

Parity-violating marginal deformation of the 3D Gross-Neveu-Thirring model

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A hybrid of the critical three-dimensional Gross-Neveu and Thirring models deformed by explicit parity-breaking operators is studied in the large- N expansion and using the renormalization group. The regime of coupling constants where the theory is stable is identified and criteria for the occurrence of fixed points in that regime are found. For a certain range of Chern-Simons level, we find stable charge-gapped phase with spontaneously broken approximate scale invariance and a parametrically light dilaton. The Chern-Simons level can be tuned to the stability edge, resulting in exact scale invariance which is spontaneously broken, accompanied by a massless dilaton. For another, narrow range of Chern-Simons levels, we find a conformal window where the theory flows to a Wilson-Fisher-like fixed point and is a novel (and rare) example of a nonsupersymmetric nontrivial parity- and time-reversal-violating three-dimensional conformal field theory with scalar, spinor, and vector fields.

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I. INTRODUCTION

The Gross-Neveu model in three dimensions has become an important paradigm. It is a relativistic quantum field theory which, for sufficiently strong coupling, exhibits a chiral symmetry-breaking quantum phase transition in a context which can be studied systematically in a large- N expansion. Moreover, with the advent of two-dimensional Dirac materials and their emergent massless Dirac fermions [1–5], its quantum phase transition has been used as a model of the transition between a Dirac semimetal and a Mott insulator that is driven by strong short-ranged interactions [6–13]. Although N is typically small in a Dirac material, the large- N limit is still thought to provide important clues about the nature of the phase diagram and the phase transition.

In some beautiful recent work using the functional renormalization group [14–17], the possibility of adding explicit chiral (or parity) symmetry-violating deformations to the critical three-dimensional Gross-Neveu model has been explored, and it has been argued to lead to interesting behavior. At first glance, explicitly breaking the symmetry which protects the mass of the fermions would have the rather bland consequence of simply gapping the fermion spectrum, resulting in an insulator, albeit, if the

deformation breaks parity and time-reversal symmetry, an insulator with a zero field Hall conductivity, similar to Haldane’s parity-violating model of graphene [18].

What was shown [14–17] was the emergence of a line of renormalization group fixed points such that, at the end points of the line, a nonzero chiral condensate could form and the fermion spectrum becomes gapped. The scale invariance of the fixed point is spontaneously broken by the vacuum expectation value of the fermion mass operator. This scale symmetry breaking is accompanied by a massless Goldstone boson, which plays the role of a dilaton. The result is the interesting scenario of a topological insulator with a gapless exciton where the exciton couples to energy and momentum density like a dilaton, and it could have important implications for the bulk properties of a Dirac material.

The functional renormalization group computations which are used to arrive at this conclusion are approximate. They are known to become exact in the large- N limit. Motivated by this, the current authors examined the simplest scenario for a parity-violating deformation of the Gross-Neveu model using a direct application of the large- N expansion [19]. It was confirmed that, indeed, in the strict infinite- N limit, the scenario described above was realized in precisely the way that it is described. There is indeed a line of fixed points parametrized by the coupling constant for an exactly marginal parity-violating operator. When the tunable coupling was tuned to an end point of its line of fixed points, the vacuum energy became independent of the chiral condensate. Selecting a nonzero value for the chiral condensate breaks scale invariance, and the spectrum of the theory contains a massless dilaton.

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However, in Ref. [19], it was also demonstrated that when one relaxes the infinite- N limit, so that N is large but finite, and the leading corrections in the $1/N$ expansion are taken into account, the line of fixed points are no longer fixed points. Rather, the coupling constant which parametrized that line obtains a beta function which is of the order of $1/N$. What is more, the energy landscape is no longer flat for any value of the chiral condensate. Most importantly, and unfortunately, the theory becomes unstable.

It is easy to understand the origin of the problem with stability. The limit of infinite chiral condensate is governed by the ultraviolet fixed points of the theory. The ultraviolet fixed points turn out to occur in the regime where the large- N effective potential is unbounded from below and the theory has runaway behavior—negative infinite energy as the chiral condensate goes to infinity.

The details of how this occurs are summarized in Ref. [19]. In that reference, the same quantum field theory model, but in $3 + \epsilon$ dimensions, with $\epsilon \sim 1/N$ was shown to be stable. In essence, the positions of the ultraviolet fixed points depend on ϵ and, if ϵ is made sufficiently large, the ultraviolet fixed points migrate back into the regime where the model is stable. When that occurs, we have an explicit, systematically analyzable example of the phenomenon where an approximate scale symmetry is spontaneously broken by the formation of a chiral condensate and the spectrum of the quantum field theory contains a pseudo-dilaton. This scenario is one of few mechanisms whereby a scalar field mass is naturally protected from large quantum corrections, and it has potentially important phenomenological applications, both in the context of the standard model of elementary particle physics [20–32] and in atomic physics, where it potentially describes an exotic new phase of matter [33,34].

As interesting as this is, its occurrence only in $3 + \epsilon$ dimensions, with $\epsilon \sim \frac{1}{N}$ makes its relevance to two (space) dimensional Dirac material, where N is small and the spacetime dimension is strictly three, obscure at best. In this paper, we shall report the results of our subsequent cursory search for nonsupersymmetric variants of the three-dimensional Gross-Neveu model whose parity-violating deformations can exhibit interesting large- N behavior and which are stable at large but finite N and at precisely three spacetime dimensions. What we find is that, for some range of its parameters, a hybrid of the three-dimensional Gross-Neveu and Thirring models perturbed by parity-violating operators has interesting stable solutions.

Introducing the Thirring interaction leads to an emergent vector field, and we shall find that the radiative corrections that the vector field mediates are essential in modifying the beta functions so that stable solutions exist. When the vector field is present, among the marginal parity- and time-reversal-violating operators is a Chern-Simons term with a coefficient $(N\kappa)/4\pi$ where we shall assume that, when

$N \sim \infty$, κ is a parameter of the order of one and, even if $N\kappa$ is an integer, in this large- N limit, κ can be varied continuously. The Chern-Simons coefficient has only finite renormalization, and only at one loop; therefore, its tree level value can be regarded as a tunable parameter. When this parameter is within a certain range, $|\kappa| \lesssim \kappa_{\text{crit}}$ with $\kappa_{\text{crit}} \sim 0.162\dots$, there is a single ultraviolet fixed point (shared by all of the marginal and relevant coupling constants), and this fixed point is inside the regime where the model is stable. The field theory with couplings tuned to this ultraviolet fixed point is an exotic parity- and time-reversal-violating conformal field theory with scalar, spinor, and vector fields. Moreover, we shall show that, in the vicinity of this fixed point, the theory flows to a gapped phase which has spontaneously broken approximate scale symmetry and the parametrically light dilaton.

There is a second, separate, smaller range of the Chern-Simons coefficient $\kappa_{\text{con},-} < |\kappa| < \kappa_{\text{con},+}$, with $\kappa_{\text{con},-} \sim 0.272\dots$ and $\kappa_{\text{con},+} \sim 0.283\dots$, where there is an infrared as well as two ultraviolet fixed points. In this entire range, only one of the ultraviolet fixed points occurs inside in the stable regime, and the other is outside. It is in this range where, as well as the quasi-scale-invariant solutions of the type that we have discussed above, we shall find a second interesting particular solution of the field theory. It involves renormalization group trajectories which are trapped between the stable ultraviolet fixed point and a Wilson-Fisher type of infrared fixed—in a conformal window. The chiral condensate vanishes and the fermions are gapless, even when the theory has explicit parity and time-reversal violation. In particular, when tuned to the infrared fixed point, the model should be a conformal field theory, one which contains scalar, spinor, and vector fields and which breaks parity and time-reversal invariance.

We note that all of the fixed points that we find in the regime where the theory is stable are conformal field theories. A very interesting phenomenon occurs when the Chern-Simons level is fine-tuned to an edge of the stability regime, $\kappa \rightarrow \pm 0.162\dots$ or $\kappa \rightarrow \pm 0.283\dots$ whence an ultraviolet fixed point occurs precisely at the stability edge. In that case, the effective potential becomes a flat function of the chiral condensate, similar to what occurs in the strict infinite- N limits of the Gross-Neveu and Gross-Neveu-Thirring models. Like in those cases, the condensate can take up any value, and, when the value is nonzero, it breaks the scale symmetry spontaneously and the spectrum contains an exactly massless dilaton. This provides a highly nontrivial realization of this phenomenon in a fully interacting nonsupersymmetric field theory which is under good analytic control if N is large enough that the large- N expansion is reliable.

In the remainder of this paper, we will describe how we obtain these results in some detail. The paper is organized as follows. In Sec. II, we shall review the essential features of the three-dimensional Gross-Neveu model. In Sec. III,

we review the results of Ref. [19] and the demonstration that, at large but finite N , the Gross-Neveu model with a parity-breaking marginal operator added to the action becomes unstable. In Sec. IV, we discuss a hybrid of the Gross-Neveu and Thirring models with parity-breaking operators, and we compute the relevant beta functions. In Sec. V, we outline the details of the large- N expansion applied to the latter model. In Sec. VI, we discuss an alternative model which is parity and time-reversal symmetric but where the Gross-Neveu coupling breaks an internal discrete symmetry. In Sec. VII, we summarize our results and conclusions.

II. THE THREE-DIMENSIONAL GROSS-NEVEU MODEL

The three-dimensional Gross-Neveu model describes self-interacting relativistic fermions:

$$S_{\text{GN}} = \int d^3x \left[i\bar{\psi}_a(x) \not{\partial} \psi_a(x) + \frac{g}{2N} (\bar{\psi}_a(x) \psi_a(x))^2 \right], \quad (1)$$

where the repeated index $a = 1, \dots, N$ is summed. Here and in the following, we will take the spacetime as three-dimensional Euclidean space. We will take the Dirac field $\psi_a(x)$ to be a complex two-component spinor representation of the Euclidean Lorentz group $\text{SO}(3)$. As well, there is a global $\text{O}(2N)$ symmetry (which can be found by writing the action with Majorana rather than complex fermions) [35].

In three dimensions, the four-fermion operator has classical dimension four, and it is an irrelevant operator. The quantum field theory described by (1) is, therefore, not renormalizable in conventional perturbation theory which would expand in the coupling constant g . However, this quantum field theory is renormalizable in the $1/N$ expansion [36–43]. This is most easily seen by introducing the Hubbard-Stratonovich scalar field $\phi(x)$ and using the action

$$S_{\text{GN}} = \int d^3x \left[i\bar{\psi}_a(x) (\not{\partial} + \phi(x)) \psi_a(x) + \frac{N}{2g} \phi^2(x) \right]. \quad (2)$$

The action (1) can be recovered from (2) by integrating out the scalar field. With the action as presented in Eq. (2), both $\psi(x)$ and $\phi(x)$ have classical dimension one, and the action contains all of the relevant and marginal local operators that are compatible with the Poincaré and $\text{O}(2N)$ symmetries as well as the discrete spacetime symmetries C , P , and T . Then, renormalization of the large- N expansion, order by order in the dimensionless constant $1/N$, does not require the addition of any counterterms with local operators which are not already present in the action (2).

The analog of chiral symmetry in three dimensions is spacetime parity and time reversal. These are symmetries of the field theory described by the actions in Eqs. (1) or (2). The parity transformation is

$$\begin{aligned} \mathbf{P}: (x_1', x_2', x_3') &= (-x_1, x_2, x_3), \\ \psi(x) &\rightarrow \gamma^1 \psi(x'), \quad \bar{\psi}(x) \rightarrow -\bar{\psi}(x') \gamma^1, \\ \phi(x) &\rightarrow -\phi(x'), \\ \bar{\psi}(x) \psi(x) &\rightarrow -\bar{\psi}(x') \psi(x'). \end{aligned} \quad (3)$$

When the coupling constant g is sufficiently large, the Gross-Neveu model (1) or (2) exhibits a phase transition whose order parameter is the chiral condensate $\langle 0 | \bar{\psi}(x) \psi(x) | 0 \rangle$ or the vacuum expectation value of the scalar $\langle 0 | \phi(x) | 0 \rangle$, both of which become nonzero. These condensates break the spacetime parity and time-reversal symmetries while leaving the $\text{O}(2N)$ flavor symmetry intact.

