

Manifestly covariant worldline actions from coadjoint orbits.

II. Twistorial descriptions

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Worldline actions for various twistor particles in Anti de Sitter (AdS) spacetimes are constructed from the coadjoint orbits of $Sp(4, \mathbb{R})$, $SU(2, 2)$, and $O^*(8)$ as constrained Hamiltonian systems. The constraints are associated with the coadjoint orbits of the dual groups, respectively identified as $O(p, M - p)$, $U(p, M - p)$, and $Sp(p, M - p)$. These actions are presented in a universal form by making use of the twistor variables taking values in $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, respectively. We find that the massless, the massive, and the conformal particles (namely the singletons) of any spins appear for the compact dual groups with $p = 0$ reproducing many of the results in earlier literature, whereas more exotic particle species, such as various tachyons, continuous spin particles, and bitemporal anti de Sitter (BdS) particles appear for the noncompact dual groups.

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I. INTRODUCTION

In this article, we continue the construction [1] of worldline particle actions to the twistor particles. To begin with, let us summarize briefly the previous work [1], where we classified the coadjoint orbits of the isometry group G of Minkowski, dS, and Anti de Sitter (AdS) spacetimes, namely $ISO(1, d - 1)$, $SO(2, d - 1)$, and $SO(1, d)$, and interpreted them as different particle species. This is based on Wigner's idea that fundamental particles carry unitary irreducible representations of the isometry group, and the orbit method allows one to construct unitary irreducible representations out of coadjoint orbits (see, e.g., [2]). This viewpoint of the relativistic particle dynamics has been proposed already in early literature, e.g., [3–5]. In this paper, we use the term *particles* to designate coadjoint orbits, even in the case where the physical property for particle interpretation is rather unclear, such as various tachyons, continuous spin particles, as well as the ones living in the bitemporal AdS space, defined as $\eta_{AB}X^AX^B = 1$ with $\eta = \text{diag}(-1, -1, 1, \dots, 1)$ that we abbreviate as bitemporal anti de Sitter (BdS) space.¹

The construction of worldline action is given by the pullback of the Kirillov-Kostant-Souriau (KKS) symplectic

potential, which we reformulate as constrained Hamiltonian systems by implementing the defining conditions of the isometry group as constraints. Depending on the choice of the coadjoint orbits of the isometry group G , we find different constrained systems where the constraints are associated with coadjoint orbits of a Lie group $\tilde{G}$, which we refer to as a dual group. This choice of name follows from the reductive dual pair correspondence that arises upon quantizing this constrained Hamiltonian system. We found that the dual group of $G = ISO(1, d - 1)$ is $\tilde{G} = Sp(2M, \mathbb{R}) \ltimes Heis_{2M}$, and the dual group of $G = SO(1, d)$ and $SO(2, d - 1)$ is $Sp(2(M + 1), \mathbb{R})$, for both cases. Here, the number M is determined by the label of the G -coadjoint orbit: for the example of a massive spinning particle, it is the number of rows in the Young diagram representing the spin.

For special dimensions, the double covers of the AdS group $SO(2, d - 1)$ are isomorphic to other classical Lie groups: $\widetilde{SO}^+(2, 3) \cong Sp(4, \mathbb{R})$, $\widetilde{SO}^+(2, 4) \cong SU(2, 2)$, and $\widetilde{SO}^+(2, 6) \cong O^*(8)$. Therefore, the worldline particle actions of these groups can be constructed by using different defining conditions, and hence, different dual groups. We find that the dual groups of $G = Sp(4, \mathbb{R})$, $SU(2, 2)$, and $O^*(8)$ are, respectively, $\tilde{G} = O(p, M - p)$, $U(p, M - p)$, and $Sp(p, M - p)$. The resulting worldline particle actions comprise various twistor particles constructed earlier in the literature, together with several new twistorial models for more general types of particles.

Descriptions of worldline particles using twistors [8,9] have a long history: massless/massive and scalar/spinning twistor particles in flat/AdS spacetime with/without supersymmetry (SUSY) have been addressed in literature. These worldline models share the same symplectic

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¹In a sense, it is also natural to regard it as the bitemporal dS space. See [6,7] for the appearance of BdS in other contexts.

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structures [10], but feature slightly different constraints. Anyway, to describe a massive particle, the constraints should depend explicitly on the mass label. On the other hand, to describe a spinning particle, often *superscalar* particles are considered where the number of SUSY is proportional to the maximum spin that the superparticle contains. In fact, it is also possible to describe a spinning particle by deforming the constraints so that they depend explicitly on the spin labels. Our construction follows the latter method.

The twistor description for worldline particle action was introduced first for the four-dimensional (4D) massless (super)conformal scalar particle in [11–13], and the model was generalized to three dimensions (3D) and six dimensions (6D) in [5,14,15]. Note that the last paper [5] makes use of the same coadjoint method that we used in this paper, so our work can be regarded as its generalization to general particle species. See also [16] for related discussions with more emphasis on the role of division algebra.

After understanding that a single complex twistor describes 4D massless scalar, it was subsequently understood that two complex twistors can describe a massive particle in Mink_4 [17]: the relevant $\text{Mink}_{3,4}$ twistor models with explicit mass and spin label are studied in [18–20] (see also more recent works in this direction, [21,22], for superscalar and spin s particles, respectively) where the constraints are quadratic in twistor variables and involve the mass and spin label as constant shifts. This is the feature that our particle models share. Different two-twistor massive particle models in Mink_3 , Mink_6 , and $\text{Mink}_{3,4,6}$ are considered respectively in [23–25]. In these models, the mass constraint is given by a determinant condition so they are quartic in twistor variables, and nonzero spins are included by SUSY instead of deforming the constraints by spin labels.

The discovery of AdS/CFT incited motivation to explore the twistor description for particles living in AdS space-time. In [26,27], a complex two-twistor model has been shown to describe massive scalar particles in AdS_5 . This model is in fact rather exceptional because in general massive AdS particles require four twistors instead of two, as shown for $\text{AdS}_{4,5,7}$ massive scalar particles in [28], and consistent with the general pattern found in the analysis of relevant oscillator representations [29]. See also [30–32] for the $\text{AdS}_5 \times S^5$ massless supertwistor particles. Two-twistor models for the massless $\text{AdS}_{4,5,7}$ superparticle are studied in [33–35] in relation with the $\text{Mink}_{3,4,6}$ massive twistors [23–25], where the AdS_5 exception is clarified further. See also [36,37] for more recent coset constructions of the Lagrangians for massive (super)twistor particles in $\text{AdS}_{4,5}$.

As we explained before, to describe a definite spin s particle, a “spin-shell” constraint must be imposed. If we

remove this constraint in a proper manner, the action can describe infinitely many particles of any allowed spins, and hence provides a worldline model for higher spin theory. Such models for massless higher spin particles have been considered in [38–42], and massive higher spin models are also studied in [43,44]. See also [45,46] for the use of classical twistor variables in constructing higher spin algebras.

Twistor descriptions are not limited to the ordinary massless and massive (spinning) particles, but they can cover also the continuous spin particles: the Mink_4 and Mink_6 continuous spin particle actions are obtained respectively in [47–50].

The current paper is organized as follows. In Sec. II, we begin with the review of the isomorphisms between $\widetilde{SO}^+(2, d)$ and “twistor groups”: $Sp(4, \mathbb{R})$, $SU(2, 2)$, and $O^*(8)$ for $d = 3, 4$, and 6. Then we demonstrate how various twistor particle actions can be obtained from coadjoint orbits of twistor groups, by implementing their defining condition as Hamiltonian constraints. The constraints are associated with the coadjoint orbits of dual groups, respectively identified with $O(p, M - p)$, $U(p, M - p)$, and $Sp(p, M - p)$, for $d = 3, 4$, and 6, where M is the number of twistors. In Sec. III, for each of $\mathfrak{so}(2, d)$ coadjoint orbits in the list of classifications obtained in [1],² we determine the system of constraints, associated with the dual groups, of the corresponding twistor actions. We find dual groups with different M and p appear for different coadjoint orbits, namely for different particle species. In Sec. IV, we summarize the results according to the dual groups and generalize them by making use of the dual pair correspondence [1]. For compact dual groups with $M = 1$, we find conformal particles in $|\mathbb{F}| + 2$ dimensions, and the $M = 2$ and 4 cases describe massless and massive particles in $\text{AdS}_{|\mathbb{F}|+3}$, respectively, while the $M = 3$ case corresponds to “very light” massive particles. Noncompact dual groups cover all other exotic particle species such as continuous spin particles, tachyons, and BdS particles. In particular, the continuous spin particles arise for the $M = 4$ and $p = 1$ case. In Sec. V, we show how the $\text{Mink}_{|\mathbb{F}|+2}$ or $(\text{A})\text{dS}_{|\mathbb{F}|+2}$ massive twistor particle actions can be obtained by restricting the massless one of $(\text{A})\text{dS}_{|\mathbb{F}|+3}$. This discussion is along the line of [35], while we make more emphases on the algebraic side, such as seesaw dual pairs. Finally, in Appendix A, we provide complex matrix representations of quaternion and their use in Poisson algebra, and Appendix B contains the explicit component expressions of the AdS generators, which can serve as a dictionary between vectorial and twistorial descriptions.

²See [51] for the 4D orbit classification based on $GL(2, \mathbb{C})$.

II. MANIFESTLY COVARIANT WORLDLINE ACTIONS FROM COADJOINT ORBITS OF A CLASSICAL LIE GROUP

Let us review the construction of manifestly covariant particle actions from coadjoint orbits of a classical Lie group [1].

A. Classical Lie groups

Any classical Lie group is a subgroup of a general linear group $GL(N, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$, which preserves a certain bilinear form: We can define any classical Lie group as

$$B(b_{(N)}, \mathbb{F}) = \{A \in GL(N, \mathbb{F}) | A^\dagger b_{(N)} A = b_{(N)}\}, \quad (2.1)$$

where $b_{(N)}$ is an element of $GL(N, \mathbb{F})$. The dagger operation is defined as the composition $A^\dagger = (A^t)^*$ of the matrix transpose t and the $\mathbb{F}$ conjugation $*$.³ Up to a $GL(N, \mathbb{F})$ conjugation, $b_{(N)}$ can be mapped to either Hermitian or anti-Hermitian with respect to $\dagger$. A Hermitian $b_{(N)}$ is in the conjugacy class of a flat metric of $(p, N-p)$ signature: $b_{(N)} = \eta_{(p, N-p)}$. An anti-Hermitian $b_{(N)}$ is in the conjugacy class of the symplectic matrix: $b_{(N)} = \Omega_{(N)}$. With different choices of $b_{(N)}$, we find the full list of classical Lie groups, besides the $GL(N, \mathbb{F})$ itself:

$$B(\eta_{(p, N-p)}, \mathbb{R}) = O(p, N-p), \quad B(\Omega_{(N)}, \mathbb{R}) = Sp(N, \mathbb{R}), \quad (2.2)$$

$$B(\eta_{(p, N-p)}, \mathbb{C}) = U(p, N-p), \quad B(\Omega_{(N)}, \mathbb{C}) \cong U\left(\frac{N}{2}, \frac{N}{2}\right), \quad (2.3)$$

$$B(\eta_{(p, N-p)}, \mathbb{H}) = Sp(p, N-p), \quad B(\Omega_{(N)}, \mathbb{H}) \cong O^*(2N). \quad (2.4)$$

In the right column, N is even for $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$, whereas it can also be an odd number for $\mathbb{H}$. This is because we can take the second element of $\mathbb{H}$ as a two-dimensional symplectic matrix: $\Omega_{(N)} = \mathbf{j}I_{(N)}$. Since $\Omega_{(N)}$ can be diagonalized as $\mathbf{i}\eta_{(N/2, N/2)}$, the group $B(\Omega_{(N)}, \mathbb{C})$ is isomorphic to $U(N/2, N/2)$. On the contrary, even though $\Omega_{(N)}$ can still be diagonalized as $\mathbf{i}\eta_{(N/2, N/2)}$, the group $B(\Omega_{(N)}, \mathbb{H})$ is not isomorphic to $Sp(N/2, N/2)$ because the $\mathbf{i}$, the first element of $\mathbb{H}$, does not commute with a quaternionic matrix. See, e.g., [[52], Prop. 9.3.2.] for relevant discussions.

³The $*$ conjugation is the identity map for $\mathbb{R}$, the complex conjugation for $\mathbb{C}$, and the quaternionic conjugation for $\mathbb{H}$.

B. Twistor groups

The classical Lie groups $B(\Omega_{(4)}, \mathbb{F})$ happen to be isomorphic to AdS groups of special dimensions,

$$\begin{aligned} B(\Omega_{(4)}, \mathbb{R}) &= Sp(4, \mathbb{R}) \cong \widetilde{SO}^+(2, 3), \\ B(\Omega_{(4)}, \mathbb{C}) &= U(2, 2) \supset SU(2, 2) \cong \widetilde{SO}^+(2, 4), \\ B(\Omega_{(4)}, \mathbb{H}) &= O^*(8) \cong \widetilde{SO}^+(2, 6), \end{aligned} \quad (2.5)$$

and they contain $GL(2, \mathbb{F})$ whose semisimple part is isomorphic with the Lorentz subgroups,

$$\begin{aligned} GL(2, \mathbb{R}) &\supset SL(2, \mathbb{R}) \cong \widetilde{SO}^+(1, 2), \\ GL(2, \mathbb{C}) &\supset SL(2, \mathbb{C}) \cong \widetilde{SO}^+(1, 3), \\ GL(2, \mathbb{H}) &\supset SL(2, \mathbb{H}) \cong \widetilde{SO}^+(1, 5). \end{aligned} \quad (2.6)$$

Here $\widetilde{SO}^+(p, q)$ designates the double cover of the orthochronous Lorentz group $SO^+(p, q)$. Hence, letting $|\mathbb{F}|$ be the real dimensions of $\mathbb{F}$, the twistor group $B(\Omega_{(4)}, \mathbb{F})$ is the double cover of the $\text{AdS}_{|\mathbb{F}|+2}$ isometry group, or equivalently the $(|\mathbb{F}| + 1)$ -dimensional conformal group. This isomorphism allows one to describe half-integral spins, which was not possible in the vectorial description based on the Lie group $O(p, q)$ [1], and it is in the heart of twistor description of particles and fields: see, e.g., [5, 28]. We shall show how the twistor actions of various AdS particles in the literature can be recovered, together with many new particle actions, from a universal expression by applying the information of the Lie group, that is $b_{(N)}$, and the coadjoint orbit $\mathcal{O}_\phi^G$, which will be specified by a representative element ϕ .

All the twistor actions will be described in an equal footing by taking the advantage of the natural inclusion as groups of 4×4 matrices,

$$B(\Omega_{(4)}, \mathbb{R}) \subset B(\Omega_{(4)}, \mathbb{C}) \subset B(\Omega_{(4)}, \mathbb{H}), \quad (2.7)$$

with the 2×2 matrix subgroups,

$$GL(2, \mathbb{R}) \subset GL(2, \mathbb{C}) \subset GL(2, \mathbb{H}), \quad (2.8)$$

which contain the Lorentz groups. This allows us to describe the corresponding twistor actions in a universal expression in terms of $4M|\mathbb{F}|$ variables, where M is the rank of the dual group fixed by the choice of the coadjoint orbit $\mathcal{O}_\phi^G$.

C. $GL(N, \mathbb{F})$ manifestly covariant worldline action

Let us now review the derivation of a manifestly covariant worldline action with a $GL(N, \mathbb{F})$ symmetry from a coadjoint orbit of the Lie group $GL(N, \mathbb{F})$. On the group manifold $GL(N, \mathbb{F})$, the matrix components of

$X^a_b \in \mathbb{F}$ of an element $X \in GL(N, \mathbb{F})$ serve a natural and global coordinate system, and the left-invariant vector fields are given by $V = V^a_b X^c_a \frac{\partial}{\partial X^c_b}$. The symplectic structure of the cotangent bundle $T^*GL(N, \mathbb{F})$ prescribes the geometric action, which reads simply

$$S[X, P, A] = \int \text{Tr}_N [P dX + A(PX - \phi)]_{\mathbb{R}}, \quad (2.9)$$

where X, P, A , and ϕ belong to $Mat_{N \times N}(\mathbb{F})$. The real part $[\bullet]_{\mathbb{R}}$ stands for

$$[A]_{\mathbb{R}} = \frac{1}{2} (A + A^\dagger), \quad (2.10)$$

with $A^\dagger := (A^t)^*$, and the $(\bullet)^*$ stands for the conjugate in $\mathbb{F}$. The real part does not do anything for $\mathbb{F} = \mathbb{R}$, but they are necessary for the correct symplectic potentials for $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{H}$. The constraint imposed by the Lagrange multiplier A can be algebraically solved to give

$$S[X] = \int \text{Tr}_N [\phi X^{-1} dX]_{\mathbb{R}}. \quad (2.11)$$

If the coadjoint matrix ϕ has a rank $M \leq N$, we can transform it into a triangular form and integrate out the components of X in the lower subspace $Mat_{N \times (N-M)}(\mathbb{F})$. This brings the reduction of the action into the form

$$S[X, P, A] = \int \text{Tr}_M [P dX + A(PX - \tilde{\phi})]_{\mathbb{R}}, \quad (2.12)$$

where X, P , and A take values in $Mat_{N \times M}(\mathbb{F})$, $Mat_{M \times N}(\mathbb{F})$, and $Mat_{M \times M}(\mathbb{F})$, respectively. The matrix $\tilde{\phi}$ is the $M \times M$ submatrix of the triangulized ϕ , and hence it belongs to $Mat_{M \times M}(\mathbb{F})$. In the end, the action describes a $G = GL(N, \mathbb{F})$ coadjoint orbit $\mathcal{O}_\phi^G$ as a reduced phase space inside $\mathbb{F}^{2MN}$ where the constraints are given through the moment maps $\tilde{\mu}(X, P) = PX \in Mat_{M \times M}(\mathbb{F})$ generating $\tilde{G} = GL(M, \mathbb{F})$ under the Poisson bracket. Note also that the moment maps $\mu(X, P) = XP \in Mat_{N \times N}(\mathbb{F})$ associated with the original $GL(N, \mathbb{F})$ symmetry Poisson-commute with $\tilde{\mu}$. The constraints $\tilde{\mu}(X, P) - \tilde{\phi} \approx 0$ are associated with the dual $\tilde{G} = GL(M, \mathbb{F})$ coadjoint orbit $\mathcal{O}_{\tilde{\phi}}^{\tilde{G}}$.

