

Graviton number radiated during binary inspiral and its link to angular momentum radiation

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The number of gravitons emitted during binary inspiral is found to be remarkably close to (though slightly larger than) the radiated angular momentum in units of $2\hbar$. This ($\sim 1\%$) closeness holds both for an infinitesimal change in the eccentricity, and for the total graviton number emitted during inspiral from a large eccentricity initial orbit down to negligible final eccentricity.

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I. INTRODUCTION

In recent papers I have pointed out that discrete orbit effects invalidate the previously widely used adiabatic approximation for the evolution of the nonrelativistic energy, semimajor axis, and period of inspiraling binary black holes [1], and I have calculated “exact” (in the nonrelativistic approximation) and approximate formulas for the inspiral time as functions of the black hole masses, initial relative velocity, and impact parameter, as well as approximate inverse functions for the impact parameter in terms of the inspiral time and other parameters [2].

Here I wish to give an “exact” formula for the number of gravitons emitted per orbital angular momentum lost by highly eccentric nonrelativistic orbits, and approximate but highly precise formulas for that ratio during the entire inspiral from a given initial eccentricity, as well as for the total number of gravitons emitted during the entire inspiral.

The result may be briefly summarized by noting that the number of gravitons emitted is slightly more than the loss of orbital angular momentum divided by $2\hbar$, the spin angular momentum of a graviton, but the excess is never much more than 0.2% for the entire inspiral emission of gravitons and angular momentum.

II. INFINITE SERIES FOR GRAVITON NUMBER EMITTED

Consider two black holes (or other effectively point masses, each much smaller than their separation distance) with masses M_1 and M_2 , total mass $M = M_1 + M_2$,

reduced mass $\mu = M_1 M_2 / M$, and dimensionless mass ratio $\eta = M_1 M_2 / (M_1 + M_2)^2 = \mu / M$, and assume they are slowly inspiraling along a sequence of nearly Keplerian orbits, each with semimajor axis a , eccentricity e , period $\tau = 2\pi\sqrt{a^3/GM}$, nonrelativistic energy $E = -GM^2\eta/(2a)$, and angular momentum $L = \sqrt{GM^3\eta^2 a(1-e^2)}$.

Calculations by Philip Carl Peters and by Peters and Jon Mathews [3–5] show that the average gravitational power per orbit radiated into the n th harmonic (angular frequency $\omega = n\sqrt{GM/a^3}$) is

$$P(n) = \frac{32}{5} \frac{G^4}{c^5} \frac{M^5 \eta^2}{a^5} g(n, e), \quad (1)$$

$$g(n, e) = \frac{n^4}{32} \left[J_{n-2}(ne) - 2eJ_{n-1}(ne) + \frac{2}{n} J_n(ne) + 2eJ_{n+1}(ne) - J_{n+2}(ne) \right]^2 + \frac{n^4}{32} (1-e^2) [J_{n-2}(ne) - 2J_n(ne) + J_{n+2}(ne)]^2 + \frac{n^2}{24} [J_n(ne)]^2. \quad (2)$$

Then the orbit-averaged total gravitational wave power in all harmonics is

$$P = \frac{32}{5} \frac{G^4}{c^5} \frac{M^5 \eta^2}{a^5} \sum_{n=1}^{\infty} g(n, e) = \frac{32}{5} \frac{G^4}{c^5} \frac{M^5 \eta^2}{a^5} \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}}. \quad (3)$$

Since each graviton has energy $\hbar\omega = n\hbar\sqrt{GM/a^3}$, the orbit-averaged number rate of graviton emission is

$$\frac{dN}{dt} = \frac{32}{5} \frac{G^{7/2}}{\hbar c^5} \frac{M^{9/2} \eta^2}{a^{7/2}} \sum_{n=1}^{\infty} \frac{g(n, e)}{n}. \quad (4)$$

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However, in this case there does not seem to be an explicit closed-form elementary function of the eccentricity e for the infinite sum of $g(n, e)/n$ over the harmonic index n .

