

Higher derivative scalar-vector-tensor theories from Kaluza-Klein reductions of Horndeski theory

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It was recently pointed out that some precise photon-Galileon couplings in four dimensions (4D)—inspired by a higher dimensional reduction—are enough to obtain a Horndeski theory that is less constrained by the stringent experimental bounds on the speed of gravitational waves. They imply the constancy of the ratio of speed of gravity to light throughout cosmic evolution. This holds even if we include the general scalar potentials $G_4(\pi, X)$ and $G_5(\pi)$. In this paper we go into the details of this 4D luminal extension of Horndeski theory including its scalar sector. We also present the complete action including the general $G_5(\pi, X)$, $G_6(\pi, X)$ scalar potentials. Thus we show all the $U(1)$ gauge invariant vector Galileons in 4D that result from a Kaluza-Klein dimensional reduction from 5D Horndeski. They provide a *consistent* coupling of a higher derivative vector to scalar modifications of gravity—namely, without inducing Ostrogradsky ghosts and keeping gauge invariance—in the aim to explore more universal couplings of dark energy to other matter, such as vectors and in particular the photon.

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I. INTRODUCTION

The strict experimental bounds on the difference between the speed of gravitational and electromagnetic waves, deduced from the event GW170817 [1], heavily constrained the application of Horndeski theory/generalized Galileons—the most general modification of GR with a higher derivative scalar in the action, but with second order equations of motion¹ [2–4]—for late time cosmology [1,5–10]. The standard approach to deal with these experimental constraints has been simply pragmatical: take advantage of the generality of Horndeski theory by choosing the four scalar potentials in the action to satisfy the sub/ luminality of

the graviton (and other physically compelling conditions in cosmological solutions). Nevertheless, it was the main point in a recent work [11] that those constraints on Horndeski theory assumed that the well-known photon of Maxwell electrodynamics *remains without modifications* in the IR scale of gravity, and that lifting this assumption offers a new range of possibilities. Indeed, it was shown in [11] that there can be photon-Galileon couplings that can account for the *exact* constancy of the ratio of speed of gravity to light throughout cosmic evolution, even with the general potentials $G_4(\pi, X)$ and $G_5(\pi)$.² Furthermore, it was also recently shown for the latter class that the same luminal property holds on more general spherically symmetric and dynamical backgrounds [14,15]. Therefore, the latter could be relevant to promising proposals in Galileons, for instance, the *fab four* to address the cosmological constant problem [16] and which were strictly constrained assuming the minimally coupled photon [7]. Now, the question of how to recover the usual Maxwell photon in laboratory tests from that proposed IR modification was also briefly discussed in [11], and a recent analysis in beyond Horndeski theory has also touched on the

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¹In this work we are only concerned with the classical solutions and thus, it is relevant that the equations of motion are of second order. In particular we do not discuss quantum corrections and the higher derivative operators that could become relevant in some specific physical situation. Nevertheless, it would be interesting to assess the quantum robustness of the aspects discussed in this work.

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²A part of the result in [11]—whose scalar sector we show below—relies on the assumption that the scalar potential G_5 depends only on π and not on X . However, one must be aware that in some physical situations where quantum corrections are relevant, even if one assumes a X independent $G_5(\pi)$, quantum corrections will generate the X dependence in the absence of a symmetry that avoids it. See for instance [12,13].

topic of experimental constraints on the new type of photon-Galileon couplings proposed in [11] and the analogous in the context of beyond Horndeski [17].

Thus, the exploration of for instance photon-(Dark Energy) couplings and more broadly Matter-(Dark Energy) couplings—if we use Galileon models to try to account for Dark Energy—can be an important new way to pick up experimentally favored directions in modified gravity for late time cosmology using laboratory tests. Indeed, the universal coupling of gravity to all forms of energy may be a motivation for the exploration of more ubiquitous couplings of IR modifications of gravity to other energy sources, specially given the possibility to account for the speed tests of GW and light with the latter [11]. In this work we expand on the results shown in the recent letter [11] without restricting to the luminal cases. We use the same analysis to show the dimensional reduction of Horndeski theory in five dimensions (5D) to obtain a broad class of $U(1)$ gauge invariant vector/ multiscalar Galileons in 4D: namely, a theory with higher derivatives of a $U(1)$ gauge invariant vector and two scalars in the action, but with second order equations of motion for all of the fields. Such types of theories had been previously obtained from different considerations [18] and also by Kaluza-Klein (KK) reductions from, for instance, Gauss-Bonnet in 5D [19,20]. Also, more recently, Horndeski scalar-vector couplings were obtained from KK reductions of Lovelock theory [21], and biscalar theories in 2D were obtained from 4D Horndeski [22]. However, given that we start from the most general scalar modification of Lovelock theory in 5D with higher derivatives in the action, but with second order equations of motion—Horndeski/ Galileons in 5D—we obtain a much *larger class* of vector-scalar couplings in 4D, while obviously preserving the second order equations of motion. Indeed, since the 5D metric solves second order equations, its components (4D metric, vector and dilaton) solve also second order equations as well as the Galileon, by construction. The task of building such gauge invariant vector Galileons and Einstein-Maxwell Lagrangians in a systematic way is daunting from other perspectives [23–27], and hence the interest of using the dimensional reduction also as a tool to expand the theory space of vector-scalar couplings while keeping gauge invariance and avoiding the Ostrogradsky ghost. We, nevertheless, do not aim for completeness, nor for a complete classification, but rather put emphasis on these theories with the interest that first, a subclass of it already leads to a luminal graviton in 4D (for very general scalar potentials and nonminimal derivative couplings to gravity) [11]; and second, because being connected to higher dimensional theories³ they could be simpler to analyze in some

³An additional interest on this regard comes from the very origin of Galileons, which were inspired by the effective theory of the DGP model [3] and further relations that have been established with higher dimensional theories [28–30], and to string theory [31].

cases, for instance, in the search for black-hole solutions in 4D [21,32] (See for instance the connection of covariant Galileons in general D dimensions [33]—a special case of our starting point, 5D generalized Galileons—to Lovelock theory in yet higher $D + N$ dimensions by KK diagonal compactifications [28]). Furthermore, vectors have been long studied for cosmological applications [34–36], and so the consistent coupling to scalar modifications of gravity—without inducing ghosts, and keeping gauge invariance—could shed light on promising theories, however, their application is beyond the scope of this work.

A derivation of these $U(1)$ gauge invariant vector/ multiscalar Galileons in 4D, which, as is also common, we denote as SVT (scalar-vector-tensor theories), is given in Sec. II.

On the other hand, given the motivation of these SVT theories for IR modifications of gravity by a *scalar field*, but also *more ubiquitously coupled to other matter* such as the photon, we start the discussion on the stability of nonsingular cosmological solutions. Indeed, modified theories of gravity such as Horndeski theory can violate the null-energy-condition (NEC) without obvious pathologies. Thus, this compelling feature has been widely used to build interesting cosmological solutions not possible in GR, such as non-singular universes, among others. However, it has been also shown in the form of a no-go theorem [37,38] that they suffer, in general, of global stability issues [39–50]. More specifically, even if far from the most physically relevant phase of cosmic evolution, instabilities will arise earlier or later in any nonsingular, nonghosty cosmological solution.⁴ This no-go was even generalized to multi-Galileons [45] and to a subclass of Horndeski theory on a spacetime with torsion [53], although it was also shown to be broken for the full Horndeski theory with both curvature and torsion [54]. It is important to mention that an alternative to avoid the no-go is to consider Beyond-Horndeski theory [46–48,50]. However, in this work we stay within the Horndeski class, namely, with second order equations of motion and we use the no-go theorem in multi-Galileons to show that the theories considered here also suffer these global instability issues. In particular, this result applies to the theories with a Luminal graviton that were proposed in [11].

In Sec. III we introduce notation for the first order perturbation on the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) cosmological background. In Sec. IV we compute the quadratic action for the tensor and vector sectors and review the cases in which the automatical luminality of GWs is achieved. In Sec. IV C—restricted to these luminal cases—we also discuss the identification of the vector mode with the photon of Maxwell electrodynamics in the same line of reasoning that it is common practice to identify the tensor modes of Horndeski with the graviton of GR.

⁴This holds at first order and assuming conventional asymptotics in the general case. See also [38,46–49,51,52].

In Sec. V we show the scalar modes and discuss the stability issues of nonsingular cosmological solutions. We end with conclusions in Sec. VI.

II. OBTAINING 4D LUMINAL HORNDESKI FROM A HIGHER DIMENSIONAL REDUCTION

The standard formulation of Horndeski theory/generalized Galileons in 4D assumes that the photon is minimally coupled to gravity. Namely, the scalar field (Galileon) couples to curvature and it is assumed that the electromagnetic 4-vector does not couple to the Galileon. Thus, the speed of the photon c_A is a constant throughout cosmic evolution (1 in our units). In [11] a new type of photon-Galileon couplings was found—inspired by a KK reduction of a higher dimensional scalar tensor theory—in such a way that *what is kept constant in time* is not the speed of light but *the ratio of speeds* of gravity c_g and light, namely

$$\frac{c_A(t)}{c_g(t)} = 1, \quad (1)$$

while $c_A(t)$ and $c_g(t)$ can vary on the *cosmological* background. Indeed, it seems clear—up to a caveat to be shown

below—that if the graviton and photon in 4D are different components of the same “gravitational wave” in 5D, then the speed of gravity and light will coincide in 4D. This turns out to be the case for a large class—but *not all*—of Horndeski theories in 4D with photon-scalar couplings derived by the KK dimensional reduction. Thus, the KK reduction is a *useful tool* to find theories with the property (1). The latter were the main topic in [11] and in this work we cover in more detail both of these types theories, those with and without a luminal graviton.

For completeness, let us review the idea proposed in [11]. Let us consider generalized Galileons formulated in 5D. It is then clear that we do not aim to compare nor discuss the fundamental status of the 5D theory against the 4D theory. Both of them are simply modifications of gravity, but as we will see, with interesting properties in 4D.

Denoting all quantities defined on the 5D manifold M_5 with hats, such as $\hat{R}$ -curvature tensors computed with the 5D metric $\hat{g}$ (with mostly minus signature)-, and choosing indices that are capital Latin letters running over the five dimensions, $M = 0, 1, 2, 3, 4$, while lowercase Greek letters run over our four dimensions, $\alpha = 0, 1, 2, 3$, the 5D Galileons can be written as [4]

$$S = \int d^5x \sqrt{\hat{g}} \mathcal{L}_\pi, \quad (2a)$$

$$\mathcal{L}_\pi = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_6 \quad (2b)$$

$$\mathcal{L}_2 = F(\pi, X), \quad (2c)$$

$$\mathcal{L}_3 = K(\pi, X) \hat{\square} \pi, \quad (2d)$$

$$\mathcal{L}_4 = -G_4(\pi, X) \hat{R} + 2G_{4X}(\pi, X) [(\hat{\square} \pi)^2 - \pi_{;MN} \pi^{;MN}], \quad (2e)$$

$$\mathcal{L}_5 = G_5(\pi, X) \hat{G}^{MN} \pi_{;MN} + \frac{1}{3} G_{5X}(\pi, X) [(\hat{\square} \pi)^3 - 3 \hat{\square} \pi \pi_{;MN} \pi^{;MN} + 2 \pi_{;MN} \pi^{;MP} \pi_{;P}{}^N], \quad (2f)$$

$$\begin{aligned} \mathcal{L}_6 = & \frac{3}{4} G_6(\pi, X) (\hat{R}^2 - 4 \hat{R}^{AB} \hat{R}_{AB} + \hat{R}^{ABCD} \hat{R}_{ABCD}) + 3 G_{6X}(\pi, X) (-\hat{R} ((\hat{\square} \pi)^2 - \pi^{;AB} \pi_{;AB}) \\ & + 4 \hat{R}^{AB} \hat{\square} \pi \pi_{;AB} - 4 \hat{R}^{AB} \pi_{;A}{}^C \pi_{;CB} - 2 \hat{R}^{ABCD} \pi_{;AC} \pi_{;BD}) + G_{6XX}(\pi, X) ((\hat{\square} \pi)^4 \\ & - 6 \pi^{;AB} \pi_{;AB} (\hat{\square} \pi)^2 + 8 \hat{\square} \pi \pi^{;AB} \pi_{;B}{}^C \pi_{;CA} + 3 (\pi^{;AB} \pi_{;AB})^2 - 6 \pi^{;AB} \pi_{;B}{}^C \pi_{;C}{}^D \pi_{;DA}), \end{aligned} \quad (2g)$$

where π is the scalar field (Galileon), $X = \hat{g}^{MN} \pi_{;M} \pi_{;N}$, $\pi_{;M} = \partial_M \pi$, F, K, G_4, G_5, G_6 are general scalar potentials, $\pi_{;MN} = \hat{\nabla}_N \hat{\nabla}_M \pi$, $\hat{\square} \pi = \hat{g}^{MN} \hat{\nabla}_N \hat{\nabla}_M \pi$, and we have defined as $G_{4X} = \partial G_4 / \partial X$ and so on.