D. $B(b_{(N)}, \mathbb{F})$ manifestly covariant worldline action

The passage from $GL(N, \mathbb{F})$ to $B(b_{(N)}, \mathbb{F})$ can be done by implementing the defining condition (2.1) of $B(b_{(N)}, \mathbb{F})$ into the action (2.11) as a constraint. The resulting action reads

$$\begin{aligned} S[X, A] &= \int \text{Tr}_N [\phi X^{-1} dX + A(X^\dagger b_{(N)} X - b_{(N)})]_{\mathbb{R}} \\ &\cong \int \text{Tr}_N [\phi b_{(N)}^{-1} X^\dagger b_{(N)} dX + A(X^\dagger b_{(N)} X - b_{(N)})]_{\mathbb{R}}, \end{aligned} \quad (2.13)$$

where $\cong$ designates the redefinition of A replacing X^{-1} by $b_{(N)}^{-1} X^\dagger b_{(N)}$. The constant matrix ϕ belongs to

$$\mathfrak{b}(b_{(N)}, \mathbb{F}) = \{\xi \in Mat_{N \times N}(\mathbb{F}) | \xi^\dagger b_{(N)} + b_{(N)} \xi = 0\}, \quad (2.14)$$

whereas X takes value in $Mat_{N \times N}(\mathbb{F})$ after introducing the constraint. If the matrix $\phi \in \mathfrak{b}(b_{(N)}, \mathbb{F})$ has a rank M , it can always be mapped to a $\tilde{b}_{(M)}$ as

$$\phi = T \tilde{b}_{(M)} T^\dagger b_{(N)}, \quad (2.15)$$

in terms of a transformation matrix $T \in Mat_{N \times M}(\mathbb{F})$. Moreover, the mapping has a very simple rule: we can always choose T such that $\tilde{b}_{(M)} = (\Omega_{(M)})^{-1}$ for $b_{(N)} = \eta_{(p, N-p)}$ and $\tilde{b}_{(M)} = (\eta_{(q, M-q)})^{-1}$ for $b_{(N)} = \Omega_{(N)}$. In the latter case, the number q depends on ϕ . The real part is concerned mostly about the quaternion case since in the real and complex cases the Lagrangian is already real up to a boundary term.

After a suitable redefinition of X and A in terms of the transformation matrix T and integrating out the nondynamical components of X in the subspace $Mat_{N \times (N-M)}(\mathbb{F})$, the action becomes

$$S[X, A] = \int \text{Tr}_M [\tilde{b}_{(M)} X^\dagger b_{(N)} dX + A(X^\dagger b_{(N)} X - \tilde{b}_{(M)}^{-1} \tilde{\phi})]_{\mathbb{R}}, \quad (2.16)$$

where X takes value in $Mat_{N \times M}(\mathbb{F})$ and $\tilde{\phi}$ is given by

$$\tilde{b}_{(M)}^{-1} \tilde{\phi} = T^\dagger b_{(N)} T. \quad (2.17)$$

Here, $\tilde{\mu}(X) = X^\dagger b_{(N)} X$ are the moment maps generating the dual symmetry $\tilde{G} = B(\tilde{b}_{(M)}, \mathbb{F})$, and hence $\tilde{\phi}$ is an element of $\mathfrak{b}(\tilde{b}_{(M)}, \mathbb{F})$. Note that the moment maps $\tilde{\mu}$ of the dual group Poisson commute with the moment maps $\mu(X) = X \tilde{b}_{(M)} X^\dagger$ generating the original symmetry $G = B(b_{(N)}, \mathbb{F})$.

E. Worldline action of twistor particles in $\text{AdS}_{|\mathbb{F}|+2}$

With the choice $b_{(4)} = \Omega_{(4)}$ and $(\tilde{b}_{(M)})^{-1} = \eta_{(q, M-q)}$, the universal action (2.16) can be written in a component form as

$$S[X, A] = \int [\eta^{IJ} (X^A_J)^* \Omega_{AB} dX^B_I + A^{IJ} ((X^A_J)^* \Omega_{AB} X^B_I - \eta_{JK} \tilde{\phi}^K_I)]_{\mathbb{R}}, \quad (2.18)$$

where η_{IJ} are the components of $\eta_{(q, M-q)}$. The dual group indices I, J take values in $\{1, \dots, M\}$ for $\eta_{(M)}$, $\{0, 1, \dots, M-1\}$ for $\eta_{(1, M-1)}$, and $\{0', 0, 1, \dots, M-2\}$ for $\eta_{(2, M-2)}$. These indices should not be confused with the spacetime coordinate indices. What transforms covariantly under the spacetime symmetry is the twistor variable with the indices A, B , which takes four values. They can be further split into the undotted and dotted bivalued indices α, β , and $\dot{\alpha}, \dot{\beta}$:

$$\Omega_{AB} = \begin{pmatrix} 0 & -\delta^{\dot{\beta}}_{\dot{\alpha}} \\ \delta^{\alpha}_{\beta} & 0 \end{pmatrix}, \quad X^A_I = \begin{pmatrix} \xi^{\alpha}_I \\ \pi_{\dot{\alpha}I} \end{pmatrix},$$

$$(X^A_I)^* = \begin{pmatrix} (\xi^{\alpha}_I)^* \\ (\pi_{\dot{\alpha}I})^* \end{pmatrix} = \begin{pmatrix} \bar{\xi}^{\dot{\alpha}}_I \\ \bar{\pi}_{\alpha I} \end{pmatrix}, \quad (2.19)$$

to result in the more familiar form of the twistor action

$$S[\xi, \pi, A] = \int [\eta^{IJ} (\bar{\pi}_{\alpha I} d\xi^{\alpha}_I - \bar{\xi}^{\dot{\alpha}}_J d\pi_{\dot{\alpha}I}) + A^{IJ} (\bar{\pi}_{\alpha I} \xi^{\alpha}_I - \bar{\xi}^{\dot{\alpha}}_J \pi_{\dot{\alpha}I} - \tilde{\phi}_{IJ})]_{\mathbb{R}}. \quad (2.20)$$

Here, $(\tilde{\phi}_{IJ})^* = -\tilde{\phi}_{JI}$ and $(A^{IJ})^* = -A^{JI}$ under the $\mathbb{F}$ conjugation. For $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$, the Lagrangian without the projection to the real part $[\bullet]_{\mathbb{R}}$ is already real up to a boundary term, but for $\mathbb{F} = \mathbb{H}$, the (conj) term is necessary for the reality of the Lagrangian. Note that M corresponds to the number of twistors.

The Poisson bracket of the action for $\mathbb{F} = \mathbb{C}$ reads

$$\{\xi^{\alpha}_I, \bar{\pi}_{\beta J}\} = \eta_{IJ} \delta^{\alpha}_{\beta}, \quad \{\bar{\xi}^{\dot{\alpha}}_I, \pi_{\dot{\beta} J}\} = \eta_{IJ} \delta^{\dot{\alpha}}_{\dot{\beta}}, \quad (2.21)$$

where the complex variables and their complex conjugate are treated independently as usual. For $\mathbb{F} = \mathbb{R}$, we may either begin with the real variables with

$$\bar{\xi}^{\dot{\alpha}}_I = \xi^{\alpha}_I, \quad \bar{\pi}_{\alpha I} = \pi_{\dot{\alpha}I}, \quad (2.22)$$

or work with the complex variables and treat the above reality conditions as second class constraints. For $\mathbb{F} = \mathbb{H}$, we may use the 2×2 Hermitian matrix representation of a quaternion, where the Hermiticity condition should be imposed as second class constraints (see Appendix A). In fact, for a universal treatment of any $\mathbb{F}$, it is convenient to make use of the following properties of the real part $[\bullet]_{\mathbb{R}}$. For any numbers $a, b \in \mathbb{F}$,

$$[ab]_{\mathbb{R}} = [ba]_{\mathbb{R}} = [a^* b^*]_{\mathbb{R}} = a_0 b_0 - \sum_{r=1}^{|\mathbb{F}|-1} a_r b_r \in \mathbb{R}, \quad (2.23)$$

where the real components a_r are given by

$$a = a_0 \in \mathbb{R}, \quad a = a_0 + a_1 i \in \mathbb{C},$$

$$a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}. \quad (2.24)$$

Then, the Poisson bracket can be expressed as

$$\{[a \xi^{\alpha}_I]_{\mathbb{R}}, [b \bar{\pi}_{\beta J}]_{\mathbb{R}}\} = \{[a \xi^{\alpha}_I]_{\mathbb{R}}, [b^* \pi_{\dot{\beta} J}]_{\mathbb{R}}\} = \frac{1}{2} \eta_{IJ} \delta^{\alpha}_{\beta} [ab]_{\mathbb{R}}, \quad (2.25)$$

for any $\mathbb{F}$ numbers a and b . The constraints imposed by the Lagrange multiplier A^{IJ} are

$$\tilde{\chi}_{IJ} = \tilde{\mu}_{IJ} - \tilde{\phi}_{IJ} \approx 0, \quad (2.26)$$

where $\tilde{\mu}_{IJ}$ are the $\mathbb{F}$ -valued functions given by

$$\tilde{\mu}_{IJ} = \bar{\pi}_{\alpha I} \xi^{\alpha}_J - \bar{\xi}^{\dot{\alpha}}_I \pi_{\dot{\alpha} J}, \quad (2.27)$$

and they are closed under the Poisson bracket,

$$\{[a \tilde{\mu}_{IJ}]_{\mathbb{R}}, [b \tilde{\mu}_{KL}]_{\mathbb{R}}\} = \frac{1}{2} [\eta_{JK} (ba) \tilde{\mu}_{IL} - \eta_{JL} (b^* a) \tilde{\mu}_{IK} - \eta_{KI} (ba^*) \tilde{\mu}_{JL} + \eta_{LI} (b^* a^*) \tilde{\mu}_{JK}]_{\mathbb{R}}. \quad (2.28)$$

These Poisson brackets define the dual algebra $\tilde{\mathfrak{g}} = \mathfrak{b}(\eta_{(q,M-q)}, \mathbb{F})$: the functions $\tilde{\mu}^{IJ}$ are the moment maps of the dual algebra.

Note that for $\mathbb{F} = \mathbb{R}$, the real part does not play any role and the numbers a, b can be factored out freely, leaving the usual Lie bracket of the orthogonal algebra $\mathfrak{so}(q, M-q)$. For $\mathbb{F} = \mathbb{C}$, the above is equivalent to the more usual expression,

$$\{\tilde{\mu}_{IJ}, \tilde{\mu}_{KL}\} = \eta_{JK}\tilde{\mu}_{IL} - \eta_{IL}\tilde{\mu}_{KJ}. \quad (2.29)$$

The moment maps $\tilde{\mu}_{IJ}$ transform under the $\mathbb{F}$ conjugate as

$$(\tilde{\mu}_{IJ})^* = -\tilde{\mu}_{JI}, \quad (2.30)$$

and hence the symmetric and antisymmetric parts $\tilde{\mu}_{(IJ)}$ and $\tilde{\mu}_{[IJ]}$ have the property

$$(\tilde{\mu}_{(IJ)})^* = -\tilde{\mu}_{(IJ)}, \quad (\tilde{\mu}_{[IJ]})^* = \tilde{\mu}_{[IJ]}. \quad (2.31)$$

This shows that each symmetric component $\tilde{\mu}_{(IJ)}$ is, respectively, zero, pure imaginary, and pure quaternion for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$. Therefore, each $\tilde{\mu}_{(IJ)}$ generates the Lie algebra $\hat{\mathbb{F}}$ that we define as $\hat{\mathbb{R}} := \mathfrak{o}(1) = 0$, $\hat{\mathbb{C}} := \mathfrak{u}(1)$, and $\hat{\mathbb{H}} := \mathfrak{sp}(1)$ (in other words, $\hat{\mathbb{F}} = \mathbb{F}/\mathbb{R}$).

The moment maps $\tilde{\mu}_{IJ}$ are the image of the co-moment map m of the dual algebra $\tilde{\mathfrak{g}} = \mathfrak{b}(\eta_{(q,M-q)}, \mathbb{F})$,

$$m\left[\frac{1}{2}(a^*R_{IJ} - aR_{JI})\right] = [a\tilde{\mu}_{IJ}]_{\mathbb{R}}, \quad \forall a \in \mathbb{F}, \quad (2.32)$$

where the basis of $\tilde{\mathfrak{g}}$ is the subset of $\mathbb{F}R_{IJ}$ ⁴ whose Lie brackets are inherited from the associative product,

$$R_{IJ}R_{KL} = \eta_{JK}R_{IL}. \quad (2.33)$$

The basis of the coadjoint space $\tilde{\mathfrak{g}}^* = \mathfrak{b}(\eta_{(q,M-q)}, \mathbb{F})^*$ can be chosen in the $\mathbb{F}\mathcal{R}^{IJ}$ basis with⁵

$$\langle a\mathcal{R}^{IJ}, bR_{KL} \rangle = [a^*b]_{\mathbb{R}}\delta_K^I\delta_L^J. \quad (2.36)$$

⁴By $\mathbb{F}R_{IJ}$, we mean the basis $\{R_{IJ}\}$, $\{R_{IJ}, iR_{IJ}\}$, and $\{R_{IJ}, \mathbf{i}R_{IJ}, \mathbf{j}R_{IJ}, \mathbf{k}R_{IJ}\}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$.

⁵It is convenient to use the defining representations of R_{IJ} and $\mathcal{R}^{IJ}$:

$$(R_{IJ})^K{}_L = \delta_I^K\eta_{JL}, \quad (\mathcal{R}^{IJ})^K{}_L = \eta^{IK}\delta_L^J, \quad (2.34)$$

so that the two are related by $\mathcal{R}^{IJ} = R^{IJ} = \eta^{IK}\eta^{JL}R_{KL}$ and satisfy

$$\langle \mathcal{R}^{IJ}, R_{KL} \rangle = \text{Tr}((\mathcal{R}^{IJ})^t R_{KL}) = \delta_K^I\delta_L^J. \quad (2.35)$$

Hence, any elements $\xi \in \tilde{\mathfrak{g}}$ and $\tilde{\phi} \in \tilde{\mathfrak{g}}^*$ can be decomposed in the above basis as

$$\xi = \xi^{IJ}R_{IJ}, \quad \tilde{\phi} = \tilde{\phi}_{IJ}\mathcal{R}^{IJ}, \quad (2.37)$$

with ξ^{IJ} and $\tilde{\phi}_{IJ}$ in $\mathbb{F}$ satisfying

$$(\xi^{IJ})^* = -\xi^{JI}, \quad (\tilde{\phi}_{IJ})^* = -\tilde{\phi}_{JI}. \quad (2.38)$$

The bilinear form between $\tilde{\phi}$ and ξ can be expressed as

$$\langle \tilde{\phi}, \xi \rangle = [(\tilde{\phi}_{IJ})^*\xi^{IJ}]_{\mathbb{R}} = \text{Tr}_M[\tilde{\phi}^\dagger \xi]_{\mathbb{R}}. \quad (2.39)$$

The constraints $\chi_{IJ} = \tilde{\mu}_{IJ} - \tilde{\phi}_{IJ} \approx 0$ are split into the first and second class constraints. The first class constraints are the combinations $[\xi^{IJ}\chi_{IJ}]_{\mathbb{R}}$ with

$$\text{ad}_{[\xi^{IJ}R_{IJ}]_{\mathbb{R}}}^*\tilde{\phi} = 0, \quad (2.40)$$

while the rest corresponding to $\tilde{G}/\tilde{G}_{\tilde{\phi}} \cong \mathcal{O}_{\tilde{\phi}}^{\tilde{G}}$ are the second class constraints. The dimension of the phase space, that is, the original coadjoint orbit $\mathcal{O}_{\tilde{\phi}}^G$, is

$$\begin{aligned} \dim(\mathcal{O}_{\tilde{\phi}}^G) &= 4M|\mathbb{F}| - \dim(\tilde{\mathfrak{g}}) - \dim(\tilde{\mathfrak{g}}_{\tilde{\phi}}) \\ &= M + \frac{M(7-M)}{2}|\mathbb{F}| - \dim(\tilde{\mathfrak{g}}_{\tilde{\phi}}), \end{aligned} \quad (2.41)$$

where the dimension of $\tilde{\mathfrak{g}}$ is

$$\dim(\tilde{\mathfrak{g}}) = \frac{M(M+1)}{2}|\mathbb{F}| - M. \quad (2.42)$$

As we shall see later, the possible values of M are 1,2,3,4.