By comparison, the angular momentum J per time radiated into gravitational waves (decreasing the orbital angular momentum L of the masses by the same amount), averaged over one orbit, is given by Peters [3,4] as

$$\frac{dJ}{dt} = \frac{32}{5} \frac{G^{7/2}}{c^5} \frac{M^{9/2} \eta^2}{a^{7/2}} \frac{1 + \frac{7}{8} e^2}{(1 - e^2)^2}. \quad (5)$$

Therefore, the ratio of the sum of magnitudes of the spin angular momenta $2\hbar$ to the magnitude of the total angular momentum vector emitted per orbit is

$$F(e) \equiv 2\hbar \frac{dN}{dJ} = \frac{2(1 - e^2)^2}{1 + \frac{7}{8} e^2} \sum_{n=1}^{\infty} \frac{g(n, e)}{n}. \quad (6)$$

One can easily see that for circular orbits, with eccentricity $e = 0$, in Eq. (2) only $J_0(0)$ is nonzero, and its value of 1 makes $g(n, 0)$ nonzero only for $n = 2$, giving $g(2, 0) = 1$ and hence $2\hbar dN/dJ = 1$ for $e = 0$. This is the case in which each graviton has angular momentum $2\hbar$ parallel to the orbital angular momentum L .

My attempt to plug Eq. (6) into *Mathematica* to evaluate $F(e)$, the spin angular momentum $2\hbar$ of a graviton multiplied by the ratio of the number of gravitons emitted to the total angular momentum emitted, did not give good convergence for $e > 0.6$, because of the large number of Bessel function terms contributing significantly. However, the referee pointed out that *Mathematica* can rapidly compute many terms in the power series in the eccentricity e for $F(e)$, from which one can calculate a good approximation for $F(e)$ all the way up to $e = 1$. On my laptop, it took 14 hours (the limit of my patience) for *Mathematica* to calculate exactly the first 501 nonzero terms, up through e^{1000} (all terms with even powers of the eccentricity e and with rational coefficients given explicitly as the ratio of integers with up to 1404 digits each, perhaps around 700,000 digits in total, which of course I shall not copy here).

Before proceeding further with the power series, it is useful to calculate the exact result for $F(1)$, in order to see how much is missing from my finite truncation of the power series.

III. HIGH ECCENTRICITY GRAVITON NUMBER EMISSION

Although I do not know how to get an exact explicit closed-form elementary function for $2\hbar dN/dJ$ as a function of the eccentricity e for $0 < e < 1$, one can find it exactly in the limit of unit eccentricity. For simplicity, define

$$f \equiv 1 - e^2. \quad (7)$$

Then for $f \ll 1$, the sum over n of $g(n, e)/n$ is dominated by terms with very large n .

One can then use Eq. (8.455.1) on page 913 of Gradshteyn and Ryzhik [6], that for real x and integer $n \gg 1$,

$$J_n(x) \approx \frac{1}{\pi} \sqrt{\frac{2(n-x)}{3x}} K_{\frac{1}{3}} \left\{ \frac{[2(n-x)]^{\frac{3}{2}}}{3\sqrt{x}} \right\}. \quad (8)$$

With $x = ne = n\sqrt{1-f} \approx n(1-f/2)$ for $f = 1 - e^2 \ll 1$, this becomes

$$J_n(ne) \approx \frac{1}{\pi} \sqrt{\frac{f}{3}} K_{\frac{1}{3}} \left(\frac{n}{3} f^{\frac{3}{2}} \right). \quad (9)$$

By using the recursion relation for the derivative of the modified Bessel functions $K_\nu(z)$, namely $K'_\nu(z) = -K_{\nu-1}(z) - (\nu/z)K_\nu(z)$, where the prime means the derivative with respect to its argument $z = (n/3)f^{3/2}$, and the fact that $K_\nu(z) = K_{-\nu}(z)$, one can get the derivative of the ordinary Bessel function with respect to its argument $x = ne$, $J'_n(x) = dJ_n(x)/dx$, as

$$J'_n(ne) \approx \frac{1}{\pi} \frac{f}{\sqrt{3}} K_{\frac{2}{3}} \left(\frac{n}{3} f^{\frac{3}{2}} \right). \quad (10)$$

Next, we can follow the use by Peters and Mathews [3] of the recurrence relations and Bessel's equation to rewrite $g(n, e)$ as their Eq. (A1), with the correction by Nikishov [7] that the factor $(4/e^2)^2$ in the 3rd line should instead be $(4/e)^2$, to get

$$\begin{aligned} 2e^4 \frac{g(n, e)}{n} &= (1 - e^2)^3 n^3 J_n^2(ne) + \left(1 - e^2 + \frac{1}{3} e^4\right) n J_n^2(ne) \\ &\quad - e(4 - 3e^2)(1 - e^2) n^2 J_n(ne) J'_n(ne) \\ &\quad + e^2(1 - e^2)^2 n^3 J_n'^2(ne) + e^2(1 - e^2) n J_n'^2(ne). \end{aligned} \quad (11)$$

Therefore, to evaluate the sum over n of $g(n, e)/n$, we need to evaluate the sums of the five series that are each a power of n multiplied by a bilinear in the corresponding Bessel function or its derivative. When the powers of n are one greater, Peters and Mathews [3] were able to evaluate the corresponding five sums explicitly as $(1 - e^2)^{-13/2}$ multiplied by finite polynomials of e in their Eqs. (A3). However, their procedure does not work when the exponents of n are those of Eq. (11), each being one less than the corresponding exponent for each sum evaluated by Peters and Mathews.