There are versions of the Gross-Neveu model, obtained by modifying the interaction, where the breaking is of an internal chiral symmetry rather than parity. Their analysis is in most aspects very similar to what we do in the following. We will give an example of such a model in Sec. VI.

When g is tuned to its critical value, the quantum field theory defined by Eq. (2) is a three-dimensional conformal field theory. This has been confirmed by explicit computation to a relatively high order of the large- N expansion [44–47]. For certain small values of N , it is also consistent with lattice simulations [48,49] and with recent work using the conformal bootstrap [50].

III. PARITY-VIOLATING DEFORMATION OF THE GROSS-NEVEU MODEL

The only classically marginal local operator that one could add to the action in (2) is a cubic term in the scalar field:

$$\delta S = \int d^3x \frac{N\lambda}{6} \phi^3(x). \quad (4)$$

Such a term breaks parity and time-reversal invariance explicitly. The possibility of adding this coupling to the critical Gross-Neveu model has been examined in some detail [19]. In this section, we shall review some of the results of that study.

Once the parity- and time-reversal-violating interaction (5) is added to the theory, generally all other relevant and marginal operators which are now unsuppressed by symmetry must be added to the action. These include a fermion mass term $\sim \int i m \bar{\psi}(x) \psi(x)$ and a scalar field tadpole $\sim \int N \lambda_1 \phi(x)$. Indeed, such terms are needed as counterterms to cancel ultraviolet divergences which occur in perturbation theory. We can always use a field redefinition $\phi(x) \rightarrow \phi(x) + \text{const}$ to remove the residual part (after counterterms have been chosen) to set one of these operators to zero. Generally, we will do this to set the fermion mass operator $\sim \int i m \bar{\psi}(x) \psi(x)$ to zero. This is

tantamount to defining the fermion mass as being proportional to the vacuum expectation value of the scalar field.

In the large- N limit, in the leading order of the $1/N$ expansion, the full effective action is given by the expression

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \Gamma[\psi, \phi] = & -\ln \det(\not{\partial} + \phi(x)) \\ & + \int d^3x \left[\frac{1}{N} \sum_{a=1}^N i\bar{\psi}_a(x)(\not{\partial} + \phi(x))\psi_a(x) \right. \\ & \left. + \lambda_1 \phi(x) + \frac{1}{2g} \phi^2(x) + \frac{\lambda}{6} \phi^3(x) \right]. \end{aligned} \quad (5)$$

The solution of the quantum field theory is found from the nonlocal classical field theory (5) by finding solutions of the field equations that are gotten by setting the first functional derivatives of the effective action to zero. Then the irreducible correlation functions are obtained by taking functional derivatives of the effective action by the classical fields and evaluating the quantities that are obtained this way on the solutions of the field equations.

If we seek a Poincaré invariant solution of (5), the spinor field must vanish and the scalar field must be a constant equal to the condensate:

$$\phi_0 = \langle 0 | \phi(x) | 0 \rangle. \quad (6)$$

The value of ϕ_0 must minimize the effective potential which, to the leading order at large N , has a very simple form. It is obtained from the effective action (5) by putting $\psi(x) = 0$ and $\phi(x) = \phi_0$ to get [19]

$$\lim_{N \rightarrow \infty} \frac{1}{N} V_{\text{eff}} = \frac{1}{2g} \phi_0^2 + \frac{1}{6\pi} |\phi_0|^3 + \lambda_1 \phi_0 + \frac{\lambda}{6} \phi_0^3. \quad (7)$$

The first two terms on the right-hand side of Eq. (7) are the contribution of the trace of the logarithm of the Dirac operator, the first term on the right-hand side of Eq. (5). That trace log is formally parity covariant, and, with a parity covariant ultraviolet regularization, it must be an even function of ϕ_0 . It contributes the first two terms on the right-hand side of Eq. (7), which are indeed even in ϕ_0 .

In addition to those terms, $1/g$ has been renormalized by a linearly divergent counterterm, and λ_1 is not renormalized at this leading order. What is more, there are no logarithmic divergences, and, as a consequence, at this leading order in N , the renormalization scale decouples. The constants λ_1 , $1/g$, and λ are freely tunable parameters.

The effective potential in Eq. (7) is bounded from below, and the field theory that it describes is stable when λ is in the range

$$-\frac{1}{\pi} < \lambda < \frac{1}{\pi}. \quad (8)$$

In this stability regime, the effective potential (7) exhibits quantum phase transitions typical of a cubic potential as the dimensionful constants λ_1 and $1/g$ are varied. There are possible phases with either zero or nonzero values of ϕ_0 which correspond to an ungapped and a gapped fermion spectrum, respectively.

There is also a scale-invariant limit of this theory where we put the dimensionful couplings to zero, $\lambda_1 = 1/g = 0$, and we get

$$\lim_{\substack{\lambda_1, 1/g \rightarrow 0 \\ N \rightarrow \infty}} \frac{1}{N} V_{\text{eff}} = \left(\frac{1}{\pi} + \text{sgn}(\phi_0) \lambda \right) \frac{|\phi_0|^3}{6}. \quad (9)$$

When $|\lambda| < 1/\pi$, this potential is minimized at $\phi_0 = 0$. In this scale-invariant theory, the fermions are massless. If we examine the two-point function of the scalar field [see Eq. (26) below], we find

$$\Delta(p) = \frac{1}{N} \frac{8}{\sqrt{p^2}} + \mathcal{O}(1/N^2). \quad (10)$$

This propagator has the two-fermion cut singularity on the half line $p^2 \in (-\infty, 0)$, but it does not have a pole singularity. In this case, the scalar is truly an auxiliary field.

If we further tune the dimensionless coupling constant to the edge of the stability regime,

$$\lambda \rightarrow \frac{1}{\pi} \quad \text{or} \quad \lambda \rightarrow -\frac{1}{\pi}, \quad (11)$$

we find

$$\lim_{\lambda \rightarrow \pm \frac{1}{\pi}} \lim_{\substack{\lambda_1, 1/g \rightarrow 0 \\ N \rightarrow \infty}} \frac{1}{N} V_{\text{eff}} = \begin{cases} \frac{2}{\pi} \frac{|\phi_0|^3}{6} & \text{sgn}(\phi_0 \lambda) = 1, \\ 0 & \text{sgn}(\phi_0 \lambda) = -1. \end{cases} \quad (12)$$

In this scenario, for a particular sign of the condensate ϕ_0 , the effective potential (12) is flat, and it is independent of the magnitude of ϕ_0 . The sign of the condensate ϕ_0 is then determined to be opposite to the sign of λ . However, the magnitude of the condensate is not determined by the potential. It can take up any value. When it takes up a nonzero value, it breaks the scale invariance of the potential and of the effective action. The set of all possible symmetry-breaking vacua, that is, all nonzero values of ϕ_0 with a given sign, form an orbit of the scale symmetry.

This scenario is a spontaneous breaking of the scale symmetry. This symmetry breaking is accompanied by a massless Goldstone boson, a dilaton. The fluctuations of the scalar field $\phi(x)$ in the flat direction are the dilaton.

The dilaton can be seen explicitly by using the effective action (5) to evaluate the scalar field two-point function. It is given by [see Eq. (26) below]

$$\Delta(p) = \frac{1}{p^2} \frac{\pi/N}{\int_0^1 d\alpha \frac{\alpha(1-\alpha)}{\sqrt{\phi_0^2 + p^2 \alpha(1-\alpha) + |\phi_0|}}} + \mathcal{O}(1/N^2). \quad (13)$$

When $|\phi_0| \neq 0$, this expression has a pole at $p^2 = 0$ as well as the fermion-antifermion cut on the interval $p^2 \in (-\infty, -4\phi_0^2)$. The spectrum now contains a massless scalar field which is interpolated by $\phi(x)$. This physical massless scalar field is the dilaton.

However, if we remain at large N , but we relax the infinite- N limit and we turn on $1/N$ corrections, the scenario that we have described above becomes unstable. The reason for this can be understood by the fact that the corrections contain logarithmic divergences. The dimensionful couplings obtain corrections to their tree-level beta functions:

$$\beta_{\lambda_1} = 2\lambda_1 + \mathcal{O}(1/N), \quad (14)$$

$$\beta_{1/g} = \frac{1}{g} + \mathcal{O}(1/N), \quad (15)$$

and the coupling λ obtains a nonzero beta function which is of the order of $1/N$ [19]:

$$\beta(\lambda) = \frac{32}{\pi^2 N} (\lambda - 8\lambda^3) + \mathcal{O}(1/N^2). \quad (16)$$

Although this beta function is small at large N , when N is finite, λ is no longer tunable. It is a running coupling which depends on the renormalization scale.

Moreover, the limit of the off-shell effective action and the effective potential for large values of the condensate, $|\phi_0| \rightarrow \infty$, like the asymptotic high-energy limit, is governed by the ultraviolet fixed points. These occur where the dimensionful parameters λ_1 and $1/g$ are set to zero, $\lambda_1 = 1/g = 0$, and, in addition, at the zeros of the beta function for λ (16) which correspond to ultraviolet fixed points. These occur at

$$\lambda_{\text{UV}}^* = \pm \frac{1}{\sqrt{8}}. \quad (17)$$

These fixed points turn out to be at values of λ which are outside of the stability regime (8); that is, they occur where

$$|\lambda_{\text{UV}}^*| = \frac{1}{\sqrt{8}} > \frac{1}{\pi}. \quad (18)$$

As a consequence, as long as these first few orders of the $1/N$ expansion are accurate, the effective potential is unbounded from below. It is minimized by a runaway condensate, $|\phi_0| \rightarrow \infty$. This conclusion holds as long as N is large enough that our formulas for the beta functions are valid.

IV. THE GROSS-NEVEU-THIRING MODEL AND ITS PARITY-VIOLATING DEFORMATION

Among the simple variants of the three-dimensional Gross-Neveu model with parity-breaking operators added,

the model in (2) coupled to an additional vector field turns out to have a stable large- N limit for some range of freely tunable parameters. We will show this explicitly in the context of the large- N expansion, and it should be valid when N is large enough that the expansion is reliable.

Before we proceed, we will review some features of the parity symmetric theory. A hybrid of the three-dimensional Gross-Neveu model and Thirring models is obtained by adding a vector-vector interaction

$$\delta S_T = - \int d^3x \frac{\eta}{2N} \left(\sum_a \bar{\psi}_a \gamma_\mu \psi_a \right)^2 \quad (19)$$

to the action of the Gross-Neveu model in Eq. (1). This interaction breaks some of the flavor symmetry explicitly. It reduces the $O(2N)$ symmetry to $U(N)$. It preserves space-time C , P , and T invariance. Just as the Gross-Neveu interaction can be rewritten using a Hubbard-Stratonovich scalar field $\phi(x)$ as in Eq. (2), the Thirring interaction (19) can be represented by introducing a Hubbard-Stratonovich vector field $A_\mu(x)$ so that the hybrid model has the Euclidean action

$$S_{GNT} = \int d^3x \left\{ i\bar{\psi}_a(x)(\not{\partial} + i\not{A}(x) + \phi(x))\psi_a(x) + \frac{N}{2g}\phi^2(x) + \frac{N}{2\eta}A_\mu^2(x) \right\}. \quad (20)$$

If we compute the effective action using a gauge-invariant regularization such as dimensional regularization, the mass term for the vector field, $\frac{N}{2\eta}A_\mu^2(x)$, is not corrected. This is guaranteed by the Ward-Takahashi identity for the $U(1)$ global symmetry, and it holds to all orders in perturbation theory. In that case, the Thirring coupling $1/\eta$ does not renormalize at all. The choice of a gauge-invariant regulator, or, alternatively, the use of counterterms which cancel gauge-variant local operators which might appear in the effective action due to higher-order radiative corrections, is a choice of short-distance physics. We will make this choice in the following. This “hidden gauge symmetry” of the Thirring model interactions is already well recognized in existing literature [51–55]. For our present case, this leaves the constant $1/\eta$ a freely tunable dimensionful constant, and the operator $\frac{N}{2\eta}A_\mu^2(x)$ the only gauge-variant term in the effective action, to all orders in $1/N$.