Now let us consider the spacetime symmetry that is generated by the moment maps,

$$\begin{aligned} L^\alpha{}_\beta &= \xi^{\alpha I}\tilde{\pi}_{\beta I}, & P^{\alpha\dot{\beta}} &= \xi^{\alpha I}\tilde{\xi}^{\dot{\beta}}{}_I, \\ K_{\alpha\dot{\beta}} &= \pi_{\dot{\alpha}}{}^I\tilde{\pi}_{\beta I}, & \bar{L}_{\dot{\alpha}}{}^{\dot{\beta}} &= \pi_{\dot{\alpha}}{}^I\tilde{\xi}^{\dot{\beta}}{}_I, \end{aligned} \quad (2.43)$$

with the Poisson brackets,

$$\{[aL^\alpha{}_\beta]_{\mathbb{R}}, [bL^\gamma{}_\delta]_{\mathbb{R}}\} = \frac{1}{2}[\delta_\delta^\alpha(ab)L^\gamma{}_\beta - \delta_\beta^\gamma(ba)L^\alpha{}_\delta]_{\mathbb{R}}, \quad (2.44)$$

$$\{[aP^{\alpha\dot{\beta}}]_{\mathbb{R}}, [bK_{\dot{\gamma}\delta}]_{\mathbb{R}}\} = \frac{1}{2}[\delta_\delta^\alpha(ab)\bar{L}_{\dot{\gamma}}{}^{\dot{\beta}} + \delta_{\dot{\gamma}}^{\dot{\beta}}(ba)L^\alpha{}_\delta]_{\mathbb{R}}, \quad (2.45)$$

$$\{[a\bar{L}_{\dot{\alpha}}{}^{\dot{\beta}}]_{\mathbb{R}}, [b\bar{L}_{\dot{\gamma}}{}^{\dot{\delta}}]_{\mathbb{R}}\} = \frac{1}{2}[\delta_{\dot{\gamma}}^{\dot{\beta}}(ba)\bar{L}_{\dot{\alpha}}{}^{\dot{\delta}} - \delta_{\dot{\alpha}}^{\dot{\delta}}(ab)\bar{L}_{\dot{\gamma}}{}^{\dot{\beta}}]_{\mathbb{R}}. \quad (2.46)$$

They are the co-moment map images of the Lie algebra $\mathfrak{g} = \mathfrak{b}(\Omega_{(4)}, \mathbb{F})$ generated by the generators of the conformal algebra.

III. CLASSIFICATION OF TWISTOR PARTICLES IN $\text{AdS}_{|\mathbb{F}|+3}$

In this section, we determine the coadjoint matrices ϕ corresponding to different particle species. For each of ϕ , we identify $(\tilde{b}_{(M)})^{-1} = \eta_{(q,M-q)}$, which defines the dual group, and the dual coadjoint matrix $\tilde{\phi}$. The classification of coadjoint orbits has already been carried out in the $\mathfrak{so}(2, d)$ basis. Therefore, we simply need to transform the $\mathfrak{so}(2, d)$ basis into the twistor basis. The change of basis can be made using the spinor representations of $\mathfrak{so}(2, d)$. We shall first find the spinor representations of Lorentz subalgebra $\mathfrak{so}(1, d-1)$ and extend them to $\mathfrak{so}(2, d)$.

Let us begin with the $d = 3$ case, $\mathfrak{so}(1, 2) \subset \mathfrak{gl}(2, \mathbb{R})$, where the gamma matrices are indeed all real and given by

$$\gamma_{\mu}^{(1,2)} = \tau_{\mu}, \quad (3.1)$$

with

$$\begin{aligned} \tau_0 &= i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \tau_1 &= \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \tau_2 &= \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (3.2)$$

The τ_{μ} matrices satisfy the (anti)commutation relations

$$[\tau_{\mu}, \tau_{\nu}] = 2\epsilon_{\mu\nu\rho}\tau^{\rho}, \quad \{\tau_{\mu}, \tau_{\nu}\} = 2\eta_{\mu\nu}, \quad (3.3)$$

with the conventions

$$\epsilon_{012} = 1, \quad \eta_{\mu\nu} = \text{diag}(-1, 1, 1). \quad (3.4)$$

This choice of basis can be extended to the Clifford algebra $\mathfrak{CI}(2, 3) \supset \mathfrak{b}(\Omega_{(4)}, \mathbb{R})$ as

$$\begin{aligned} \gamma_{0'}^{(2,3)} &= \hat{\tau}_0 \otimes 1_{(2)}, & \gamma_{\mu}^{(2,3)} &= \hat{\tau}_2 \otimes \tau_{\mu}, \\ \gamma_3^{(2,3)} &= \hat{\tau}_1 \otimes 1_{(2)}, \end{aligned} \quad (3.5)$$

while keeping the reality of the gamma matrices, or to $\mathfrak{CI}(1, 5) \supset \mathfrak{sl}(2, \mathbb{H})$ with the gamma matrices

$$\gamma_{\mu}^{(1,5)} = \tau_{\mu} \otimes \check{\tau}_1, \quad \gamma_{\omega}^{(1,5)} = \mathbf{q}_{\omega-3} 1_{(2)} \otimes \check{\tau}_0 \quad [\omega = 4, 5, 6], \quad (3.6)$$

where $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = (\mathbf{i}, \mathbf{j}, \mathbf{k})$ are the quaternion bases. The first element $\mathbf{i}$ can serve as $\sqrt{-1}$ for the subalgebra $\mathfrak{CI}(1, 3) \supset \mathfrak{sl}(2, \mathbb{C})$. We introduced different notations $\hat{\tau}_{\mu} = \check{\tau}_{\mu} = \tau_{\mu}$ to make the inclusion structure more clear. Finally the largest Clifford algebra $\mathfrak{CI}(2, 6) \supset \mathfrak{b}(\Omega_{(4)}, \mathbb{H})$ is realized with the gamma matrices,

$$\begin{aligned} \gamma_{0'}^{(2,6)} &= \hat{\tau}_0 \otimes 1_{(2)} \otimes \check{\tau}_1, & \gamma_{\mu}^{(2,6)} &= \hat{\tau}_2 \otimes \tau_{\mu} \otimes \check{\tau}_1, \\ \gamma_3^{(2,6)} &= \hat{\tau}_1 \otimes 1_{(2)} \otimes \check{\tau}_1, & \gamma_{\omega}^{(2,6)} &= \mathbf{q}_{\omega-3} \hat{1}_{(2)} \otimes 1_{(2)} \otimes \check{\tau}_0. \end{aligned} \quad (3.7)$$

These gamma matrices are 8×8 with $\mathbb{H}$ entries, but the associated Lie algebra generators are all proportional to $\check{1}_{(2)}$ or $\check{\tau}_2$ (no dependence on $\check{\tau}_0$ and $\check{\tau}_1$), and hence the last tensor product entry with $\check{\bullet}$ can be dropped. In the end, we identified the spinor representations of all $\mathfrak{so}(2, 6) \cong \mathfrak{b}(\Omega_{(4)}, \mathbb{H})$ generators. For the realization of $\mathfrak{b}(\Omega_{(4)}, \mathbb{H})$ with the standard form of $\Omega_{(4)}$,

$$\Omega_{(4)} = \hat{\tau}_0 \otimes 1_{(2)} = \begin{pmatrix} 0 & 1_{(2)} \\ -1_{(2)} & 0 \end{pmatrix}, \quad (3.8)$$

we have to intertwine the representations as

$$J_{AB} = Q^{-1} \frac{[\gamma_A, \gamma_B]}{4} Q \quad [A, B = 0', \mu, \omega], \quad (3.9)$$

where the intertwiner Q is given by

$$Q = \begin{pmatrix} 1_{(2)} & 0 \\ 0 & \tau_0 \end{pmatrix}. \quad (3.10)$$

See Appendix B for the explicit component expressions of J_{AB} . The dual basis $\mathcal{J}^{AB}$ of $\mathfrak{b}(\Omega_{(4)}, \mathbb{F})^*$ can also be represented by 4×4 matrices as

$$\mathcal{J}^{AB} = J^{BA}, \quad (3.11)$$

and satisfy

$$\langle \mathcal{J}^{AB}, J_{CD} \rangle = \text{Tr}(\mathcal{J}^{AB} J_{CD}) = \delta_{[C}^{[A} \delta_{D]}^{B]}. \quad (3.12)$$

In the following, we find simple matrix expressions of various coadjoint vectors ϕ by making use of the matrix representation of $\mathcal{J}^{AB}$ introduced above. We then diagonalize the matrices ϕ to identify the dual groups and the corresponding dual coadjoint vectors $\tilde{\phi}$. The matrix representation of each $\tilde{\phi}$ determines the worldline action of the corresponding twistor particle: it is enough to put the data of $\tilde{\phi}$ into the general expression (2.20) of the twistor action.

Before enlisting our results, let us remark that we assume none of the parameters of ϕ or $\tilde{\phi}$ vanish. To address such a limiting case, we perform separate analyses. Moreover, we limit ourselves to the coadjoint orbits with at most two independent labels for simplicity. This will exclude, for instance, a large class of mixed symmetry spin particles.

A. Particles with timelike momenta

We begin with the particle species whose representative momenta are timelike.

- (1) Let us consider first the case of massive scalar particles. The corresponding coadjoint orbit is given by the representative vector

$$\phi = m\mathcal{J}^{0'0}, \quad (3.13)$$

and it can be written in the 4×4 matrix form as

$$\phi\Omega_{(4)}^{-1} = \frac{m}{2} \begin{pmatrix} 1_{(2)} & 0 \\ 0 & 1_{(2)} \end{pmatrix}. \quad (3.14)$$

From the above, we identify that the dual group $\tilde{G}$ is $B(\eta_{(4)}, \mathbb{F})$, and the dual coadjoint orbit is given by the coadjoint element

$$\eta_{(4)}\tilde{\phi} = \frac{m}{2} \begin{pmatrix} 0 & 1_{(2)} \\ -1_{(2)} & 0 \end{pmatrix}, \quad (3.15)$$

which immediately specifies the twistor particle action through (2.20). We remind the reader that the constraints analysis of particle action can be universally replaced by the analysis of the dual stabilizer. For that, it is useful to express the matrix form (3.15) of the coadjoint element in terms of the dual basis $\mathcal{R}^{IJ}$ as

$$\tilde{\phi} = \frac{|m|}{2} (\mathcal{R}^{[13]} + \mathcal{R}^{[24]}). \quad (3.16)$$

Then, it becomes fairly straightforward to analyze the stabilizer $\tilde{\mathfrak{g}}_{\tilde{\phi}}$. We find that $\tilde{\mathfrak{g}}_{\tilde{\phi}}$ is generated by

$$\begin{aligned} R_{[13]} + R_{[24]}, & \quad R_{[13]} - R_{[24]}, \\ R_{[12]} + R_{[34]}, & \quad R_{[14]} + R_{[23]}, \end{aligned} \quad (3.17)$$

$$\begin{aligned} \mathbf{q}_\alpha(R_{11} + R_{33}), & \quad \mathbf{q}_\alpha(R_{22} + R_{44}), \\ \mathbf{q}_\alpha(R_{(12)} + R_{(34)}), & \quad \mathbf{q}_\alpha(R_{(14)} - R_{(23)}), \end{aligned} \quad (3.18)$$

where $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = (\mathbf{i}, \mathbf{j}, \mathbf{k})$ are the quaternion basis, and hence the generators containing $\mathbf{q}_\alpha$ are all absent for $\mathbb{F} = \mathbb{R}$, and only those with $\mathbf{q}_1 = \mathbf{i}$ are present for $\mathbb{F} = \mathbb{C}$. Note that the stabilizer algebra $\tilde{\mathfrak{g}}_{\tilde{\phi}}$ is isomorphic to $\mathfrak{u}(2)$, $\mathfrak{u}(2) \oplus \mathfrak{u}(2)$ and $\mathfrak{u}(4)$ for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, respectively. Since $\dim(\tilde{\mathfrak{g}}_{\tilde{\phi}}) = 4|\mathbb{F}|$, we find the dimension of the coadjoint orbit, according to (2.41), as

$$\dim(\mathcal{O}_\phi^G) = 2|\mathbb{F}| + 4. \quad (3.19)$$

The phase space dimension of a scalar particle in AdS_d is $2(d-1)$, that is, 6, 8, and 12 for $d = 4, 5$, and 7, respectively, and hence matches the dimensions (3.19) for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$.

- (2) Let us move to the case of spinning particles. The coadjoint orbit with timelike momentum and space-like spin is given by

$$\phi = m\mathcal{J}^{0'0} + s\mathcal{J}^{12} \quad (3.20)$$

or, in 4×4 matrix form, by

$$\phi\Omega_{(4)}^{-1} = \frac{1}{2} \begin{pmatrix} m1_{(2)} & s\tau_0 \\ -s\tau_0 & m1_{(2)} \end{pmatrix}. \quad (3.21)$$

We can diagonalize this matrix to identify the dual group $\tilde{G}$ and the corresponding dual orbit. At this step, we find three subcases to which we accord different particle interpretations.

- (a) The case with $|m| > |s|$ is interpreted as a massive spinning particle. The dual group $\tilde{G}$ is $B(\eta_{(4)}, \mathbb{F})$, the same as the scalar case, while the dual coadjoint orbit is given by the coadjoint element,

$$\begin{aligned} \tilde{\phi} &= (\eta_{(4)})^{-1} \frac{1}{2} \begin{pmatrix} |m+s|\tau_0 & 0 \\ 0 & |m-s|\tau_0 \end{pmatrix} \\ &= \frac{|m+s|}{2} \mathcal{R}^{[12]} + \frac{|m-s|}{2} \mathcal{R}^{[34]}. \end{aligned} \quad (3.22)$$

The dual stabilizer is

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathfrak{u}(1)_{R_{[12]}} \oplus \hat{\mathbb{F}}_{R_{11}+R_{22}} \oplus \mathfrak{u}(1)_{R_{[34]}} \oplus \hat{\mathbb{F}}_{R_{33}+R_{44}}, \quad (3.23)$$

where $\hat{\mathbb{F}}_X$ is the Lie algebra $\hat{\mathbb{R}} = \mathfrak{o}(1) = 0$, $\hat{\mathbb{C}} = \mathfrak{u}(1)$ and $\hat{\mathbb{H}} = \mathfrak{sp}(1)$ generated by $\mathbf{q}_\alpha X$. In this spinning case, the dimension of dual stabilizer is reduced to $\dim(\tilde{\mathfrak{g}}_{\tilde{\phi}}) = 2|\mathbb{F}|$. Consequently, the dimension of the coadjoint orbit is increased to

$$\dim(\mathcal{O}_\phi^G) = 4|\mathbb{F}| + 4. \quad (3.24)$$

Since the phase space dimension of a massive spinning particle in AdS_d is $2(2d-4)$, it matches the dimension (3.24) in four, five, and seven dimensions.

- (b) The case with $|m| = |s|$ is interpreted as a massless spinning particle. In this case, the dual group $\tilde{G}$ reduces to $B(\eta_{(2)}, \mathbb{F})$ with

$$\tilde{\phi} = (\eta_{(2)})^{-1} |s|\tau_0 = |s|\mathcal{R}^{[12]}. \quad (3.25)$$

The dual stabilizer is also half of the massive case

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathfrak{u}(1)_{R_{[12]}} \oplus \hat{\mathbb{F}}_{R_{11}+R_{22}}, \quad (3.26)$$

with $\dim(\tilde{\mathfrak{g}}_{\tilde{\phi}}) = |\mathbb{F}|$. The dimension of the coadjoint orbit is

$$\dim(\mathcal{O}_{\tilde{\phi}}^G) = 4|\mathbb{F}| + 2 \quad (3.27)$$

and matches the phase space dimension of a massless spinning particle in AdS_d , that is, $2(2d-5)$, in four, five, and seven dimensions.

- (c) The case with $|m| < |s|$ can be better interpreted as a particle of mass s and timelike spin m living in BdS. The dual group is $B(\eta_{(2,2)}, \mathbb{F})$, while the dual orbit is given by the same coadjoint element (3.22) with the same dual stabilizer (3.23).

B. Particles with lightlike momenta

Let us move to the case where the momentum of the particle is null, namely lightlike. As we will show shortly, this case includes the massless scalar and continuous spin particles, together with the massless limit of massive tachyons.

- (1) The coadjoint orbit of the massless scalar particle is given by

$$\phi\Omega_{(4)}^{-1} = E\mathcal{J}^{0+}\Omega_{(4)}^{-1} = \frac{E}{4} \begin{pmatrix} 1_{(2)} & 1_{(2)} \\ 1_{(2)} & 1_{(2)} \end{pmatrix}, \quad (3.28)$$

with $\mathcal{J}^{0+} = \frac{1}{2}(\mathcal{J}^{00} + \mathcal{J}^{03})$. The dual group is $\tilde{G} = B(\eta_{(2)}, \mathbb{F})$, and the dual coadjoint orbit is the trivial orbit,

$$\tilde{\phi} = 0. \quad (3.29)$$

In this case, the dual stabilizer coincides with the dual group itself, and hence $\dim(\tilde{\mathfrak{g}}_{\tilde{\phi}}) = 3|\mathbb{F}| - 2$. Consequently, the dimension of the coadjoint orbit,

$$\dim(\mathcal{O}_{\tilde{\phi}}^G) = 2|\mathbb{F}| + 4, \quad (3.30)$$

coincides with the phase space dimension of a scalar particle in $\text{AdS}_{|\mathbb{F}|+3}$.