Nevertheless, for the leading behavior for the sum of $g(n, e)/n$ for small $f = 1 - e^2$, we can use the

approximations of Eqs. (9) and (10). Since for $f \ll 1$ the sums are dominated by terms with large values of n and do not change rapidly with n , we can replace the sums by integrals over n to get the leading terms for $f \ll 1$. The integrals are given by Eq. (6.576.4) on page 676 of Gradshteyn and Ryzhik [6], which for equal arguments for the two modified Bessel functions and for integer exponent p gives

$$\begin{aligned} & \int_0^\infty x^p K_\mu(ax) K_\nu(ax) dx \\ &= \frac{2^{p-2}}{p! a^{p+1}} \Gamma\left(\frac{p+1+\mu+\nu}{2}\right) \Gamma\left(\frac{p+1-\mu-\nu}{2}\right) \\ & \times \Gamma\left(\frac{p+1+\mu-\nu}{2}\right) \Gamma\left(\frac{p+1-\mu+\nu}{2}\right). \end{aligned} \quad (12)$$

Letting $x = n$ and $a = (1/3)f^{3/2} = (1/3)(1-e^2)^{3/2} \ll 1$, employing the triplication formula for the gamma function, and using the approximations of Eqs. (9) and (10) allows us to calculate that for $1 - e^2 \ll 1$ and for positive integer exponents $p = 2k + 1 > 0$ or $p = 2k > 0$, the leading terms in powers of $1 - e^2$ are

$$\begin{aligned} \sum_{n=1}^\infty n^{2k+1} J_n(ne) J_n(ne) &\approx \frac{2^{2k} k! (3k+1)!}{3^{k+1/2} \pi (2k+1)!} (1-e^2)^{-3k-2}, \\ \sum_{n=1}^\infty n^{2k} J_n(ne) J_n(ne) &\approx \frac{(6k-1)!!}{2^{2k+1} 3^k (2k)!!} (1-e^2)^{-3k-1/2}, \\ \sum_{n=1}^\infty n^{2k+1} J_n(ne) J'_n(ne) &\approx \frac{(6k+1)!!}{2^{2k+2} 3^k (2k)!!} (1-e^2)^{-3k-3/2}, \\ \sum_{n=1}^\infty n^{2k} J_n(ne) J'_n(ne) &\approx \frac{2^{2k-1} k! (3k-1)!}{3^{k-1/2} \pi (2k)!} (1-e^2)^{-3k}, \\ \sum_{n=1}^\infty n^{2k+1} J'_n(ne) J'_n(ne) &\approx \frac{2^{2k} (3k+2) k! (3k)!}{3^{k+1/2} \pi (2k+1)!} (1-e^2)^{-3k-1}, \\ \sum_{n=1}^\infty n^{2k} J'_n(ne) J'_n(ne) &\approx \frac{(6k+1)(6k-3)!!}{2^{2k+1} 3^k (2k)!!} (1-e^2)^{-3k+1/2}. \end{aligned} \quad (13)$$

One may note that

$$\begin{aligned} & (3p-1) \sum_{n=1}^\infty n^p J'_n(ne) J'_n(ne) \\ & \approx (3p+1)(1-e^2) \sum_{n=1}^\infty n^p J_n(ne) J_n(ne). \end{aligned} \quad (14)$$

