}, \theta_{d-2}, \dots, \theta_2, \varphi)$ where $\theta_i \in [0, \pi]$ and $\varphi \in [0, 2\pi]$.

The scalar spherical harmonics on S^{d-1} are classified by $(d-1)$ quantum numbers

$$l = (l_{d-1}, l_{d-2}, \dots, l_1), \quad l_i = 0, 1, 2, \dots \tag{B2}$$

obeying

$$l_{d-1} \geq l_{d-2} \geq \dots \geq l_2 \geq |l_1| \geq 0.$$

The first quantum number in the list (B2), i.e. l_{d-1} , parametrizes the Laplace eigenvalue¹⁹

$$-\nabla_{\text{sph}}^2 Y_l = l_{d-1}(l_{d-1} + d - 2) Y_l, \quad l_{d-1} = 0, 1, 2, \dots \tag{B3}$$

¹⁹We denoted $k = l_{d-1}$ in (4.3) and $l = l_{d-1}$ in Appendix C.

They take the explicit form [21]

$$Y_l(\Omega) = \prod_{r=1}^{d-2} \left(C(l_{d-r}, l_{d-r-1}) (\sin \theta_{d-r})^{-(d-r-2)/2} \right. \\ \left. \times P_{l_{d-r} + \frac{d-r-2}{2}}^{-(l_{d-r-1} + \frac{d-r-2}{2})}(\cos \theta_{d-r}) \right) \times \frac{1}{\sqrt{2\pi}} e^{l_1 \varphi}, \quad (\text{B4})$$

where

$$C(l_{d-1}, l_{d-2}) = \left(\frac{2l_{d-1} + d - 2}{2} \frac{(l_{d-1} + l_{d-2} + d - 3)!}{(l_{d-1} - l_{d-2})!} \right)^{1/2}. \quad (\text{B5})$$

In (B4), $P_\nu^{-\mu}$ are the associated Legendre function of the first kind [55]. The normalization factors ensure that

$$\int_{\text{sph}} (Y_l(\Omega))^* Y_{l'}(\Omega) = \delta_{ll'}. \quad (\text{B6})$$

The harmonics on S^{d-1} can be written in terms of S^{d-2} harmonics $Y_{\hat{l}}(\hat{\Omega})$ where $\hat{l} = (l_{d-2}, l_{d-3}, \dots, l_1)$ and $\hat{\Omega} = (\theta_{d-2}, \dots, \theta_2, \varphi)$ as

$$Y_l(\Omega) = C(l_{d-1}, l_{d-2}) (\sin \theta_{d-1})^{-(d-3)/2} P_{l_{d-1} + \frac{d-3}{2}}^{-(l_{d-2} + \frac{d-3}{2})} \\ \times (\cos \theta_{d-1}) Y_{\hat{l}}(\hat{\Omega}). \quad (\text{B7})$$

APPENDIX C: SHIFT-SYMMETRIC SCALARS IN GLOBAL DS

1. Mode solutions for shift-symmetric scalars on dS_d

Shift-symmetric scalars on dS_d satisfy (de Sitter radius $\ell^2 = 1$)

$$\nabla^2 \Phi = -n(n + d - 1) \Phi, \quad n = 0, 1, 2, \dots \quad (\text{C1})$$

To find the solution to this equation in global coordinates (4.16) we make the ansatz

$$\Phi(t, \Omega) = f(t) Y_l(\Omega) \quad \text{with} \quad l = (l, l_{d-2}, \dots, l_1). \quad (\text{C2})$$

Here $Y_l(\Omega)$ are the scalar spherical harmonics on S^{d-1} (see Appendix B). Inserting the ansatz (C2) in (C1) one finds²⁰

$$f(t) = \frac{1}{(\cosh t)^{\frac{d-1}{2}}} \left(c_1 P_{l + \frac{d-1}{2}}^{n + \frac{d-1}{2}}(\tanh t) + c_2 Q_{l + \frac{d-1}{2}}^{n + \frac{d-1}{2}}(\tanh t) \right) \quad (\text{C3})$$

Discrete/exceptional UIRs follow for particular choices of $c_{1,2}$ for $f(t)$. The precise choice of coefficients relating (C3) to (C5) was originally spelled out in [56] using the Whipple formula (see Sec. V.5 in [31,57]).

$$f'' + (d - 1) \tanh t f' \\ + \left(-n(n + d - 1) + \frac{l(l + d - 2)}{\cosh^2 t} \right) f = 0, \quad (\text{C4})$$

where the prime denotes differentiation with respect to t .