The critical theory that is obtained by tuning the dimensionful coupling constants to their ultraviolet fixed points $1/g \rightarrow 0$, $1/\eta \rightarrow 0$ is a conformal field theory. This is in line with known results [48–59].

The only Poincaré and gauge-invariant marginal operators that we can add to the action of the hybrid Gross-Neveu-Thirring model are the cubic scalar self-coupling of the scalar field $\int N \frac{\lambda}{6} \phi^3(x)$ and a Chern-Simons term

$\int Ni \frac{\kappa}{4\pi} A dA$ for the vector field. These terms violate parity and time-reversal symmetry explicitly. The action with them added is

$$\begin{aligned} S = \int d^3x \left\{ i\bar{\psi}_a(x)(\not{\partial} + i\mathbf{A}(x) + \phi(x))\psi_a(x) \right. \\ + N\lambda_1\phi(x) + \frac{N}{2g}\phi^2(x) + N\frac{\lambda}{6}\phi^3(x) \\ \left. + \frac{N}{2\eta}A_\mu^2(x) + Ni \frac{\kappa}{4\pi} \epsilon_{\mu\nu\lambda} A_\mu(x) \partial_\nu A_\lambda(x) \right\}. \end{aligned} \quad (21)$$

As for the deformed Gross-Neveu model, we can always choose counterterms which set the fermion mass operator $\int i m \bar{\psi}(x)\psi(x)$ to zero. As it was there, this is equivalent to the field redefinition $\phi(x) \rightarrow \phi(x) + \text{const}$ so the vacuum expectation value of the scalar field is proportional to the fermion mass.

The effective potential to the leading and next-to-leading orders in N is given by the expression

$$\begin{aligned} \frac{1}{N} \mathbf{V}_{\text{eff}}[\phi_0] = \lambda_1 \phi_0 + \frac{\phi_0^2}{2g} + \frac{\lambda \phi_0^3}{6} + \frac{|\phi_0|^3}{6\pi} \\ + \frac{1}{2N} \int \frac{d^3p}{(2\pi)^3} \{ \ln[\Delta^{-1}(p)] + \text{Tr} \ln[\Delta_{\mu\nu}^{-1}(p)] \} \\ + \text{counterterms} + \mathcal{O}(1/N^2). \end{aligned} \quad (22)$$

The scalar and vector field propagators Δ and $\Delta_{\mu\nu}$ are given in Eqs. (26)–(34) below.

The first line in Eq. (22) is the leading order of the effective potential at large N . At this lowest order of $1/N$ perturbation theory, the vector interaction does not contribute to the effective potential at all. The leading-order effective potential is identical to that of the deformed Gross-Neveu model which we have already described in the discussion around Eqs. (7)–(13). This includes the tunability of the coupling constants in the effective action and the effective potential and the fact that the effective potential is stable in the regime $|\lambda| \leq 1/\pi$. As in the deformed Gross-Neveu model, the theory becomes scale invariant when the dimensionful parameters are tuned to zero: $\lambda_1 = 1/g = 1/\eta = 0$.

Also, as in the deformed Gross-Neveu model, when the remaining coupling is tuned to the edge of the stability regime, $|\lambda| = 1/\pi$, the effective potential becomes flat. A condensate can form, and when it does, it breaks the scale symmetry. In that case, the scalar field propagator is identical to the one quoted in Eq. (13), where it exhibits the massless dilaton pole. The vector field propagator is [see Eqs. (30)–(34) below] that of a topologically massive photon.

The second line in Eq. (22) contains the leading corrections to the large- N limit. It has the contributions

from the fluctuations of the scalar and the vector field. What is now different from the deformed Gross-Neveu model that we described in the previous section is the contribution of the vector field contained in the trace-log term with the vector field propagator.

The vector interaction contributes to the renormalization of the theory and, in particular, to the beta function for λ which now has the form

$$\begin{aligned} \beta(\lambda, \kappa) = -\frac{1}{2\pi^2 N} [(8\lambda)^3 + \zeta(\kappa)(8\lambda) + \iota(\kappa)] + \mathcal{O}(1/N^2), \\ \zeta(\kappa) = -8 \frac{(\frac{\kappa}{2\pi})^4 + 10(\frac{\kappa}{2\pi})^2(\frac{1}{16})^2 - 3(\frac{1}{16})^4}{[(\frac{\kappa}{2\pi})^2 + (\frac{1}{16})^2]^2}, \\ \iota(\kappa) = 4 \left(\frac{\kappa}{2\pi} \right) \left(\frac{1}{16} \right)^2 \frac{3(\frac{1}{16})^2 - 5(\frac{\kappa}{2\pi})^2}{[(\frac{\kappa}{2\pi})^2 + (\frac{1}{16})^2]^3}. \end{aligned} \quad (23)$$

At this order, this beta function depends only on λ and the Chern-Simons level κ , and it is independent of the dimensionful constants λ_1 , $1/g$, and $1/\eta$. Also, we note that the Chern-Simons coefficient κ does not have a renormalization group flow. It can be treated as a tunable parameter.

It is interesting to note that, in the limit as $\kappa \rightarrow \infty$ where we might expect the vector field to decouple, as its topological mass becomes large, the beta function in Eq. (23) indeed reproduces the deformed Gross-Neveu model beta function in Eq. (16) whose ultraviolet fixed points are outside of the stability regime (and we have already argued that the field theory is unstable in that case).

On the other hand, the limit as $\kappa \rightarrow 0$ of the beta function (23) produces

$$\lim_{\kappa \rightarrow 0} \beta(\lambda, \kappa) = -\frac{(8\lambda)}{2\pi^2 N} [(8\lambda)^2 + 24] + \mathcal{O}(1/N^2), \quad (24)$$

which has only one ultraviolet fixed point which is an ultraviolet fixed point located at $\lambda_{\text{UV}}^* = 0$. This fixed point is inside the stability regime, and we will argue that, in this case, the quantum field theory, as well as being a conformal field theory when it is tuned to this fixed point, has an interesting, stable, gapped solution. In this sense, inclusion of the Thirring model coupling, in addition to the Gross-Neveu coupling, has given us a stable deformed field theory, at least where $\kappa \sim 0$. In that field theory, the particle spectrum has fermions with mass gap $\sim |\phi_0|$, a vector field with a topological mass which is also $\sim |\phi_0|$, and a light pseudodilaton scalar field with mass $\sim \frac{1}{N} |\phi_0|$. We expect this to remain so for some range of values of the parameter κ near zero. Here, the scalar field is a spin zero exciton—a bound state of a fermion and an antifermion.

The beta function is graphed for various values of κ in Figs. 1–3. For the range of κ that is depicted in the graph in

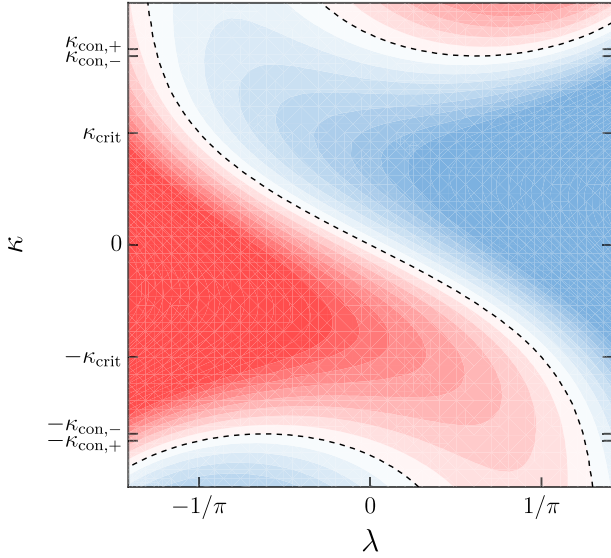


FIG. 1. The beta function $\beta(\lambda, \kappa)$ as a function of λ , plotted on the horizontal axis, and κ , plotted on the vertical axis, is depicted. The beta function is positive in the reddish region, negative in the bluish region, and zero along the dashed lines. For fixed values of κ , the central dashed line is ultraviolet fixed points, and those fixed points are inside the stability regime, $-1/\pi < \lambda < 1/\pi$, for the range of kappa $-\kappa_{\text{crit}} < \kappa < \kappa_{\text{crit}}$ where $\kappa_{\text{crit}} \sim 0.162\dots$ is noted on the vertical axis. The second regime where a stable solution exists is near the top and the bottom of the figure. There, the cubic nature of the beta function provides two more zeros; for a fixed value of κ , they are an ultraviolet and infrared fixed point, and the ultraviolet fixed point is inside the stability regime for $\kappa_{\text{con},-} < |\kappa| < \kappa_{\text{con},+}$ with $\kappa_{\text{con},-} \sim 0.272\dots$ and $\kappa_{\text{con},+} \sim 0.283\dots$

Fig. 2, we can see that, as κ sweeps across that range, the ultraviolet fixed point moves across the stable region. It is contained inside the stable region when

$$-\kappa_{\text{crit}} \lesssim \kappa \lesssim \kappa_{\text{crit}},$$

where $\kappa_{\text{crit}} \sim 0.162\dots$. Stable solutions exist when κ is in this range.

There is another, separate, more narrow regime with $\kappa_{\text{con},-} < |\kappa| < \kappa_{\text{con},+}$ and $\kappa_{\text{con},-} \sim 0.272\dots$, $\kappa_{\text{con},+} \sim 0.283\dots$, where the beta function has three zeros, two ultraviolet fixed points, and an infrared fixed point and one of the two ultraviolet fixed points is in the stable regime. It is seen in the diagram in Fig. 1 and further depicted in the diagram in Fig. 3. For this case, if we initiate the renormalization group flow of λ in the domain of attraction of the ultraviolet fixed point which is in the stable regime, we can obtain a stable gapped solution of the theory. In the next section, we will argue that, for renormalization group flows toward the infrared, away from this ultraviolet fixed point, the two different possible directions that the flow can go have markedly different behaviors. If λ flows to the strong coupling side of the fixed point, there is a stable gapped

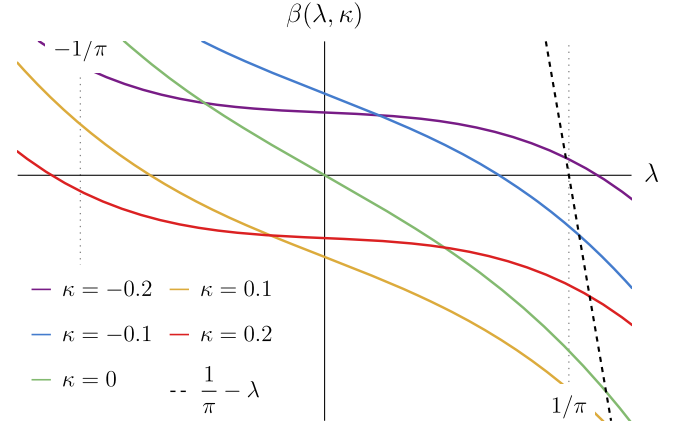


FIG. 2. The β function $\beta(\lambda, \kappa)$ is plotted on the vertical axis versus λ , which is plotted on the horizontal axis, for various values of κ , depicted by the separate curves. The intersection of a curve with the horizontal axis is the ultraviolet fixed point for that value of κ . The stability bounds are denoted by the faint vertical dotted lines. We can see that the fixed point is in the stable region, $-1/\pi < \lambda_{\text{UV}}^* < 1/\pi$ when $-0.162 \lesssim \kappa \lesssim 0.162$. The slanted, dashed line is a graphic of the equation of motion (58). The equation of motion is satisfied when that line intersects the beta function. Note that this is always slightly outside of the stability regime. This is needed so that the solution has lower energy than the trivial solution at $|\phi_0| = 0$ and is thereby a global minimum of the effective potential.

solution of the theory very much as it was in the previous stability regime that we have described in the paragraphs above. It results in a phase where all of the fields are gapped and the scalar field is parametrically light.