- (2) The coadjoint orbit with lightlike momentum and spacelike spin, corresponding to the end point of the massive tachyonic particle, is given by

$$\begin{aligned} \phi\Omega_{(4)}^{-1} &= (E\mathcal{J}^{0+} + s\mathcal{J}^{12})\Omega_{(4)}^{-1} \\ &= \frac{1}{2} \begin{pmatrix} \frac{E}{2} 1_{(2)} & \frac{E}{2} 1_{(2)} + s\tau_0 \\ \frac{E}{2} 1_{(2)} - s\tau_0 & \frac{E}{2} 1_{(2)} \end{pmatrix}. \end{aligned} \quad (3.31)$$

The dual group is $\tilde{G} = B(\eta_{(2,2)}, \mathbb{F})$, and the dual coadjoint orbit is given by

$$\begin{aligned} \tilde{\phi} &= (\eta_{(2,2)})^{-1} \frac{1}{4} \begin{pmatrix} -(2|s| - \epsilon)\tau_0 & -\epsilon 1_{(2)} \\ \epsilon 1_{(2)} & (2|s| + \epsilon)\tau_0 \end{pmatrix} \\ &= -|s|(\mathcal{R}^{[++]} + \mathcal{R}^{[--]}) + \epsilon\mathcal{R}^{[+-]}. \end{aligned} \quad (3.32)$$

The dual stabilizer is

$$\begin{aligned} \tilde{\mathfrak{g}}_{\tilde{\phi}} &\cong \mathbb{R}_{R_{[+'] + R_{[-'] - 2R_{[+']}}} \oplus \mathfrak{u}(1)_{R_{[+'] + R_{[-']}}} \\ &\oplus \hat{\mathbb{F}}_{R_{(+') + R_{(+)}} \oplus \hat{\mathbb{F}}_{R_{+'] - R_{-'] - R_{++} + R_{--}}, \end{aligned} \quad (3.33)$$

where the second line is trivial for $\mathbb{F} = \mathbb{R}$ and isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$ and $\mathfrak{sl}(2, \mathbb{C})$ for $\mathbb{F} = \mathbb{C}$ and $\mathbb{H}$, respectively. Here, the light-cone indices, $\mathcal{V}^{\pm'} = \frac{1}{2}(\mathcal{V}^{0'} \pm \mathcal{V}^{2'})$ and $\mathcal{V}^{\pm} = \frac{1}{2}(\mathcal{V}^0 \pm \mathcal{V}^1)$, are used also for the dual group indices. Note that $\epsilon \cong \lambda\epsilon$ for any $\lambda \in \mathbb{R}_{\neq 0}$: it can be rescaled using the mutual boost in the $+/-$ and $+'/-'$ coordinates.

The presence of ϵ parameters shows that the corresponding orbit is the special semisimple orbit discussed in Sec. 7.2 of [1]: when the dual group is regarded as the spacetime symmetry, the dual orbit corresponds to the particle with entangled mass and spin, and the parameter ϵ is associated with a nilpotent coadjoint element.

- (3) The coadjoint orbit with lightlike momentum and lightlike spin, corresponding to the continuous spin particle, is given by

$$\begin{aligned} \phi\Omega_{(4)}^{-1} &= (E\mathcal{J}^{0+} + \epsilon\mathcal{J}^{-2})\Omega_{(4)}^{-1} \\ &= \frac{1}{4} \begin{pmatrix} E1_{(2)} + \epsilon\tau_1 & E1_{(2)} - \epsilon\tau_1 \\ E1_{(2)} - \epsilon\tau_1 & E1_{(2)} + \epsilon\tau_1 \end{pmatrix}. \end{aligned} \quad (3.34)$$

In this case, the dual group $\tilde{G}$ is $B(\eta_{(1,3)}, \mathbb{F})$, and the dual orbit is given by

$$\tilde{\phi} = (\eta_{(1,3)})^{-1} \frac{\rho}{2} \begin{pmatrix} \tau_0 & 0 \\ 0 & -\tau_0 \end{pmatrix} = \frac{\rho}{2} (\mathcal{R}^{[01]} - \mathcal{R}^{[23]}), \quad (3.35)$$

with the stabilizer

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathbb{R}_{R_{[01]}} \oplus \hat{\mathbb{F}}_{R_{00}-R_{11}} \oplus \mathfrak{u}(1)_{R_{[23]}} \oplus \hat{\mathbb{F}}_{R_{22}+R_{33}}. \quad (3.36)$$

Here, $\rho = \sqrt{|E\epsilon|}$ is the genuine label of the dual coadjoint orbit, which cannot be freely rescaled.

C. Particles with spacelike momenta

The particles with spacelike momenta, namely tachyons, have infinitely many degrees of freedom⁶ except for the scalar case. For the spinning cases, there are several different kinds.

- (1) The coadjoint orbit of the tachyonic scalar particle is given by

$$\phi\Omega_{(4)}^{-1} = \mu\mathcal{J}^{01}\Omega_{(4)}^{-1} = \frac{\mu}{2} \begin{pmatrix} -\tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}. \quad (3.37)$$

The dual group is $\tilde{G} = B(\eta_{(2,2)}, \mathbb{F})$, and the dual orbit is given by

$$\tilde{\phi} = (\eta_{(2,2)})^{-1} \frac{|\mu|}{2} \begin{pmatrix} 0 & \tau_2 \\ -\tau_2 & 0 \end{pmatrix} = \frac{|\mu|}{2} (\mathcal{R}^{[0'1]} - \mathcal{R}^{[0'2]}), \quad (3.38)$$

with the stabilizer $\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathfrak{gl}(2, \mathbb{F})$ generated by

$$\begin{aligned} R_{[0'1]} - R_{[0'2]}, & \quad R_{[0'0]} - R_{[12]}, \\ R_{[0'1]} + R_{[0'2]}, & \quad R_{[0'2]} - R_{[01]}, \end{aligned} \quad (3.39)$$

$$\begin{aligned} \mathbf{q}_\alpha(R_{0'0'} - R_{11}), & \quad \mathbf{q}_\alpha(R_{00} - R_{22}), \\ \mathbf{q}_\alpha(R_{(0'0)} - R_{(12)}), & \quad \mathbf{q}_\alpha(R_{(0'2)} + R_{(01)}). \end{aligned} \quad (3.40)$$

- (2) The coadjoint orbit with spacelike momentum and timelike spin is given by

$$\phi\Omega_{(4)}^{-1} = (\mu\mathcal{J}^{01} + \nu\mathcal{J}^{02})\Omega_{(4)}^{-1} = \frac{1}{2} \begin{pmatrix} -\mu\tau_2 & -\nu\tau_1 \\ -\nu\tau_1 & \mu\tau_2 \end{pmatrix}. \quad (3.41)$$

Up to conjugation, we can take $\mu \geq \nu > 0$.

- (a) For $\mu > \nu$, the dual group $\tilde{G}$ is $B(\eta_{(2,2)}, \mathbb{F})$, and the dual coadjoint orbit is given by

$$\begin{aligned} \tilde{\phi} &= (\eta_{(2,2)})^{-1} \frac{1}{2} \begin{pmatrix} 0 & \mu\tau_0 + \nu\tau_1 \\ \mu\tau_0 - \nu\tau_1 & 0 \end{pmatrix} \\ &= \frac{\mu + \nu}{2} \mathcal{R}^{[0'2]} - \frac{\mu - \nu}{2} \mathcal{R}^{[01]}, \end{aligned} \quad (3.42)$$

with the stabilizer

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathbb{R}_{R_{[01]}} \oplus \hat{\mathbb{F}}_{R_{00}-R_{11}} \oplus \mathbb{R}_{R_{[0'2]}} \oplus \hat{\mathbb{F}}_{R_{0'0'}-R_{22}}. \quad (3.43)$$

- (b) For a short tachyon with $\mu = \nu$, the dual group $\tilde{G}$ is $B(\eta_{(1,1)}, \mathbb{F})$, and the dual coadjoint orbit is given by

$$\tilde{\phi} = (\eta_{(1,1)})^{-1} \nu\tau_0 = \nu\mathcal{R}^{[01]}, \quad (3.44)$$

with the stabilizer

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathbb{R}_{R_{[01]}} \oplus \hat{\mathbb{F}}_{R_{00}-R_{11}}. \quad (3.45)$$

- (3) The coadjoint orbit with spacelike momentum and spacelike spin is given by

$$\begin{aligned} \phi\Omega_{(4)}^{-1} &= (\mu\mathcal{J}^{01} + s\mathcal{J}^{23})\Omega_{(4)}^{-1} \\ &= \frac{1}{2} \begin{pmatrix} -\mu\tau_2 + s\tau_1 & 0 \\ 0 & \mu\tau_2 + s\tau_1 \end{pmatrix}. \end{aligned} \quad (3.46)$$

The dual group $\tilde{G}$ is $B(\eta_{(2,2)}, \mathbb{F})$, and the dual coadjoint orbit is given by

$$\begin{aligned} \tilde{\phi} &= (\eta_{(2,2)})^{-1} \frac{1}{2} \begin{pmatrix} |s|\tau_0 & |\mu|\tau_1 \\ -|\mu|\tau_1 & |s|\tau_0 \end{pmatrix} \\ &= \frac{|s|}{2} (\mathcal{R}^{[0'0]} + \mathcal{R}^{[12]}) + \frac{|\mu|}{2} (\mathcal{R}^{[0'2]} + \mathcal{R}^{[01]}), \end{aligned} \quad (3.47)$$

with the stabilizer

$$\begin{aligned} \tilde{\mathfrak{g}}_{\tilde{\phi}} &\cong \mathfrak{u}(1)_{R_{[0'0]}+R_{[12]}} \oplus \mathbb{R}_{R_{[0'2]}+R_{[01]}} \\ &\oplus \hat{\mathbb{F}}_{R_{0'0'}+R_{00}-R_{11}-R_{22}} \oplus \hat{\mathbb{F}}_{R_{(0'1)}-R_{(02)}}. \end{aligned} \quad (3.48)$$

Note that the second line is trivial for $\mathbb{F} = \mathbb{R}$ and isomorphic to $\mathfrak{u}(1) \oplus \mathbb{R}$ and $\mathfrak{sl}(2, \mathbb{C})$ for $\mathbb{F} = \mathbb{C}$ and $\mathbb{H}$, respectively. The dual coadjoint vector $\tilde{\phi}$ can be interpreted as the coadjoint vector of the one-parameter family of a continuous spinning particle when the dual group $\tilde{G}$ is regarded as the spacetime symmetry.

- (4) The coadjoint orbit with spacelike momentum and lightlike spin is given by

$$\begin{aligned} \phi\Omega_{(4)}^{-1} &= (\mu\mathcal{J}^{01} + \epsilon\mathcal{J}^{2+})\Omega_{(4)}^{-1} \\ &= \frac{1}{2} \begin{pmatrix} 0 & 0 & \mu & -\epsilon \\ 0 & 0 & 0 & \mu \\ \mu & 0 & 0 & 0 \\ -\epsilon & \mu & 0 & 0 \end{pmatrix}, \end{aligned} \quad (3.49)$$

with $\mathcal{J}^{2\pm} = \frac{1}{2}(\mathcal{J}^{20} \pm \mathcal{J}^{23})$. The dual group $\tilde{G}$ is $B(\eta_{(2,2)}, \mathbb{F})$, and the dual orbit is given by

⁶Quantizations of these particles will lead to irreps, which are realized by infinite component fields, such as in the continuous spin case.

$$\begin{aligned}\tilde{\phi} &= (\eta_{(2,2)})^{-1} \frac{1}{4} \begin{pmatrix} \varepsilon\tau_0 & 2|\mu|\tau_1 + \varepsilon\tau_0 \\ -2|\mu|\tau_1 + \varepsilon\tau_0 & \varepsilon\tau_0 \end{pmatrix} \\ &= |\mu|(\mathcal{R}^{[-'+]} - \mathcal{R}^{[+ '-]}) + \varepsilon\mathcal{R}^{[+ '+]},\end{aligned}\quad (3.50)$$

with the stabilizer

$$\begin{aligned}\tilde{\mathfrak{g}}_{\tilde{\phi}} &\cong \mathbb{R}_{R_{[+ '+]}} \oplus \mathbb{R}_{R_{[+ '-] - R_{[- '+]}}} \\ &\oplus \hat{\mathbb{F}}_{R_{+ '+ -} - R_{++}} \oplus \hat{\mathbb{F}}_{R_{+ '- -} + R_{(+ -)}}.\end{aligned}\quad (3.51)$$

The second line is trivial for $\mathbb{F} = \mathbb{R}$ and isomorphic to $\mathbb{R} \oplus \mathbb{R}$ and $\mathfrak{iso}(3)$ for $\mathbb{F} = \mathbb{C}$ and $\mathbb{H}$, respectively. Here, $\mathcal{V}^{\pm} = \frac{1}{2}(\mathcal{V}^0 \pm \mathcal{V}^2)$ and $\mathcal{V}^{\pm'} = \frac{1}{2}(\mathcal{V}^{0'} \pm \mathcal{V}^1)$, and $\varepsilon \cong \lambda\varepsilon$ for any $\lambda \in \mathbb{R}_{\neq 0}$.

The dual coadjoint vector $\tilde{\phi}$ can be understood as the coadjoint vector of the entangled short tachyon if the dual group is regarded as the space-time symmetry.

D. Particles with entangled mass and spin

Next, we consider the cases that we referred to as *entangled mass and spin*, in the sense that the representative coadjoint vector is given by the sum of two coadjoint elements in a subtle relation. For detailed discussion, we refer to our previous work [1].

- (1) The coadjoint orbit of one parameter family of continuous spin particle is given by

$$\begin{aligned}\phi\Omega_{(4)}^{-1} &= (s(\mathcal{J}^{00} + \mathcal{J}^{12}) + \nu(\mathcal{J}^{01} - \mathcal{J}^{02}))\Omega_{(4)}^{-1} \\ &= \frac{1}{2} \begin{pmatrix} s1_{(2)} - \nu\tau_2 & s\tau_0 + \nu\tau_1 \\ -s\tau_0 + \nu\tau_1 & s1_{(2)} + \nu\tau_2 \end{pmatrix}.\end{aligned}\quad (3.52)$$

The dual group $\tilde{G}$ is $B(\eta_{(1,3)}, \mathbb{F})$, and the dual coadjoint orbit is given by

$$\tilde{\phi} = (\eta_{(1,3)})^{-1} \begin{pmatrix} -|\nu|\tau_0 & 0 \\ 0 & |s|\tau_0 \end{pmatrix} = |\nu|\mathcal{R}^{[01]} - |s|\mathcal{R}^{[23]},\quad (3.53)$$

with the stabilizer

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathbb{R}_{R_{[01]}} \oplus \hat{\mathbb{F}}_{R_{00} - R_{11}} \oplus \mathfrak{u}(1)_{R_{[23]}} \oplus \hat{\mathbb{F}}_{R_{22} - R_{33}}.\quad (3.54)$$

Depending on the value of ν , the cases with $\nu > s$, $\nu < s$ and $\nu = s$ may be interpreted as massive, tachyonic and massless particles of continuous spin, and the last case coincides with the usual continuous spin particle treated in (3.35).

- (2) The coadjoint orbit of an entangled massless particle is given by

$$\begin{aligned}\phi\Omega_{(4)}^{-1} &= (s(\mathcal{J}^{00} + \mathcal{J}^{12}) \\ &\quad + \varepsilon(\mathcal{J}^{0'0} + \mathcal{J}^{0'1} - \mathcal{J}^{12} - \mathcal{J}^{02}))\Omega_{(4)}^{-1} \\ &= \begin{pmatrix} \frac{s}{2} & 0 & 0 & \frac{s}{2} \\ 0 & \frac{s}{2} + \varepsilon & -\frac{s}{2} + \varepsilon & 0 \\ 0 & -\frac{s}{2} + \varepsilon & \frac{s}{2} + \varepsilon & 0 \\ \frac{s}{2} & 0 & 0 & \frac{s}{2} \end{pmatrix}.\end{aligned}\quad (3.55)$$

The dual group $\tilde{G}$ is $B(\eta_{(3)}, \mathbb{F})$ for $s\varepsilon > 0$ and $B(\eta_{(1,2)}, \mathbb{F})$ for $s\varepsilon < 0$. In both cases, the dual coadjoint orbit is given by

$$\tilde{\phi} = (\eta_{(p,q)})^{-1} \begin{pmatrix} 0 & |s| & 0 \\ -|s| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -|s|\mathcal{R}^{[12]},\quad (3.56)$$

with the stabilizer

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathfrak{u}(1)_{R_{[12]}} \oplus \hat{\mathbb{F}}_{R_{11} + R_{22}} \oplus \hat{\mathbb{F}}_{R_{\bullet\bullet}}.\quad (3.57)$$

Here, the index $\bullet$ indicates the spatial direction for the $s\varepsilon > 0$ case, and the temporal direction for $s\varepsilon < 0$.

These cases correspond to the large remnant orbits of massive spinning orbits in the massless limit approaching from $m > s$ and $m < s$, respectively.