The appropriate *positive norm modes* for the quantization of a scalar field in dS were introduced originally in [56,58]. For the case of tachyons they are (see also [21,24,31,59])

$$\Phi_l(t, \Omega) = (\cosh t)^{-(d-2)/2} P_{n + \frac{d-2}{2}}^{-(l + \frac{d-2}{2})} (i \sinh t) Y_l(\Omega), \quad l > n \quad (\text{C5})$$

with the KG inner product, in global coordinates, taking the form

$$(\Phi, \Psi) = i (\cosh t)^{d-1} \int_{\text{sph}} d\Omega (\Phi^* \partial_t \Psi - (\partial_t \Phi)^* \Psi). \quad (\text{C6})$$

The argument of the Legendre function in (C5) can be understood as arising from the Wick rotation (4.15) on the $\cos \theta \mapsto i \sinh t$ which appears in the spherical harmonics (B4).

Some comments are in order:

- (1) $P_\nu^{-\mu}$ are the associated Legendre function of the first kind which can be expressed in terms of the Gauss hypergeometric function as [55]

$$P_\nu^{-\mu}(z) = \frac{1}{\Gamma(\mu + 1)} \left(\frac{1 - z}{1 + z} \right)^{\mu/2} \\ \times F \left(-\nu, \nu + 1; \mu + 1; \frac{1 - z}{2} \right), \quad (\text{C7})$$

thus,

$$\Phi_l(t, \Omega) \\ = \frac{1}{\Gamma(l + \frac{d}{2})} (\cosh t)^{-(d-2)/2} \left(\frac{1 - i \sinh t}{1 + i \sinh t} \right)^{\frac{l + \frac{d-2}{2}}{2}} \\ \times F \left(-n - \frac{d-2}{2}, n + \frac{d}{2}; l + \frac{d}{2}; \frac{1 - i \sinh t}{2} \right) Y_l(\Omega). \quad (\text{C8})$$

- (2) In the short wavelength limit, $l \gg 1$, the modes (C5) behave as [58,60]²¹

$$\frac{\partial}{\partial t} \Phi_l(t, \Omega) \sim -il \frac{\Phi_l(t, \Omega)}{\cosh t}, \quad \text{for } l \gg 1, \quad (\text{C9})$$

²¹See the discussion around Eq. (5.73) in [31].

which is interpreted as a generalized positive frequency behavior. Notice that $\partial_\tau = \cosh t \partial_t$ is the proper time of a comoving observer in de Sitter space.

- (3) The scalar product, of any two modes solutions (C5) is [21]

$$(\Phi_l, \Phi_{l'}) = \frac{2}{\Gamma(l-n)\Gamma(l+n+d-1)} \delta_{ll'}. \quad (\text{C10})$$

This important result leads to the following consequences:

- (1) *Physical modes*: correspond to the set of (complex) modes with $l > n$, they have positive norm.
- (2) *Gauge modes*: correspond to the set of $l \leq n$ modes (real up to a phase), they have zero norm.

Important features of the set $\{\Phi_l\}$ built with P -functions are

- (1) Gauge modes transform among themselves under $\text{SO}(d, 1)$ [21], we will summarize this below. As expected, the number of gauge modes coincides with the number of independent symmetric polynomials appearing in (4.6).²² They are zero-norm modes because for $l \leq n$ the associated Legendre P -functions appearing in (C5) are proportional to their complex conjugates [see (C12) below].
- (2) Physical modes transform among themselves and into gauge modes (see below). It was shown in [21] that the physical modes form a UIR of $\text{SO}(d, 1)$ [21].
- (3) For $l > n$ the Wronskians in Section 14.2(iv) in [38] show that the complex conjugates $(\Phi_l)^*$ are *negative norm* states

$$(\Phi_l^*, \Phi_{l'}) = -(\Phi_l, \Phi_{l'}), \quad (\Phi_l^*, \Phi_l) = 0.$$

2. Seed modes

As we mentioned earlier, in the range $l \leq n$ the complex conjugates of the P -functions in (C5) are not independent from the P -functions [cf. (C12)]. The second set of independent solutions to (C1) is

$$\tilde{\Phi}_l(t, \Omega) = (\cosh t)^{-(d-2)/2} Q_{n+\frac{d-2}{2}}^{l+\frac{d-2}{2}}(i \sinh t) Y_l(\Omega), \quad (\text{C11})$$

where $Q_\nu^\mu(z)$ are the associated Legendre functions of the second kind, analytic in the cut complex plane [38,55].