Now, let us consider the 5D Galileons (2) compactified on a circle. We write the metric as usual in terms of $h_{\mu\nu}$, a 4-vector A_μ and a Dilaton ϕ , assuming the “cylinder condition”⁵ [55],

$$\hat{g}_{MN} = \begin{pmatrix} h_{\mu\nu} - \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & -\phi^2 \end{pmatrix}, \quad (4)$$

and with inverse,

$$\hat{g}^{MN} = \begin{pmatrix} h^{\mu\nu} & h^{\mu\nu} A_\nu \\ h^{\mu\nu} A_\mu & -\frac{1}{\phi^2} + h^{\mu\nu} A_\mu A_\nu \end{pmatrix}. \quad (5)$$

As usual we identify $h_{\mu\nu}$ as the metric on our 4D spacetime. Now, at least in the Luminal theories shown in [11]—considered as IR modifications of gravity—we will identify A_μ with the photon because, first, it is a $U(1)$ gauge invariant theory in 4D and second, because the vector modes propagate at exactly the same speed as the tensor modes of Horndeski gravity, which is suggested by recent observations [1, 5–7]. This is further discussed in Sec. IV C, although this identification is irrelevant for the most part of this work.

With the metric (4) and the cylinder condition, normalizing our fields to reabsorb the multiplicative constant in the action⁶ $\int dx^4$, we rewrite the 5D Galileon (2) as a 4D theory (See the Appendix A for details)

$$S = \int d^4x \sqrt{-h} \phi (\mathcal{L}_\pi + \mathcal{L}_\phi + \mathcal{L}_A), \quad (6)$$

where $\mathcal{L}_\pi$ takes the form of (2), but removing the hat of all the quantities, namely, computing all curvature tensors and covariant derivatives with the 4D metric h (the decomposition of five-dimensional scalar structures in terms of four-dimensional ones can be found in the Appendix A). In other words, $\mathcal{L}_\pi$ takes the form of the usual Galileons in 4D plus an extra contribution proportional to the G_6 scalar potential and its derivatives. Let us note, however, that the G_6 terms can be reabsorbed in the usual 4D Horndeski theory. Indeed, due to the fact that Horndeski theory is the most general scalar-tensor theory of gravity in four-dimensional spacetime with second-order equations of motion, after reduction, we can get rid of all G_6 summands and its derivatives by the following redefinition of scalar potentials:

$$F \rightarrow \hat{F} + \frac{3}{2} X^2 \int (G_{6\pi\pi\pi\pi} X^{-1} + 2G_{6X\pi\pi\pi\pi}) dX - \frac{9}{2} G_{6\pi\pi\pi\pi} X^2 + \frac{3}{2} X \int G_{6X\pi\pi\pi\pi} X dX, \quad (7a)$$

$$K \rightarrow \hat{K} + \frac{9}{2} X \int (G_{6\pi\pi\pi} X^{-1} + 2G_{6X\pi\pi\pi}) dX - \frac{21}{2} X G_{6\pi\pi\pi} + \frac{3}{2} \int X G_{6X\pi\pi\pi} dX, \quad (7b)$$

$$G_4 \rightarrow \hat{G}_4 + \frac{3}{2} X \int (G_{6\pi\pi} X^{-1} + 2G_{6\pi\pi X}) dX - 3G_{6\pi\pi} X, \quad (7c)$$

$$G_5 \rightarrow \hat{G}_5 + 3 \int (G_{6\pi} X^{-1} + 2G_{6\pi X}) dX \quad (7d)$$

thus the Lagrangian $\mathcal{L}_\pi$ can be written explicitly as 4D Horndeski with only four independent scalar potentials $\mathcal{L}_\pi(\hat{F}, \hat{K}, \hat{G}_4, \hat{G}_5)$ (this is more easily checked at the level of the equations of motion). On the other hand, we define $\mathcal{L}_\phi$ such that in it we include all contributions to the action without the 4-vector A^μ and also not contributing to Horndeski theory in 4D, $\mathcal{L}_\pi(\hat{F}, \hat{K}, \hat{G}_4, \hat{G}_5)$. Namely, it takes the form,

⁵We assume that M_5 can be decomposed as a direct product of a pseudo-Riemannian manifold, M_4 , and the circle, S^1 , such that $M_5 = M_4 \times S^1$. This decomposition allows us to expand fields on M_5 as Fourier series in the additional coordinate, x^4 . Specifically, we have:

$$\rho = \sum_n \rho_n e^{inx^4} \quad (3)$$

where ρ represents any field on M_5 , and the coefficients ρ_n are functions of the coordinates x^μ on M^4 . If there are any terms in the series other than ρ_0 , the fields defined on M_5 would give rise to an infinite number of massive fields after reduction on M^4 . To avoid this, we impose the condition that all quantities are independent of the fifth dimension, namely $\rho = \rho(x^\mu)$.

⁶Namely, in the dimensional reduction we integrate the action (2) on the coordinate x^4 because all fields of the theory do not depend on this coordinate.

$$\mathcal{L}_\phi = \mathcal{L}_{K\phi} + \mathcal{L}_{4\phi} + \mathcal{L}_{5\phi} + \mathcal{L}_{6\phi}, \quad (8a)$$

$$\mathcal{L}_{K\phi} = \frac{1}{\phi} K \phi^{;\alpha} \pi_{;\alpha}, \quad (8b)$$

$$\mathcal{L}_{4\phi} = \frac{2}{\phi} G_4(\Box\phi) + \frac{4}{\phi} G_{4X}(\Box\pi) \phi^{;\alpha} \pi_{;\alpha}, \quad (8c)$$

$$\mathcal{L}_{5\phi} = \frac{1}{\phi} G_5((\Box\phi)(\Box\pi) - \phi^{;\alpha\beta} \pi_{;\alpha\beta}) - \frac{1}{2\phi} G_5 R \phi^{;\alpha} \pi_{;\alpha} + \frac{1}{\phi} G_{5X} \phi^{;\alpha} \pi_{;\alpha} ((\Box\pi)^2 - \pi_{;\alpha\beta} \pi^{;\alpha\beta}), \quad (8d)$$

$$\begin{aligned} \mathcal{L}_{6\phi} = & \frac{6}{\phi} G_6 G^{\alpha\beta} \phi_{;\alpha\beta} + \frac{4}{\phi} G_{6XX} \phi^{;\alpha} \pi_{;\alpha} ((\Box\pi)^3 - (\Box\pi) \pi_{;\alpha\beta} \pi^{;\alpha\beta} + 2 \pi_{;\alpha\beta} \pi^{;\beta\gamma} \pi^{;\gamma\alpha}) \\ & + \frac{6}{\phi} G_{6X} ((\Box\phi)(\Box\pi)^2 - (\Box\phi) \pi_{;\alpha\beta} \pi^{;\alpha\beta} - 2(\Box\pi) \phi^{;\alpha\beta} \pi_{;\alpha\beta}) + \frac{12}{\phi} G_{6X} G^{\alpha\beta} \pi_{;\alpha\beta} \phi^{;\gamma} \pi_{;\gamma} + \frac{12}{\phi} G_{6X} \phi^{;\alpha\beta} \pi_{;\beta\gamma} \pi^{;\gamma\alpha}. \end{aligned} \quad (8e)$$

Finally, we write the action for the 4-vector as

$$\mathcal{L}_A = \mathcal{L}_{4A} + \mathcal{L}_{5A} + \mathcal{L}_{6A}, \quad (9a)$$

$$\mathcal{L}_{4A} = -\frac{\phi^2}{4} G_4 F_{\alpha\beta} F^{\alpha\beta} + \phi^2 G_{4X} F_{\alpha\gamma} F^{\alpha}{}_{\beta} \pi^{;\beta} \pi^{;\gamma} \quad (9b)$$

$$\begin{aligned} \mathcal{L}_{5A} = & G_5 \phi^2 \left(\frac{1}{2} F^{\alpha\beta} F_{\alpha}{}^{\gamma} \pi_{;\beta\gamma} - \frac{1}{8} F^{\alpha\beta} F_{\alpha\beta} \Box\pi + \frac{1}{2} F^{\alpha\beta} \nabla^{\gamma} F_{\alpha\gamma} \pi_{;\beta} \right) + G_5 \phi^2 \left(\frac{3}{2\phi} F^{\alpha\beta} F_{\alpha}{}^{\gamma} \phi_{;\beta} \pi_{;\gamma} - \frac{3}{8\phi} F^{\alpha\beta} F_{\alpha\beta} \phi^{;\gamma} \pi_{;\gamma} \right) \\ & + G_{5X} \phi^2 \left(\frac{1}{2} F^{\alpha\beta} F_{\alpha}{}^{\gamma} (\Box\pi) \pi_{;\beta} \pi_{;\gamma} - \frac{1}{2} F^{\alpha\beta} F^{\gamma\delta} \pi_{;\alpha\gamma} \pi_{;\beta\delta} \right) \end{aligned} \quad (9c)$$