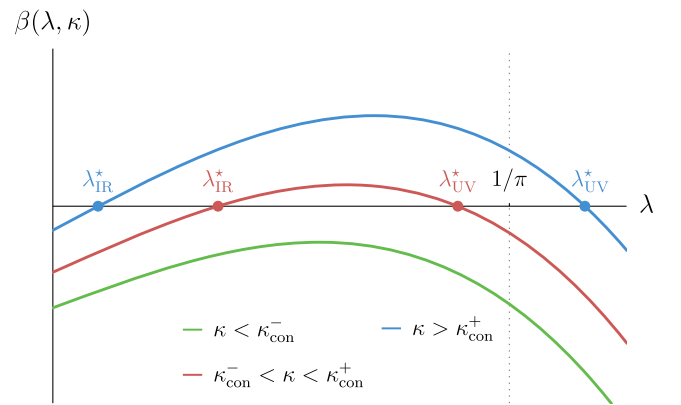


FIG. 3. The β function $\beta(\lambda, \kappa)$ is plotted on the vertical axis versus λ , which is plotted on the horizontal axis, for three values of $\kappa \sim \kappa_{\text{con},\pm}$, depicted by the three separate curves. The intersections with the horizontal axis are fixed points. The stability bounds are denoted by the faint vertical dotted lines. The ultraviolet fixed point is in the stable region $-1/\pi < \lambda_{\text{UV}}^* < 1/\pi$ when the Chern-Simons coefficient is in the narrow range $\kappa_{\text{con},-} \lesssim \kappa \lesssim \kappa_{\text{con},+}$ which are depicted on the vertical axis and which have the approximate values $\kappa_{\text{con},-} \sim 0.272\dots$ and $\kappa_{\text{con},+} \sim 0.283\dots$

On the other hand, if λ flows away from the ultraviolet fixed point to the weaker coupling side—where the infrared fixed point is located—we find a truly exotic phase of this system. Toward the infrared, the coupling λ will flow until it encounters the infrared fixed point. Remarkably, in this phase, even though the symmetries which protect the fermion mass are explicitly broken, the condensate vanishes, $\phi_0 = 0$, and the fermion is massless. This solution describes a “conformal window” with an infrared fixed point which is a very close analog to the Wilson-Fisher fixed point—it is a repulsive fixed point for the dimensionful couplings λ_1 , $1/g$, and $1/\eta$ —which must, therefore, be fine-tuned to reach it—and an attractive fixed point for the coupling λ . In this unexpected place, we find a conformal field theory. This interesting fact certainly deserves further study beyond the scope of what we can present in this paper.

Then, finally, in both of the cases that we have described above, we can fine-tune the Chern-Simons level so that the ultraviolet fixed point sits precisely at the edge of the stability regime. In this case, as we shall argue in the next section, the effective potential for the condensate becomes flat, independent of the condensate. The field theory sitting at this fixed point is scale invariant, and, when the condensate takes up a nonzero value, the scale invariance is spontaneously broken. This symmetry breaking is accompanied by a massless dilaton. This as a realization of the phenomenon which we described in the introduction and which we argued generically occurs only in the infinite- N limit of the deformed Gross-Neveu or Gross-Neveu-Thirring models, in what is now the fully interacting theory with order $1/N$ corrections turned on. We have not checked that this phenomenon survives the next order corrections in the large- N expansion. However, if we conjecture that it does survive, this fine-tuned version of the deformed Gross-Neveu-Thirring model should, for some range of large N , have an exact scale symmetry-breaking solution of the sort that was envisioned in the functional renormalization group literature which motivated the current work [14–17]. Instead of occurring at the edge of a line of fixed points, it occurs when the isolated ultraviolet fixed point lies on the stability edge.

In the next section, we will present some of the details of the large- N expansion applied to this model, including some of the details of how we arrived at the conclusions of this section.

V. DETAILS OF THE LARGE- N EXPANSION

In this section, we shall present the details of the computations which led us to the conclusions which we have outlined in the previous sections.

The effective action at the leading order in large N is given by the expression

$$\begin{aligned} \lim_{N \rightarrow \infty} \Gamma[\psi, \phi, A] = & -N \text{Tr} \ln [i \not{\partial} + \not{A} + i \phi] \\ & + \int d^3x \left\{ i \bar{\psi}_a(x) (\not{\partial} + i \not{A}(x) + \phi(x)) \psi_a(x) \right. \\ & + N \lambda_1 \phi(x) + \frac{N}{2g} \phi^2(x) + N \frac{\lambda}{6} \phi^3(x) \\ & \left. + \frac{N}{2\eta} A_\mu^2(x) + N i \frac{\kappa}{4\pi} \epsilon_{\mu\nu\lambda} A_\mu(x) \partial_\nu A_\lambda(x) \right\}. \end{aligned} \quad (25)$$

We extract a factor of N from the Chern-Simons level, $N\kappa$. Since we will always assume that N is very large, if $N\kappa$ is quantized, we can still take κ to be a continuously varying parameter, and we will assume that this parameter does not grow as we scale N to infinity.

We will look for Poincaré-invariant solutions of this theory described by (25) which have the classical fields $A_\mu(x) = 0$, $\psi(x) = 0$, and $\bar{\psi}(x) = 0$ and for which $\phi(x) = \phi_0$ is a spacetime-position-independent constant. The effective potential for ϕ_0 , to the leading and next-to-leading orders in $1/N$, is then given by the expression in Eq. (22) above, which we recopy here for the readers’ convenience:

$$\begin{aligned} \frac{1}{N} \mathbf{V}_{\text{eff}}[\phi_0] = & \lambda_1 \phi_0 + \frac{\phi_0^2}{2g} + \frac{\lambda \phi_0^3}{6} + \frac{|\phi_0|^3}{6\pi} \\ & + \frac{1}{2N} \int \frac{d^3p}{(2\pi)^3} \{ \ln [\Delta^{-1}(p)] + \text{Tr} \ln [\Delta_{\mu\nu}^{-1}(p)] \} \\ & + \text{counterterms} + \mathcal{O}(1/N^2). \end{aligned}$$

The first line in this equation is simply the effective action (25) evaluated on the classical fields, of which only ϕ_0 is nonzero. The remaining variable ϕ_0 should be determined to minimize $\mathbf{V}_{\text{eff}}[\phi_0]$. The second line in the above equation is due to the fluctuations of the fields $\phi(x)$ and $A_\mu(x)$ about their classical values, and it has the form of the trace logs of the inverse propagators of the scalar and vector fields. Those inverse propagators can be obtained by taking two functional derivatives of the effective action in Eq. (25) and then evaluating the quadratic forms obtained on the classical fields.

The inverse propagator for the scalar field in the background field ϕ_0 is given by the second functional derivative of the effective action (25) by ϕ , evaluated on the background fields $A_\mu(x) = 0 = \psi(x) = \bar{\psi}(x)$, $\phi(x) = \phi_0$,

$$\begin{aligned} \Delta^{-1}(x, y) = & N \left(\frac{1}{g} + \lambda \phi_0 \right) \delta(x - y) \\ & - N \text{Tr} \left(x \left| \frac{1}{i \not{\partial} + i \phi_0} \right| y \right) \left(y \left| \frac{1}{i \not{\partial} + i \phi_0} \right| x \right), \end{aligned} \quad (26)$$

which is easily evaluated to give the momentum space function

$$\begin{aligned}\Delta^{-1}(p) &= N \left[\frac{1}{g} + \lambda \phi_0 + \frac{1}{\pi} \int_0^1 d\alpha \sqrt{\phi_0^2 + p^2 \alpha(1-\alpha)} \right] \\ &= N \left[\frac{1}{g} + \lambda \phi_0 + \frac{|\phi_0|}{2\pi} + \frac{4\phi_0^2 + p^2}{4\pi p} \arctan \frac{p}{2|\phi_0|} \right].\end{aligned}\quad (27)$$

The inverse propagator for the vector field in the background field ϕ_0 is obtained in a similar way as

$$\begin{aligned}\Delta_{\mu\nu}^{-1}(x, y) &= N \text{Tr} \gamma_\mu \left(x \left| \frac{1}{i\partial + i\phi_0} \right| y \right) \gamma_\nu \left(y \left| \frac{1}{i\partial + i\phi_0} \right| x \right) \\ &\quad + \frac{N}{\eta} \delta_{\mu\nu} \delta(x - y),\end{aligned}\quad (28)$$

which, in momentum space, must take the form

$$\Delta_{\mu\nu}^{-1}(p) = \Pi_e \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) - \Pi_o \epsilon_{\mu\lambda\nu} p_\lambda + \frac{N}{\eta} \frac{p_\mu p_\nu}{p^2} \quad (29)$$

so that the vector field propagator is

$$\Delta_{\mu\nu}(p) = \frac{\Pi_e(p) \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + \Pi_o(p) \epsilon_{\mu\lambda\nu} p_\lambda}{\Pi_e(p)^2 + \Pi_o(p)^2 p^2} + \frac{\eta}{N} \frac{p_\mu p_\nu}{p^2}. \quad (30)$$

The one-loop integrals in Eq. (28) are easily evaluated to give the parity-even part:

$$\Pi_e(p) = \frac{N}{\eta} + \frac{N}{2\pi} \int_0^1 d\alpha \frac{\alpha(1-\alpha)p^2}{\sqrt{\alpha(1-\alpha)p^2 + \phi_0^2}} \quad (31)$$

$$= N \left[\frac{1}{\eta} + \frac{p^2 - 4\phi_0^2}{8\pi p} \arctan \frac{p}{2|\phi_0|} + \frac{|\phi_0|}{4\pi} \right] \quad (32)$$

and the parity-odd part

$$\Pi_o(p) = N \left[\frac{\kappa}{2\pi} - \frac{\phi_0}{4\pi} \int_0^1 \frac{d\alpha}{\sqrt{\alpha(1-\alpha)p^2 + \phi_0^2}} \right] \quad (33)$$

$$= N \left[\frac{\kappa}{2\pi} - \frac{\phi_0}{2\pi p} \arctan \frac{p}{2|\phi_0|} \right], \quad (34)$$

respectively. Here, we have used a regularization to define the formally linearly divergent one-loop integrals which

lead to the expressions above. If the regularization respects the Ward-Takahashi identity for the global U(1) symmetry (dimensional regularization is a good example of such), the divergent terms drop out, and the remaining integrations are ultraviolet finite and result in the expressions (31)–(34).