²²As an example consider $n = 1$ in (C1). The condition $l \leq n$ gives as possible harmonics

$$l \in \left\{ (0, \dots, 0), (1, 0, \dots, 0), \underbrace{(1, 1, 0, \dots, 0), (1, 1, 1, \dots, 1)}_{d-2 \text{ terms}}, (1, 1, \dots, -1) \right\}$$

which comprise $d + 1$ distinct gauge modes. This agrees with the $d + 1$ embedding space coordinates.

There are several alternative representations for the Q -functions which will not be relevant to our discussion. For our purposes, we consider $Q_\nu^\mu(z)$ to be given by the analytic extension, beyond the unit circle, of Eq. (14.3.20) in [38].

The Q -functions satisfy the following properties:

- (i) Zero norm: being $\mu, \nu \in \mathbb{R}$, when evaluated at $z = i \sinh t$ we have that $(Q_\nu^\mu(z))^* = Q_\nu^\mu(-z)$. Then, using Eq. (14.24.2) in [38] we conclude that for $\text{Im}(z) < 0$

$$Q_\nu^\mu(-z) = -e^{-i\nu\pi} Q_\nu^\mu(z) \rightsquigarrow (Q_\nu^\mu(z))^* \propto Q_\nu^\mu(z).$$

- (ii) Linear dependence: for $l > n$, Q_ν^μ are customarily disregarded, as they are not independent from $\{P_\nu^{-\mu}(i \sinh t), P_\nu^{-\mu}(-i \sinh t)\}$. This can be seen from (Eq. (14.24.1) in [38])

$$\begin{aligned} P_\nu^{-\mu}(-z) - e^{-i\pi\nu} P_\nu^{-\mu}(z) \\ = \frac{2}{\Gamma(\mu - \nu) \Gamma(\mu + \nu + 1)} e^{-i\mu\pi} Q_\nu^\mu(z), \quad \text{Im}(z) > 0. \end{aligned} \quad (\text{C12})$$

A similar relation holds for $\text{Im}(z) < 0$.

- (iii) Linear independence: for $l \leq n$, the set of linear independent solutions to (C1) is $\{P_\nu^{-\mu}(i \sinh t), Q_\nu^\mu(i \sinh t)\}$. Independence follows from Eq. (14.2.6) in [38].

3. de Sitter isometry and mode expansions

In this section, we study the transformation properties of the mode functions $\{\Phi_l, \tilde{\Phi}_l\}$ in (C5) and (C11) under $\text{SO}(d, 1)$. The transformation properties for the P -modes were originally uncovered in [21]. Here we extend it to the (seed) Q -modes (C11).

We start by mentioning that the l -quantum number in (B3) is preserved under the action of $\text{SO}(d) \subset \text{SO}(d, 1)$. However, under a $\text{SO}(1, 1) \subset \text{SO}(d, 1)$ boost, the l -quantum number shifts by one. For concreteness, we write the sphere metric in (B1) as

$$d\Omega_{d-1}^2 = d\vartheta^2 + \sin^2 \vartheta d\Omega_{d-2}^2$$

and consider, without loss of generality, the $\text{SO}(1, 1)$ de Sitter Killing vector

$$\xi = \cos \vartheta \partial_t - \tanh t \sin \vartheta \partial_\vartheta. \quad (\text{C13})$$

Our aim is to study its action on the set of modes

$$\Psi_l(t, \Omega) = f_l(t) Y_l(\Omega),$$

where

$$f_l(t) \in \left\{ (\cosh t)^{-(d-2)/2} P_{n+\frac{d-2}{2}}^{-(l+\frac{d-2}{2})}(i \sinh t), (\cosh t)^{-(d-2)/2} Q_{n+\frac{d-2}{2}}^{l+\frac{d-2}{2}}(i \sinh t) \right\}.$$

We show below that (C13) leads to a simple mixing of modes which can be schematically summarized as

$$l = (l, l_{d-2}, \dots, l_1) \mapsto l' = (l \pm 1, l_{d-2}, \dots, l_1).$$

Moreover, the important results are

- (i) Gauge modes $l \leq n$ (P -modes) do not mix with $l > n$ modes [see (C27)].
- (ii) Seed modes with $l \leq n$ (Q -modes) can be connected with $l > n$ [see (C28)]. However, from (C12) we conclude the result involves a combination of positive and negative norm P -modes.