$$\begin{aligned} \mathcal{L}_{6A} = & G_6 \phi^2 \left(\frac{3}{8} F_{\alpha\beta} F^{\alpha\beta} R - \frac{9}{4\phi} F_{\alpha\beta} F^{\alpha\beta} (\Box\phi) + \frac{9\phi^2}{64} F_{\alpha\beta} F_{\gamma\delta} F^{\alpha\beta} F^{\gamma\delta} + 3 F_{\alpha\beta} F_{\gamma}{}^{\alpha} R^{\gamma\beta} - \frac{9}{2\phi} F_{\alpha\beta} F_{\gamma}{}^{\alpha} \phi^{;\gamma\beta} - \frac{9\phi^2}{32} F_{\alpha\beta} F_{\gamma}{}^{\alpha} F_{\delta}{}^{\beta} F^{\gamma\delta} \right. \\ & - \frac{9}{\phi^2} F_{\alpha\beta} F_{\gamma}{}^{\alpha} \phi^{;\beta} \phi^{;\gamma} - \frac{9}{\phi} F_{\alpha}{}^{\beta} \nabla^{\gamma} F_{\beta\gamma} \phi^{;\alpha} + \frac{3}{2} \nabla^{\alpha} F_{\alpha\beta} \nabla^{\gamma} F_{\gamma}{}^{\beta} + \frac{9}{4} R^{\alpha\gamma\beta\delta} F_{\alpha\beta} F_{\gamma\delta} - \frac{3}{\phi} F^{\alpha\beta} \nabla^{\gamma} F_{\alpha\beta} \phi_{;\gamma} + \frac{3}{\phi} F_{\alpha}{}^{\beta} \nabla^{\alpha} F_{\beta\gamma} \phi^{;\gamma} \\ & - \frac{9}{2\phi^2} F_{\alpha\beta} F^{\alpha\beta} \phi_{;\gamma} \phi^{;\gamma} - \frac{15}{16} \nabla^{\alpha} F_{\beta\gamma} \nabla_{\alpha} F^{\beta\gamma} + \frac{3}{8} \nabla^{\alpha} F_{\beta\gamma} \nabla^{\beta} F_{\alpha}{}^{\gamma} \left. \right) + G_{6X} \left(-\frac{3}{4} F_{\alpha\beta} F^{\alpha\beta} (\Box\pi)^2 - \frac{9}{2\phi} F_{\alpha\beta} F^{\alpha\beta} (\Box\pi) \phi^{;\gamma} \pi_{;\gamma} \right. \\ & + \frac{3}{2} R F_{\alpha\beta} F_{\gamma}{}^{\alpha} \pi^{;\beta} \pi^{;\gamma} + \frac{3}{4} F_{\alpha\beta} F^{\alpha\beta} \pi_{;\gamma\delta} \pi^{;\gamma\delta} + \frac{9\phi^2}{8} F_{\alpha\beta} F_{\gamma\delta} F_{\gamma}{}^{\alpha} F^{\gamma\delta} \pi^{;\beta} \pi^{;\gamma} - 6 F_{\alpha\beta} F_{\gamma}{}^{\alpha} (\Box\pi) \pi^{;\gamma\beta} - \frac{9}{\phi} F_{\alpha\beta} F_{\gamma}{}^{\alpha} \pi^{;\gamma\beta} \phi^{;\delta} \pi_{;\delta} \\ & - \frac{18}{\phi} F_{\alpha\beta} F_{\gamma}{}^{\alpha} (\Box\pi) \phi^{;\gamma} \pi^{;\beta} - 6 F_{\alpha}{}^{\beta} (\Box\pi) \nabla^{\gamma} F_{\beta\gamma} \pi^{;\alpha} + 6 F_{\alpha\beta} F_{\gamma}{}^{\alpha} \nabla^{\gamma} \pi_{;\delta} \pi^{;\beta\delta} + 3 F_{\alpha\beta} F_{\gamma\delta} R^{\alpha\gamma} \pi^{;\beta} \pi^{;\delta} - \frac{9\phi^2}{4} F_{\alpha\beta} F_{\gamma}{}^{\alpha} F_{\delta}{}^{\beta} F_{\gamma}{}^{\delta} \pi^{;\gamma} \pi^{;\delta} \\ & - \frac{18}{\phi} F_{\alpha\beta} F_{\gamma\delta} \pi^{;\alpha\gamma} \phi^{;\beta} \pi^{;\delta} + 6 F_{\alpha\beta} \nabla^{\gamma} F_{\gamma\delta} \pi^{;\alpha\delta} \pi^{;\beta} - \frac{9}{2} F_{\alpha\beta} F_{\gamma\delta} \pi^{;\alpha\gamma} \pi^{;\beta\delta} + 6 F_{\alpha}{}^{\beta} \nabla^{\gamma} F_{\beta\delta} \pi_{;\gamma}{}^{\delta} \pi^{;\alpha} + \frac{18}{\phi} F_{\alpha\beta} F_{\gamma}{}^{\alpha} \pi^{;\gamma\delta} \phi_{;\delta} \pi^{;\beta} \left. \right) \\ & - 3\phi^2 G_{6XX} (F_{\alpha\beta} F_{\gamma}{}^{\alpha} \pi^{;\beta} \pi^{;\gamma} (\Box\pi)^2 + 2 F_{\alpha\beta} F_{\gamma\delta} (\Box\pi) \pi^{;\alpha\gamma} \pi^{;\beta} \pi^{;\delta} - F_{\alpha\beta} F_{\gamma}{}^{\alpha} \pi_{;\delta\kappa} \pi^{;\delta\kappa} \pi^{;\beta} \pi^{;\gamma} - 2 F_{\alpha\beta} F_{\gamma\delta} \pi_{;\alpha}{}^{\kappa} \pi_{;\kappa}{}^{\gamma} \pi^{;\beta} \pi^{;\delta}) \end{aligned} \quad (9d)$$

$\mathcal{L}_{4A}$ and the first line of $\mathcal{L}_{5A}$ are the Lagrangians for the photon such that the graviton of Horndeski theory—in the case of a frozen Dilaton—is luminal even with the general scalar potentials $G_4(\pi, X)$ and $G_5(\pi)$ (namely, if $G_{5,X} = 0$ and so on), respectively. In the case with a general dilaton, a Luminal graviton is also automatically obtained if we also consider the second line in $\mathcal{L}_{5A}$. The remaining parts of

the Lagrangian contain $G_6(\pi, X)$ and $G_{5X}(\pi, X)$. As we will see below, in the case that the latter two scalar potentials are present, the graviton and the vector modes propagate in general at *different speeds*, unless there is a specific choice of scalar potentials and background solutions. Lagrangians containing “nonluminal” contributions will be generally named $\mathcal{L}_{NL}$. An exception to the latter is a branch of background solutions where luminality of all perturbations

but a scalar mode is achieved even with $G_5(\pi, X)$ and $G_6(\pi, X)$. This is shown in Sec. V C.

III. NOTATION FOR THE LINEARIZED ANALYSIS

We are concerned with IR modifications of GR and of its accompanying photon that could be relevant for dark energy (From now on it will be assumed that we discuss 4D theories, unless otherwise stated). Thus, we will examine the evolution of the FLRW spatially flat, isotropic and homogeneous background, and its stability against first order perturbations of the tensor, vector and scalar modes.

A. Decomposition into irreducible components

We write the perturbed metric and 4-vector as

$$ds^2 = h_{\mu\nu} dx^\mu dx^\nu = (\eta_{\mu\nu} + \delta h_{\mu\nu}) dx^\mu dx^\nu, \quad (10)$$

$$A^\mu = A^{(0)\mu} + \delta A^\mu, \quad (11)$$

while the Galileon $\pi(x^\mu)$ and the dilaton $\phi(x^\mu)$ are split as $\pi(t) + \delta\pi(x^\mu)$ and $\phi(t) + \delta\phi(x^\mu)$ respectively, and π, ϕ will be understood as background fields or not depending on the context in linearized expressions.

The background, spatially flat FLRW metric is written as usual

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2), \quad (12)$$

where $a(t)$ is the scale factor. On the other hand, the transverse vector background $A^{(0)i}$ is chosen identically to zero by isotropy considerations, and the other components of $A^{(0)\mu}$ also vanish on-shell.

The decomposition of perturbations $\delta h_{\mu\nu}, \delta A^\mu, \delta\pi$ and $\delta\phi$ into helicity components in the general case has the form

$$\delta h_{00} = 2\Phi \quad (13a)$$

$$\delta h_{0i} = -\partial_i \beta + Z_i^T, \quad (13b)$$

$$\delta h_{ij} = -2\Psi\delta_{ij} - 2\partial_i \partial_j E - (\partial_i W_j^T + \partial_j W_i^T) + h_{ij}^T, \quad (13c)$$

$$\delta A_0 = \gamma, \quad (13d)$$

$$\delta A_i = \partial_i \alpha + A_i, \quad (13e)$$

$$\delta\pi = \chi, \quad (13f)$$

$$\delta\phi = \varphi, \quad (13g)$$

where $\Phi, \beta, \Psi, E, \chi, \varphi, \alpha, \gamma$ are scalar fields, Z_i^T, W_i^T, A_i are transverse two-component vector fields ($\partial_i Z_i^T = \partial_i W_i^T = \partial_i A_i = 0$) and h_{ij}^T is a transverse traceless two-component tensor.

The evolution of the cosmological background is determined by three equations $\mathcal{E}_f = \partial\mathcal{L}/\partial f = 0$ where f is one of the following: $\pi, \phi, h_{00}, h_{ij}$. There is a redundancy due to little gauge invariance, and thus, we can express, for instance, $\mathcal{E}_\pi$ in terms of the others and their time derivatives. By looking at the equations of the background fields and the linearized expressions, it becomes evident that it is useful to define $H_\phi = \dot{\phi}/\phi$ in analogy with our Hubble parameter $H = \dot{a}/a$.

B. From 5D diffeomorphisms, to 4D little gauge and $U(1)$ gauge invariant vector

The initial 5D theory in Eq. (2) is invariant under small coordinate transformations in M_5

$$x^M \rightarrow x^M - \xi^M. \quad (14)$$

As usual, upon dimensional reduction the invariance under 5D diffeomorphisms turns into invariance under 4D diffeomorphisms and $U(1)$ gauge transformations of the vector field A_μ . All fields under the small coordinate shift (14) are transformed as follows:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\alpha \eta_{\mu\nu} \xi^\alpha + \partial_\mu \xi^\alpha \eta_{\nu\alpha} + \partial_\nu \xi^\alpha \eta_{\mu\alpha}, \quad (15a)$$

$$A_\mu \rightarrow A_\mu + \xi^\alpha \partial_\alpha A_\mu^{(0)} + A_\alpha^{(0)} \partial_\mu \xi^\alpha + \partial_\mu \xi^\alpha, \quad (15b)$$

$$\varphi \rightarrow \varphi + \xi^\alpha \partial_\alpha \phi, \quad (15c)$$

$$\chi \rightarrow \chi + \xi^\alpha \partial_\alpha \pi, \quad (15d)$$

where $\xi^\mu = (\xi_0, \xi_T^i + \delta^{ij} \partial_j \xi_S)^T$.

IV. LUMINAL GRAVITATIONAL WAVES WITH $\mathcal{L}_{4A}$ AND $\mathcal{L}_{5A}$ VS NONLUMINALITY WITH $\mathcal{L}_{NL}$

A. The graviton

Expanding to second order the action (6) with (10)–(13), and using the equations for the background fields and its derivatives to vanish the h_{ij}^2 term, we obtain the quadratic action for the tensor perturbation of the metric

$$S_{\text{tensor}}^{(2)} = \int dt d^3x a^3 \phi \left[\mathcal{G}_\tau (\dot{h}_{ij})^2 - \frac{\mathcal{F}_\tau}{a^2} (\vec{\nabla} h_{ij})^2 \right], \quad (16)$$

where $\mathcal{G}_\tau$ and $\mathcal{F}_\tau$ are functions of the background fields $\pi(t), \phi(t), a$. They are given in the Appendix C. They take the same form as in the usual Horndeski theory, up to $H_\phi = \dot{\phi}/\phi$ contributions.

Let us recall that the squared speed of the tensor modes

$$c_g^2 = \frac{\mathcal{F}_\tau}{\mathcal{G}_\tau}, \quad (17)$$

is different from unity even in the Horndeski theory case, namely, even if $H_\phi = 0$. For instance, in the case $H_\phi = 0$ and with zero scalar potential G_5 , the speed takes the well known⁷ form with $\mathcal{F}_\tau = 2G_4$ and $\mathcal{G}_\tau = 2(G_4 - 2\dot{\pi}^2 G_{4X})$. Thus, only if G_4 is a function of π -but not of X - can the speed of the graviton be exactly unity. In other words, because for a minimally coupled photon the speed of light is constant ($c_A^2 = 1$ in our units), Horndeski theories with the minimal photon are, in general, heavily constrained by the precise experimental bounds on the speed of gravitational waves and light [7],

$$\left| \frac{c_g}{c_A} - 1 \right| \leq 5 \times 10^{-16}. \quad (18)$$

A way around this strict constraint is to consider Horndeski theory complemented by the photon-scalar couplings shown in the actions (9b) and (9c) for the general scalar potentials $G_4(\pi, X)$ and $G_5(\pi)$. In those cases, even for nonzero H_ϕ , the speed of the photon changes with time in exactly the same way as the speed of graviton so that $\frac{c_g(t)}{c_A(t)} = 1$ is preserved throughout cosmic evolution. The key aspect is that the effective metric of both the graviton and the photon are modified by the same gravitoelectric Galileon. Indeed, let us now compute the perturbations to the cosmological background in the vector sector.