- (3) The coadjoint orbit of an entangled short tachyon is given by

$$\begin{aligned}\phi\Omega_{(4)}^{-1} &= (\nu(\mathcal{J}^{0'2} + \mathcal{J}^{01}) \\ &\quad + \varepsilon(\mathcal{J}^{0'0} + \mathcal{J}^{0'2} - \mathcal{J}^{21} - \mathcal{J}^{01}))\Omega_{(4)}^{-1} \\ &= \frac{1}{2} \begin{pmatrix} \varepsilon 1_{(2)} + (\varepsilon + \nu)\tau_1 & \varepsilon\tau_0 - (\varepsilon - \nu)\tau_2 \\ -\varepsilon\tau_0 - (\varepsilon - \nu)\tau_2 & \varepsilon 1_{(2)} - (\varepsilon + \nu)\tau_1 \end{pmatrix}.\end{aligned}\quad (3.58)$$

The dual group $\tilde{G}$ is $B(\eta_{(1,2)}, \mathbb{F})$. The dual coadjoint orbit is given by

$$\tilde{\phi} = (\eta_{(p,q)})^{-1} \begin{pmatrix} 0 & |\nu| & 0 \\ -|\nu| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -|\nu|\mathcal{R}^{[01]},\quad (3.59)$$

with the dual stabilizer

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathbb{R}_{R_{[01]}} \oplus \hat{\mathbb{F}}_{R_{00} - R_{11}} \oplus \hat{\mathbb{F}}_{R_{22}}.\quad (3.60)$$

This case corresponds to the large remnant orbit of a tachyonic spinning orbit in the massless limit.

E. Particles in BdS spacetime

Let us move to consider the particles in bitemporal AdS (BdS) spacetime defined as the hypersurface $\eta_{AB}X^AX^B = +L^2$ with the metric $\eta_{AB} = \text{diag}(- - + \cdots +)$. BdS was identified originally in the vectorial description where the hypersurface equation arises as a constraint. The BdS particles have many subcases, including the massive spinning particle with $|m| < |s|$ treated before.

- (1) The coadjoint orbit of the BdS “massive” scalar is given by

$$\phi\Omega_{(4)}^{-1} = m\mathcal{J}^{12}\Omega_{(4)}^{-1} = \frac{m}{2} \begin{pmatrix} 0 & \tau_0 \\ -\tau_0 & 0 \end{pmatrix}. \quad (3.61)$$

The dual group is $\tilde{G} = B(\eta_{(2,2)}, \mathbb{F})$, and the dual coadjoint orbit is given by

$$\tilde{\phi} = (\eta_{(2,2)})^{-1} \frac{|m|}{2} \begin{pmatrix} \tau_0 & 0 \\ 0 & \tau_0 \end{pmatrix} = \frac{|m|}{2} (\mathcal{R}^{[0'0]} + \mathcal{R}^{[12]}). \quad (3.62)$$

The dual stabilizer $\tilde{\mathfrak{g}}_{\tilde{\phi}}$ generated by

$$\begin{aligned} R_{[0'0]} + R_{[12]}, & \quad R_{[0'0]} - R_{[12]}, \\ R_{[0'1]} - R_{[02]}, & \quad R_{[0'2]} + R_{[01]}, \end{aligned} \quad (3.63)$$

$$\begin{aligned} \mathfrak{q}(R_{0'0'} + R_{00}), & \quad \mathfrak{q}_\alpha(R_{11} + R_{22}), \\ \mathfrak{q}_\alpha(R_{(0'1)} - R_{(02)}), & \quad \mathfrak{q}_\alpha(R_{(0'2)} + R_{(01)}) \end{aligned} \quad (3.64)$$

is isomorphic to $\mathfrak{u}(1, 1)$, $\mathfrak{u}(1, 1) \oplus \mathfrak{u}(1, 1)$, and $\mathfrak{u}(2, 2)$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$.

- (2) The coadjoint orbit of the BdS “massless” scalar is given by

$$\phi\Omega_{(4)}^{-1} = E\mathcal{J}^{1+}\Omega_{(4)}^{-1} = \frac{E}{2} \begin{pmatrix} 0 & \tau_0 - \tau_2 \\ -\tau_0 + \tau_2 & 0 \end{pmatrix}, \quad (3.65)$$

where $\mathcal{J}^{1+} = \frac{1}{2}(\mathcal{J}^{10} + \mathcal{J}^{13})$. The dual group is $\tilde{G} = B(\eta_{(1,1)}, \mathbb{F})$ and its dual coadjoint orbit is the trivial orbit

$$\tilde{\phi} = 0. \quad (3.66)$$

The BdS spinning particles exist only in $\text{AdS}_{d>4}$, and hence we restrict $\mathbb{F}$ to $\mathbb{C}$ and $\mathbb{H}$.

- (3) The coadjoint orbit of a BdS spinning particle with spacelike momentum and spacelike spin is given by

$$\phi\Omega_{(4)}^{-1} = (m\mathcal{J}^{12} + s\mathcal{J}^{34})\Omega_{(4)}^{-1} = \frac{1}{2} \begin{pmatrix} i s \tau_0 & m \tau_0 \\ -m \tau_0 & i s \tau_0 \end{pmatrix}. \quad (3.67)$$

- (a) In the massive case with $m \neq s$, the dual group is $B(\eta_{(2,2)}, \mathbb{F})$, and the dual coadjoint orbit is given by

$$\begin{aligned} \tilde{\phi} &= (\eta_{(2,2)})^{-1} \frac{i}{2} \text{diag}(|m+s|, -|m-s|, \\ &\quad -|m-s|, |m+s|) \\ &= \frac{i}{2} (|m+s|(\mathcal{R}^{0'0'} + \mathcal{R}^{22}) \\ &\quad - |m-s|(\mathcal{R}^{00} + \mathcal{R}^{11})). \end{aligned} \quad (3.68)$$

For $\mathbb{F} = \mathbb{C}$, the stabilizer is

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathfrak{u}(1)_{iR_{0'0'}} \oplus \mathfrak{u}(1)_{iR_{00}} \oplus \mathfrak{u}(1)_{iR_{11}} \oplus \mathfrak{u}(1)_{iR_{22}}, \quad (3.69)$$

while for $\mathbb{F} = \mathbb{H}$, the stabilizer is

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathfrak{u}(1, 1)_{0'2} \oplus \mathfrak{u}(1, 1)_{01}, \quad (3.70)$$

where $\mathfrak{u}(1, 1)_{IJ} = \text{Span}_{\mathbb{R}}\{iR_{IJ}, jR_{IJ}, kR_{IJ}\}$.

- (b) In the massless case with $m = s$, the dual group is $\tilde{G} = B(\eta_{(1,1)}, \mathbb{F})$, and the dual coadjoint orbit is given by

$$\tilde{\phi} = (\eta_{(1,1)})^{-1} i|s| = i|s|(\mathcal{R}^{00} + \mathcal{R}^{11}). \quad (3.71)$$

For $\mathbb{F} = \mathbb{C}$, the stabilizer is

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathfrak{u}(1)_{R_{00}} \oplus \mathfrak{u}(1)_{R_{11}}, \quad (3.72)$$

while for $\mathbb{F} = \mathbb{H}$, the stabilizer is

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathfrak{u}(1, 1)_{01}. \quad (3.73)$$

- (4) The coadjoint orbit of a BdS spinning particle with lightlike momentum and spacelike spin is given by

$$\begin{aligned} \phi\Omega_{(4)}^{-1} &= (E\mathcal{J}^{1+} + s\mathcal{J}^{23})\Omega_{(4)}^{-1} \\ &= \frac{1}{4} \begin{pmatrix} s\tau_1 & -E(\tau_1 + \tau_2) \\ E(\tau_1 - \tau_2) & s\tau_1 \end{pmatrix}, \end{aligned} \quad (3.74)$$

where $\mathcal{J}^{1+} = \frac{1}{2}(\mathcal{J}^{10} + \mathcal{J}^{14})$. The dual group is $B(\eta_{(2,2)}, \mathbb{F})$ and the dual coadjoint orbit is given by

$$\begin{aligned}\tilde{\phi} &= (\eta_{(2,2)})^{-1} \frac{1}{4} \begin{pmatrix} i(|s|\tau_2 + \varepsilon 1_{(2)}) & -\varepsilon 1_{(2)} \\ \varepsilon 1_{(2)} & i(-|s|\tau_2 + \varepsilon 1_{(2)}) \end{pmatrix} \\ &= \frac{i|s|}{2} (\mathcal{R}^{zz} + \mathcal{R}^{\bar{z}\bar{z}} - \mathcal{R}^{ww} - \mathcal{R}^{\bar{w}\bar{w}}) + i\varepsilon (\mathcal{R}^{z\bar{z}} + \mathcal{R}^{w\bar{w}}).\end{aligned}\quad (3.75)$$

The dual stabilizer for $\mathbb{F} = \mathbb{C}$ is

$$\begin{aligned}\tilde{\mathfrak{g}}_{\tilde{\phi}} &\cong \mathfrak{u}(1)_{i(R_{zz} + R_{\bar{z}\bar{z}})} \oplus \mathfrak{u}(1)_{iR_{(z\bar{z})}} \\ &\oplus \mathfrak{u}(1)_{i(R_{ww} + R_{\bar{w}\bar{w}})} \oplus \mathfrak{u}(1)_{iR_{(w\bar{w})}},\end{aligned}\quad (3.76)$$

and the dual stabilizer for $\mathbb{F} = \mathbb{H}$ is generated by

$$\begin{aligned}i(R_{zz} + R_{\bar{z}\bar{z}}), iR_{(z\bar{z})}, i(R_{ww} + R_{\bar{w}\bar{w}}), iR_{(w\bar{w})}, \\ jR_{(z\bar{w})}, jR_{(\bar{z}w)}, kR_{(z\bar{w})}, kR_{(\bar{z}w)}.\end{aligned}\quad (3.77)$$

Here, $\mathcal{V}^{z/\bar{z}} = \frac{1}{2}(\mathcal{V}^0 \pm i\mathcal{V}^1)$ and $\mathcal{V}^{w/\bar{w}} = \frac{1}{2}(\mathcal{V}^0 \pm i\mathcal{V}^2)$ can be viewed as a light-cone basis for complex and quaternionic space: $\mathcal{R}^{z\bar{z}}$ and $\mathcal{R}^{w\bar{w}}$ correspond to the nilpotent matrices $R^{z\bar{z}}$ and $R^{w\bar{w}}$, namely $(R^{z\bar{z}})^2 = 0$, $(R^{w\bar{w}})^2 = 0$, and $[R^{z\bar{z}}, R^{w\bar{w}}] = 0$. Note that $\varepsilon \cong \lambda\varepsilon$ for any $\lambda \in \mathbb{R}_+$, and its sign is determined by the sign of $E\varepsilon$.

- (5) The coadjoint orbit of a BdS spinning particle with lightlike momentum and lightlike spin is given by

$$\begin{aligned}\phi\Omega_{(4)}^{-1} &= (E\mathcal{J}^{1+} + \varepsilon\mathcal{J}^{2+})\Omega_{(4)}^{-1} \\ &= \frac{1}{4} \begin{pmatrix} -E\tau_2 - \varepsilon\tau_1 & -(E + i\varepsilon)\tau_2 \\ -(E - i\varepsilon)\tau_2 & -E\tau_2 + \varepsilon\tau_1 \end{pmatrix}.\end{aligned}\quad (3.78)$$

The dual group is $B(\eta_{(1,1)}, \mathbb{F})$, and the dual coadjoint orbit is nilpotent and given by

$$\tilde{\phi} = (\eta_{(1,1)})^{-1} \frac{1}{4} \begin{pmatrix} i\varepsilon & \varepsilon \\ -\varepsilon & i\varepsilon \end{pmatrix} = i\varepsilon \mathcal{R}^{\bar{z}z}, \quad (3.79)$$

with $\mathcal{V}^{z/\bar{z}} = \frac{1}{2}(\mathcal{V}^0 \pm i\mathcal{V}^1)$, and $\varepsilon \cong \lambda\varepsilon$ for any $\lambda \in \mathbb{R}_+$. The sign of ε is fixed by the sign of $E\varepsilon$. The dual stabilizer for $\mathbb{F} = \mathbb{C}$ is

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathfrak{u}(1)_{i(R_{zz} + R_{\bar{z}\bar{z}})} \oplus \mathfrak{R}_{i(R_{zz} - 2R_{z\bar{z}} + R_{\bar{z}\bar{z}})}, \quad (3.80)$$

and the dual stabilizer for $\mathbb{F} = \mathbb{H}$ is

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathfrak{u}(1)_{iR_{z\bar{z}}} \oplus \mathfrak{iso}(1, 1), \quad (3.81)$$

where $\mathfrak{iso}(1, 1)$ is generated by

$$i(R_{zz} - R_{z\bar{z}} + R_{\bar{z}\bar{z}}), jR_{z\bar{z}}, kR_{z\bar{z}}. \quad (3.82)$$

- (6) The coadjoint orbit of a BdS particle with spacelike momentum and doubly lightlike spin is given by

$$\begin{aligned}\phi\Omega_{(4)}^{-1} &= (m\mathcal{J}^{12} + \varepsilon\mathcal{J}^{++})\Omega_{(4)}^{-1} \\ &= \frac{1}{4} \begin{pmatrix} 0 & 2m\tau_0 \\ -2m\tau_0 & \varepsilon 1_{(2)} + i\varepsilon\tau_0 \end{pmatrix}.\end{aligned}\quad (3.83)$$

Then the dual group is $B(\eta_{(2,2)}, \mathbb{F})$, and the dual coadjoint orbit is given by

$$\begin{aligned}\tilde{\phi} &= (\eta_{(2,2)})^{-1} \frac{i}{2} \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & -m + \varepsilon & 0 & -\varepsilon \\ 0 & 0 & -m & 0 \\ 0 & -\varepsilon & 0 & m + \varepsilon \end{pmatrix} \\ &= \frac{im}{2} (\mathcal{R}^{0'0'} - \mathcal{R}^{11} - 4\mathcal{R}^{(++)}) + 2i\varepsilon \mathcal{R}^{--},\end{aligned}\quad (3.84)$$

where $\mathcal{V}^{\pm} = \frac{1}{2}(\mathcal{V}^0 \pm \mathcal{V}^2)$. The dual stabilizer for $\mathbb{F} = \mathbb{C}$ is

$$\begin{aligned}\tilde{\mathfrak{g}}_{\tilde{\phi}} &\cong \mathfrak{u}(1)_{i(R_{0'0'} - R_{11})} \oplus \mathfrak{u}(1)_{iR_{(+-)}} \\ &\oplus \mathfrak{u}(1)_{i(2R_{++} + 2R_{--} + R_{(++)} + R_{(--)})} \oplus \mathfrak{heis}_2,\end{aligned}\quad (3.85)$$

where $\mathfrak{heis}_2$ is generated by

$$i(R_{0'0'} + R_{11}), \quad R_{[0'1]}, \quad iR_{(0'1)}, \quad (3.86)$$

and the dual stabilizer for $\mathbb{F} = \mathbb{H}$ is generated by

$$\begin{aligned}R_{[0'1]}, \quad iR_{0'0'}, \quad iR_{11}, \quad iR_{(+-)}, \quad iR_{(0'1)}, \\ jR_{(0'-)}, \quad kR_{(0'-)}, \quad jR_{(1+)}, \quad kR_{(1+)}, \\ i(2R_{++} + 2R_{--} + R_{(++)} + R_{(--)})\end{aligned}\quad (3.87)$$

Note that $\varepsilon \cong \lambda\varepsilon$ for any $\lambda \in \mathbb{R}_+$ and its sign is given by the sign of $m\varepsilon$.

- (7) The coadjoint orbit of a BdS particle with lightlike momentum and doubly lightlike spin is given by

$$\begin{aligned}\phi\Omega_{(4)}^{-1} &= (E\mathcal{J}^{1+} + \varepsilon\mathcal{J}^{++})\Omega_{(4)}^{-1} \\ &= \frac{1}{4} \begin{pmatrix} 0 & -E(\tau_2 - \tau_0) \\ -E(\tau_2 + \tau_0) & -\varepsilon(1_{(2)} + \tau_1) \end{pmatrix}.\end{aligned}\quad (3.88)$$

The dual group is $B(\eta_{(2,1)}, \mathbb{F})$, and the dual coadjoint orbit is nilpotent and given by

$$\tilde{\phi} = (\eta_{(2,1)})^{-1} \frac{\varepsilon}{2} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \varepsilon \mathcal{R}^{[0'+]}, \quad (3.89)$$

with $\mathcal{V}^\pm = \frac{1}{2}(\mathcal{V}^0 \pm \mathcal{V}^1)$. The dual stabilizer for $\mathbb{F} = \mathbb{C}$ is

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathbb{R}_{R_{[0' -]}} \oplus \mathfrak{u}(1)_{iR_{(++)}} \oplus \mathfrak{u}(1)_{iR_{--}}, \quad (3.90)$$

and the dual stabilizer for $\mathbb{F} = \mathbb{H}$ is

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} \cong \mathbb{R}_{R_{[0' -]}} \oplus \mathfrak{iso}(3), \quad (3.91)$$

where $\mathfrak{iso}(3)$ is generated by imaginary generators

$$\mathfrak{q}_\alpha R_{(++)}, \mathfrak{q}_\alpha R_{--}. \quad (3.92)$$

Note that $\varepsilon \cong \lambda \varepsilon$ for any $\lambda \in \mathbb{R}_{\neq 0}$.