Then the leading terms for the sums with n , n^2 , n^3 , and n^4 become

$$\begin{aligned} \sum_{n=1}^\infty n J_n(ne) J_n(ne) &\approx \frac{1}{\sqrt{3}\pi} (1-e^2)^{-2}, \\ \sum_{n=1}^\infty n J_n(ne) J'_n(ne) &\approx \frac{1}{4} (1-e^2)^{-3/2}, \\ \sum_{n=1}^\infty n J'_n(ne) J'_n(ne) &\approx \frac{2}{\sqrt{3}\pi} (1-e^2)^{-1}, \\ \sum_{n=1}^\infty n^2 J_n(ne) J_n(ne) &\approx \frac{5}{16} (1-e^2)^{-7/2}, \\ \sum_{n=1}^\infty n^2 J_n(ne) J'_n(ne) &\approx \frac{2}{\sqrt{3}\pi} (1-e^2)^{-3}, \\ \sum_{n=1}^\infty n^2 J'_n(ne) J'_n(ne) &\approx \frac{7}{16} (1-e^2)^{-5/2}, \\ \sum_{n=1}^\infty n^3 J_n(ne) J_n(ne) &\approx \frac{16}{3\sqrt{3}\pi} (1-e^2)^{-5}, \\ \sum_{n=1}^\infty n^3 J_n(ne) J'_n(ne) &\approx \frac{35}{32} (1-e^2)^{-9/2}, \\ \sum_{n=1}^\infty n^3 J'_n(ne) J'_n(ne) &\approx \frac{20}{3\sqrt{3}\pi} (1-e^2)^{-4}, \\ \sum_{n=1}^\infty n^4 J_n(ne) J_n(ne) &\approx \frac{1155}{256} (1-e^2)^{-13/2}, \\ \sum_{n=1}^\infty n^4 J_n(ne) J'_n(ne) &\approx \frac{80}{3\sqrt{3}\pi} (1-e^2)^{-6}, \\ \sum_{n=1}^\infty n^4 J'_n(ne) J'_n(ne) &\approx \frac{1365}{256} (1-e^2)^{-11/2}. \end{aligned} \quad (15)$$

As a check, the 4th, 6th, 8th, 10th, and 12th formulas above agree with the leading terms of what Peters and Mathews [3] calculated in their Eqs. (A3). Then the 1st, 3rd, 5th, 7th, and 9th formulas above can be inserted into Eq. (11) to give, again for $f \equiv 1 - e^2 \ll 1$,

$$\sum_{n=1}^\infty \frac{g(n, e)}{n} \approx \frac{31}{6\sqrt{3}\pi} (1-e^2)^{-2} \quad (16)$$

This can then be inserted into Eq. (6) to give, in the limit of unit eccentricity ($e = 1$),

$$F(1) = 2\hbar \frac{dN}{dJ}(e=1) = \frac{248}{45\sqrt{3}\pi} \approx 1.012811600479. \quad (17)$$

It is rather remarkable that even for eccentricity $e = 1$ the number of gravitons emitted is so close to the angular momentum emitted divided by the spin angular momentum $2\hbar$ of each graviton.

IV. INDEPENDENT DERIVATION FOR A PARABOLIC ORBIT

As a check of the derivation above, I used the independently derived energy spectrum for a parabolic orbit (eccentricity $e = 1$) that has been given by Turner [8], by Berry and Gair [9], and by Gröbner, Jetzer, Haney, Tiwari, and Ishibashi [10], all of whom got results that I could show are equivalent to the simplified form below.

Their results can be written in terms of the total mass $M = M_1 + M_2$, the dimensionless mass ratio $\eta = M_1 M_2 / (M_1 + M_2)^2$, the frequency f_r of the gravitational radiation, the periastron distance r_p (which would be $r_p = a(1 - e)$ for an elliptical orbit with semimajor axis $a = r_p / (1 - e)$, except that here I am taking the limit $e \rightarrow 1$ with fixed r_p , sending the semimajor axis to infinity), and the frequency f_c of a circular orbit of the same radius as the periastron distance r_p ,

$$f_c = \frac{1}{2\pi} \sqrt{\frac{GM}{r_p^3}}, \quad (18)$$

Both the frequency f_r of the radiation and the frequency f_c of a circular orbit of the same radius as the periastron distance r_p are in units of cycles per second, so f_c and f_r are not to be confused with the dimensionless $f \equiv 1 - e^2$. Using also a conveniently normalized dimensionless frequency I shall call x (which is the argument of each modified Bessel function of the second kind below, $K_{1/3}$ and $K_{2/3}$),

$$x = \sqrt{\frac{8}{9}} \frac{f_r}{f_c}, \quad (19)$$

the gravitational wave energy dE emitted per orbit and per frequency interval df_r is

$$\frac{dE}{df_r} = \frac{16}{5} \frac{G^3}{c^5} \frac{M^4 \eta^2}{r_p^2} [(27x^4 + x^2)K_{1/3}(x)^2 - 9x^3 K_{1/3}(x)K_{2/3}(x) + 27x^4 K_{2/3}(x)^2]. \quad (20)$$