B. Vector sector and the photon

Noticing that the combination

$$V_i = \dot{W}_i - 2HW_i + Z_i$$

is gauge-invariant by the transformations

$$Z_i \rightarrow Z_i - a^2 \xi_T^i, \quad (19a)$$

$$W_i \rightarrow W_i + a^2 \xi_T^i, \quad (19b)$$

obtained from (15a), and expanding to second order the action (6) with (10)–(13), the vector sector can be written as

$$\begin{aligned} \delta S_{\text{vector}}^2 = \int dt d^3x a^3 \phi \left(\frac{\phi^2}{a^2} (\mathcal{G}_V(\dot{\vec{A}})^2) - \frac{\phi^2}{a^2} \mathcal{F}_V \left(\frac{\vec{\nabla}}{a} \times \vec{A} \right)^2 \right. \\ \left. + \mathcal{K} \left(\frac{\vec{\nabla}}{a} \times \vec{V} \right)^2 \right), \end{aligned} \quad (20)$$

where $\mathcal{G}_V, \mathcal{F}_V, \mathcal{K}$ are functions of the background fields. They are given in the Appendix C.

From (20) it is clear that the vector perturbations of the metric $\vec{V}$ and the vector perturbations of the 4-vector A^μ

decouple. The former is nondynamical, and thus we can write the final action for the vector modes as

$$\delta S_{\text{vector}}^2 = \int dt d^3x a^3 \phi^3 \left(\mathcal{G}_V(\dot{\vec{A}})^2 - \mathcal{F}_V \left(\frac{\vec{\nabla}}{a} \times \vec{A} \right)^2 \right), \quad (21)$$

such that they propagate with the speed squared

$$c_A^2 = \frac{\mathcal{F}_V}{\mathcal{G}_V}. \quad (22)$$

As anticipated from the discussion in the previous section, for the theories (6) with only $F(\pi, X)$, $K(\pi, X)$, $G_4(\pi, X)$, and $G_5(\pi)$ general scalar potentials, we can note that

$$\frac{\mathcal{F}_V}{\mathcal{G}_V} = \frac{\mathcal{F}_\tau}{\mathcal{G}_\tau} \quad (23)$$

and thus, the speed of the graviton and the 4-vector are the same throughout cosmic evolution $\frac{c_g}{c_A} = 1$ [11]. Now, provided the experimental bounds that set the speed of GWs very close to the speed of light, it is reasonable—at least in these cases—to identify the 4-vector with the photon relevant for cosmic evolution. We discuss this identification from other perspective in Sec. IV C. We will generally denote these as theories with an exactly “luminal graviton” or luminal extensions of Horndeski theory.

It is worth to note that a recent work [17] found that, at least in a cosmological background, the $G_4(\pi, X)$ Luminal extensions of Horndeski theory -namely, with nonminimal photon-scalar couplings $\mathcal{L}_{4A}$ and with $G_5 \equiv 0$ - can be reproduced by disformal transformations of already known luminal *beyond Horndeski theories* with a minimally coupled photon. Thus, in this sense and as far as it is known, only on a cosmological background, the $\mathcal{L}_{4A}$ Luminal extension of Horndeski maps to a beyond Horndeski theory.

C. The 4-vector and the photon with $\mathcal{L}_{4A}$ and $\mathcal{L}_{5A}$

The common practice in IR modifications of GR, and in particular, in Horndeski theory, is to identify the tensor modes of perturbations about cosmological background with the graviton of GR. This identification is clearly motivated in first place, by the two propagated polarizations and in second place, by the fact that the modifications of GR due to the Horndeski scalar are screened in solar system tests. See for instance [56–59]. The latter is specially important provided the stringent experimental bounds, otherwise, there would be little point in naming them “IR modifications of gravity” (the theory valid in the solar system) in the first place.

Similarly, with regards to the vector-Galileon couplings discussed before $\mathcal{L}_{4A}, \mathcal{L}_{5A}$, there is as a matter of fact, an experimental motivation to try to identify that $U(1)$ gauge invariant vector coupled to the Horndeski scalar with the

⁷Let us recall that if $H_\phi = 0$ the scalar potential G_6 can be fully reabsorbed by (7).

photon: the strict bounds on the similarity between the speeds of gravitational waves and light by the event GW170817 [1]. These suggest that a graviton even in modified gravity theories *may* propagate at the same speed as light. Assuming the latter, $\mathcal{L}_{4A}, \mathcal{L}_{5A}$ together with Horndeski theory (in the case of a constant dilaton) expand the theory space of allowed⁸ theories in what respects these speed tests, at least within the Horndeski class.⁹ Now, how to match the IR scalar modification of the familiar photon of Maxwell electrodynamics to laboratory tests is an important challenge. This was briefly discussed in connection with the Vainshtein screening of GR modifications in [11]. And a related discussion on experimental bounds

for these sort of scalar-photon couplings was advanced in [17]. Although this question merits more investigation, it falls way beyond the scope of this work.

V. SCALAR MODES, A LUMINAL BRANCH OF SOLUTIONS AND THE STABILITY ANALYSIS OF THE FLRW BACKGROUND

A. Scalar sector

Expanding to second order the action (6) with (10)–(13), and partly using the gauge freedom to set the longitudinal component E to vanish, the quadratic action for the scalar perturbations has the form

$$\begin{aligned} \delta S_{\text{scalar}}^2 = \int dt d^3x a^3 \phi & \left(A_1 (\dot{\Psi})^2 + A_2 \frac{(\vec{\nabla}\Psi)^2}{a^2} + A_3 \Phi^2 + A_4 \Phi \frac{\vec{\nabla}^2 \beta}{a^2} + A_5 \dot{\Psi} \frac{\vec{\nabla}^2 \beta}{a^2} + A_6 \Phi \dot{\Psi} \right. \\ & + A_7 \Phi \frac{\vec{\nabla}^2 \Psi}{a^2} + A_8 \Phi \frac{\vec{\nabla}^2 \chi}{a^2} + A_9 \frac{\vec{\nabla}^2 \beta}{a^2} \dot{\chi} + A_{10} \chi \ddot{\Psi} + A_{11} \Phi \dot{\chi} + A_{12} \chi \frac{\vec{\nabla}^2 \beta}{a^2} + A_{13} \chi \frac{\vec{\nabla}^2 \Psi}{a^2} \\ & + A_{14} (\dot{\chi})^2 + A_{15} \frac{(\vec{\nabla}\chi)^2}{a^2} + A_{17} \Phi \dot{\chi} + A_{18} \chi \dot{\Psi} + A_{19} \Psi \dot{\chi} + A_{20} \chi^2 + C_1 \dot{\Psi} \dot{\phi} + C_2 \dot{\Psi} \dot{\varphi} + C_4 \chi \ddot{\varphi} \\ & + C_5 \chi \dot{\phi} + C_7 \dot{\varphi} \frac{\vec{\nabla}^2 \beta}{a^2} + C_8 \varphi \frac{\vec{\nabla}^2 \beta}{a^2} + C_9 \Phi \dot{\varphi} + C_{10} \varphi \frac{\vec{\nabla}^2 \chi}{a^2} + C_{11} \varphi \frac{\vec{\nabla}^2 \Psi}{a^2} + C_{12} \varphi \frac{\vec{\nabla}^2 \Phi}{a^2} \\ & \left. + \phi^2 \left(B_1 \frac{(\vec{\nabla}\dot{\alpha})^2}{a^2} + B_2 \gamma \frac{\vec{\nabla}^2 \alpha}{a^2} - 2B_1 \dot{\gamma} \frac{\vec{\nabla}^2 \alpha}{a^2} + B_1 \frac{(\vec{\nabla}\gamma)^2}{a^2} \right) \right), \end{aligned} \quad (24)$$

where coefficients A_i, B_i, C_i are combinations of the Lagrangian functions and their derivatives, written in terms of the background fields. The explicit form of these coefficients can be found in the Appendix C.

A first important aspect is that the scalar perturbations of A^μ , namely α, γ do not mix with the scalar perturbations of metric, Galileon or dilaton. We will use this fact later on to discuss the stability of the scalar modes.

The equations of motion obtained from the variations on β, Φ, γ , and α are constraints. The equations from the variations on γ and α are equivalent. They can be written as,

$$\delta\Phi: \frac{\vec{\nabla}^2 \beta}{a^2} = -\frac{1}{A_4} \left(A_{17} \chi + A_8 \frac{\vec{\nabla}^2 \chi}{a^2} + A_{11} \dot{\chi} + 2A_3 \Phi + C_{12} \frac{\vec{\nabla}^2 \varphi}{a^2} + C_9 \dot{\varphi} + A_7 \frac{\vec{\nabla}^2 \Psi}{a^2} + A_6 \dot{\Psi} \right) \quad (25a)$$

$$\delta\beta: \Phi = -\frac{1}{A_4} (A_{12} \chi + A_9 \dot{\chi} + C_8 \varphi + C_7 \dot{\varphi} + A_5 \dot{\Psi}), \quad (25b)$$

$$\delta\gamma: \frac{\vec{\nabla}^2 \gamma}{a^2} = \frac{\vec{\nabla}^2 \alpha}{a^2}. \quad (25c)$$

The γ constraint turns out to be trivial due to the relation $\frac{1}{2}B_2 + \dot{B}_1 + B_1(3H_\phi + H) = 0$.

Now let us use the residual gauge freedom and set $\chi = 0$. Using the constraints (25) back in the quadratic Lagrangian (24), we integrate out Φ, β, γ . The terms containing α are canceled automatically. The resulting action contains two dynamic variables, Ψ and φ :

⁸See however [60] for an alternative.