We note that the parity-odd part of the irreducible vector field two-point function has the low- and high-energy behaviors

$$\begin{aligned}\lim_{\frac{p}{|\phi_0|} \rightarrow \infty} \Pi_o(p) &= N \frac{\kappa}{2\pi}, \\ \lim_{\frac{p}{|\phi_0|} \rightarrow 0} \Pi_o(p) &= N \left[\frac{\kappa}{2\pi} - \frac{1/2}{2\pi} \text{sgn}(\phi_0) \right].\end{aligned}\quad (35)$$

The low-energy limit exhibits the shift of the Chern-Simons level $k \rightarrow k - \frac{N}{2} \text{sgn}(\phi_0)$ and $\kappa \rightarrow \kappa - \frac{1}{2} \text{sgn}(\phi_0)$ resulting from integrating out N species of massive complex Dirac fields [60,61]. Moreover, as long as the charged fields have a mass gap, in this case as long as $|\phi_0|$ is nonzero, there is a no-renormalization theorem which asserts that the shift of the low-energy limit in Eq. (35) does not obtain contributions beyond one loop [62,63]. Here, the only one-loop order contributions are included in the leading order in large N , which is contained in the result in Eq. (35). Among other things, this implies that we require no Chern-Simons counterterms to renormalize the two-point function of the vector field, or any of the other correlation functions, at any order of perturbation theory.

The high-energy limits $|p| \gg |\phi_0|$ of the inverse propagators are

$$\lim_{\frac{p}{|\phi_0|} \rightarrow \infty} \frac{1}{N} \Delta^{-1}(p) = \left[\frac{p}{8} \right], \quad (36)$$

$$\begin{aligned}\lim_{\frac{p}{|\phi_0|} \rightarrow \infty} \frac{1}{N} \Delta_{\mu\nu}^{-1}(p) &= \left[\frac{p}{16} \right] \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + \frac{\kappa}{2\pi} \epsilon_{\mu\lambda\nu} p_\lambda \\ &\quad + \frac{1}{\eta} \frac{p_\mu p_\nu}{p^2},\end{aligned}\quad (37)$$

$$\lim_{\frac{p}{|\phi_0|} \rightarrow \infty} \det \frac{1}{N} \Delta_{\mu\nu}^{-1}(p) = p^2 \left[\left(\frac{\kappa}{2\pi} \right)^2 + \left(\frac{1}{16} \right)^2 \right] \cdot \frac{1}{\eta}, \quad (38)$$

respectively.

The order $1/N$ corrections in (22) are ultraviolet divergent, and they require regularization and renormalization. To this end, we define the integral by imposing an ultraviolet hard cutoff Λ on the magnitude of the momentum integration variable. We then perform an asymptotic expansion of the integrands about the large $p/|\phi_0|$ limit

to identify the ultraviolet divergent parts of the integral. Then we add and subtract the divergent parts to obtain the expression for the scalar field contribution:

$$\begin{aligned} & \frac{1}{2} \int^\Lambda \frac{d^3 p}{(2\pi)^3} (\ln [\Delta^{-1}(p)] - \ln [\Delta^{-1}(p)]_{\phi_0=0}) \\ &= \frac{1}{2} \int^\Lambda \frac{d^3 p}{(2\pi)^3} \ln \left[\frac{\Delta^{-1}(p)}{N[\frac{p}{8}]} \right] \\ &= C_1 \phi_0 + C_2 \phi_0^2 + C_3 \phi_0^3 + C_4 |\phi_0|^3 \\ &+ \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \left[\ln \left[\frac{\Delta^{-1}(p)}{N[\frac{p}{8}]} \right] - (\dots) \right]. \end{aligned} \quad (39)$$

The terms with $C_1, \dots, C_4$ are the ultraviolet divergent contributions. Their coefficients have the explicit expressions

$$C_1 = \frac{1}{4\pi^2} \int_0^\Lambda dp \left[\frac{512\lambda}{g^2(p + |\phi_0|)} - \frac{64\lambda}{g} + 8p\lambda \right], \quad (40)$$

$$C_2 = \frac{1}{4\pi^2} \int_0^\Lambda dp \left[\frac{512\lambda^2 - 32}{g(p + |\phi_0|)} - 32\lambda^2 + 4 \right], \quad (41)$$

$$C_3 = \frac{1}{4\pi^2} \int_0^\Lambda dp \left[\frac{512\lambda^3 - 96\lambda}{3(p + |\phi_0|)} \right], \quad (42)$$

$$C_4 = \frac{1}{4\pi^2} \int_0^\Lambda dp \left[-\frac{32}{3\pi(p + |\phi_0|)} \right]. \quad (43)$$

The last term on the right-hand side of Eq. (39) has the ultraviolet divergent terms subtracted. As it is now finite, the ultraviolet cutoff is removed.

There are a few notable features of the above development:

- (i) The terms with $C_1, \dots, C_4$ contain quadratic, linear, and logarithmic divergences, all of which have been defined by imposing the cutoff Λ .
- (ii) We have cut off the infrared limit of the logarithmically divergent integrals in $C_1, \dots, C_4$ by replacing a factor of p in the denominator by $p + |\phi_0|$. We are free to do this, as these terms are both added and subtracted in Eq. (39). We do this in order to minimize our introduction of dimensionful parameters in the last line in Eq. (39).
- (iii) The divergent terms should be canceled by adding the appropriate counterterms to the original action. However, these divergent terms are not all analytic in ϕ_0 , since the term $|\phi_0|^3$ appears with a logarithmically divergent coefficient C_4 . This singular term can be removed only by a wave-function renormalization for the scalar field, $\phi \rightarrow Z^{\frac{1}{2}}\phi$. This is made possible by the existence of the leading-order term $\frac{|\phi_0|^3}{6\pi}$ in the effective potential (22). It is easy to confirm independently that the wave-function renormalization that is necessary to render the $|\phi_0|^3$ terms finite in the effective potential has the same

singular contribution at the order of $1/N$ as the scalar field wave-function renormalization, which is computed more generally from Feynman diagrams and which is needed to renormalize the correlation functions of the field theory.

- (iv) Once the wave-function renormalization is taken care of, the remaining divergent terms renormalize the tree-level terms with λ_1 , $1/g$, $1/\eta$, and λ in the action. The logarithmic divergences will give rise to or correct leading-order beta functions for these couplings. For the dimensionful couplings, which already have tree-level beta functions,

$$\begin{aligned} \beta_{\lambda_1} &= 2\lambda_1 + \mathcal{O}(1/N), \\ \beta_{1/g} &= \frac{1}{g} + \mathcal{O}(1/N), \\ \beta_{1/\eta} &= \frac{1}{\eta}. \end{aligned} \quad (44)$$

These are the tree-level beta functions associated with the classical scaling dimensions of the associated operators plus order $1/N$ corrections. Our main point here is that, since the $1/N$ contributions are small and we can check that they are all proportional to the constants λ_1 , $1/g$, and $1/\eta$, their ultraviolet fixed point remains positioned at $\lambda_1 = 1/g = 1/\eta = 0$.

- (v) In the ultraviolet finite fourth line in Eq. (39), we can remove the ultraviolet cutoff. That line is then a function of the dimensionful parameters $1/g$ and ϕ_0 only. This becomes particularly simple when we put $1/g \rightarrow 0$, whence it is a function of ϕ_0 only, and dimensional analysis tells us that it must have the form

$$\begin{aligned} & \lim_{1/g \rightarrow 0} \frac{1}{2N} \int \frac{d^3 p}{(2\pi)^3} \left[\ln \left[\frac{\Delta^{-1}(p)}{N[\frac{p}{8}]} \right] - (\dots) \right] \\ &= \frac{\xi_1}{N} \phi_0^3 + \frac{\xi_2}{N} |\phi_0|^3. \end{aligned} \quad (45)$$

We shall not need explicit expressions for the quantities ξ_1 and ξ_2 since, in our subtraction scheme, they will be absorbed by counterterms.

The contribution of the fluctuations of the vector field to the effective potential is given by

$$\begin{aligned} & \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} (\text{Tr} \ln [\Delta_{\mu\nu}^{-1}(p)] - \text{Tr} \ln [\Delta_{\mu\nu}^{-1}(p)]_{\phi_0=0}) \\ &= \frac{1}{2} \int^\Lambda \frac{d^3 p}{(2\pi)^3} \ln \left[\frac{\Pi_e(p)^2 + p^2 \Pi_o(p)^2}{N^2[(\frac{1}{16})^2 + (\frac{k}{2\pi})^2] p^2} \right] \\ &= D_1 \phi_0 + D_2 \phi_0^2 + D_3 \phi_0^3 + D_4 |\phi_0|^3 + D_5 \phi_0 |\phi_0| \\ &+ \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \left[\ln \left[\frac{\Pi_e(p)^2 + p^2 \Pi_o(p)^2}{N^2[(\frac{1}{16})^2 + (\frac{k}{2\pi})^2] p^2} \right] - (\dots) \right], \end{aligned} \quad (47)$$

where the coefficients are given by

$$\begin{aligned}
 D_1 &= \int_0^\Lambda \frac{dp}{4\pi^2} \left\{ \frac{2048\pi^3\kappa}{\eta(64\kappa^2 + \pi^2)^2} - \frac{64\pi\kappa p}{64\kappa^2 + \pi^2} \right. \\
 &\quad \left. + \frac{1048576\pi^3\kappa^3 - 49152\pi^5\kappa}{\eta^2(64\kappa^2 + \pi^2)^3(p + |\phi_0|)} \right\}, \\
 D_2 &= \int_0^\Lambda \frac{dp}{4\pi^2} \left\{ \frac{8\pi^4 - 1536\pi^2\kappa^2}{(64\kappa^2 + \pi^2)^2} \right. \\
 &\quad \left. + \frac{98304\pi^4\kappa^2 - 384\pi^6 - 524288\pi^2\kappa^4}{\eta(64\kappa^2 + \pi^2)^3(p + |\phi_0|)} \right\}, \\
 D_3 &= \int_0^\Lambda \frac{dp}{4\pi^2} \left\{ \frac{512\pi^5\kappa - 163840\pi^3\kappa^3/3}{(64\kappa^2 + \pi^2)^3(p + |\phi_0|)} \right\}, \\
 D_4 &= \int_0^\Lambda \frac{dp}{4\pi^2} \left\{ \frac{10240\pi\kappa^2 - 96\pi^3 + 32\pi(64\kappa^2 + \pi^2)/3}{(64\kappa^2 + \pi^2)^2(p + |\phi_0|)} \right\}, \\
 D_5 &= \int_0^\Lambda \frac{dp}{4\pi^2} \left\{ \frac{256\kappa}{64\kappa^2 + \pi^2} - \frac{8192\pi^2\kappa}{\eta(64\kappa^2 + \pi^2)^2(p + |\phi_0|)} \right\}.
 \end{aligned}$$

There are some notable features about the contribution in Eq. (47).

- (i) The D_5 term on the right-hand side of Eq. (47), that is, $\sim \Lambda\phi_0|\phi_0|$, is not analytic in the scalar field. This divergence is a symptom of a singular contribution to the fermion mass operator $\sim \int d^3x i \frac{1}{N} \delta m \bar{\psi}(x) \psi(x)$ which, in our renormalization scheme, must be canceled by a field translation $\phi_0 \rightarrow \phi_0 + \delta\phi_0$. Under this field translation, the leading-order $|\phi_0|^3$ term in the effective potential changes as

$$\frac{N}{6\pi} \left| \phi_0 + \frac{\delta m}{N} \right|^3 \approx \frac{N}{6\pi} |\phi_0|^3 + \frac{\delta m}{2\pi} \phi_0 |\phi_0|.$$

This provides the counterterm that is needed to cancel the ultraviolet divergent nonanalytic $\sim \Lambda\phi_0|\phi_0|$ term which arises in the next-to-leading-order effective potential (47).