Details: to analyze the action of symmetries, following [21] we re-express the Lie derivative $\mathcal{L}_\xi$ as

$$\mathcal{L}_\xi \Psi_l = \frac{1}{2l + d - 2} (T^{(+)} + T^{(-)}) \Psi_l$$

with

$$T^{(+)} = D^{(+)} \times \tilde{D}^{(+)}, \quad (\text{C14})$$

where

$$\begin{aligned} D^{(+)} &= \partial_t - l \tanh t, \\ \tilde{D}^{(+)} &= \sin \vartheta \partial_\vartheta + (l + d - 2) \cos \vartheta, \end{aligned} \quad (\text{C15})$$

and

$$T^{(-)} = D^{(-)} \times \tilde{D}^{(-)}, \quad (\text{C16})$$

where

$$\begin{aligned} D^{(-)} &= \partial_t + (l + d - 2) \tanh t, \\ \tilde{D}^{(-)} &= -\sin \vartheta \partial_\vartheta + l \cos \vartheta. \end{aligned} \quad (\text{C17})$$

The virtue of the decomposition $T = D \times \tilde{D}$ is that it factorizes the action on the spatial slices, $\tilde{D}$, from the action along time, D .²³

Using several properties of the associated Legendre functions one finds:

²³It is interesting to notice that the operators $D^{(\pm)}, \tilde{D}^{(\pm)}$ in (C26) are related to the ladder operators $\mathcal{D}, \mathcal{D}^s$ defined in the main text. For example, $\mathcal{D} = -\cosh t D^{(+)}$, etc. However, while $\mathcal{D}, \mathcal{D}^s$ shift the mass eigenvalue keeping the sphere quantum numbers remain unchanged, the operators $D^{(\pm)}, \tilde{D}^{(\pm)}$ shift the (principal) angular quantum number.

- (1) $\tilde{D}^{(+)}$ raises the principal quantum number $l \mapsto l + 1$

$$\begin{aligned} \tilde{D}^{(+)} Y_l(\Omega) &= \frac{C(l, l_{d-2})}{C(l+1, l_{d-2})} (l + l_{d-2} + d - 2) \\ &\times Y_{l^+}(\Omega), \end{aligned} \quad (\text{C18})$$

here $l^+ = (l + 1, l_{d-2}, \dots, l_1)$ and the normalization factors are defined in (B5).

- (2) $\tilde{D}^{(-)}$ lowers the principal quantum number $l \mapsto l - 1$

$$\tilde{D}^{(-)} Y_l(\Omega) = \frac{C(l, l_{d-2})}{C(l-1, l_{d-2})} (l - l_{d-2}) Y_{l^-}(\Omega), \quad (\text{C19})$$

here $l^- = (l - 1, l_{d-2}, \dots, l_1)$.

- (3) $D^{(+)}$ shifts $l \mapsto l + 1$ in the P -function

$$\begin{aligned} D^{(+)} \left((\cosh t)^{-\frac{d-2}{2}} P_{n+\frac{d-2}{2}}^{-(l+\frac{d-2}{2})}(i \sinh t) \right) \\ = k_P^{(+)} (\cosh t)^{-\frac{d-2}{2}} P_{n+\frac{d-2}{2}}^{-(l+1+\frac{d-2}{2})}(i \sinh t) \end{aligned} \quad (\text{C20})$$

where

$$k_P^{(+)} = i(n - l)(n + l + d - 1). \quad (\text{C21})$$

Acting on the Legendre Q -functions, one obtains

$$\begin{aligned} D^{(+)} \left((\cosh t)^{-\frac{d-2}{2}} Q_{n+\frac{d-2}{2}}^{l+\frac{d-2}{2}}(i \sinh t) \right) \\ = k_Q^{(+)} (\cosh t)^{-\frac{d-2}{2}} Q_{n+\frac{d-2}{2}}^{l+1+\frac{d-2}{2}}(i \sinh t), \end{aligned} \quad (\text{C22})$$