⁹Namely, while keeping second order equations of motion. See however [17] for a recent discussion in beyond Horndeski theory.

$$\delta S_{\text{scalar}}^2 = \int dt d^3x a^3 \phi \left((\dot{\Psi})^2 \mathcal{G}_S^\Psi + \mathcal{G}_S^{\Psi\phi} \dot{\Psi} \dot{\phi} + (\dot{\phi})^2 \mathcal{G}_S^\phi - \mathcal{F}_S^\Psi \frac{(\vec{\nabla}\Psi)^2}{a^2} - \mathcal{F}_S^{\Psi\phi} \frac{\vec{\nabla}\Psi \cdot \vec{\nabla}\phi}{a} - \mathcal{F}_S^\phi \frac{(\vec{\nabla}\phi)^2}{a^2} + \mathcal{R} \dot{\Psi} \dot{\phi} + M_\phi \phi^2 \right), \quad (26)$$

where

$$\mathcal{G}_S^\Psi = -A_1 + \frac{4A_3A_1^2}{9A_4^2}, \quad \mathcal{G}_S^{\Psi\phi} = \frac{2(2A_3C_1 + 3A_4C_9)A_1}{9A_4^2}, \quad \mathcal{G}_S^\phi = \frac{1}{9} \frac{A_3C_1^2}{A_4^2} + \frac{1}{3} \frac{C_1C_9}{A_4}, \quad (27a)$$

$$\mathcal{F}_S^\Psi = -A_2 - \frac{1}{a\phi} \frac{d}{dt} \left[\frac{2A_1^2\phi a}{9A_4} \right], \quad \mathcal{F}_S^{\Psi\phi} = C_{11} - \frac{1}{a^2} \frac{d}{dt} \left[\frac{2A_1C_1a^2}{9A_4} \right], \quad \mathcal{F}_S^\phi = -\frac{\phi}{a^3} \frac{d}{dt} \left[\frac{a^3C_1^2}{18\phi A_4} \right], \quad (27b)$$

$$\mathcal{R} = \frac{4A_1A_3C_2}{9A_4^2}, \quad M_\phi = -\frac{1}{9a^4} \frac{d}{dt} \left[\frac{A_3C_1C_2a^4}{A_4^2} \right] - \frac{1}{6\phi a^3} \frac{d}{dt} \left[\frac{C_2C_9\phi a^3}{A_4} \right], \quad (27c)$$

Defining a kinetic matrix $\mathcal{G}_S$ and a “gradient” matrix $\mathcal{F}_S$,

$$\mathcal{G}_S = \begin{pmatrix} \mathcal{G}_S^\Psi & \frac{1}{2}\mathcal{G}_S^{\Psi\phi} \\ \frac{1}{2}\mathcal{G}_S^{\Psi\phi} & \mathcal{G}_S^\phi \end{pmatrix}, \quad \mathcal{F}_S = \begin{pmatrix} \mathcal{F}_S^\Psi & \frac{1}{2}\mathcal{F}_S^{\Psi\phi} \\ \frac{1}{2}\mathcal{F}_S^{\Psi\phi} & \mathcal{F}_S^\phi \end{pmatrix}, \quad (28)$$

it is easily verified that $\mathcal{G}_S$ has nonzero determinant, and thus there are two degrees of freedom,

$$\det \mathcal{G}_S = -\frac{1}{9} \frac{A_1}{A_4^2} (A_1C_9^2 + A_3C_1^2 + 3A_4C_1C_9). \quad (29)$$

Furthermore, a field rotation that diagonalizes the kinetic matrix does not diagonalize $\mathcal{F}_S$. Thus, the speeds of propagation of the two scalar modes can be found as eigenvalues of the matrix $\mathcal{G}_S^{-1}\mathcal{F}_S$. They are not factorized and their expressions are cumbersome, but we can express their sum and product in terms of the determinant of both matrices using the property that the trace of a matrix is the sum of its eigenvalues, and the determinant of a matrix is the product of eigenvalues

$$(c_S^{(1)})^2 (c_S^{(2)})^2 = \frac{\det \mathcal{F}_S}{\det \mathcal{G}_S}, \quad (30a)$$

$$(c_S^{(1)})^2 + (c_S^{(2)})^2 = \frac{1}{\det \mathcal{G}_S} \left(\mathcal{G}_S^\Psi \mathcal{F}_S^\phi - \frac{1}{2} \mathcal{G}_S^{\Psi\phi} \mathcal{F}_S^{\Psi\phi} + \mathcal{G}_S^\phi \mathcal{F}_S^\Psi \right), \quad (30b)$$

where,

$$\det \mathcal{F}_S = \frac{\phi}{18a^3} \left(A_2 + \frac{2}{9a\phi} \frac{d}{dt} \left[\frac{\phi a A_1^2}{A_4} \right] \right) \frac{d}{dt} \left[\frac{C_1^2 a^3}{A_4 \phi} \right] - \frac{1}{4} \left(-C_{11} + \frac{2}{9a^2} \frac{d}{dt} \left[\frac{a^2 A_1 C_1}{A_4} \right] \right)^2. \quad (31)$$

A particular branch of background solutions where we solve explicitly the speeds of the scalars is given below, in Sec. V C.

B. Stability of a nonsingular cosmological background against perturbations

Let us now discuss the stability of the FLRW background. Let us first assume a definite sign for the dilaton background $\phi(t)$. This is a natural assumption because otherwise at the vanishing of $\phi(t)$ the 5D metric would be degenerate. Indeed, let us recall that the components of the 4-vector background are also vanishing on-shell and by isotropy.

Now, requiring that the tensor and vector modes are not ghosts we demand $\mathcal{G}_\tau > 0$ and $\mathcal{G}_V > 0$. Furthermore, to avoid gradient instabilities we require that $\mathcal{F}_\tau > 0$ and $\mathcal{F}_V > 0$. Finally, we assume conventional asymptotics in that $\mathcal{F}_\tau$ has a nonzero lower bound in the asymptotic past and future (See for instance [38,51]). Let us note in passing that for the Luminal extensions of Horndeski theory—namely, in the cases with $\mathcal{L}_{4A}$ and $\mathcal{L}_{5A}$ but without G_6 or X dependence on G_5 —only two conditions are really required. Indeed, in that case the no-ghost and stability of the graviton obviously imply the same for the photon.

On the other hand, in the scalar sector it must be required that $\det \mathcal{G}_S > 0$ and $\det \mathcal{F}_S > 0$.

However, let us now show that well-known no-go theorems [37,38] for all-time stable, *nonsingular* cosmological solutions also apply in this case, even with the general Galileon and dilaton. Thus even if local stability can be achieved at some time for some model, it is certain that—assuming conventional asymptotics and not special cases (See [38,51,52])—a gradient instability will arise earlier or later in the evolution.

To this end let us recall that by the analysis of the previous section, the scalar perturbations of A^μ decouple from the scalar perturbations of the metric, Galileon and Dilaton. In fact, the former are nondynamical. Thus the

quadratic action for the scalar sector obtained from (6) is equivalent on-shell to a quadratic action obtained by setting A^μ to zero, or $\mathcal{L}_A \rightarrow 0$ in (6). Namely, we can obtain the same scalar dynamics considering only the $\mathcal{L}_\pi$ and $\mathcal{L}_\phi$ in (6). Now, let us recall that using the redefinition of scalar potentials (7) in $\mathcal{L}_\pi$, we can rewrite this part of the action only as a function of $\hat{F}, \hat{K}, \hat{G}_4, \hat{G}_5$ (namely, without G_6). On the other hand, upon the redefinition (7) we obtain $\mathcal{L}_\phi = \mathcal{L}_\phi(\hat{F}, \hat{K}, \hat{G}_4, \hat{G}_5, G_6)$. Now, following [45,61] let us define the following multiscalar potentials

$$G_2 = \hat{F}(\pi, X) + \frac{1}{\phi} \hat{K}(\pi, X) \phi^{;\alpha} \pi_{;\alpha}, \quad (32a)$$

$$G_3^\pi = \hat{K}(\pi, X) + \frac{4}{\phi} \hat{G}_{4X} \phi^{;\alpha} \pi_{;\alpha}, \quad (32b)$$

$$G_3^\phi = \frac{2}{\phi} \hat{G}_4(\pi, X), \quad (32c)$$

$$\tilde{G}_4 = \hat{G}_4 + \frac{1}{2\phi} \hat{G}_5(\pi, X) \phi^{;\alpha} \pi_{;\alpha}, \quad (32d)$$

$$G_5^\pi = \hat{G}_5(\pi, X) + \frac{9}{\phi} G_{6X} \phi^{;\alpha} \pi_{;\alpha}, \quad (32e)$$

$$G_5^\phi = \frac{6}{\phi} G_6, \quad (32f)$$

Thus we can rewrite $\mathcal{L}_\pi + \mathcal{L}_\phi$ as

$$\begin{aligned} \mathcal{L} = & G_2 - G_3^I \square \phi^I + G_4 R + G_{4, \langle IJ \rangle} (\square \phi^I \square \phi^J \\ & - \nabla_\mu \nabla_\nu \phi^I \nabla^\mu \nabla^\nu \phi^J) + G_5^I G^{\mu\nu} \nabla_\mu \nabla_\nu \phi^I \\ & - \frac{1}{6} G_{5, (JK)}^I [\square \phi^I \square \phi^J \square \phi^K - 3 \square \phi^I \nabla_\mu \nabla_\nu \phi^J \nabla^\mu \nabla^\nu \phi^K] \\ & + 2 \nabla_\mu \nabla_\nu \phi^I \nabla^\nu \nabla^\lambda \phi^J \nabla_\lambda \nabla^\mu \phi^K, \end{aligned} \quad (33)$$

where, in our case, ϕ^I are the fields π and ϕ , $X^{IJ} := h^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J$, and the index $\langle IJ \rangle$ denotes the symmetrized derivative $f_{, \langle IJ \rangle} := (\partial f / \partial X^{IJ} + \partial f / \partial X^{JI}) / 2$. Thus, the no-go theorem proved in [45] applies to the theories discussed in this note.

C. A “luminal” branch of background solutions

At this point it is relevant to discuss a cross-check of all the computations. This discussion will reveal an interesting aspect of the action (6) when considered within the general context of vector-scalar Galileons, or SVT theories.

Since the 4D metric, the 4-vector and the Dilaton are different components of the same 5D metric, it would be expected that, for instance, their speeds are the same. As we saw before, this happens in many cases, but not all. In particular, the speed of the 4-vector and the graviton is

different if the scalar potential G_5 depends on X or G_6 is present. The reason behind this difference is that we have compactified one dimension and broken isotropy in the spatial dimensions. As a *cross-check of our results*, let us restore isotropy and homogeneity in the spacelike dimensions in 5D by setting $\phi(t) = a(t)$ (in fact, the discussion below only requires us to assume $H_\phi = H$), in other words, by considering FLRW in 5D. In that case another of the equations for the background fields becomes redundant—namely, the equation for the dilaton background ϕ ($\mathcal{E}_\phi = 0$)—and thus, as expected, only $\mathcal{E}_{h_{00}} = 0, \mathcal{E}_{h_{11}} = 0$ are independent. Furthermore, we find in that case that both *the speeds of the graviton and the 4-vector coincide* even including the general potentials $G_5(\pi, X), G_6(\pi, X)$. Finally, the action coefficients are simplified as

$$C_1 = A_1, \quad C_2 = C_8 = 0, \quad C_9 = -A_4, \quad C_{11} = -2A_2, \quad (34)$$

and the speeds for the scalar modes can be explicitly written as

$$(c_S^{(1)})^2 = A_4^2 \frac{\frac{1}{a^2} \frac{d}{dt} \left[\frac{a^2 A_1^2}{A_4} \right] + 6A_2}{4A_1 A_4^2 - 2A_1^2 A_3}, \quad (35a)$$

$$(c_S^{(2)})^2 = -3 \frac{A_2}{A_1} = \frac{\mathcal{F}_\tau}{\mathcal{G}_\tau}. \quad (35b)$$

Thus, we verify that the speed of one of the scalars—the extra scalar mode due to the dilaton— $(c_S^{(2)})^2$, coincides with the speed of the tensor and vector modes for all scalar potentials $G_5(\pi, X)$ and $G_6(\pi, X)$.

Let us stress, however, that this special case *lacks physical significance* within the context of the KK dimensional reduction. Indeed, for the latter it was natural to assume that the radius of the compactified fifth dimension is small, thus necessarily breaking 5D spatial isotropy and homogeneity, and such that there is in principle no reason to assume that $H_\phi = H$.

On the other hand, let us discuss the later result more broadly: Within the context of all higher derivative SVT, the last result simply says that the theory (6) is a subclass with a luminal branch of background solutions. Namely, without discussing its connection to higher dimensions, consider the action (6) as an SVT theory for the metric, two higher derivative scalars ϕ and π —here regarded on equal footing—and for a $U(1)$ gauge invariant higher derivative vector A^μ , such that the equations of motion are of second order. This theory has the defining property that it has a particular *branch of cosmological background solutions* ($\phi(t) = a(t)$) *on which luminosity of all perturbations, with the exception of a scalar mode, is achieved for all scalar potentials up to $G_6(\pi, X)$* . Despite its unphysical

character in connection with the higher dimensional origin, this case could be, however, relevant on the search for vector-Galileon couplings from a broader perspective, beyond our specific method of KK reductions.