- (ii) As in the case of the scalar correction, the logarithmically divergent terms in Eq. (47) also contain terms of the form

$$\sim |\phi_0|^3 \ln \frac{\Lambda}{|\phi_0|}.$$

As in the case of the scalar contribution that we have discussed above, cancellation of these ultraviolet singularities is achieved by wave-function renormalization. This contributes singular terms to the scalar field wave-function renormalization factor Z in

$$\phi(x) \rightarrow Z^{\frac{1}{2}} \phi(x).$$

- (iii) The vector field correction in (47) will contribute to the beta functions of λ_1 and $1/g$ at the order of $1/N$. It turns out that the contributions are all proportional

to one of the constants λ_1 or $1/\eta$. This means that the beta functions for all three of the dimensionful couplings are proportional to the dimensionful couplings, with their signs determined by the dominant tree-level terms. They still have an ultraviolet fixed point located at $\lambda_1 = 1/g = 1/\eta = 0$.

- (iv) As it was for the scalar field contribution, the last terms in Eq. (47) are now ultraviolet finite. The cutoff can be removed, and, when $\lambda_1 = 1/g = 1/\eta = 0$, the only dimensionful quantity they depend on is ϕ_0 , whence simple dimensional analysis tells us that

$$\begin{aligned}
 &\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[\ln \left[\frac{\Pi_e(p)^2 + p^2 \Pi_o(p)^2}{N^2 \left[\left(\frac{1}{16} \right)^2 + \left(\frac{k}{2\pi} \right)^2 \right] p^2} \right] - (\dots) \right] \\
 &= \tilde{\zeta}_1 \phi_0^3 + \tilde{\zeta}_2 |\phi_0|^3
 \end{aligned}$$

and where $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ are finite numbers. As before, we shall not need to compute them as, in our subtraction scheme, they will be absorbed by counterterms.

In the following, we will seek a particular gapped solution of the field theory which involves a simplification of the renormalization group flow. To find this solution, we begin with the ultraviolet limit where $|\phi_0| \rightarrow \infty$ and where the dimensionful coupling constants are tuned to zero, which is their ultraviolet fixed point $\lambda_1 = 0$, $1/g = 0$, $1/\eta = 0$. In this limit, λ is also in the vicinity of its ultraviolet fixed point λ_{UV}^* . We assume that this ultraviolet fixed point is in the stable regime. Then, as we lower $|\phi_0|$ from infinity, we keep the dimensionful couplings tuned precisely to their ultraviolet fixed point, so that they do not flow. On the other hand, λ flows as we lower $|\phi_0|$. We proceed to lower $|\phi_0|$ until we arrive at that value of λ and ϕ_0 where the equation of motion is satisfied. This gives us a particular solution of the theory. More general solutions would be obtained by letting all of the couplings flow as we lower $|\phi_0|$. We will not consider that more general scenario here.

Let us proceed by assuming that the power law singularities $\sim \Lambda^3$, $\sim \Lambda^2$, and $\sim \Lambda$ are canceled by counterterms and then the renormalized couplings λ_1 , $1/g$, and $1/\eta$ are tuned to zero but that we have yet to perform the wave-function and coupling constant λ renormalizations. Combining the leading order and the corrections that we have described above, the resulting, partially renormalized effective potential is given by the expression

$$\begin{aligned}
 V_{\text{eff}}[\phi_0] &= N \left(\frac{|\phi_0|^3}{6\pi} + \lambda \frac{\phi_0^3}{6} \right) \\
 &+ \frac{|\phi_0|^3}{4\pi^2} \left(\frac{10240\pi\kappa^2 - 96\pi^3}{(64\kappa^2 + \pi^2)^2} - \frac{32}{3\pi} \frac{64\kappa^2}{64\kappa^2 + \pi^2} \right) \ln \frac{\Lambda}{|\phi_0|} \\
 &+ \frac{\phi_0^3}{4\pi^2} \left(\frac{512}{3} \lambda^3 - 32\lambda + \frac{512\pi^3\kappa}{3} \frac{3\pi^2 - 320\kappa^2}{(64\kappa^2 + \pi^2)^3} \right) \ln \frac{\Lambda}{|\phi_0|} \\
 &+ \mathcal{O}(1/N).
 \end{aligned} \tag{48}$$

To complete the renormalization of the effective potential, we introduce a positive parameter μ with the dimension of mass and which will play the part of the renormalization scale. The construction that we shall use in the following is very close to the one originally pioneered by Coleman and Weinberg [64] in their analysis of dynamical symmetry breaking. Accordingly, we shall use the renormalization scheme where

$$\mathbf{V}_{\text{eff}}[\phi_0] \Big|_{|\phi_0|=\mu} = N \left(\frac{1}{\pi} + \text{sgn}(\phi_0) \lambda \right) \frac{\mu^3}{6}. \quad (49)$$

This scheme fixes the wave-function renormalization and the renormalization of λ uniquely. From these, we deduce the renormalization group functions

$$\begin{aligned} \gamma &= -\frac{1}{2Z} \mu \frac{\partial}{\partial \mu} Z \\ &= -\frac{1}{4\pi^2 N} \left(\frac{\frac{5}{8} \left(\frac{\kappa}{2\pi} \right)^2 - \frac{3}{8} \left(\frac{1}{16} \right)^2}{\left[\left(\frac{\kappa}{2\pi} \right)^2 + \left(\frac{1}{16} \right)^2 \right]^2} - \frac{\frac{32}{3} \left(\frac{\kappa}{2\pi} \right)^2}{\left[\left(\frac{\kappa}{2\pi} \right)^2 + \left(\frac{1}{16} \right)^2 \right]} \right) \\ &\quad + \mathcal{O}(1/N^2) \end{aligned} \quad (50)$$

and the beta function for λ which we have quoted in Eq. (23) and which, for the reader's convenience, we recopy here:

$$\begin{aligned} \beta(\lambda, \kappa) &= -\frac{1}{2\pi^2 N} [(8\lambda)^3 + \zeta(\kappa)(8\lambda) + \iota(\kappa)], \\ \zeta(\kappa) &= -8 \frac{\left(\frac{\kappa}{2\pi} \right)^4 + 10 \left(\frac{\kappa}{2\pi} \right)^2 \left(\frac{1}{16} \right)^2 - 3 \left(\frac{1}{16} \right)^4}{\left[\left(\frac{\kappa}{2\pi} \right)^2 + \left(\frac{1}{16} \right)^2 \right]^2}, \\ \iota(\kappa) &= 4 \left(\frac{\kappa}{2\pi} \right) \left(\frac{1}{16} \right)^2 \frac{3 \left(\frac{1}{16} \right)^2 - 5 \left(\frac{\kappa}{2\pi} \right)^2}{\left[\left(\frac{\kappa}{2\pi} \right)^2 + \left(\frac{1}{16} \right)^2 \right]^3}. \end{aligned}$$

To obtain a renormalization group improved effective potential, we proceed by demanding that the renormalized effective potential obey the renormalization group equation

$$[\mu \partial_\mu + \gamma \phi_0 \partial_{\phi_0} + \beta(\lambda, \kappa) \partial_\lambda] \mathbf{V}_{\text{eff}}[\phi_0] = 0. \quad (51)$$

Alongside this, we may utilize the fact that the effective potential has mass dimension 3, leading to the expression

$$[\mu \partial_\mu + \phi_0 \partial_{\phi_0} - 3] \mathbf{V}_{\text{eff}}[\phi_0] = 0. \quad (52)$$

Together, these yield the equation

$$\left[\mu \partial_\mu + 3 \frac{\gamma}{1-\gamma} + \frac{\beta(\lambda, \kappa)}{1-\gamma} \partial_\lambda \right] \mathbf{V}_{\text{eff}}[\phi_0] = 0. \quad (53)$$

The effective potential that solves the above equation, together with the boundary condition (49), is

$$\mathbf{V}_{\text{eff}}[\phi_0] = N \tilde{z}(t)^3 \frac{|\phi_0|^3}{6} \left(\frac{1}{\pi} + \text{sgn}(\phi_0) \tilde{\lambda}(t) \right), \quad (54)$$

where

$$t \equiv \ln \left(\frac{\mu}{|\phi_0|} \right), \quad \tilde{z}(t) = \exp \left(- \int_0^t ds \frac{\gamma(s)}{1-\gamma(s)} \right), \quad (55)$$

and the cubic coupling obeys

$$\frac{d\tilde{\lambda}(t)}{dt} = -\frac{\beta(\tilde{\lambda}(t), \kappa)}{1-\gamma(t)}, \quad \tilde{\lambda}(0) = \lambda. \quad (56)$$

Here, we have denoted the wave-function factor and the running coupling by $\tilde{z}(t)$ and $\tilde{\lambda}(t)$ since, because of the $\frac{1}{1-\gamma}$ factor, they satisfy a slightly nonstandard flow equation. We emphasize that the fixed points of the flow of $\tilde{\lambda}$ are the same as those of λ .

If the trivial strong coupling fixed points of the dimensional couplings survive at all orders, the expression for the effective potential in Eq. (54) is essentially exact. It describes the full effective potential of this model when $\lambda_1 = 1/g = 1/\eta = 0$. The only approximation that we shall make going forward is our partial knowledge of the renormalization group functions—we have determined them only to their first nontrivial order $\sim 1/N$.

In particular, at this point we can address the issue of global stability of the effective potential. The limit at $|\phi_0| \rightarrow \infty$ is the asymptotic limit of the flow, $t \rightarrow -\infty$, toward ultraviolet fixed points of the beta function, if they exist. In order for the potential to be positive and bounded from below in this limit, we require that the renormalization group flow is attracted to an ultraviolet fixed point which is in the stability regime, $|\lambda| \leq 1/\pi$. This is the criterion for global stability. Examining the beta function in Eq. (23) (and Figs. 1–3), we see that is so only for one of two ranges of the parameter κ , specified in those figures. We must, therefore, assume that κ is in one of those ranges.

The physical values that ϕ_0 must take up are at minima of the effective potential, with the lowest minimum being the stable one. The extrema of the effective potential occur at zeros of the first derivative, those values of ϕ_0 for which

$$\partial_{\phi_0} \mathbf{V}_{\text{eff}}[\phi_0] = 0. \quad (57)$$

From this equation, by taking the derivative of Eq. (54) explicitly, we learn that the extrema occur either at $\phi_0 = 0$ or at that value of ϕ_0 where $\tilde{\lambda}(t)$ solves the equation

$$\frac{1}{\pi} + \tilde{\lambda}(t) \text{sgn}(\phi_0) = -\frac{\beta(\tilde{\lambda}(t), \kappa)}{3} \text{sgn}(\phi_0). \quad (58)$$

The latter Eq. (58) is interesting from the point of view of our large- N approximate solution of the theory, since in that case $\beta \sim 1/N$ and the solution occurs where the running coupling has run, within accuracy $1/N$, to the edge of the

stability regime at $|\tilde{\lambda}| = 1/\pi$. In fact, generally, for this solution to be the global minimum, that is, for

$$\mathbf{V}_{\text{eff}}[\phi_0] \leq \mathbf{V}_{\text{eff}}[0] = 0, \quad (59)$$

ϕ_0 should occur where $\tilde{\lambda}$ has run to a point outside of the stability regime. Equation (58) tells us that this is so when its right-hand side is negative, that is, where

$$\beta(\tilde{\lambda}(t), \kappa) \text{sgn}(\phi_0) > 0 \quad (60)$$

or when ϕ_0 and $\beta(\tilde{\lambda}(t), \kappa)$ have the same signs.