This then gives the number of gravitons emitted during one parabolic orbit of periastron distance r_p and orbital angular momentum $L = \sqrt{2GM^3 \eta^2 r_p}$ as

$$\begin{aligned} \Delta N &= \int \frac{dE}{hf_r} = \frac{1}{h} \int_0^\infty \frac{dE}{df_r} \frac{dx}{x} = \frac{248}{15\sqrt{3}} \frac{G^3}{\hbar c^5} \frac{M^4 \eta^2}{r_p^2} \\ &= \frac{992}{15\sqrt{3}} \frac{G^5 M^{10} \eta^6}{\hbar c^5 L^4}. \end{aligned} \quad (21)$$

One can compare this with the calculations of Peters [4,5] for the angular momentum radiated during one parabolic orbit

$$\Delta J = 6\pi \frac{G^3 M^4 \eta^2}{c^5 r_p^2} = 24\pi \frac{G^5 M^{10} \eta^6}{c^5 L^4}. \quad (22)$$

This then gives $F(e)$ for eccentricity $e = 1$ as

$$F(1) = 2\hbar \frac{\Delta N}{\Delta J}(e=1) = \frac{248}{45\sqrt{3}\pi} \approx 1.012811600479, \quad (23)$$

just as we found before from the sum of $g(n, e)/n$ in the limit that $1 - e^2$ was taken to zero.

V. GRAVITON NUMBERS EMITTED DURING INSPIRAL

Next, we would like to use the exact answer for $F(1) = 248/(45\sqrt{3}\pi)$ at unit eccentricity along with the truncated series expansion in powers of e^2 for $e < 1$ to calculate to high precision the total number of gravitons emitted during the inspiral from arbitrary initial orbital angular momentum L_i and initial eccentricity e_i . Since the fractional change in the orbital angular momentum and in the eccentricity during each orbit is small (unlike the case for the semimajor axis a , nonrelativistic orbit energy $E = -GM^2\eta/(2a)$, and orbital period $\tau = 2\pi\sqrt{a^3/(GM)}$, which can have large fractional changes during orbits of large eccentricity [1,2]), we can use Eq. (5.11) of [5] to get the ratio of the orbital angular momentum $L(e)$, when the eccentricity is e , to L_1 , what it would have been at eccentricity $e = 1$ if the inspiral had started there,

$$\begin{aligned} \lambda(e) &\equiv \frac{L(e)}{L_1} = e^{6/19} \left(\frac{304 + 121e^2}{425} \right)^{435/2299} \\ &= (1 - f)^{3/19} \left(1 - \frac{121}{425} f \right)^{435/2299}, \end{aligned} \quad (24)$$

where now we return to $f \equiv 1 - e^2$. Then if an inspiral starts at initial orbital angular momentum L_i and eccentricity e_i , the orbital angular momentum during the inspiral as the eccentricity decreases below e_i is

$$L(e) = L_i \frac{\lambda(e)}{\lambda(e_i)}. \quad (25)$$

In order to calculate the number of gravitons

$$N(L_i, e_i) = [L_i/(2\hbar)]\nu(e_i) \quad (26)$$

emitted during the inspiral from initial orbital angular momentum L_i and eccentricity e_i to final eccentricity very close to zero (assuming the initial orbit is highly non-relativistic) and final angular momentum much smaller than the initial value, we need a good approximation for

$$F(e) = 2\hbar \frac{dN}{dJ}(e) \quad (27)$$

that we can use in a numerical integration to get the total number of gravitons emitted. Then one can calculate the number of gravitons emitted in going from initial orbital angular momentum L_i at initial eccentricity e_i to final eccentricity indistinguishable from 0 as

$$N(L_i, e_i) \equiv \frac{L_i}{2\hbar} \nu(e_i) = \frac{L_i}{2\hbar \lambda(e_i)} \int_0^{e_i} F(e) \frac{d\lambda}{de} de, \quad (28)$$

or

$$\nu(e_i) \equiv \frac{2\hbar N(L_i, e_i)}{L_i} = \int_0^{e_i} \frac{F(e) d\lambda}{\lambda(e_i) de} de = \int_0^{\lambda(e_i)} \frac{F(e(\lambda))}{\lambda(e_i)} d\lambda, \quad (29)$$

which is the sum of the spin angular momentum magnitudes of the gravitons emitted ($2\hbar$ times the number N of gravitons emitted) divided by the magnitude $J = L_i$ of the total angular momentum vector emitted in the gravitational radiation during nonrelativistic inspiral from an initial eccentricity e_i down to a final eccentricity indistinguishable from zero when the orbit becomes relativistic and then rapidly leads to merger, with its final angular momentum being assumed to be much less than the initial orbital angular momentum L_i .