VI. CONCLUSIONS

We expanded on the results shown in a recent letter [11] and showed a broad class of $U(1)$ gauge invariant vector/multiscalar Galileons in 4D: namely, a theory with higher derivatives of a $U(1)$ gauge invariant vector and two scalars in the action, but with second order equations of motion for all of the fields. It was obtained from the dimensional reduction of Galileons in five dimensions (5D). We did not aim for completeness, nor for a complete classification of $U(1)$ gauge invariant vector Galileons. Nevertheless, the theories shown in this work provide a consistent way to couple vectors to IR scalar modifications of gravity—without inducing ghosts, and keeping gauge invariance—in the aim to explore more universal couplings of the scalar of dark energy to other matter. Furthermore, they are a specially relevant class of vector Galileons because first, a subclass of these theories leads to a luminal graviton in 4D even with general $G_4(\pi, X)$ and $G_5(\pi)$ scalar potentials,

second, for the whole class of theories up to $G_6(\pi, X)$ there is a *particular branch* of background solutions where most modes (tensor, vector and one scalar) propagate at the same speed, and third, because being connected to higher dimensional theories they could be simpler to analyze in some cases, for instance, in the look for black-hole solutions in 4D.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DETAILS FOR THE DIMENSIONAL REDUCTION

In this appendix we collect the expressions for all five-dimensional scalar structures in terms of four-dimensional ones:

$$\hat{R} = R - \frac{2}{\phi} \square \phi + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \phi^2, \quad (\text{A1})$$

$$\hat{\square} \pi = \square \pi + \frac{1}{\phi} \phi^{;\alpha} \pi_{;\alpha}, \quad (\text{A2})$$

$$\pi_{;MN} \pi^{;MN} = \pi_{;\mu\nu} \pi^{;\mu\nu} - \frac{1}{2} F_{\alpha\beta} F^{\alpha\gamma} \pi^{;\beta} \pi_{;\gamma} \phi^2 + \frac{1}{\phi^2} (\phi^{;\alpha} \pi_{;\alpha})^2, \quad (\text{A3})$$

$$\pi_{;MN} \pi^{;MP} \pi_{;P}{}^N = \pi_{;\mu\nu} \pi^{;\mu\kappa} \pi_{;\kappa}{}^\nu - \frac{3\phi^2}{4} F^{\alpha\beta} F^{\gamma\delta} \pi_{;\alpha} \pi_{;\gamma} \pi_{;\beta\delta} - \frac{3\phi}{4} F^{\alpha\beta} F_{\alpha\delta} \pi_{;\beta} \pi^{;\delta} \phi_{;\gamma} \pi^{;\gamma} + \frac{1}{\phi^3} (\phi^{;\alpha} \pi_{;\alpha})^3, \quad (\text{A4})$$

$$\begin{aligned} \hat{G}^{MN} \pi_{;MN} &= G^{\alpha\beta} \pi_{;\alpha\beta} + \frac{1}{\phi} \square \phi \square \pi - \frac{1}{\phi} \phi^{;\alpha\beta} \pi_{;\alpha\beta} - \frac{\phi^2}{8} F_{\alpha\beta} F^{\alpha\beta} \square \pi + \frac{\phi^2}{2} F^{\alpha\beta} F_{\alpha\gamma} \pi_{;\beta}{}^\gamma + \frac{3\phi}{2} F^{\alpha\beta} F_{\alpha\delta} \phi_{;\beta} \pi^{;\delta} + \frac{\phi^2}{2} F^{\alpha\beta} \nabla^\gamma F_{\alpha\gamma} \pi_{;\beta} \\ &\quad - \frac{3\phi}{8} F_{\alpha\beta} F^{\alpha\beta} \pi^{;\gamma} \phi_{;\gamma} - \frac{1}{2\phi} R \phi^{;\alpha} \pi_{;\alpha}, \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} \pi^{;AB} \pi_{;BC} \pi^{;CD} \pi_{;AD} &= \pi^{;\alpha\beta} \pi_{;\beta\gamma} \pi^{;\gamma\delta} \pi_{;\delta\alpha} - \phi^2 F^{\alpha\beta} F^{\gamma\delta} \pi_{;\alpha}{}^\epsilon \pi_{;\gamma\epsilon} \pi_{;\beta\delta} + \frac{\phi^4}{8} (F^{\alpha\beta} F_{\alpha\delta} \pi_{;\beta} \pi^{;\delta})^2 - \phi F^{\alpha\beta} F^{\gamma\delta} \pi_{;\alpha\gamma} \pi_{;\beta\delta} \phi^{;\epsilon} \pi_{;\epsilon} \\ &\quad - F^{\alpha\beta} F_{\alpha\gamma} \pi^{;\gamma} (\phi_{;\delta} \pi^{;\delta})^2 + \frac{1}{\phi^4} (\phi_{;\alpha} \pi^{;\alpha})^4, \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} \hat{R}_{MN} \hat{R}^{MN} &= R^{\alpha\beta} R_{\alpha\beta} - \frac{2}{\phi} R^{\alpha\beta} \phi_{;\alpha\beta} + F^{\alpha\beta} F_{\alpha}{}^\gamma R_{\beta\gamma} \phi^2 + \frac{1}{\phi^2} \phi^{;\alpha\beta} \phi_{;\alpha\beta} - \phi F^{\alpha\beta} F_{\alpha}{}^\gamma \phi_{;\beta\gamma} + \frac{\phi^4}{4} F^{\alpha\beta} F_{\alpha}{}^\gamma F_{\beta}{}^\delta F_{\gamma\delta} - \frac{9}{2} F^{\alpha\beta} F_{\alpha}{}^\gamma \phi_{;\beta} \phi_{;\gamma} \\ &\quad - 3\phi F^{\alpha\beta} \nabla^\gamma F_{\alpha\gamma} \phi_{;\beta} + \frac{\phi^2}{2} \nabla^\alpha F_{\alpha}{}^\beta \nabla^\gamma F_{\beta\gamma} + \frac{1}{\phi^2} (\square \phi)^2 + \frac{\phi}{2} F^{\alpha\beta} F_{\alpha\beta} \square \phi + \frac{\phi^4}{16} F^{\alpha\beta} F_{\alpha\beta} F^{\gamma\delta} F_{\gamma\delta}, \end{aligned} \quad (\text{A7})$$

$$\begin{aligned}
\hat{R}^{ABCD}\hat{R}_{ABCD} = & R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} + 3\phi^2 F^{\alpha\beta}F^{\gamma\delta}R_{\alpha\gamma\beta\delta} + \frac{3}{8}F^{\alpha\beta}F_{\alpha\beta}F^{\gamma\delta}F_{\gamma\delta}\phi^4 + \frac{5}{8}F^{\alpha\beta}F_{\alpha}{}^{\gamma}F_{\beta}{}^{\delta}F_{\gamma\delta}\phi^4 - \frac{5}{4}\nabla^{\gamma}F^{\alpha\beta}\nabla_{\gamma}F_{\alpha\beta}\phi^2 \\
& - 4\phi F^{\alpha\beta}\phi_{;\gamma}(\nabla^{\gamma}F_{\alpha\beta} - \nabla_{\alpha}F_{\beta}{}^{\gamma}) - 6(F^{\alpha\beta}\phi_{;\gamma})^2 - 6F^{\alpha\beta}F_{\alpha}{}^{\gamma}\phi_{;\beta}\phi_{;\gamma} + \frac{\phi^2}{2}\nabla^{\gamma}F^{\alpha\beta}\nabla_{\alpha}F_{\gamma\beta} \\
& + \frac{4}{\phi^2}\phi^{;\alpha\beta}\phi_{;\alpha\beta} + 2\phi F^{\alpha\beta}F_{\alpha}{}^{\gamma}\phi_{;\beta\gamma},
\end{aligned} \tag{A8}$$

$$\begin{aligned}
\hat{R}^{AB}\pi_{;AB} = & -\frac{1}{\phi}\phi^{;\alpha\beta}\pi_{;\alpha\beta} - \frac{1}{\phi^2}(\Box\phi)\phi^{;\alpha}\pi_{;\alpha} + R^{\alpha\beta}\pi_{;\alpha\beta} + \frac{\phi^2}{2}F^{\alpha\beta}F_{\alpha}{}^{\gamma}\pi_{;\beta\gamma} + \frac{\phi^2}{2}F^{\alpha\beta}\nabla^{\gamma}F_{\alpha\gamma}\pi_{;\beta} + \frac{3\phi}{2}F^{\alpha\beta}F_{\alpha}{}^{\gamma}\phi_{;\beta}\pi_{;\gamma} \\
& - \frac{\phi}{4}F^{\alpha\beta}F_{\alpha\beta}\phi^{;\gamma}\pi_{;\gamma},
\end{aligned} \tag{A9}$$

$$\begin{aligned}
\hat{R}^{AB}\pi_{;A}{}^C\pi_{;BC} = & R^{\alpha\beta}\pi_{;\alpha}{}^{\gamma}\pi_{;\beta\gamma} - \frac{1}{\phi}\phi^{;\alpha\beta}\pi_{;\alpha}{}^{\gamma}\pi_{;\beta\gamma} - \frac{1}{\phi^3}(\phi^{;\alpha}\pi_{;\alpha})^2(\Box\phi) \\
& + F^{\alpha\beta}F_{\alpha}{}^{\gamma}\left(\frac{\phi^2}{2}\pi_{;\beta}{}^{\delta}\pi_{;\gamma\delta} - \frac{\phi^4}{8}F_{\beta}{}^{\delta}F_{\gamma}{}^{\kappa}\pi_{;\delta}\pi_{;\kappa} + \frac{3}{2}\phi_{;\beta}\phi^{;\delta}\pi_{;\gamma}\pi_{;\delta} + \frac{\phi}{4}(\Box\phi)\pi_{;\beta}\pi_{;\gamma}\right) \\
& + F^{\alpha\beta}F^{\gamma\delta}\left(-\frac{\phi^2}{4}R_{\alpha\gamma}\pi_{;\beta}\pi_{;\delta} + \frac{\phi}{4}\phi_{;\alpha\gamma}\pi_{;\beta}\pi_{;\delta} + \frac{3\phi}{2}\pi_{;\alpha\gamma}\phi_{;\beta}\pi_{;\delta} + \frac{\phi^4}{16}F_{\alpha\beta}F_{\gamma}{}^{\kappa}\pi_{;\delta}\pi_{;\kappa}\right) \\
& - \frac{\phi^2}{2}F^{\alpha\beta}\nabla^{\gamma}F_{\gamma}{}^{\delta}\pi_{;\alpha\delta}\pi_{;\beta} + \frac{1}{2}F^{\alpha\beta}\nabla^{\gamma}F_{\alpha\gamma}\phi^{;\delta}\pi_{;\beta}\pi_{;\delta}\phi + \frac{1}{4}(F^{\alpha\beta}\phi^{;\gamma}\pi_{;\gamma})^2,
\end{aligned} \tag{A10}$$

$$\begin{aligned}
\hat{R}^{ABCD}\pi_{;AC}\pi_{;BD} = & R^{\alpha\beta\gamma\delta}\pi_{;\alpha\gamma}\pi_{;\beta\delta} - \frac{2}{\phi^2}\phi^{;\alpha\beta}\pi_{;\alpha\beta}\phi^{;\gamma}\pi_{;\gamma} + F^{\alpha\beta}\left(\frac{3\phi^2}{4}F^{\gamma\delta}\pi_{;\alpha\gamma}\pi_{;\beta\delta} + \phi^2\nabla^{\gamma}F_{\alpha}{}^{\delta}\pi_{;\beta}\pi_{;\gamma\delta} + 3\phi F_{\alpha}{}^{\gamma}\phi^{;\delta}\pi_{;\beta}\pi_{;\gamma\delta}\right. \\
& \left. - \frac{\phi}{2}F_{\alpha}{}^{\gamma}\phi^{;\delta}\pi_{;\delta}\pi_{;\beta\gamma}\phi - \frac{\phi^4}{8}F_{\alpha}{}^{\gamma}F_{\beta}{}^{\delta}F_{\gamma}{}^{\kappa}\pi_{;\delta}\pi_{;\kappa}\right) - \frac{1}{2}F^{\alpha\beta}F^{\gamma\delta}\phi_{;\alpha\gamma}\pi_{;\beta}\pi_{;\delta}\phi.
\end{aligned} \tag{A11}$$