We can confirm that, in this case, Eq. (58) is actually an equation for a local minimum by examining the second derivative:

$$\begin{aligned} & \left. \frac{\partial^2}{\partial |\phi_0|^2} \mathbf{V}_{\text{eff}}[\phi_0] \right|_{\frac{\partial}{\partial \phi_0} \mathbf{V}_{\text{eff}}[\phi_0]=0} \\ &= N \frac{\tilde{z}^3(t)}{(1-\gamma(t))^2} \frac{|\phi_0|}{2} \beta(\tilde{\lambda}(t), \kappa) (1 + \beta'(\tilde{\lambda}(t), \kappa)) \text{sgn}(\phi_0). \end{aligned} \quad (61)$$

This is indeed positive when Eq. (60) is satisfied if, in addition to that,

$$\partial_{\tilde{\lambda}} \beta(\tilde{\lambda}(t), \kappa) > -1$$

at the position of the solution. This is certainly the case for our perturbative solution where $\beta' \sim 1/N$.

In the cases where there is a single ultraviolet fixed point in the stability regime, the task of finding the particular solution of this theory has been reduced to solving Eq. (58), which is an equation for the coupling constant. We began at infinite N by observing that tuning λ to the edge of the stability regime was interesting. We find at the next to leading order that λ is no longer tunable; however, it must satisfy Eq. (58). Since, in that equation, $\beta \sim 1/N$, this implies that λ self-tunes to roughly the same place. Beginning the flow to the infrared at a place which is a slightly larger or slightly smaller value than λ_{UV}^* simply chooses the sign of ϕ_0 , and the flow proceeds until λ is just outside of the stability range, at one edge or the other depending on the sign of ϕ_0 .

In the case where there are an ultraviolet and an infrared fixed point, the flow to the infrared which begins on the strong coupling side of the infrared fixed point, the side which is away from the infrared fixed point, also flows to the edge of the stability range as described above. A flow to the infrared which begins on the weaker coupling side of the ultraviolet fixed point, on the other hand, flows to the infrared fixed point where $\phi_0 = 0$. This flow describes the conformal window. In this case, the coupling constant tunes itself to the infrared fixed point, and the theory should be a conformal field theory. This theory occurs at the infrared fixed point which is a quantum analog of the Wilson-Fisher fixed point of classical statistical models where, like those,

it is of mixed stability. It occurs at the ultraviolet fixed points of the couplings $\lambda_1, \frac{1}{g}$, and $\frac{1}{\eta}$ and at the infrared fixed point of λ .

Finally, there is the interesting possibility where we tune κ so that an ultraviolet fixed point sits precisely at the stability edge. The values of κ where this occurs are the extreme ones $|\kappa| = 0.162\dots, 0.283\dots$. In these cases, when the coupling constants are tuned to the ultraviolet fixed point, the effective potential in (54) is flat and, consistent with this, because the beta function vanishes and $|\tilde{\lambda}| = 1/\pi$, Eq. (58) is satisfied for any value of ϕ_0 which has sign opposite to that of λ . In this case, the condensate, ϕ_0 can take up any value which has the correct overall sign. If this value is nonzero, it breaks the scale symmetry spontaneously. The scalar field becomes massless, playing the role of the dilaton. It is the only massless field. The fermion spectrum is gapped and the vector field has a topological mass.

VI. A PARITY-CONSERVING VARIANT

If N is even, we can formulate a variant of the Gross-Neveu interaction which drives a phase transition which breaks an internal discrete symmetry rather than parity and time-reversal invariance. This version of the Gross-Neveu interaction would have the form

$$\mathbf{S}_{\text{GN}} = \int d^3x \frac{g}{2N} \left(\sum_{a=1}^{N/2} (\bar{\psi}_a \psi_a - \bar{\chi}_a \chi_a) \right)^2, \quad (62)$$

where we retain the notation ψ_a for the first $N/2$ spinor fields and we rename the remainder of them as χ_a . With the Thirring interaction added, and after introducing the scalar and vector Hubbard-Stratonovich fields, and adding the relevant and marginal symmetry-breaking operators, the action can be written as

$$\begin{aligned} \mathbf{S} = \int d^3x \Big\{ & i\bar{\psi}_a(x)(\not{\partial} + i\not{A}(x) + \phi(x))\psi_a(x) \\ & + i\bar{\chi}_a(x)(\not{\partial} + i\not{A}(x) - \phi(x))\chi_a(x) + N\lambda_1\phi(x) \\ & + \frac{N}{2g}\phi^2(x) + N\frac{\lambda}{6}\phi^3(x) + \frac{N}{2\eta}A_\mu^2(x) \Big\} \end{aligned} \quad (63)$$

where the index a on ψ_a and χ_a runs over $a = 1, \dots, N/2$. This model has $U(N/2) \times U(N/2)$ flavor symmetry. It also has an internal Z_2 symmetry:

$$\begin{aligned} Z_2: \quad & \phi(x) \rightarrow -\phi(x), \\ & A_\mu(x) \rightarrow A_\mu(x), \\ & \psi_a(x) \rightarrow \chi_a(x), \\ & \chi_a(x) \rightarrow \psi_a(x), \end{aligned}$$

which is broken explicitly by the introduction of the couplings $\lambda_1\phi(x)$ and $+\frac{\lambda}{6}\phi^3(x)$. The discrete spacetime symmetries C , P , and T remain intact. The Yukawa coupled field $\phi(x)$ is now a scalar rather than a pseudoscalar. The Chern-Simons term for the vector field is absent.

This model shares many features with the parity-violating version of the previous section, including an identical effective potential for the vacuum expectation value ϕ_0 at the leading order of large N and a beta function for the Z_2 violating coupling λ which, at the order of $1/N$, is identical to that of the previous model with $\kappa = 0$, the one quoted in Eq. (24) which we recopy here:

$$\beta(\lambda) = -\frac{(8\lambda)}{2\pi^2 N}[(8\lambda)^2 + 24] + \mathcal{O}(1/N^2).$$

This beta function has a single ultraviolet fixed point located at $\lambda_{UV}^* = 0$, which is inside the stable regime. This is a parity-preserving model which will also have a stable gapped phase with a pseudodilaton.

VII. CONCLUSION AND DISCUSSION

We have studied a variant of the three-dimensional Gross-Neveu model—the Gross-Neveu-Thirring model—whose deformation by adding marginal and relevant parity and time-reversal symmetry-breaking operators has, for a range of its parameters, interesting stable solutions—conformal field theories when the coupling constants are tuned to fixed points which are in the stable regime—and interesting stable gapped phases otherwise. The gapped phases are characterized by the presence of a light scalar, an exciton which plays the role of a pseudodilaton. Its mass is a factor of $1/N$ smaller than that of the spinor and vector field masses.

A second type of solution that we have found is an ungapped phase which exists in a conformal window, where the spinor field is massless and there exists a nontrivial infrared fixed point where the model is a conformal field theory. It would be very interesting to understand whether this latter phase or a close analog could occur in a realistic setting like a Dirac material. Graphene itself has a large dynamical regime where many of its properties are scale symmetric to a good approximation. This is usually attributed to weak coupling—the fact that the many possible interactions would be ineffective in violating scale symmetry. This is in slight contradiction with the fact that the Coulomb interaction, for example, is relatively strong and does not have a convergent perturbation theory. A scale symmetric phase trapped in the window

between infrared and ultraviolet fixed points would be a fascinating option there.

Finally, when the Chern-Simons level is fine-tuned in such a way that an ultraviolet fixed point sits precisely at the stability edge, we find a rare example of a non-supersymmetric quantum field theory with exact scale invariance, a conformal field theory, where the scale invariance can be spontaneously broken so that it has a gap in the fermion spectrum, and a topologically massive vector field but accompanied by a massless dilaton. What is more, this interesting example occurs in the context of an interacting field theory which is systematically analyzable in a controlled approximation, the $1/N$ expansion.

The instability which plagues the deformed Gross-Neveu model is very similar to the one which occurs for the large- N $(\phi^2)^3$ vector model in three dimensions. The coupling constant in that model also has an ultraviolet fixed point was originally found by Pisarski [65] and which occurs outside of the regime in which the theory would be stable. This destabilizes the apparent strong coupling Bardeen-Moshe-Bander phase of the model that was originally found to be there in the infinite- N limit [66]. A symptom of the instability is that the dilaton, which was massless in the strict large- N limit, gets a tachyonic mass at the order of $1/N$ [67]. It would be interesting to adapt some of the technique which we have developed here to better understand that theory.

After this work was completed, the authors learned of other recent work where the beta functions of the model that we consider as well as some other interesting cases are computed [68]. We agree with that work on issues where we overlap. We also note that the existence of fixed points of the renormalization group flow in more general theories with non-Abelian Chern-Simons terms have been examined for a large array of theories and many such fixed points have been found [69]. It would be very interesting to reexamine those examples to see whether fixed points of those more general theories can also occur at the stability edge and can then have gapped phases with a dilaton, in addition to the scale-invariant theories which occur at the fixed points.

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DATA AVAILABILITY

No data were created or analyzed in this study.