F. Particles on the conformal boundary

The twistor group $B(\Omega_{(4)}, \mathbb{F})$ is not only the isometry group of $\text{AdS}_{|\mathbb{F}|+3}$ but also the conformal group in $|\mathbb{F}| + 2$ dimensions. The conformal particles living in $|\mathbb{F}| + 2$ dimensions have *very small* phase space compared to the other particles living in AdS or BdS. In this section, we consider the twistor action for such conformal particles.

- (1) The coadjoint orbit corresponding to the conformal scalar particle is given by

$$\phi \Omega_{(4)}^{-1} = \epsilon \mathcal{J}^{++} \Omega_{(4)}^{-1} = \frac{\epsilon}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.93)$$

The dual group is $\tilde{G} = B(\eta_{(1)}, \mathbb{F})$, and the dual coadjoint orbit is the trivial orbit

$$\tilde{\phi} = 0. \quad (3.94)$$

- (2) The coadjoint orbit of a spinning singleton is given by

$$\begin{aligned} \phi \Omega_{(4)}^{-1} &= s(\mathcal{J}^{0'0} \pm \mathcal{J}^{12} + \mathcal{J}^{34} \mp \mathcal{J}^{56}) \Omega_{(4)}^{-1} \\ &= \frac{s}{2} \begin{pmatrix} 1_{(2)} + i\tau_0 & \mp (i1_{(2)} - \tau_0) \\ \pm (i1_{(2)} - \tau_0) & 1_{(2)} + i\tau_0 \end{pmatrix}. \end{aligned} \quad (3.95)$$

The dual group is $\tilde{G} = B(\eta_{(1)}, \mathbb{F})$, and the dual coadjoint orbit is a single point,

$$\tilde{\phi} = \pm 2|s|i. \quad (3.96)$$

Note here that we use complex entries for the coadjoint element ϕ , and hence the corresponding

coadjoint orbit does not exist for $\mathbb{F} = \mathbb{R}$, as is also manifest from the use of the elements $\mathcal{J}^{34}$ and $\mathcal{J}^{56}$.

For $\mathbb{F} = \mathbb{C}$, all sign choices correspond to distinct cases: they are the massless helicity $\pm s$ particles. Note also that we included the $\mathfrak{u}(1)$ part of $\mathfrak{b}(\Omega_{(4)}, \mathbb{C}) \cong \mathfrak{u}(2, 2)$ through $\mathcal{J}^{56}$ in a way that the dual group becomes minimal, that is, $B(\eta_{(1)}, \mathbb{C}) \cong U(1)$, but other choices of $\mathfrak{u}(1)$ are equally viable with the dual group $B(\eta_{(1,3)}, \mathbb{C}) \cong U(1, 3)$.

For $\mathbb{F} = \mathbb{H}$, the two sign choices are, in fact, equivalent: they are related by a π rotation by, for instance, J^{15} , which is a quaternionic matrix. Then, the opposite helicity can be described by the same action but with a different isomorphism between $B(\Omega_{(4)}, \mathbb{H}) = O^*(8)$ and $\widetilde{SO}^+(2, 6)$.

IV. CLASSIFICATION OF TWISTOR PARTICLES ACCORDING TO THE DUAL GROUPS

Let us summarize the results of classification according to the dual group $\tilde{G} = B(\eta_{(p, M-p)}, \mathbb{F})$. Since the twistor group is $G = B(\Omega_{(4)}, \mathbb{F})$, the possible values of M are restricted to 1, 2, 3, and 4.

A. Compact dual group

All the ordinary twistor particles, namely the massive and massless, twistor particles in $\text{AdS}_{|\mathbb{F}|+3}$, as well as the conformal twistor particles in $|\mathbb{F}| + 2$ dimensions, are obtained with compact dual groups $\tilde{G} = B(\eta_{(M)}, \mathbb{F})$.

Even though we have limited our focus to the symmetric spin case, we can fully generalize the analysis when the dual group is compact. This is thanks to the mileage that we have in the representation theory side: we know all the unitary irreducible representations associated with the dual pair correspondence between $B(\Omega_{(4)}, \mathbb{F})$ and $B(\eta_{(M)}, \mathbb{F})$ [53] (also [54–56]). Let us summarize the results here: the correspondences read for $\mathbb{F} = \mathbb{R}$

$$\left(\frac{\bar{n}+1}{2}; \frac{\bar{n}}{2}\right)_{Sp(4, \mathbb{R})} \leftrightarrow [\bar{n}]_{O(1)}, \quad (4.1)$$

$$(s+1; s)_{Sp(4, \mathbb{R})} \leftrightarrow [2s]_{O(2)}, \quad (4.2)$$

$$\left(s + \frac{3}{2}; s\right)_{Sp(4, \mathbb{R})} \leftrightarrow [2s]_{O(3)}, \quad (4.3)$$

$$\left(s + \frac{5}{2}; s\right)_{Sp(4, \mathbb{R})} \leftrightarrow [2s+1, 1]_{O(3)}, \quad (4.4)$$

$$(s+n+2; s)_{Sp(4, \mathbb{R})} \leftrightarrow [2s+n, n]_{O(4)}; \quad (4.5)$$

for $\mathbb{F} = \mathbb{C}$

$$(s+1; s, \pm s)_{SU(2,2)} \leftrightarrow [\pm 2s]_{U(1)}, \quad (4.6)$$

$$(s_1+2; s_1, s_2)_{SU(2,2)} \leftrightarrow [s_1+s_2, s_2-s_1]_{U(2)}, \quad (4.7)$$

$$(s+n+2; s, \pm s)_{SU(2,2)} \leftrightarrow [\pm(2s+n), \pm n]_{U(2)}, \quad (4.8)$$

$$(s_1+n+3; s_1, s_2)_{SU(2,2)} \leftrightarrow \begin{cases} [s_1+s_2+n, n, s_2-s_1]_{U(3)} \\ [s_1+s_2, -n, s_2-s_1-n]_{U(3)} \end{cases}, \quad (4.9)$$

$$(s_1+n+k+4; s_1, s_2)_{SU(2,2)} \leftrightarrow [s_1+s_2+n, n, -k, s_2-s_1-k]_{U(4)}; \quad (4.10)$$

and finally for $\mathbb{F} = \mathbb{H}$

$$(s+2; s, s, \pm s)_{O^*(8)} \leftrightarrow [2s]_{Sp(1)}, \quad (4.11)$$

$$(s_1+4; s_1, s_2, \pm s_2)_{O^*(8)} \leftrightarrow [s_1+s_2, s_1-s_2]_{Sp(2)}, \quad (4.12)$$

$$(s_1+s_2-s_3+6; s_1, s_2, \pm s_3)_{O^*(8)} \leftrightarrow [s_1+s_2, s_1-s_3, s_2-s_3]_{Sp(3)}, \quad (4.13)$$

$$(s_1+s_2-s_3+8+2n; s_1, s_2, \pm s_3)_{O^*(8)} \leftrightarrow [s_1+s_2+s_3+n, s_1+n, s_2+n, s_3+n]_{Sp(4)}. \quad (4.14)$$

In the last case of $\mathbb{F} = \mathbb{H}$, the $\pm$ sign corresponds to two possible isomorphisms between $O^*(8)$ and $\widetilde{SO}^+(2, 6)$. The label of $B(\eta_{(4)}, \mathbb{F})$ is $(\Delta; s_1, \dots)$ where Δ is the minimum of the global energy in AdS or equivalently the conformal weight, and $(s_1, \dots)$ is the Young diagram representing the compact rotational subgroup $\widetilde{SO}(|\mathbb{F}|+2)$. The label $[\ell_1, \dots]$ is the Young diagram representing the compact dual group $B(\eta_{(M)}, \mathbb{F})$, that is, the eigenvalues of the Cartan subalgebra generators for the highest weight state. From these results, we can extract the information about their classical counterpart.

Below, we summarize the classifications we found in the previous section, which cover only symmetric spin cases. Then, we complete the classifications by making use of the results (4.1)–(4.14) obtained in the dual pair correspondence.

(1) $B(\eta_{(1)}, \mathbb{F})$

For $M = 1$, we find conformal twistor particles in $|\mathbb{F}|+2$ dimensions with the action,

$$S = \int [\bar{\pi}_\alpha d\xi^\alpha - \bar{\xi}^{\dot{\alpha}} d\pi_{\dot{\alpha}} + A(\bar{\pi}_\alpha \xi^\alpha - \bar{\xi}^{\dot{\alpha}} \pi_{\dot{\alpha}} - 2is)]_{\mathbb{R}}, \quad (4.15)$$

which reproduces earlier results [5,11–13,15,57]. The constraints are associated with the moment maps generating $\hat{\mathfrak{f}} \cong \mathfrak{h}(\eta_{(1)}, \mathbb{F})$ symmetry.

For $\mathbb{F} = \mathbb{R}$, these constraints disappear, and the system describes the conformal scalar in three dimensions. In fact, the same particle action also describes the conformal spinor in three dimensions. The distinction between the conformal scalar and spinor can also be made based on the dual group $O(1) = \mathbb{Z}_2$. Since the latter is a finite group, the classical particle action itself cannot distinguish the two, but when we perform quantization, the $\mathbb{Z}_2$ symmetry can be imposed as constraints. The even and odd parts will result in the conformal scalar and the conformal spinor, respectively.

For $\mathbb{F} = \mathbb{C}$, the action describes 4D conformal particles with helicity s : positive and negative signs of s correspond to positive and negative helicity, respectively.

For $\mathbb{F} = \mathbb{H}$, it describes 6D conformal antichiral particles with spin $(s, -s)$. In this case, the two signs of s are equivalent and describe the same particle. For the description of a chiral particle with spin (s, s) , we can use the same action, but a different isomorphism between $\widetilde{SO}^+(2, 6)$ and $O^*(8)$.

The dual coadjoint orbit is trivial for $\mathbb{F} = \mathbb{R}$ as the dual coadjoint space itself is trivial. For $\mathbb{F} = \mathbb{C}$, the dual orbit consists of a single point. For $\mathbb{F} = \mathbb{H}$, it is $S^2 \cong Sp(1)/U(1)$.

- (2) $B(\eta_{(2)}, \mathbb{F})$ For $M = 2$, we find the massless spin s particles in $\text{AdS}_{|\mathbb{F}|+3}$, with the action

$$\begin{aligned}
S = & \int [\bar{\pi}_{\alpha 1} d\xi^{\alpha}_1 - \bar{\xi}^{\alpha}_1 d\pi_{\dot{\alpha} 1} + A^{11}(\bar{\pi}_{\alpha 1} \xi^{\alpha}_1 - \bar{\xi}^{\alpha}_1 \pi_{\dot{\alpha} 1}) \\
& + \bar{\pi}_{\alpha 2} d\xi^{\alpha}_2 - \bar{\xi}^{\alpha}_2 d\pi_{\dot{\alpha} 2} + A^{22}(\bar{\pi}_{\alpha 2} \xi^{\alpha}_2 - \bar{\xi}^{\alpha}_2 \pi_{\dot{\alpha} 2}) \\
& + A^{(12)}(\bar{\pi}_{\alpha(1} \xi^{\alpha}_{2)} - \bar{\xi}^{\alpha}_{(1} \pi_{\dot{\alpha} 2)}) \\
& + A^{[12]}(\bar{\pi}_{\alpha[1} \xi^{\alpha}_{2]} - \bar{\xi}^{\alpha}_{[1} \pi_{\dot{\alpha} 2]} - s)]_{\mathbb{R}}. \quad (4.16)
\end{aligned}$$

Each of the constraints $\tilde{\mu}_{11}$, $\tilde{\mu}_{22}$, and $\tilde{\mu}_{(12)}$ generates $\hat{\mathbb{F}}$, whereas the moment map $\tilde{\mu}_{[12]}$ generates $U(1) \cong O(2)$ (remember $\tilde{\mu}_{IJ} = \bar{\pi}_{\alpha I} \xi^{\alpha}_J - \bar{\xi}^{\alpha}_I \pi_{\dot{\alpha} J}$). This action reproduces the earlier results of $\text{AdS}_{|\mathbb{F}|+3}$ massless particles [33–35], where the difference is that the nonzero spin is introduced by SUSY, instead of shifting a constraint by spin label s .

For $\mathbb{F} = \mathbb{R}$, the constraints associated with $\tilde{\mu}_{11}$, $\tilde{\mu}_{22}$, and $\tilde{\mu}_{(12)}$ are absent, whereas the $U(1)$ constraint $\tilde{\mu}_{[12]} \approx s$ fixes the spin. The dual coadjoint orbit is simply a single point in this case.

For $\mathbb{F} = \mathbb{C}$, the spin is fixed by the $U(1)$ of the $SU(2)$ generated by $\tilde{\mu}_{11} - \tilde{\mu}_{22}$ and $\tilde{\mu}_{12}$, defining a coadjoint orbit $SU(2)/U(1) \cong S^2$, whereas the diagonal $U(1)$ generated by $\tilde{\mu}_{11} + \tilde{\mu}_{22}$ has a trivial coadjoint orbit; that is, the constraint is given without any constant shift.

For $\mathbb{F} = \mathbb{H}$, the dual group $B(\eta_{(2)}, \mathbb{H})$ is $Sp(2)$, and the constraints define a six-dimensional coadjoint orbit $Sp(2)/[U(1) \times Sp(1)]$.

In fact, for $\mathbb{F} = \mathbb{C}$ and $\mathbb{H}$, any system of constraints can be mapped to

$$\begin{aligned}
& A^{11}(\tilde{\mu}_{11} - i\ell_1) + A^{22}(\tilde{\mu}_{22} - i\ell_2) + A^{(12)}\tilde{\mu}_{(12)} \\
& + A^{[12]}\tilde{\mu}_{[12]}, \quad (4.17)
\end{aligned}$$

with $\ell_1 \geq \ell_2$, using a suitable transformation of $B(\eta_{(2)}, \mathbb{F})$, that is, $U(2)$ or $Sp(2)$. Note that for $\mathbb{F} = \mathbb{H}$ we can change the sign of ℓ_i by, for instance, an adjoint action of j on the ii component, and hence positive and negative ℓ_i are equivalent: both signs correspond to the same orbit. On the contrary, for $\mathbb{F} = \mathbb{C}$, positive and negative ℓ_i correspond to distinct orbits.

The massless spin s action (4.16) corresponds to the case with $[\ell_1, \ell_2] = [s, -s]$, while we may consider the more general case with $[\ell_1, \ell_2]$.

Let us first consider the case with $\ell_2 \leq 0$. According to (4.7) and (4.12), the action with the constraints (4.17) describes a massless particle of mixed symmetry spin (s_1, s_2) in AdS_5 and $(s_1, s_2, \pm s_2)$ in AdS_7 , for $s_1 > |s_2|$. The two signs reflect the two possible isomorphisms between $O^*(8)$ and $\widetilde{SO}^+(2, 6)$. The labels (s_1, s_2) and $[\ell_1, \ell_2]$ are related by

$$[\ell_1, \ell_2] = [s_1 + s_2, s_2 - s_1] = s_1[1, -1] + s_2[1, 1]. \quad (4.18)$$

Note that for $s_1 = |s_2|$ the action with (4.18) describes, in fact, a massive particle: the quantization of the model would generate a small finite mass.

Let us now turn to the remaining case of $\ell_2 > 0$. For $\mathbb{F} = \mathbb{H}$, a positive ℓ_2 is equivalent to a negative ℓ_2 , as we explained above. Therefore, it is sufficient to consider the $\mathbb{F} = \mathbb{C}$ case only. According to (4.8), this new case corresponds to a massive particle of mass $m > s$ and spin $(s, \pm s)$, where $[\ell_1, \ell_2]$ is parametrized by

$$[\ell_1, \ell_2] = [\pm m + s, \pm m - s] = s[1, -1] \pm m[1, 1]. \quad (4.19)$$

Note that in the case of a massless particle of symmetric spin $(s, 0)$ (4.16), the $U(1)$ part of the constraints does not have a constant shift. In the case of a massless particle of mixed symmetry spin (s_1, s_2) (4.18) or a massive particle of mixed symmetry spin $(s, \pm s)$ (4.19), the $U(1)$ part has the shift $2s_2$ or $\pm 2m$. In other words, the first spin label s_1 is always fixed by the $SU(2)$ orbit data s , whereas the $U(1)$ orbit position, say n , determines the second spin label as $s_2 = \frac{|n|}{2}$ for $\frac{|n|}{2} < s$. For $\frac{|n|}{2} \geq s$, it rather determines the mass value as $m = \frac{|n|}{2}$ while the second spin label is fixed by $s_2 = \text{sgn}(n)s$. The latter type of massive twistor action in AdS_5 was considered in [26] for the scalar case with $s = 0$, and its quantization was discussed in [58]. See, e.g., [35] for an early account for the AdS_5 exception. Let us remark that the distinction between the two cases $|n| \geq 2s$ and $|n| < 2s$ may invoke subtle issues when spin is introduced by SUSY, and this point has not been addressed in earlier literature.