Now we can return to the series expansion of $F(e)$ that *Mathematica* calculates for $F(e)$ given in Eq. (6) with $g(n, e)$ written in terms of Bessel functions by Eq. (2) from Peters and Mathews [3–5].

The first seven terms in the power series

$$F(e) = \sum_{n=0}^{\infty} F_{2n} e^{2n}, \quad (30)$$

give

$$F(e) = 1 + \frac{e^2}{48} - \frac{3e^4}{128} + \frac{59e^6}{2304} - \frac{4603e^8}{221184} + \frac{558239e^{10}}{29491200} - \frac{34385443e^{12}}{2123366400} + O(e^{14}), \quad (31)$$

which for $e = 1$ sum to be almost 0.78% smaller than the exact result of

$$F(1) = \frac{248}{45\sqrt{3}\pi} \approx 1.012811600479 \quad (32)$$

given above. However, the average of the sum of the first 6 and first 7 terms for $e = 1$ is too large by only about 0.021% and so is the same as the exact answer when both are rounded to 4 decimal digits.

One curious fact about the power series of Eq. (30) for $F(e)$ is that the coefficients F_{2n} for positive odd and even values of n (the powers of e^2 in the series) oscillate in sign up through the coefficient of e^{296} (i.e., except for the positive value of $F_0 = 1$, for natural numbers $m \leq 73$, all $F_{2(2m+1)} > 0$ for $2n = 2(2m+1) \leq 294$, whereas all $F_{2(2m)} < 0$ for $n = 2m > 0$, but thereafter (e.g., for all $2n > 294$) all coefficients (all F_{2n} for all odd or even values of $n > 147$) appear to stay negative, and the ratio of successive coefficients slowly approach 1 with negligible differences between the slow changes in the even-odd ratios and the odd-even ratios (using even and odd for the n that is the exponent of e^2 and that is half the subscript $2n$ of the coefficient F_{2n} of e^{2n}). For example, $F_{996}/F_{994} \approx 0.994723968$, $F_{998}/F_{996} \approx 0.994732533$, and $F_{1000}/F_{998} \approx 0.994741074$.

Let the sum of the first 501 terms I had *Mathematica* calculate for $F(e)$ be

$$F_{1000}(e) = \sum_{n=0}^{500} F_{2n} e^{2n}. \quad (33)$$

This gave (here rounded to 13 digits)

$$F_{1000}(1) = 1.012811605063 \\ = F(1) + 0.00000004585. \quad (34)$$

Mainly from the coefficients of the last 4 terms of $F_{1000}(e)$, namely $F_{994} = -1.662918024 \times 10^{-11}$, $F_{996} = -1.654144416 \times 10^{-11}$, $F_{998} = -1.645431265 \times 10^{-11}$, and $F_{1000} = -1.636778064 \times 10^{-11}$ (here rounded to 10 digits; *Mathematica* gives the exact coefficients as the ratios of integers with up to 1404 digits each for $2n$ up through 1000), but also checking that it gave a reasonably good approximation for F_{400} , I found what appeared to be a reasonably good approximate fitting function for F_{2n} for $n > 500$:

$$F_{2n} \sim -0.0000915/n^{5/2}. \quad (35)$$

This gave

$$\sum_{n=501}^{\infty} F_{2n} \sim -0.00000005448 \quad (36)$$

and hence

$$\sum_{n=0}^{\infty} F_{2n} \sim 1.012811599615 = F(1) - 0.00000000863, \quad (37)$$

overshooting the correction needed for $F_{1000}(1)$ by about 16% but nevertheless giving the correct first 10 digits of $F(1)$ when rounded.

I then used the values *Mathematica* calculated for the sum of the first 501 terms of $F_{1000}(e)$ of Eq. (33) for values of e in steps of 0.05 from 0 through 0.95 (where I calculated that the contribution of higher order terms would be less than 10^{-32}) and then used the exact $F(1)$ for $e = 1$. The results are given in Table I.

$$R_{54}(e) \approx (1 - 0.7752642998739374e^2 - 0.6784219118736136e^4 + 0.6489052965752502e^6 - 0.11239271572245049e^8 + 0.0026877471640758543e^{10}) / (1 - 0.796097644179696e^2 - 0.6383986668380869e^4 + 0.6179346051204716e^6 - 0.09900589203608591e^8). \quad (38)$$