where

$$k_Q^{(+)} = 1. \quad (\text{C23})$$

- (4) Proceeding similarly, the $D^{(-)}$ action can be summarized as

$$D^{(-)} f_l = k_f^{(-)} f_{l-1}, \quad (\text{C24})$$

where $k_f^{(\pm)}$ depends on whether the $D^{(-)}$ acts on P - or Q -functions. The factors $k_f^{(-)}$ are given by

$$k_P^{(-)} = -i \quad \text{and} \quad k_Q^{(-)} = (n - l + 1)(n + l + d - 2).$$

Combining the results above we have

$$T^{(+)}\Psi_l = \frac{C(l, l_{d-2})}{C(l+1, l_{d-2})} (l + l_{d-2} + d - 2) k_f^{(+)} \Psi_{l^+},$$

$$T^{(-)}\Psi_l = \frac{C(l, l_{d-2})}{C(l-1, l_{d-2})} (l - l_{d-2}) k_f^{(-)} \Psi_{l^-}. \quad (C25)$$

Long story short, we conclude that

$$\begin{aligned} \mathcal{L}_\xi \Psi_l &= \frac{1}{2l + d - 2} (T^{(+)} + T^{(-)}) \Psi_l \\ &= \frac{C(l, l_{d-2})}{2l + d - 2} \left(\frac{l + l_{d-2} + d - 2}{C(l+1, l_{d-2})} k_f^{(+)} \Psi_{l^+} \right. \\ &\quad \left. + \frac{l - l_{d-2}}{C(l-1, l_{d-2})} k_f^{(-)} \Psi_{l^-} \right). \end{aligned} \quad (C26)$$

Transformation of P -modes: specializing to Φ_l in (C5) one obtains, forgetting the factor in front of the parentheses in (C26),

$$\begin{aligned} \mathcal{L}_\xi \Phi_l &\propto \frac{l + l_{d-2} + d - 2}{C(l+1, l_{d-2})} (n - l)(n + l + d - 1) \Phi_{l^+} \\ &\quad - \frac{l - l_{d-2}}{C(l-1, l_{d-2})} \Phi_{l^-}. \end{aligned} \quad (C27)$$

The $(n - l)$ factor multiplying Φ_{l^+} implies that gauge modes ($l \leq n$) cannot be turned into physical modes ($l > n$) under the action of dS isometries. Concomitantly it follows that all gauge modes ($l \leq n$) mix among themselves. However, physical modes can be connected with gauge modes [due to the last term in (C27)].

Transformation of Q -modes: for the seed modes (C11) we obtain

$$\begin{aligned} \mathcal{L}_\xi \tilde{\Phi}_l &\propto \frac{l + l_{d-2} + d - 2}{C(l+1, l_{d-2})} \tilde{\Phi}_{l^+} + \frac{l - l_{d-2}}{C(l-1, l_{d-2})} \\ &\quad \times (n - l + 1)(n + l + d - 2) \tilde{\Phi}_{l^-}. \end{aligned} \quad (C28)$$

Some comments are in order: contrary to (C27), the first term in (C28) never vanishes. This implies that Q -modes with $l \leq n$ can be connected with physical modes ($l > n$) under the action of dS isometries. However, using (C12) is important to stress that the result is a P -mode combination of positive and negative norm.

APPENDIX D: VECTOR SPHERICAL HARMONICS IN S^3 AND LADDER OPERATORS

Transverse vector harmonics (TVH) on S^3 were thoroughly discussed in [22]. We show here how to obtain the full spectrum using ladder operators.