(3) $B(\eta_{(3)}, \mathbb{F})$

For $M = 3$, we find the case of entangled mass and spin (3.55) with $s\epsilon > 0$. The action is given simply by

$$\begin{aligned}
S = & \int [\delta^{IJ}(\bar{\pi}_{\alpha I} d\xi^{\alpha}_J - \bar{\xi}^{\alpha}_J d\pi_{\dot{\alpha} I}) + A^{(IJ)}\tilde{\mu}_{(IJ)} \\
& + A^{[12]}(\tilde{\mu}_{[12]} - s) + A^{[23]}\tilde{\mu}_{[23]} + A^{[31]}\tilde{\mu}_{[31]}]_{\mathbb{R}}, \quad (4.20)
\end{aligned}$$

where $I, J = 1, 2, 3$. This action describes the very light massive particle of spin s : the corresponding orbit of $\text{AdS}_{|\mathbb{F}|+3}$ is the massive part of the massless limit of the massive spinning orbit, which arises together with the massless orbit. Let us refer to this special orbit as the orbit of “very light mass.” The

very light massive orbits have the same classical Casimir values as the massless ones, so we cannot distinguish the very light massive particles from the massless ones in terms of the classical mass parameter. If we quantize the particle actions, the classical mass parameters will acquire a small quantum correction, from which we can distinguish the two cases.

For $\mathbb{F} = \mathbb{C}$ and $\mathbb{H}$, we can perform a $B(\eta_{(3)}, \mathbb{F})$ rotation to transform the constraints in a way that the part with nontrivial shifts is

$$A^{11}(\tilde{\mu}_{11} - i\ell_1) + A^{22}(\tilde{\mu}_{22} - i\ell_2) + A^{33}(\tilde{\mu}_{33} - i\ell_3), \quad (4.21)$$

with $\ell_1 \geq \ell_2 \geq \ell_3$. Again, for $\mathbb{F} = \mathbb{H}$, the sign of ℓ_i can be set to be non-negative. Note that the case (4.20) corresponds to $[\ell_1, \ell_2, \ell_3] = [s, 0, -s]$, while a more general case $[\ell_1, \ell_2, \ell_3] = [s_1 + s_2, -s_2, -s_1]$ corresponds to the light massive particle of mixed symmetry spin (s_1, s_2) , according to the first case of (4.9). For $\mathbb{F} = \mathbb{H}$, we can turn on even the third spin label s_3 , according to (4.13), to describe the particles of mixed symmetry spin (s_1, s_2, s_3) by the action with the constraints,

$$[\ell_1, \ell_2, \ell_3] = [s_1 + s_2, s_1 - s_3, s_2 - s_3]. \quad (4.22)$$

For $\mathbb{F} = \mathbb{C}$, the label $m \geq s_1$ in

$$[\ell_1, \ell_2, \ell_3] = \begin{cases} [m + s_2, m - s_1, s_2 - s_1] \\ [s_1 + s_2, -m + s_1, s_2 - m] \end{cases} \quad (4.23)$$

corresponds to the mass of the AdS_5 particle of mixed symmetry spin (s_1, s_2) , according to (4.9). Note that the two options for $[\ell_1, \ell_2, \ell_3]$ describe exactly the same particle, and they can be distinguished only by the $U(1)$ shift value $2(s_2 \mp (s_1 - m))$.

(4) $B(\eta_{(4)}, \mathbb{F})$

For $M = 4$, we find the action for a particle of mass m and spin s as

$$\begin{aligned} S = & \int \left[\delta^{IJ} (\bar{\pi}_{\dot{a}I} d\xi^{\dot{a}}_J - \bar{\xi}^{\dot{a}}_J d\pi_{\dot{a}I}) + A^{(IJ)} \tilde{\mu}_{(IJ)} \right. \\ & + A^{[12]} \left(\tilde{\mu}_{[12]} - \frac{|m+s|}{2} \right) + A^{[34]} \left(\tilde{\mu}_{[34]} - \frac{|m-s|}{2} \right) \\ & \left. + A^{[13]} \tilde{\mu}_{[13]} + A^{[14]} \tilde{\mu}_{[14]} + A^{[23]} \tilde{\mu}_{[23]} + A^{[24]} \tilde{\mu}_{[24]} \right]_{\mathbb{R}}, \end{aligned} \quad (4.24)$$

where $I, J = 1, 2, 3, 4$. For $s = 0$, this action reproduces the scalar twistor model [28], where the dual group symmetry was referred to as R -symmetry.

Again, for $\mathbb{F} = \mathbb{C}$ and $\mathbb{H}$, with a $B(\eta_{(4)}, \mathbb{F})$ rotation, the action can be transformed so that the constraints with nontrivial shifts are

$$A^{11}(\tilde{\mu}_{11} - i\ell_1) + A^{22}(\tilde{\mu}_{22} - i\ell_2) + A^{33}(\tilde{\mu}_{33} - i\ell_3) + A^{44}(\tilde{\mu}_{44} - i\ell_4), \quad (4.25)$$

where the case (4.24) corresponds to $[\ell_1, \ell_2, \ell_3, \ell_4] = [\frac{s+m}{2}, \frac{-s+m}{2}, -\frac{-s+m}{2}, -\frac{s+m}{2}]$. For $\mathbb{F} = \mathbb{C}$, this can be generalized to

$$[\ell_1, \ell_2, \ell_3, \ell_4] = \left[\frac{s_1 + 2s_2 + m + n}{2}, \frac{-s_1 + m + n}{2}, -\frac{-s_1 + m - n}{2}, -\frac{s_1 - 2s_2 + m - n}{2} \right], \quad (4.26)$$

and the corresponding action describes a AdS_5 particle of mass m and mixed symmetry spin (s_1, s_2) , according to (4.10). Note that n only enters the $U(1)$ constraint as the shift $2(s_2 + n)$. For $\mathbb{F} = \mathbb{H}$, the general case

$$[\ell_1, \ell_2, \ell_3, \ell_4] = \left[\frac{s_1 + s_2 + s_3 + m}{2}, \frac{s_1 - s_2 - s_3 + m}{2}, \times \frac{s_2 - s_3 - s_1 + m}{2}, \frac{s_3 - s_1 - s_2 + m}{2} \right] \quad (4.27)$$

describes an AdS_7 particle of mass m and mixed symmetry spin $(s_1, s_2, \pm s_3)$, according to (4.14). Again, the $\pm$ signs correspond to two isomorphisms between $O^*(8)$ and $\widetilde{SO}^+(2, 6)$.

B. Noncompact dual group

In the case of noncompact dual groups, we find various exotic particle species. In this case, we have very limited information about unitary irreducible representations because they are not of the highest weight type and the relevant dual pair correspondence is much harder to analyze. We can regard the situation as the mileage of the classical analysis for the representation theory. Let us remind the reader again that we have limited to the case with two labels for simplicity, but the full cases can be treated by the same method.

(1) $B(\eta_{(1,1)}, \mathbb{F})$

In this case, we find short massless spinning tachyon (3.44). This is to be compared with the massless spinning particles with $B(\eta_{(2)}, \mathbb{F})$, which is

the shortening point of the massive spinning particle. For $\mathbb{F} = \mathbb{C}$ and $\mathbb{H}$, we also have BdS particles: the short massless one with spacelike spin (3.71) and the one with lightlike momentum and spin (3.78). Note that for $\mathbb{F} = \mathbb{C}$ with $B(\eta_{(1,1)}, \mathbb{C}) = U(1, 1)$, the three cases (3.44), (3.71), and (3.78) correspond, respectively, to the two-sheet hyperboloid, one-sheet hyperboloid, and cone coadjoint orbits of $SU(1, 1)$.

(2) $B(\eta_{(2,1)}, \mathbb{F})$

This case includes the maximal nilpotent orbit, namely the BdS particle of lightlike momentum and doubly lightlike spin (3.88). Besides it, we find also large remnant orbits (3.56) and (3.59) appearing in the short massless limit of the BdS particle (the $|m| < |s|$ case of a massive spinning particle) and the tachyon (3.41). The latter two cases are to be compared with the case of $B(\eta_{(3)}, \mathbb{F})$, namely the large remnant orbit appearing in the massless limit of massive spinning orbit.

(3) $B(\eta_{(2,2)}, \mathbb{F})$

This case includes all kinds of “long” tachyons: the ones with timelike spin (3.42), spacelike spin (3.47), and lightlike spin (3.50). Moreover, it contains all kinds of “long” BdS particles: the one appearing as the $|m| < |s|$ subcase of a massive spinning particle, the ones with spacelike momentum and spacelike spin (3.68), lightlike momentum and spacelike spin (3.75), and spacelike momentum and doubly lightlike spin (3.84).

(4) $B(\eta_{(3,1)}, \mathbb{F})$

Finally, this case contains the massive, massless, and tachyonic continuous spin particles (3.53). The action reads

$$\begin{aligned} S = & \int [\eta^{IJ} (\bar{\pi}_{\alpha I} d\xi^{\alpha}_J - \bar{\xi}^{\alpha}_J d\pi_{\alpha I}) + A^{(IJ)} \tilde{\mu}_{(IJ)} \\ & + A^{[01]} (\tilde{\mu}_{[01]} - |\nu|) - A^{[23]} (\tilde{\mu}_{[23]} - |s|) \\ & + A^{[02]} \tilde{\mu}_{[02]} + A^{[03]} \tilde{\mu}_{[03]} + A^{[12]} \tilde{\mu}_{[12]} + A^{[13]} \tilde{\mu}_{[13]}]_{\mathbb{R}}, \end{aligned} \quad (4.28)$$

where $I, J = 0, 1, 2, 3$ and $\eta^{IJ} = \text{diag}(-1, +1, +1, +1)$. It can be viewed as the twistorial counterpart of the vectorial description of the continuous spin particles [1], which are related to a variety of continuous spin fields identified by Metsaev [59]. See also [60] for other vectorial models of (A)dS₄ continuous spin particles.

V. DIMENSIONAL REDUCTION

Many of the twistor literatures concern massless and massive particles in flat spacetime, $\text{Mink}_{|\mathbb{F}|+2}$. In our construction, only conformal particles are defined in $|\mathbb{F}| + 2$ dimensions, namely $\text{Mink}_{|\mathbb{F}|+2}$, $\text{AdS}_{|\mathbb{F}|+2}$, and $\text{dS}_{|\mathbb{F}|+2}$

boundaries of $\text{AdS}_{|\mathbb{F}|+3}$. These conformal particles are massless but their spins are rather special carrying the $(|\mathbb{F}| - 1)$ -row box Young diagram representation. In fact, a more general type of particles in $\text{Mink}_{|\mathbb{F}|+2}$, $\text{AdS}_{|\mathbb{F}|+2}$, and $\text{dS}_{|\mathbb{F}|+2}$ can be obtained from $\text{AdS}_{|\mathbb{F}|+3}$ particles by restricting the isometry $B(\Omega_{(4)}, \mathbb{F}) \cong \widetilde{SO}^+(2, |\mathbb{F}| + 2)$ to the subgroups H_{Λ} , that is, $H_{\Lambda=0} \cong I\widetilde{SO}^+(1, |\mathbb{F}| + 1)$, $H_{\Lambda<0} \cong \widetilde{SO}^+(2, |\mathbb{F}| + 1)$, and $H_{\Lambda>0} \cong \widetilde{SO}^+(1, |\mathbb{F}| + 2)$. Note that the conformal particles, namely singletons, are precisely the ones that stay irreducible under this restriction. The restriction of $G = B(\Omega_{(4)}, \mathbb{F})$ to the subgroup H_{Λ} defines a new dual group $\tilde{H}_{\Lambda}$ that includes the original dual group $\tilde{G} = B(\eta_{(q, M-q)}, \mathbb{F})$. This situation can be depicted by the so-called seesaw diagram:

$$\begin{array}{ccc} G = B(\Omega_{(4)}, \mathbb{F}) & & \tilde{H}_{\Lambda} \\ \cup & \begin{array}{c} \nearrow \quad \searrow \\ \nwarrow \quad \swarrow \end{array} & \cup \\ H_{\Lambda} & & \tilde{G} = B(\eta_{(q, M-q)}, \mathbb{F}) \end{array} \quad (5.1)$$

The seesaw duality asserts that the restriction of unitary irreducible representations on the left-hand side is governed by the restriction on the right-hand side, and vice versa. More precisely, for the irreps π_{λ}^G of G and $\pi_{\tau}^{\tilde{H}_{\Lambda}}$ of $\tilde{H}_{\Lambda}$ branching respectively to H_{Λ} and $\tilde{G}$ as

$$\pi_{\lambda}^G = \bigoplus_{\sigma} \mathcal{N}_{\lambda}^{\sigma} \pi_{\sigma}^{H_{\Lambda}}, \quad \pi_{\tau}^{\tilde{H}_{\Lambda}} = \bigoplus_{\rho} \tilde{\mathcal{N}}_{\tau}^{\rho} \pi_{\rho}^{\tilde{G}}, \quad (5.2)$$

the multiplicities $\mathcal{N}_{\lambda}^{\sigma}$ and $\tilde{\mathcal{N}}_{\tau}^{\rho}$ are related by

$$\mathcal{N}_{\lambda}^{\sigma} = \tilde{\mathcal{N}}_{\phi(\sigma)}^{\theta(\lambda)}, \quad (5.3)$$

where θ and ϕ are the bijective maps of the $(G, \tilde{G})$ irreps and the $(H_{\Lambda}, \tilde{H}_{\Lambda})$ irreps, respectively. Here, we consider the classical counterpart of the seesaw dual pair, which we may refer to as *symplectic seesaw reduction*. The irreps correspond to coadjoint orbits, and the restriction of an irrep down to the irreps of a subgroup corresponds to the foliation of coadjoint orbits into the coadjoint orbits of the subgroup. Note that the restriction to $H_{\Lambda=0}$ to obtain massive twistor particles has been discussed already in [35]. Here, we generalize the idea with extra emphasis on seesaw dual pairs.

Let us consider the reductive subgroup $GL(2, \mathbb{F})$ of $B(\Omega_{(4)}, \mathbb{F})$,

$$L^{\alpha}_{\beta} = \xi^{\alpha I} \bar{\pi}_{\beta I}, \quad \bar{L}_{\dot{\alpha}}^{\dot{\beta}} = \pi_{\dot{\alpha}}^I \bar{\xi}_{\dot{\beta} I}. \quad (5.4)$$

We may separate each of L^{α}_{β} and $\bar{L}_{\dot{\alpha}}^{\dot{\beta}}$ into the traceless and trace parts: the traceless part can be expressed as

$$L_{(\alpha\beta)} = \xi_{(\alpha}^I \bar{\pi}_{\beta)I}, \quad \bar{L}_{(\dot{\alpha}\dot{\beta})} = \pi_{(\dot{\alpha}}^I \bar{\xi}_{\dot{\beta})I}, \quad (5.5)$$

by lowering indices with the Levi-Civita symbols, which are Lorentz invariant tensors only for $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$. One combination of the traces is the center of $GL(2, \mathbb{F})$,

$$D = L^\alpha_\alpha + \bar{L}_{\dot{\alpha}}^{\dot{\alpha}} = \xi^{\alpha I} \bar{\pi}_{\alpha I} + \pi_{\dot{\alpha}}^I \bar{\xi}_{\dot{\alpha} I}, \quad (5.6)$$

which is nothing but the dilatation generator. The other combination of the traces,

$$E = L^\alpha_\alpha - \bar{L}_{\dot{\alpha}}^{\dot{\alpha}} = \xi^{\alpha I} \bar{\pi}_{\alpha I} - \pi_{\dot{\alpha}}^I \bar{\xi}_{\dot{\alpha} I}, \quad (5.7)$$

takes value in $\hat{\mathbb{F}}$: it vanishes for $\mathbb{F} = \mathbb{R}$ and generates the central $U(1)$ for $\mathbb{F} = \mathbb{C}$. For $\mathbb{F} = \mathbb{H}$, the three real generators in E are parts of the Lorentz subgroup $SL(2, \mathbb{H})$. Since we consider the $(|\mathbb{F}| + 2)$ -dimensional isometry subgroup of $B(\Omega_4, \mathbb{F})$, the dilatation generator D will be discarded from the $\mathfrak{h}_\Lambda$. Moreover, among the translation and special conformal generators,

$$P^{\alpha\dot{\beta}} = \xi^{\alpha I} \bar{\xi}_{\dot{\beta} I}, \quad K_{\dot{\alpha}\beta} = \pi_{\dot{\alpha}}^I \bar{\pi}_{\beta I}, \quad (5.8)$$

we retain only the linear combinations

$$P^{\alpha\dot{\beta}} + \Lambda K^{\dot{\beta}\alpha}. \quad (5.9)$$

As we consider a subgroup H_Λ , the dual group enhances from $\tilde{G}$ to $\tilde{H}_\Lambda$. This means that on top of the $\tilde{G}$ generators,

$$\tilde{\mu}_{IJ} = \bar{\pi}_{\alpha I} \xi^\alpha_J - \bar{\xi}^{\dot{\alpha}}_I \pi_{\dot{\alpha} J}, \quad (5.10)$$

we have additional generators that commute with H_Λ . It is not difficult to enlist all possible candidates because the invariance under $L_{(\alpha\beta)}$ and $\bar{L}_{(\dot{\alpha}\dot{\beta})}$ implies that the α, β indices should be contracted: the possible moment maps are

$$\xi^\alpha_I \xi^\alpha_J, \quad \bar{\pi}_{\alpha I} \bar{\pi}_{\alpha J}, \quad \bar{\pi}^\alpha_I \bar{\pi}_{\alpha J}, \quad (5.11)$$

and their $*$ conjugates. For $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$, we can proceed to check the translational invariance to find that the additional dual generators are

$$M_{IJ} = \xi^\alpha_I \xi_{\alpha J} - \Lambda \bar{\pi}^\alpha_I \bar{\pi}_{\alpha J}, \quad \bar{M}_{IJ} = \bar{\xi}^{\dot{\alpha}}_I \bar{\xi}_{\dot{\alpha} J} - \Lambda \pi^{\dot{\alpha}}_I \pi_{\dot{\alpha} J}. \quad (5.12)$$

For $\mathbb{F} = \mathbb{R}$, the original dual generators form $O(p, M - p)$, and real M_{IJ} are antisymmetric and hence span a $[M(M - 1)/2]$ -dimensional space. The whole dual group becomes a real form of $O_M \times O_M$. For the $p = 0$ example, we find the seasaw pairs for $\Lambda < 0$ and $\Lambda > 0$ as

$$\begin{array}{ccc} Sp(4, \mathbb{R}) & & O(M)^2 \\ \cup & \begin{array}{c} \nwarrow \nearrow \\ \nearrow \nwarrow \end{array} & \cup \\ Sp(2, \mathbb{R})^2 & & O(M) \end{array}, \quad \begin{array}{ccc} Sp(4, \mathbb{R}) & & O(M, \mathbb{C}) \\ \cup & \begin{array}{c} \nwarrow \nearrow \\ \nearrow \nwarrow \end{array} & \cup \\ Sp(2, \mathbb{C}) & & O(M) \end{array}. \quad (5.13)$$

For $\mathbb{F} = \mathbb{C}$, the original dual generators form $U(p, M - p)$, and complex M_{IJ} are antisymmetric and hence span a $[M(M - 1)]$ -dimensional space. The whole dual group becomes a real form of O_{2M} . For the $p = 0$ example, we find the seasaw pairs for $\Lambda < 0$ and $\Lambda > 0$ as

$$\begin{array}{ccc} U(2, 2) & & O(2M) \\ \cup & \begin{array}{c} \nwarrow \nearrow \\ \nearrow \nwarrow \end{array} & \cup \\ Sp(4, \mathbb{R}) & & U(M) \end{array}, \quad \begin{array}{ccc} U(2, 2) & & O^*(2M) \\ \cup & \begin{array}{c} \nwarrow \nearrow \\ \nearrow \nwarrow \end{array} & \cup \\ Sp(1, 1) & & U(M) \end{array}. \quad (5.14)$$

The $M = 1$ case provides the conformal particles having a very special restriction property: it branches into at most two irreps of the isometry subgroup. For $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$, it happens that the conformal particles cover all massless particles.