This approximation for $F(e)$ was then inserted into the integral in Eq. (29), for 21 values of the initial eccentricity e_i (from $e_i = 0$ through $e_i = 1$, in steps of 0.05) for $\nu(e_i) \equiv 2\hbar N(L_i, e_i)/L_i$, the ratio of the number of gravitons emitted (times the spin angular momentum per graviton) to the total angular momentum emitted when starting with eccentricity e_i and ending with eccentricity 0. These results are also given in Table I. In particular, they give for initial eccentricity $e_i = 1$ (i.e., a parabolic orbit before capture, in the limit of infinitesimally small v/c for the initial relative approach velocity),

$$\nu(1) \approx 1.002\,268\,666\,992, \quad (39)$$

which, except for the last three digits, agrees with the value $1.002\,268\,666\,2 \pm 10^{-10}$ given by Youngjoo Chung and Hyun Seok Yang [11] in their paper written after the first version of my paper was posted on the arXiv but before I followed the helpful suggestions of the *Physical Review D* referee to use *Mathematica* to calculate the series expansion of $F(e)$ rather than directly to sum the series of products of Bessel function that I could not get to work well for $e > 0.6$.

One possible explanation for my slightly higher value for $\nu(1)$ is that my rational function approximation $R_{54}(e)$, while being forced to give $R_{54}(1) = F(1)$, has its second derivative with respect to e at $e = 1$ being 0.234 398, whereas Chung and Yang evaluated the second derivative as approximately 0.316 259, so the actual $F(e)$ probably dips slightly below my $R_{54}(e)$ for $0.95 < e < 1$. My *guess* is that near $e = 1$, a series in increasing powers of $f = 1 - e^2$ have the first fractional power with exponent $5/2$ (since the $1/2$ and $3/2$ powers would give divergent second derivatives at $e = 1$, whereas the series expansion of $(1 - e^2)^{5/2}$ seems to fit rather well to the rate at which the F_{2n} coefficients are slowly decreasing), so that the series might be

Motivated by Padé approximants (though this is not a Padé approximant about any single value of e), I tried fitting the 21 values of $F(e)$ to $R_{mn}(e^2)$, the ratio of a polynomial of degree m in e^2 in the numerator and a polynomial of degree n in the denominator, while forcing the resulting rational function to give $R_{mn}(0) = F(0) = 1$ and $R_{mn}(1) = F(1) = 248/(45\sqrt{3}\pi)$. I found that the following rational function fit all of the 21 values of $F(e)$ with a maximum error less than 2×10^{-10} :

TABLE I. Here the first column gives the eccentricity e for the quantity in the second column or the initial eccentricity e_i for the quantity in the third column. The second column gives the corresponding value of $F(e) \equiv 2\hbar dN/dJ$, the spin $2\hbar$ of a graviton multiplied by the number rate of graviton emission and divided by the rate of gravitational wave angular momentum. Values greater than 1 indicate that some gravitons do not carry away their spin angular momentum parallel to the angular momentum of the inspiralling black holes, though the average angular momentum carried away is parallel to the angular momentum of the source. The third column gives $\nu(e_i) \equiv 2\hbar N(L_i, e_i)/L_i$, $2\hbar$ times the total number $N(L_i, e_i)$ of gravitons emitted during the entire inspiral (assumed predominantly non-relativistic) from initial eccentricity e_i and divided by the total initial angular momentum L_i that gets almost entirely radiated in gravitational waves.

Eccentricity e or e_i	$F(e) \equiv 2\hbar dN/dJ$	$\nu(e_i) \equiv 2\hbar N(L_i, e_i)/L_i$
0	1	1
0.05	1.000 051 937 248 266 6	1.000 007 095 491 596
0.10	1.000 206 014 984 741 8	1.000 028 300 979 309 4
0.15	1.000 457 171 226 169	1.000 063 376 721 958 2
0.20	1.000 797 420 821 035 7	1.000 111 933 789 773
0.25	1.001 216 482 055 099 3	1.000 173 449 703 012 4
0.30	1.001 702 562 632 638 1	1.000 247 289 726 417
0.35	1.002 243 234 498 758 6	1.000 332 733 310 394 4
0.40	1.002 826 330 769 453	1.000 429 005 148 430 4
0.45	1.003 440 812 046 700 2	1.000 535 310 385 708
0.50	1.004 077 570 623 173 5	1.000 650 873 684 189 7
0.55	1.004 730 166 532 577 4	1.000 774 982 150 000 7
0.60	1.005 395 517 387 786 7	1.000 907 032 603 969
0.65	1.006 074 595 391 762 5	1.001 046 584 409 174 3
0.70	1.006 773 225 189 354 2	1.001 193 420 225 659 2
0.75	1.007 503 139 278 446 6	1.001 347 618 982 500 8
0.80	1.008 283 567 365 280 4	1.001 509 648 783 810 2
0.85	1.009 143 902 430 908 8	1.001 680 494 170 096
0.90	1.010 128 683 140 872	1.001 861 846 995 886
0.95	1.011 308 420 283 312 5	1.002 056 429 748 589
1.00	1.012 811 600 478 551 32	1.002 268 666 991 671 8