TVH are classified by the eigenvalues of the two Casimirs of $SO(4)$

$$\mathcal{C} = \mathbf{P}_i^2 + \mathbf{J}_i^2 \quad \text{and} \quad \tilde{\mathcal{C}} = -\mathbf{P}_i \mathbf{J}_i. \quad (D1)$$

The possible quantum numbers $(k, \pm)$ for UIRs are

$$\begin{aligned} \mathcal{C} A_\mu^{k\pm} &= -\frac{(k+1)^2}{\ell^2} A_\mu^{k\pm}, \quad \tilde{\mathcal{C}} A_\mu^{k\pm} = \mp \frac{(k+1)}{\ell} A_\mu^{k\pm}, \\ k &= 1, 2, \dots \end{aligned} \quad (D2)$$

Here each $k \in \mathbb{N}$ gives a different $SO(4)$ UIR. Making $k \mapsto \mathcal{L}_k$ in (D1) one finds (see [22])

$$\mathcal{C} A_\mu^{k\pm} = \left(\nabla^2 - \frac{2}{\ell^2} \right) A_\mu^{k\pm}, \quad \tilde{\mathcal{C}} A_\mu^{k\pm} = \epsilon_\mu^{\alpha\beta} \nabla_\alpha A_\beta^{k\pm} \quad (D3)$$

here $\epsilon_{\mu\nu\rho}$ is the covariant Levi + Civita tensor. Comparing (D2) and (D3) with the definition of Δ in (4.31) we have $\Delta = -k$ and $\Delta^s = 2 + k$. Since the lowest possible eigenvalue corresponds to $k = 1$, we verify from (3.3) that the six Killing vectors of the 3-sphere, spelled in (A2) and (A3), are the lowest eigenvectors $(1, \pm)$ of the vector Laplacian. We can rewrite (4.32) as

$$\mathcal{D} A_\mu^{k\pm} = c^\rho \nabla_\rho A_\mu^{k\pm} - k \sigma_c A_\mu^{k\pm} + \frac{1}{k} \nabla_\mu (c^\rho A_\rho^{k\pm}). \quad (D4)$$

Analogously, for the shadow ladder we have

$$\mathcal{D}^s A_\mu^{k\pm} = c^\rho \nabla_\rho A_\mu^{k\pm} + (2 + k) \sigma_c A_\mu^{k\pm} - \frac{1}{2 + k} \nabla_\mu (c^\rho A_\rho^{k\pm}). \quad (D5)$$

From (4.43) we get

$$\begin{aligned} [\ell^2 \nabla^2, \mathcal{D}] A_\mu^{k\pm} &= (1 + 2k) \mathcal{D} A_\mu^{k\pm} \\ [\ell^2 \nabla^2, \mathcal{D}^s] A_\mu^{k\pm} &= -(3 + 2k) \mathcal{D}^s A_\mu^{k\pm}, \end{aligned} \quad (D6)$$

which lead to

$$\begin{aligned} \mathcal{C}(\mathcal{D} A^{k\pm}) &= -\frac{k^2}{\ell^2} \mathcal{D} A^{k\pm} \\ \mathcal{C}(\mathcal{D}^s A^{k\pm}) &= -\frac{(k+2)^2}{\ell^2} \mathcal{D}^s A^{k\pm}. \end{aligned} \quad (D7)$$

This means that $\mathcal{D}^s, \mathcal{D}$ rise/lower the k -quantum number [cf. (D2)]

$$\begin{aligned} \mathcal{D}: \quad k &\mapsto k - 1 \\ \mathcal{D}^s: \quad k &\mapsto k + 1 \end{aligned}$$

which can be equivalently phrased as (4.8).

In addition we find that

$$\begin{aligned} [\tilde{\mathcal{C}}, D]A^{k\pm} &= \pm \mathcal{D}A^{k\pm} \\ [\tilde{\mathcal{C}}, \mathcal{D}^s]A^{k\pm} &= \mp \mathcal{D}^s A^{k\pm}. \end{aligned} \quad (\text{D8})$$

These transformations imply that

$$\tilde{\mathcal{C}}(\mathcal{D}A^{k\pm}) = \mp \frac{k}{\ell} \mathcal{D}A^{k\pm} \quad (\text{D9})$$

$$\tilde{\mathcal{C}}(\mathcal{D}^s A^{k\pm}) = \mp \frac{k+2}{\ell} \mathcal{D}^s A^{k\pm}. \quad (\text{D10})$$

Since no UIR with $k = 0$ exists, it is reassuring to verify that

$$\mathcal{D}A^{k\pm}|_{k=1} = 0.$$

This follows from (4.44) and the fact that the $(k = 1, \pm)$ are the six Killing vectors of S^3 .

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