The $M = 2$ case provides all massive particle species in $|\mathbb{F}| + 2$ dimensions, that is, those appearing in the dimensional reduction of the most general “massless particle” in $\text{AdS}_{|\mathbb{F}|+3}$. Here, we used the quotation marks for massive and massless because these cover not only the standard

massive and massless particles with compact dual groups but also the analog ones in tachyonic spectra or BdS with noncompact dual groups.

To reiterate, for $\Lambda \neq 0$, the restriction (5.1) to 3D and 4D isometry groups leads, respectively, to the dual pairs

$$(Sp(2, \mathbb{R})^2, O(p, 2-p) \times O(q, 2-q)),$$

$$(Sp(2, \mathbb{C}), O(2, \mathbb{C})) \quad (5.15)$$

and

$$(Sp(4, \mathbb{R}), O(p, 4-p)), \quad (Sp(1, 1), O^*(4)). \quad (5.16)$$

Therefore, one simply recovers dual pairs of another type, which describe all possible massive particles in (A)dS₃ and (A)dS₄, respectively. For $\Lambda = 0$, the restriction provides an equally good description, even though it does not correspond to a reductive dual pair (as the Poincaré group is not reductive).

Note that the restriction (5.1) to $H_{\Lambda=0}$ reproduces the 3D and 4D massive twistor models [18–22] with the dual groups containing respectively $O(2)$ and $U(2)$: these flat space twistor models can be regarded as the flat limit of (A)dS massive twistors described by the dual pairs (5.16) with $p = 0$. The restriction also reproduces the 4D continuous spin twistor model [47–49], with the dual group containing $U(1, 1)$. This model can be regarded as the flat limit of the AdS continuous spin twistor described by $(Sp(4, \mathbb{R}), O(3, 1))$.

$$\begin{array}{ccc} O^*(8) & & U(2M) \\ \nearrow & \nwarrow & \nearrow \\ H_{\Lambda < 0} & \left[\begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array} \right] & \tilde{H}_{\Lambda < 0} \\ \nwarrow & \nearrow & \nwarrow \\ U(2, 2) & & Sp(M) \end{array},$$

Note that the isometry groups $\widetilde{SO}^+(2, 5)$ or $\widetilde{SO}^+(1, 6)$ of (A)dS₆ are not isomorphic to another twistor group, which prevents one from finding a dual pair containing them in the same symplectic group in general. Indeed, dual pairs involving, say $O(2, 5)$ for the sake of definiteness, are of the form

$$(O(2, 5), Sp(2K, \mathbb{R})) \subset Sp(14K, \mathbb{R}) \quad (5.18)$$

for some K , whereas the twistorial setup for $\mathbb{F} = \mathbb{H}$ consists of dual pairs

$$(O^*(8), Sp(M)) \subset Sp(16M, \mathbb{R}), \quad (5.19)$$

The particle description provided by the restriction (5.1) has the manifest covariance of the full bulk Lorentz symmetry, as opposed to the original particle descriptions based on $(B(\Omega_{(4)}, \mathbb{F}), B(\eta_{(p, M-p)}, \mathbb{F}))$ having only manifest boundary Lorentz covariance: for the example of massive particles in AdS₄, the original description makes use of a real twistor having only manifest 3D boundary Lorentz covariance, while the restriction (5.1) makes use of complex twistors having manifest covariance of the 4D bulk Lorentz symmetry. Note, however, that the two descriptions are identical: they are simply related by a change of variables.

Let us also mention that the quantization of twistor models leads to the oscillator representations governed by dual pair correspondences. These representations are isomorphic to the spinor-helicity representations: the massless [61–64] and massive [65–67] in flat space, as well as their (A)dS counterparts [68–71].

For $\mathbb{F} = \mathbb{H}$, the situation is different: we should require the invariance under E (5.7) too, since it is a part of the Lorentz algebra. We find that M_{IJ} no longer commute with E . If we had included M_{IJ} , which are $[M(2M-1)]$ -dimensional, together with the original dual generators $Sp(p, M-p)$, the enhanced dual group would have been a real form of GL_{2N} , which is dual to a real form of GL_4 : for instance, $U(2, 2)$ containing the AdS₅ group or $GL(2, \mathbb{H})$ containing the dS₅ group, instead of the six-dimensional isometry group. The 6D isometry group H_{Λ} is a subgroup of $O^*(8)$, at the same time including $SU(2, 2)$ and $SL(2, \mathbb{H})$, but not $U(2, 2)$ and $GL(2, \mathbb{H})$:

$$\begin{array}{ccc} O^*(8) & & GL(M, \mathbb{H}) \\ \nearrow & \nwarrow & \nearrow \\ H_{\Lambda > 0} & \left[\begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array} \right] & \tilde{H}_{\Lambda > 0} \\ \nwarrow & \nearrow & \nwarrow \\ GL(2, \mathbb{H}) & & Sp(M) \end{array}. \quad (5.17)$$

so that in order for the first pair to be contained in the second one, the rank K and M should verify $14K = 16M$, which forbids low values for them.⁷

The impossibility of recovering H_{Λ} with $\mathbb{F} = \mathbb{H}$ as part of a dual pair, which in turn forms a seesaw pair with $(O^*(8), Sp(M))$, may be understood by recalling that the

⁷A possible alternative is to allow for another dual pair, $(H', \tilde{H}') \subset Sp(2K', \mathbb{R})$ such that $7K + K' = 8M$. For instance, one could be tempted to consider the pair $H' = Sp(1)$ and $\tilde{H}' = O(1) \cong \mathbb{Z}_2$, but its direct product with the dual pair containing $O(2, 5)$ for $K = M$ is not embedded in the $O^*(8)$ dual pair. Let us also point out that $O(2, 5) \times SU(2)$ is the bosonic subgroup of $F(4)$, which appears as the symmetry group of Romans supergravity [72]. This may suggest that the seesaw mechanism discussed here also works in 6D in the supersymmetric context.

setup of Howe duality is quite peculiar from the point of view of dualities in representation theory. In general, a duality in the context of representation theory designates a situation wherein a *reducible* representation V of a given Lie group G is decomposed as a direct sum of tensor products of irreps of G and irreps of its centralizer, say $\mathcal{A}$, in a certain subalgebra of the endomorphism algebra of V , such that pairs of irreps appear only once, and each G -irrep appears with a unique irrep of the $\mathcal{A}$. The dual pair correspondence, or Howe duality, corresponds to V being the Fock space of N pairs of (bosonic) oscillators, and the relevant subalgebra of its endomorphism algebra being the *Weyl algebra*. On top of that, if G is a reductive subgroup of $Sp(2N, \mathbb{R})$, then its centralizer in the Weyl algebra can be characterized using invariant theory for classical groups: since by assumption G is a product of real forms of the general linear, orthogonal, or symplectic groups of various ranks, its invariants are generated by products of the elementary invariant tensors, which are the Kronecker delta, a metric of a given signature and the canonical symplectic form. Since $Sp(2N, \mathbb{R})$ is canonically embedded in the Weyl algebra as its subspace of quadratic operators (in the oscillators), so is G , and by the previous comment, G -invariant operators are necessarily products of quadratic ones, the latter being built using the elementary invariant tensors to contract two oscillators. In other words, the centralizer of G in the Weyl algebra is generated by quadratic invariants, and therefore characterizing this centralizer boils down to identifying the centralizer of G in $Sp(2N, \mathbb{R})$ —the problem can be reduced to studying the quadratic subspace of the Weyl algebra, i.e., the image of the symplectic group in it. This latter aspect is the specificity of Howe duality, and what makes it so appealing and efficient in many contexts. However, the general problem of decomposing a space under the joint action of a group G and its centralizer (in a certain subalgebra of endomorphism), though still well-posed, generically cannot be reduced to the analysis of the centralizer of G in another finite-dimensional Lie group such as the symplectic group.

Consequently, one should not expect that the centralizer of any Lie group G in the Weyl algebra is generated by a quadratic operator only, even when G is embedded in quadratic operators: the number of oscillators should be such that it allows the action of the elementary G -invariant tensors, without leaving too big of an invariant subspace. Such conditions were studied by Howe in his seminal papers [55,56] containing the classification of reductive dual pairs. The case of H_Λ with $\mathbb{F} = \mathbb{H}$ is illustrative of this difficulty, as even if it is contained as a subgroup in $O^*(8)$, the Fock space relevant for the latter does not allow for a decomposition appropriate for invariant tensors of the former (due to the dimensional mismatch mentioned at the beginning of the previous paragraph). This explains why the centralizer of H_Λ for $\mathbb{F} = \mathbb{H}$ is generated by a subgroup contained in the same symplectic group as $O^*(8)$. In fact, one should expect its centralizer to be

generated by operators of order higher than quadratic, as is suggested by the existence of a quartic operator which can be used to define the mass of particles. Indeed, nonquadratic but quartic mass constraints have been considered in [23–25], and it was explained in [35] how these massive $\text{Mink}_{3,4,6}$ particles can be obtained from the massless $\text{AdS}_{4,5,7}$ particles by adding the mass constraint. Note also the recent 6D flat continuous spin twistor model [50] can be understood as a restriction of an exotic AdS_7 massless particle with the dual pair $(O^*(8), Sp(1, 1))$ by a quartic mass constraint.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: QUATERNION-SYMPLECTIC STRUCTURE

In this section, we provide an alternative way to handle the quaternion-symplectic structure. Instead of expanding quaternions by the typical basis i, j, k , we may use the 2×2 complex matrix representation of quaternion:

$$q = \begin{pmatrix} q^1_1 & q^1_2 \\ q^2_1 & q^2_2 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} \bar{q}^1_1 & \bar{q}^1_2 \\ \bar{q}^2_1 & \bar{q}^2_2 \end{pmatrix}, \quad (\text{A1})$$

with $\bar{q}^i_j = (q^j_i)^*$ and additional relations

$$q^1_1 = (q^2_2)^*, \quad q^1_2 = -(q^2_1)^*. \quad (\text{A2})$$

In this complex matrix description, the dot product is given by the trace

$$p \bullet q = \frac{1}{2} \bar{p}^i_j q^j_i = \frac{1}{2} \bar{q}^i_j p^j_i. \quad (\text{A3})$$

The relations (A2) imposed on $\xi^\alpha_{I^i j}$ and $\pi_{\dot{a}I^i j}$ can be considered as second class constraints, and hence can be imposed after all the computations where we use the Poisson bracket (not the Dirac bracket) given by

$$\{\xi^\alpha_{I^i k}, \bar{\pi}_{\beta J^j l}\} = \eta_{IJ} \delta^\alpha_\beta \delta^i_l \delta^j_k, \quad \{\bar{\xi}^{\dot{\alpha}}_{I^i k}, \pi_{\beta J^j l}\} = \eta_{IJ} \delta^{\dot{\alpha}}_\beta \delta^i_l \delta^j_k, \quad (\text{A4})$$

where $i, j, k, l = 1, 2$.

APPENDIX B: ADS GENERATORS

In this appendix, we provide the explicit component expressions of the $\mathfrak{h}(\Omega_{(4)}, \mathbb{F})$ generators.

Generators of $\mathfrak{sp}(4, \mathbb{R})$

$$J_{0'0} = \frac{1}{2} \begin{pmatrix} 0 & 1_{(2)} \\ -1_{(2)} & 0 \end{pmatrix}, \quad J_{0'1} = \frac{1}{2} \begin{pmatrix} 0 & \tau_2 \\ \tau_2 & 0 \end{pmatrix}, \quad J_{0'2} = -\frac{1}{2} \begin{pmatrix} 0 & \tau_1 \\ \tau_1 & 0 \end{pmatrix}, \quad J_{0'3} = \frac{1}{2} \begin{pmatrix} 1_{(2)} & 0 \\ 0 & -1_{(2)} \end{pmatrix}, \quad (\text{B1})$$

$$J_{01} = \frac{1}{2} \begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix}, \quad J_{02} = \frac{1}{2} \begin{pmatrix} -\tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}, \quad J_{03} = -\frac{1}{2} \begin{pmatrix} 0 & 1_{(2)} \\ 1_{(2)} & 0 \end{pmatrix}, \quad (\text{B2})$$

$$J_{12} = -\frac{1}{2} \begin{pmatrix} \tau_0 & 0 \\ 0 & \tau_0 \end{pmatrix}, \quad J_{13} = -\frac{1}{2} \begin{pmatrix} 0 & \tau_2 \\ -\tau_2 & 0 \end{pmatrix}, \quad J_{23} = \frac{1}{2} \begin{pmatrix} 0 & \tau_1 \\ -\tau_1 & 0 \end{pmatrix}. \quad (\text{B3})$$

Generators of $\mathfrak{u}(2, 2)$

$$J_{0'4} = \frac{i}{2} \begin{pmatrix} 0 & \tau_0 \\ \tau_0 & 0 \end{pmatrix}, \quad J_{34} = \frac{i}{2} \begin{pmatrix} 0 & -\tau_0 \\ \tau_0 & 0 \end{pmatrix}, \quad J_{14} = -\frac{i}{2} \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}, \quad (\text{B4})$$

$$J_{04} = -\frac{i}{2} \begin{pmatrix} \tau_0 & 0 \\ 0 & -\tau_0 \end{pmatrix}, \quad J_{24} = -\frac{i}{2} \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad J_{56} = \frac{i}{2} \begin{pmatrix} 1_{(2)} & 0 \\ 0 & 1_{(2)} \end{pmatrix}. \quad (\text{B5})$$

Generators of $\mathfrak{o}^*(8)$

$$J_{0'5} = \frac{j}{2} \begin{pmatrix} 0 & \tau_0 \\ \tau_0 & 0 \end{pmatrix}, \quad J_{35} = \frac{j}{2} \begin{pmatrix} 0 & -\tau_0 \\ \tau_0 & 0 \end{pmatrix}, \quad J_{15} = -\frac{j}{2} \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}, \quad (\text{B6})$$

$$J_{05} = -\frac{j}{2} \begin{pmatrix} \tau_0 & 0 \\ 0 & -\tau_0 \end{pmatrix}, \quad J_{25} = -\frac{j}{2} \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad J_{46} = -\frac{j}{2} \begin{pmatrix} 1_{(2)} & 0 \\ 0 & 1_{(2)} \end{pmatrix}, \quad (\text{B7})$$

$$J_{0'6} = \frac{k}{2} \begin{pmatrix} 0 & \tau_0 \\ \tau_0 & 0 \end{pmatrix}, \quad J_{36} = \frac{k}{2} \begin{pmatrix} 0 & -\tau_0 \\ \tau_0 & 0 \end{pmatrix}, \quad J_{16} = -\frac{k}{2} \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}, \quad (\text{B8})$$

$$J_{06} = -\frac{k}{2} \begin{pmatrix} \tau_0 & 0 \\ 0 & -\tau_0 \end{pmatrix}, \quad J_{26} = -\frac{k}{2} \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad J_{45} = \frac{k}{2} \begin{pmatrix} 1_{(2)} & 0 \\ 0 & 1_{(2)} \end{pmatrix}. \quad (\text{B9})$$

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