$$F(e) = F(1) - A(1 - e^2) + B(1 - e^2)^2 + C(1 - e^2)^{5/2} + D(1 - e^2)^3 + E(1 - e^2)^{7/2} + O[(1 - e^2)^4], \quad (40)$$

with apparently positive constants A and $B \approx (0.316\,259 - 2A)/8$ that could be evaluated from the unpublished details (e.g., the value of the first derivative of $F(e)$ at $e = 1$) of the calculations of Chung and Yang, but unknown values and signs for the constants C , D , E , and higher-order terms. It would be interesting to know whether there are any exact

closed-form elementary function expressions for all of these constants, analogous to $F(1) = 248/(45\sqrt{3}\pi)$.

I then went on to calculate a Padé-like rational approximation for $\nu(e_i)$ and found that what I shall call $S_{44}(e)$, with quartic polynomials in e^2 in both the numerator and denominator, and forced to match precisely $\nu(0) = 1$ at $e = 0$ but not to match precisely $\nu(1)$ at $e = 1$ that is only known numerically, gives values that match all 21 of my numerical integration values of $\nu(e)$ with less error than 10^{-10} . This rational function approximation for $\nu(e_i)$ is

$$S_{44}(e) = (1 - 0.6519269962486678e^2 - 0.18662235012002779e^4 + 0.06891379641025736e^6 + 0.014312419008719319e^8)/(1 - 0.6547679041192125e^2 - 0.18367526879529064e^4 + 0.06814782751928151e^6 + 0.014418380573790177e^8). \quad (41)$$

VI. CONCLUSIONS

For two black holes (or other effectively point masses) inspiraling from an initially highly nonrelativistic Keplerian orbit (e.g., initial periastris distance of closest approach much greater than the Schwarzschild radius of the total mass) with initial orbital angular momentum L_i and eccentricity e_i , approximate but highly precise formulas are given for the spin angular momentum of the graviton ($2\hbar$) multiplied by the number N of gravitons emitted and divided by the magnitude J of the angular momentum of the gravitational radiation, both at the differential level ($F(e) = 2\hbar dN/dJ$) for an infinitesimal change in the eccentricity, and at the total level ($\nu(e_i) = 2\hbar N/L_i$) for the total graviton number N emitted during inspiral to negligible final eccentricity from initial orbital angular momentum L_i and eccentricity e_i . It is calculated that at unit eccentricity, $F(1) = 248/(45\sqrt{3}\pi) \approx 1.012811600479$ (so that the number of gravitons emitted near this eccentricity is a bit less than 1.3% more than the angular momentum emitted divided by the spin angular momentum of the graviton, $2\hbar$). The total number of gravitons emitted during the total inspiral is always less than 1.002268667 times the angular momentum emitted divided by $2\hbar$ and only reaches near this value when the initial eccentricity is very near 1.

Note added. After the first version of this paper was posted on the arXiv, Youngjoo Chung and Hyun Seok Yang [11] extended the results in a tour de force to find alternative exact infinite series for $F(e)$ and $\nu(e_i)$ which did not have Bessel functions in them but instead have (rather complicated) finite sums of rational numbers multiplying the even powers of the eccentricity e in a convergent series

expansion for $F(e)$, and an infinite sum of fractional powers of e multiplying hypergeometric functions of $-121e_i^2/304$ for $\nu(e_i)$. Unlike my initial failure to be able numerically to sum the Bessel function series for $F(e)$ for $e > 0.6$, they were able to sum their series for both $F(e)$ and $\nu(e_i)$ to high precision for any e between 0 and 1 inclusive and find $\nu(1) \approx 1.002\,268\,666\,2$. Now that I have followed the helpful advice of the referee, I have been able, using *Mathematica*, to calculate the series expansion of $F(e)$ up through the one-thousandth power of the eccentricity e and independently get many very similar results to what Chung and Yang calculated, as well as now highly precise 3-line rational approximations for $F(e)$ and $\nu(e_i)$ and tables of 21 of each of their values as the eccentricity is increased from 0 to 1 in steps of 0.05.

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DATA AVAILABILITY

No observational or experimental data were created or analyzed in this study.

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