APPENDIX B: DYNAMICS OF THE COSMOLOGICAL BACKGROUND

The equations of motion for the background fields, from the Lagrangian (6) have the form

$$\begin{aligned}
\delta h^{00}: F = & 2F_X\dot{\pi}^2 - K_{\pi}\dot{\pi}^2 + 2K_X\dot{\pi}^3(H_{\phi} + 3H) - 6G_4H(H_{\phi} + H) + 24G_{4X}\dot{\pi}^2H(H + H_{\phi}) + 24G_{4XX}H\dot{\pi}^4(H_{\phi} + H) \\
& - 2G_{4\pi}\dot{\pi}(H_{\phi} + 3H) - 4G_{4X\pi}\dot{\pi}^3(H_{\phi} + 3H) + 10G_{5X}H^2\dot{\pi}^3(3H_{\phi} + H) + 4G_{5XX}H^2\dot{\pi}^5(3H_{\phi} + H) \\
& - 9G_{5\pi}H\dot{\pi}^2(H + H_{\phi}) - 6G_{5X\pi}H\dot{\pi}^4(H_{\phi} + H) - 18G_6H_{\phi}H^3 + 108G_{6X}\dot{\pi}^2H_{\phi}H^3 - 18G_{6\pi}H^2\dot{\pi}(3H_{\phi} + H) \\
& - 72G_{6X\pi}H^2\dot{\pi}^3(3H_{\phi} + H) + 216G_{6XX}\dot{\pi}^4H_{\phi}H^3 + 48G_{6XXX}H_{\phi}H^3\dot{\pi}^6 - 24G_{6XX\pi}H^2\dot{\pi}^5(3H_{\phi} + H),
\end{aligned} \tag{B1}$$

$$\begin{aligned}
\delta h^{ii}: 0 = & F - 2K_X\ddot{\pi}\dot{\pi}^2 - K_{\pi}\dot{\pi}^2 + 2G_4(2\dot{H} + 3H^2 + 2HH_{\phi} + \dot{H}_{\phi} + H_{\phi}^2) - 4G_{4X}\dot{\pi}^2(3H^2 + 2HH_{\phi} + 2\dot{H} + \dot{H}_{\phi} + H_{\phi}^2) \\
& - 4G_{4X}\dot{\pi}\ddot{\pi}(2H + H_{\phi}) + 2G_{4\pi}(\ddot{\pi} + \dot{\pi}(H_{\phi} + 2H)) + 2G_{4\pi\pi}\dot{\pi}^2 + 4G_{4\pi X}\ddot{\pi}\dot{\pi}^2 - 4G_{4\pi X}\dot{\pi}^3(H_{\phi} + 2H) \\
& - 8G_{4XX}\ddot{\pi}\dot{\pi}^3(H_{\phi} + 2H) + G_{5\pi}\dot{\pi}^2(3H^2 + \dot{H}_{\phi} + H_{\phi}^2 + 2\dot{H} + 2HH_{\phi}) + 2G_{5\pi}\dot{\pi}\ddot{\pi}(H_{\phi} + 2H) \\
& - 4G_{5X}\dot{\pi}^3(H\dot{H} + H^3 + 2H_{\phi}H^2 + H_{\phi}\dot{H} + H\dot{H}_{\phi} + HH_{\phi}^2) - 6G_{5X}\ddot{\pi}H\dot{\pi}^2(2H_{\phi} + H) - 2G_{5\pi X}H\dot{\pi}^4(H + 2H_{\phi}) \\
& + 2G_{5\pi X}\ddot{\pi}\dot{\pi}^3(H_{\phi} + 2H) - 4G_{5XX}H\ddot{\pi}\dot{\pi}^4(H + 2H_{\phi}) + G_{5\pi\pi}\dot{\pi}^3(H_{\phi} + 2H) + 24G_{6\pi XX}H\ddot{\pi}\dot{\pi}^4(2H_{\phi} + H) \\
& + 6G_6H(\dot{H}_{\phi}H + HH_{\phi}^2 + 2H_{\phi}\dot{H} + 2H_{\phi}H^2) + 6G_{6\pi}H\ddot{\pi}(2H_{\phi} + H) + 6G_{6\pi}\dot{\pi}(2H\dot{H}_{\phi} + 2HH_{\phi}^2 + 2H\dot{H} \\
& + 2H^3 + 5H_{\phi}H^2 + 2H_{\phi}\dot{H}) + 6G_{6\pi\pi}H\dot{\pi}^2(2H_{\phi} + H) - 36G_{6X}\dot{\pi}\ddot{\pi}H_{\phi}H^2 \\
& - 24G_{6X}H\dot{\pi}^2(2H_{\phi}\dot{H} + 2H_{\phi}H^2 + \dot{H}_{\phi}H + HH_{\phi}^2) + 48G_{6\pi X}H\ddot{\pi}\dot{\pi}^2(2H_{\phi} + H) \\
& + 24G_{6\pi X}\dot{\pi}^3(H_{\phi}\dot{H} + H_{\phi}H^2 + H\dot{H}_{\phi} + HH_{\phi}^2 + H\dot{H} + H^3) - 144G_{6XX}\ddot{\pi}\dot{\pi}^3H_{\phi}H^2 + 12G_{6\pi XX}H\dot{\pi}^4(2H_{\phi} \\
& + H) - 24G_{6\pi XX}\dot{\pi}^5H_{\phi}H^2 - 24G_{6XX}\dot{\pi}^4(2HH_{\phi}\dot{H} + 2H_{\phi}H^3 + \dot{H}_{\phi}H^2 + H^2H_{\phi}^2) - 48\ddot{\pi}\dot{\pi}^5G_{6XXX}H_{\phi}H^2,
\end{aligned} \tag{B2}$$

$$\begin{aligned}
\delta\phi: 0 = & G_4(-2HH_\phi + \dot{H} - \dot{H}_\phi - H_\phi^2 + 3H^2) + G_{4\pi}\dot{\pi}(H - H_\phi) + 4G_{4XX}\ddot{\pi}\dot{\pi}^3(H_\phi - H) \\
& + 2G_{4X}\dot{\pi}^2(-3H^2 + 2HH_\phi - \dot{H} + \dot{H}_\phi + H_\phi^2) - 2G_{4X}\dot{\pi}\ddot{\pi}(H - H_\phi) + 2G_{4\pi X}\dot{\pi}^3(H_\phi - H) \\
& + \frac{1}{2}G_{5\pi}\dot{\pi}^2(3H^2 - 2HH_\phi - \dot{H}_\phi - H_\phi^2 + \dot{H}) + G_{5\pi}\dot{\pi}\ddot{\pi}(H - H_\phi) - \frac{1}{2}G_{5\pi\pi}\dot{\pi}^3(H_\phi - H) \\
& + 6G_{5X}H\ddot{\pi}^2(H_\phi - H) + 2G_{5X}\dot{\pi}^3(2H_\phi H^2 - 3H^3 - 2H\dot{H} + H_\phi\dot{H} + HH_\phi + HH_\phi^2) \\
& + G_{5\pi X}\dot{\pi}^3(H_\phi - H)(2H\ddot{\pi} - \ddot{\pi}) - 4G_{5XX}H\ddot{\pi}\dot{\pi}^4(H - H_\phi) \\
& + 3G_6H(-2H_\phi H^2 - \dot{H}_\phi H - HH_\phi^2 - 2H_\phi\dot{H} + 3\dot{H}H + 3H^3) + 18G_{6X}\dot{\pi}H^2\ddot{\pi}(H_\phi - H) \\
& + 12G_{6X}H\dot{\pi}^2(2H_\phi H^2 + 2H_\phi\dot{H} + \dot{H}_\phi H + HH_\phi^2 - 3\dot{H}H - 3H^3) - 48G_{6\pi X}H\ddot{\pi}^2(H_\phi - H) \\
& + 3G_{6\pi}\dot{\pi}(-5H_\phi H^2 + 7H^3 - 2H\dot{H}_\phi - 2HH_\phi^2 + 4H\dot{H} - 2H_\phi\dot{H}) - 6G_{6\pi}\dot{\pi}\ddot{\pi}(H_\phi - H) \\
& - 6G_{6\pi\pi}H\dot{\pi}^2(H_\phi - H) + 12G_{6XX}H\dot{\pi}^4(2H_\phi H^2 + 2H_\phi\dot{H} + \dot{H}_\phi H + HH_\phi^2 - 3\dot{H}H - 36H^3) \\
& + 12G_{6\pi X}\dot{\pi}^3(-H_\phi H^2 + 2H^3 - H_\phi\dot{H} - HH_\phi - HH_\phi^2 + 2H\dot{H}) + 72G_{6XX}H^2\ddot{\pi}\dot{\pi}^3(H_\phi - H) \\
& + 12G_{6\pi XX}H\dot{\pi}^4(H_\phi - H)(H\ddot{\pi} - 2\ddot{\pi}) - 12G_{6\pi\pi X}H\dot{\pi}^4(H_\phi - H) + 24G_{6XXX}H^2\ddot{\pi}\dot{\pi}^5(H_\phi - H), \tag{B3}
\end{aligned}$$

where we denote $H = \dot{a}/a$ and $H_\phi = \dot{\phi}/\phi$, dot denotes the derivative with respect to the cosmic time t .

APPENDIX C: COEFFICIENTS OF THE QUADRATIC ACTION

In the tensor sector we defined $\mathcal{G}_\tau = \frac{A_5}{2}$ and $\mathcal{F}_\tau = A_2$, and in the vector sector

$$\mathcal{G}_V = B_1, \quad \mathcal{F}_V = \frac{1}{8}C_{11} \quad \mathcal{K} = 12A_1. \tag{C1}$$