- [1] G. W. Semenoff, Condensed-matter simulation of a three-dimensional anomaly, *Phys. Rev. Lett.* **53**, 2449 (1984).
- [2] S. Raghu, X. L. Qi, C. Honerkamp, and S. C. Zhang, Topological Mott insulators, *Phys. Rev. Lett.* **100**, 156401 (2008).
- [3] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, The electronic properties of graphene, *Rev. Mod. Phys.* **81**, 109 (2009).
- [4] V. N. Kotov, B. Uchoa, V. M. Pereira, F. Guinea, and A. H. Castro Neto, Electron-electron interactions in graphene: Current status and perspectives, *Rev. Mod. Phys.* **84**, 1067 (2012).
- [5] T. O. Wehling, A. M. Black-Schaffer, and A. V. Balatsky, Dirac materials, *Adv. Phys.* **63**, 1 (2014).
- [6] I. F. Herbut, Interactions and phase transitions on graphene's honeycomb lattice, *Phys. Rev. Lett.* **97**, 146401 (2006).
- [7] I. F. Herbut, V. Juricic, and B. Roy, Theory of interacting electrons on the honeycomb lattice, *Phys. Rev. B* **79**, 085116 (2009).
- [8] I. F. Herbut, V. Juricic, and O. Vafek, Relativistic Mott criticality in graphene, *Phys. Rev. B* **80**, 075432 (2009).
- [9] V. Juricic, I. F. Herbut, and G. W. Semenoff, Coulomb interaction at the metal-insulator critical point in graphene, *Phys. Rev. B* **80**, 081405 (2009).
- [10] C. Weeks and M. Franz, Interaction-driven instabilities of a Dirac semimetal, *Phys. Rev. B* **81**, 085105 (2010).
- [11] G. W. Semenoff, Chiral symmetry breaking in graphene, *Phys. Scr.* **T146**, 014016 (2012).
- [12] J. Ostmeier, E. Berkowitz, S. Krieg, T. A. Lähde, T. Luu, and C. Urbach, Semimetal–Mott insulator quantum phase transition of the Hubbard model on the honeycomb lattice, *Phys. Rev. B* **102**, 245105 (2020).
- [13] I. F. Herbut, Wilson-Fisher fixed points in presence of Dirac fermions, *Mod. Phys. Lett. B* **38**, 2430006 (2024).
- [14] C. Cresswell-Hogg and D. F. Litim, Line of fixed points in Gross-Neveu theories, *Phys. Rev. Lett.* **130**, 201602 (2023).
- [15] C. Cresswell-Hogg and D. F. Litim, Critical fermions with spontaneously broken scale symmetry, *Phys. Rev. D* **107**, L101701 (2023).
- [16] C. Cresswell-Hogg and D. F. Litim, Scale symmetry breaking and generation of mass at quantum critical points, *J. High Energy Phys.* **07** (2024) 066.
- [17] C. Cresswell-Hogg and D. F. Litim, Fermion mass generation without symmetry breaking, *Phys. Rev. D* **110**, 065014 (2024).
- [18] F. D. M. Haldane, Model for a quantum Hall effect without Landau levels: Condensed-matter realization of the “parity anomaly”, *Phys. Rev. Lett.* **61**, 2015 (1988).
- [19] G. W. Semenoff and R. A. Stewart, Dilaton in a multicritical 3+epsilon-D parity violating field theory, *Phys. Lett. B* **853**, 138691 (2024).
- [20] W. A. Bardeen, C. N. Leung, and S. T. Love, The dilaton and chiral symmetry breaking, *Phys. Rev. Lett.* **56**, 1230 (1986).
- [21] K. I. Kondo, H. Mino, and K. Yamawaki, Critical line and dilaton in scale invariant QED, *Phys. Rev. D* **39**, 2430 (1989).
- [22] D. D. Dietrich, F. Sannino, and K. Tuominen, Light composite Higgs from higher representations versus electroweak precision measurements: Predictions for CERN LHC, *Phys. Rev. D* **72**, 055001 (2005).
- [23] W. D. Goldberger, B. Grinstein, and W. Skiba, Distinguishing the Higgs boson from the dilaton at the Large Hadron Collider, *Phys. Rev. Lett.* **100**, 111802 (2008).
- [24] O. Antipin, M. Mojaza, and F. Sannino, Light dilaton at fixed points and ultra light scale super Yang Mills, *Phys. Lett. B* **712**, 119 (2012).
- [25] B. Bellazzini, C. Csaki, J. Hubisz, J. Serra, and J. Terning, A Higgslike dilaton, *Eur. Phys. J. C* **73**, 2333 (2013).
- [26] D. Orlando, S. Reffert, and F. Sannino, Near-conformal dynamics at large charge, *Phys. Rev. D* **101**, 065018 (2020).
- [27] L. Del Debbio and R. Zwicky, Dilaton and massive hadrons in a conformal phase, *J. High Energy Phys.* **08** (2022) 007.
- [28] T. Appelquist, J. Ingoldby, and M. Piai, Dilaton effective field theory, *Universe* **9**, 10 (2023).
- [29] R. Zwicky, QCD with an infrared fixed point and a dilaton, *Phys. Rev. D* **110**, 014048 (2024).
- [30] J. Ingoldby (Lattice Strong Dynamics Collaboration), Hidden conformal symmetry from eight flavors, *Proc. Sci. LATTICE2023* (2024) 091 [arXiv:2401.00267].
- [31] J. Bersini, A. D’Alise, C. Gambardella, and F. Sannino, Scaling results for charged sectors of near conformal QCD, *Phys. Rev. D* **109**, 125015 (2024).
- [32] R. Zwicky, The dilaton improves Goldstones, *arXiv:2306.12914*.
- [33] G. W. Semenoff and F. Zhou, Dynamical violation of scale invariance and the dilaton in a cold Fermi gas, *Phys. Rev. Lett.* **120**, 200401 (2018).
- [34] G. W. Semenoff, Dilaton in a cold Fermi gas, *arXiv:1808.03861*.
- [35] To see $O(2N)$ of the Gross-Neveu model explicitly, we can write the Euclidean Lagrangian for real Fermions $\varphi^* = \varphi$ as

$$\mathcal{L} = \sum_{a=1}^{2N} \frac{1}{2} \varphi_a \sigma^2 [\vec{\sigma} \cdot \vec{\nabla}] \varphi_a - \frac{g}{8N} \left(\sum_{a=1}^{2N} \varphi_a \sigma^2 \varphi_a \right)^2,$$

where σ^μ are Pauli matrices. The Thirring interaction would be

$$\frac{\eta}{2N} \left(\sum_{a=1}^N \varphi_a \sigma^2 \vec{\sigma} \varphi_{a+N} \right)^2,$$

which is invariant under $SP(2N)$ rather than $SO(2N)$. When they are both included, the symmetry reduces to the common subgroup of $SO(2N)$ and $SP(2N)$ which is $U(N)$. To go back to the complex representation, we pair $\psi_a = \frac{1}{\sqrt{2}}(\varphi_a + i\varphi_{a+N})$ and $\bar{\psi}_a = \frac{1}{\sqrt{2}}(\varphi_a^T - i\varphi_{a+N}^T)(-i\sigma^2)$, and the operators written with complex fermion versus Majorana spinors are

$$\sum_{a=1}^N \int i\bar{\psi}_a(x) \not{\partial} \psi_a(x) = \sum_{a=1}^{2N} \int \frac{1}{2} \varphi_a^T \sigma^2 \not{\partial} \varphi_a,$$

$$\sum_{a=1}^N i\bar{\psi}_a \psi_a = \sum_{a=1}^{2N} \frac{1}{2} \varphi_a \sigma^2 \varphi_a,$$

$$\sum_{a=1}^N \bar{\psi}_a \gamma^\mu \psi_a = \sum_{a=1}^{2N} \frac{i}{2} (-\varphi_a \sigma^2 \gamma^\mu \varphi_{a+N} + \varphi_{a+N} \sigma^2 \gamma^\mu \varphi_a).$$

- [36] G. Parisi, The theory of nonrenormalizable interactions. 1. The large N expansion, *Nucl. Phys.* **B100**, 368 (1975).
- [37] K. Gawedzki and A. Kupiainen, Renormalizing the non-renormalisable, *Phys. Rev. Lett.* **55**, 363 (1985).
- [38] K. Gawedzki and A. Kupiainen, Renormalization of a nonrenormalizable quantum field theory, *Nucl. Phys.* **B262**, 33 (1985).
- [39] B. Rosenstein, B. J. Warr, and S. H. Park, The four Fermi theory is renormalizable in $(2 + 1)$ -dimensions, *Phys. Rev. Lett.* **62**, 1433 (1989).
- [40] G. W. Semenoff and L. C. R. Wijewardhana, Dynamical mass generation in $3 - D$ four fermion theory, *Phys. Rev. Lett.* **63**, 2633 (1989).
- [41] J. Zinn-Justin, Four-fermion interaction near four dimensions, *Nucl. Phys.* **B367**, 105 (1991).
- [42] C. de Calan, P. A. Faria da Veiga, J. Magnen, and R. Seneor, Constructing the three-dimensional Gross-Neveu model with a large number of flavor components, *Phys. Rev. Lett.* **66**, 3233 (1991).
- [43] S. Hands, A. Kocic, and J. B. Kogut, Four Fermi theories in fewer than four-dimensions, *Ann. Phys. (N.Y.)* **224**, 29 (1993).
- [44] A. N. Vasilev, S. E. Derkachev, N. A. Kivel, and A. S. Stepanenko, The $1/n$ expansion in the Gross-Neveu model: Conformal bootstrap calculations of the index η in order $1/n^3$, *Theor. Math. Phys. (Engl. Transl.)* **94**, 127 (1993).
- [45] J. A. Gracey, Computation of the critical exponent η at $O(1/n^3)$ in the four-Fermi model in arbitrary dimension, *Int. J. Mod. Phys. A* **09**, 727 (1994).
- [46] N. Zerf, L. N. Mihaila, P. Marquard, I. F. Herbut, and M. M. Scherer, Four-loop critical exponents for the Gross-Neveu-Yukawa models, *Phys. Rev. D* **96**, 096010 (2017).
- [47] J. A. Gracey, Large N critical exponents for the chiral Heisenberg Gross-Neveu universality class, *Phys. Rev. D* **97**, 105009 (2018).
- [48] L. Karkkainen, R. Lacaze, P. Lacock, and B. Petersson, Critical behavior of the three-dimensional Gross-Neveu and Higgs-Yukawa models, *Nucl. Phys.* **B415**, 781 (1994); *Nucl. Phys.* **B438**, 650(E) (1995).
- [49] S. Chandrasekharan and A. Li, Quantum critical behavior in three dimensional lattice Gross-Neveu models, *Phys. Rev. D* **88**, 021701 (2013).
- [50] R. S. Erramilli, L. V. Iliesiu, P. Kravchuk, A. Liu, D. Poland, and D. Simmons-Duffin, The Gross-Neveu-Yukawa archipelago, *J. High Energy Phys.* **02** (2023) 036.
- [51] M. Gomes, R. S. Mendes, R. F. Ribeiro, and A. J. da Silva, Gauge structure, anomalies and mass generation in a three-dimensional Thirring model, *Phys. Rev. D* **43**, 3516 (1991).
- [52] D. K. Hong and S. H. Park, Large N analysis of $(2 + 1)$ -dimensional Thirring model, *Phys. Rev. D* **49**, 5507 (1994).
- [53] S. Hands, $O(1/N_f)$ corrections to the Thirring model in $2 < d < 4$, *Phys. Rev. D* **51**, 5816 (1995).
- [54] T. Itoh, Y. Kim, M. Sugiura, and K. Yamawaki, Thirring model as a gauge theory, *Prog. Theor. Phys.* **93**, 417 (1995).
- [55] K. I. Kondo, Thirring model as a gauge theory, *Nucl. Phys.* **B450**, 251 (1995).
- [56] L. Janssen and H. Gies, Critical behavior of the $(2 + 1)$ -dimensional Thirring model, *Phys. Rev. D* **86**, 105007 (2012).
- [57] S. Hands, Towards critical physics in $2 + 1d$ with $U(2N)$ -invariant fermions, *J. High Energy Phys.* **11** (2016) 015.
- [58] L. Janssen and Y. C. He, Critical behavior of the QED_3 -Gross-Neveu model: Duality and deconfined criticality, *Phys. Rev. B* **96**, 205113 (2017).
- [59] B. Ihrig, L. Janssen, L. N. Mihaila, and M. M. Scherer, Deconfined criticality from the QED_3 -Gross-Neveu model at three loops, *Phys. Rev. B* **98**, 115163 (2018).
- [60] A. J. Niemi and G. W. Semenoff, Axial anomaly induced fermion fractionization and effective gauge theory actions in odd dimensional space-times, *Phys. Rev. Lett.* **51**, 2077 (1983).
- [61] A. N. Redlich, Parity violation and gauge noninvariance of the effective gauge field action in three-dimensions, *Phys. Rev. D* **29**, 2366 (1984).
- [62] S. R. Coleman and B. R. Hill, No more corrections to the topological mass term in QED in three-dimensions, *Phys. Lett.* **159B**, 184 (1985).
- [63] G. W. Semenoff, P. Sodano, and Y. S. Wu, Renormalization of the statistics parameter in three-dimensional electrodynamics, *Phys. Rev. Lett.* **62**, 715 (1989).
- [64] S. R. Coleman and E. J. Weinberg, Radiative corrections as the origin of spontaneous symmetry breaking, *Phys. Rev. D* **7**, 1888 (1973).
- [65] R. D. Pisarski, Fixed point structure of $(\phi_6)_3$ in three-dimensions at large N , *Phys. Rev. Lett.* **48**, 574 (1982).
- [66] W. A. Bardeen, M. Moshe, and M. Bander, Spontaneous breaking of scale invariance and the ultraviolet fixed point in $O(N)$ symmetric $((\phi^2)^3$ in three-dimensions) theory, *Phys. Rev. Lett.* **52**, 1188 (1984).
- [67] H. Omid, G. W. Semenoff, and L. C. R. Wijewardhana, Light dilaton in the large N tricritical $O(N)$ model, *Phys. Rev. D* **94**, 125017 (2016).
- [68] L. Di Pietro, E. Lauria, and P. Niro, Vacuum stability, fixed points, and phases of QED_3 at large N_f , *Phys. Rev. D* **108**, L061902 (2023).
- [69] O. Aharony, S. Jain, and S. Minwalla, Flows, fixed points and duality in Chern-Simons-matter theories, *J. High Energy Phys.* **12** (2018) 058.