These coefficients and those shown in the quadratic action for the scalar sector are,

$$\begin{aligned}
A_1 = & -6G_4 + 12G_{4X}(\dot{\pi})^2 - 3G_{5\pi}(\dot{\pi})^2 + 6G_{5X}(\dot{\pi})^3(H + H_\phi) + 72G_{6XX}HH_\phi(\dot{\pi})^4 \\
& + 72G_{6X}HH_\phi(\dot{\pi})^2 - 36G_{6X\pi}(\dot{\pi})^3(H_\phi + H) - 18G_6HH_\phi - 18G_{6\pi}\dot{\pi}(H + H_\phi), \tag{C2}
\end{aligned}$$

$$\begin{aligned}
A_2 = & 2G_4 - G_{5\pi}(\dot{\pi})^2 - 2G_{5X}(\ddot{\pi})(\dot{\pi})^2 - 12G_{6X}H_\phi^2(\dot{\pi})^2 + 12G_{6X\pi}(\ddot{\pi})(\dot{\pi})^2 \\
& - 12G_{6X\pi}H_\phi(\dot{\pi})^3 - 24G_{6XX}H_\phi(\ddot{\pi})(\dot{\pi})^3 - 12G_{6X}H_\phi\ddot{\pi}(\ddot{\pi}) - 12G_{6X}\dot{H}_\phi(\dot{\pi})^2 \\
& + 6G_6\dot{H}_\phi + 6G_6H_\phi^2 + 6G_{6\pi\pi}(\dot{\pi})^2 + 6G_{6\pi}H_\phi\dot{\pi} + 6G_{6\pi}(\ddot{\pi}), \tag{C3}
\end{aligned}$$

$$\begin{aligned}
A_3 = & F_X(\dot{\pi})^2 + 4K_X(\dot{\pi})^3(H_\phi + 3H) - K_\pi(\dot{\pi})^2 - 2G_{4\pi}\dot{\pi}(H_\phi + 3H) + 42G_{4X}H^2(\dot{\pi})^2 \\
& + 42G_{4X}HH_\phi(\dot{\pi})^2 + 96G_{4XX}HH_\phi(\dot{\pi})^4 + 96G_{4XX}H^2(\dot{\pi})^4 - 10G_{4X\pi}(\dot{\pi})^3(H_\phi + 3H) \\
& - 6G_4HH_\phi - 6G_4H^2 - 27G_{5X\pi}HH_\phi(\dot{\pi})^4 - 27G_{5X\pi}H^2(\dot{\pi})^4 + 78G_{5XX}H_\phi H^2(\dot{\pi})^5 \\
& + 26G_{5XX}H^3(\dot{\pi})^5 - 18G_{5\pi}H^2(\dot{\pi})^2 - 18G_{5\pi}HH_\phi(\dot{\pi})^2 + 90G_{5X}H_\phi H^2(\dot{\pi})^3 + 30G_{5X}H^3(\dot{\pi})^3 \\
& + 456G_{6XXX}H_\phi H^3(\dot{\pi})^6 - 504G_{6XX\pi}H_\phi H^2(\dot{\pi})^5 - 168G_{6XXX}H^3(\dot{\pi})^5 + 972G_{6XX}H_\phi H^3(\dot{\pi})^4 \\
& - 702G_{6X\pi}H_\phi H^2(\dot{\pi})^3 - 234G_{6X\pi}H^3(\dot{\pi})^3 + 306G_{6X}H_\phi H^3(\dot{\pi})^2 - 36G_6H_\phi H^3 - 108G_{6\pi}H_\phi\dot{\pi}H^2 \\
& - 36G_{6\pi}\dot{\pi}H^3 + 2F_{XX}(\dot{\pi})^4 - K_{X\pi}(\dot{\pi})^4 + 2K_{XX}(\dot{\pi})^5(H_\phi + 3H) + 24G_{4XXX}HH_\phi(\dot{\pi})^6 \\
& + 24G_{4XXX}H^2(\dot{\pi})^6 - 4G_{4XX\pi}(\dot{\pi})^5(H_\phi + 3H) - 6G_{5XX\pi}HH_\phi(\dot{\pi})^6 - 6G_{5XX\pi}H^2(\dot{\pi})^6 \\
& + 12G_{5XXX}H_\phi H^2(\dot{\pi})^7 + 4G_{5XXX}H^3(\dot{\pi})^7 + 48G_{6XXX}H_\phi H^3(\dot{\pi})^8 - 72G_{6XXX\pi}H_\phi H^2(\dot{\pi})^7 \\
& - 24G_{6XXX\pi}H^3(\dot{\pi})^7, \tag{C4}
\end{aligned}$$

$$\begin{aligned}
A_4 = & 2K_X(\dot{\pi})^3 - 2G_4(2H + H_\phi) + 8G_{4X}(\dot{\pi})^2(2H + H_\phi) - 4G_{4X\pi}(\dot{\pi})^3 - 2G_{4\pi}\dot{\pi} \\
& + 8G_{4XX}(\dot{\pi})^4(H_\phi + 2H) + 10G_{5X}H^2(\dot{\pi})^3 + 20G_{5X}HH_\phi(\dot{\pi})^3 - 3G_{5\pi}(\dot{\pi})^2(2H + H_\phi) \\
& - 2G_{5X\pi}(\dot{\pi})^4(2H + H_\phi) + 8G_{5XX}HH_\phi(\dot{\pi})^5 + 4G_{5XX}H^2(\dot{\pi})^5 + 216G_{6XX}H_\phi H^2(\dot{\pi})^4 \\
& + 48G_{6XX\pi}H_\phi H^2(\dot{\pi})^6 - 48G_{6XX\pi}HH_\phi(\dot{\pi})^5 - 24G_{6XX\pi}H^2(\dot{\pi})^5 + 108G_{6X}H_\phi H^2(\dot{\pi})^2 \\
& - 144G_{6X\pi}HH_\phi(\dot{\pi})^3 - 72G_{6X\pi}H^2(\dot{\pi})^3 - 18G_6H_\phi H^2 - 36G_{6\pi}HH_\phi\dot{\pi} - 18G_{6\pi}\dot{\pi}H^2,
\end{aligned} \tag{C5}$$

$$A_5 = -\frac{2}{3}A_1, \tag{C6}$$

$$A_6 = -3A_4, \tag{C7}$$

$$A_7 = \frac{2}{3}A_1, \tag{C8}$$

$$\begin{aligned}
A_8 = & 2K_X(\dot{\pi})^2 - 2G_{4\pi} + 4G_{4X}\dot{\pi}(2H + H_\phi) + 8G_{4XX}(\dot{\pi})^3(H_\phi + 2H) - 4G_{4X\pi}(\dot{\pi})^2 \\
& + 12G_{5X}HH_\phi(\dot{\pi})^2 + 6G_{5X}H^2(\dot{\pi})^2 - 2G_{5\pi}\dot{\pi}(2H + H_\phi) - 2G_{5X\pi}(\dot{\pi})^3(H_\phi + 2H) \\
& + 8G_{5XX}HH_\phi(\dot{\pi})^4 + 4G_{5XX}H^2(\dot{\pi})^4 + 48G_{6XX\pi}H_\phi H^2(\dot{\pi})^5 - 48G_{6XX\pi}HH_\phi(\dot{\pi})^4 \\
& - 24G_{6XX\pi}H^2(\dot{\pi})^4 + 36G_{6X}H_\phi\dot{\pi}H^2 + 144G_{6XX}H_\phi H^2(\dot{\pi})^3 - 48G_{6X\pi}H^2(\dot{\pi})^2 \\
& - 96G_{6X\pi}HH_\phi(\dot{\pi})^2 - 12G_{6\pi}HH_\phi - 6G_{6\pi}H^2,
\end{aligned} \tag{C9}$$

$$A_9 = -A_8, \tag{C10}$$

$$A_{10} = -3A_8, \tag{C11}$$

$$\begin{aligned}
A_{11} = & -2F_X\dot{\pi} - 4F_{XX}(\dot{\pi})^3 + 2K_{X\pi}(\dot{\pi})^3 + 2K_\pi\dot{\pi} - 4K_{XX}(\dot{\pi})^4(H_\phi + 3H) \\
& - 6K_X(\dot{\pi})^2(H_\phi + 3H) - 36G_{4X}HH_\phi\dot{\pi} - 36G_{4X}\dot{\pi}H^2 - 144G_{4XX}HH_\phi(\dot{\pi})^3 \\
& - 144G_{4XX}H^2(\dot{\pi})^3 + 16G_{4X\pi}(\dot{\pi})^2(H_\phi + 3H) + 2G_{4\pi}(H_\phi + 3H) - 48G_{4XXX}HH_\phi(\dot{\pi})^5 \\
& - 48G_{4XXX}H^2(\dot{\pi})^5 + 8G_{4XX\pi}(\dot{\pi})^4(H_\phi + 3H) - 120G_{5XX}H_\phi H^2(\dot{\pi})^4 - 40G_{5XX}H^3(\dot{\pi})^4 \\
& - 90G_{5X}H_\phi H^2(\dot{\pi})^2 - 30G_{5X}H^3(\dot{\pi})^2 + 42G_{5X\pi}H^2(\dot{\pi})^3 + 18G_{5\pi}\dot{\pi}H^2 + 42G_{5X\pi}HH_\phi(\dot{\pi})^3 \\
& + 18G_{5\pi}HH_\phi\dot{\pi} + 12G_{5XX\pi}HH_\phi(\dot{\pi})^5 + 12G_{5XX\pi}H^2(\dot{\pi})^5 - 24G_{5XXX}H_\phi H^2(\dot{\pi})^6 \\
& - 8G_{5XXX}H^3(\dot{\pi})^6 - 96G_{6XXX}H_\phi H^3(\dot{\pi})^7 - 720G_{6XXX}H_\phi H^3(\dot{\pi})^5 + 144G_{6XXX\pi}H_\phi H^2(\dot{\pi})^6 \\
& + 48G_{6XXX\pi}H^3(\dot{\pi})^6 + 792G_{6XX\pi}H_\phi H^2(\dot{\pi})^4 + 264G_{6XX\pi}H^3(\dot{\pi})^4 - 1080G_{6XX}H_\phi H^3(\dot{\pi})^3 \\
& - 180G_{6X}H_\phi\dot{\pi}H^3 + 756G_{6X\pi}H_\phi H^2(\dot{\pi})^2 + 252G_{6X\pi}H^3(\dot{\pi})^2 + 18G_{6\pi}H^3 + 54G_{6\pi}H_\phi H^2,
\end{aligned} \tag{C12}$$

$$\begin{aligned}
A_{12} = & 2F_X\dot{\pi} + 2K_X(\dot{\pi})^2(H_\phi + 3H) - 2K_\pi\dot{\pi} + 12G_{4X}HH_\phi\dot{\pi} + 12G_{4X}\dot{\pi}H^2 \\
& + 2G_{4\pi\pi}\dot{\pi} - 4G_{4X\pi}(\dot{\pi})^2(5H + 2H_\phi) - 2G_{4\pi}H + 24G_{4XX}HH_\phi(\dot{\pi})^3 + 24G_{4XX}H^2(\dot{\pi})^3 \\
& + 6G_{5X}H^3(\dot{\pi})^2 + 18G_{5X}H_\phi H^2(\dot{\pi})^2 - 6G_{5\pi}\dot{\pi}H^2 + (2H + H_\phi)G_{5\pi\pi}(\dot{\pi})^2 - 8G_{5X\pi}H^2(\dot{\pi})^3 \\
& - 10G_{5X\pi}HH_\phi(\dot{\pi})^3 - 6G_{5\pi}HH_\phi\dot{\pi} + 12G_{5XX}H_\phi H^2(\dot{\pi})^4 + 4G_{5XX}H^3(\dot{\pi})^4 + 48G_{6XXX}H_\phi H^3(\dot{\pi})^5 \\
& + 144G_{6XX}H_\phi H^3(\dot{\pi})^3 - 96G_{6XX\pi}H_\phi H^2(\dot{\pi})^4 - 24G_{6XX\pi}H^3(\dot{\pi})^4 + 36G_{6X}H_\phi\dot{\pi}H^3 \\
& + 24G_{6X\pi}HH_\phi(\dot{\pi})^3 + 12G_{6X\pi}H^2(\dot{\pi})^3 - 168G_{6X\pi}H_\phi H^2(\dot{\pi})^2 - 48G_{6X\pi}H^3(\dot{\pi})^2 \\
& + 12G_{6\pi\pi}HH_\phi\dot{\pi} + 6G_{6\pi\pi}\dot{\pi}H^2 - 6G_{6\pi}H^3 - 12G_{6\pi}H_\phi H^2,
\end{aligned} \tag{C13}$$

$$\begin{aligned}
A_{13} = & -4G_{4\pi} + 8G_{4X}\dot{\pi}(H + H_\phi) + 8G_{4X}\ddot{\pi} + 8G_{4X\pi}(\dot{\pi})^2 + 16G_{4XX}\ddot{\pi}(\dot{\pi})^2 \\
& - 4G_{5\pi}\dot{\pi}(H + H_\phi) - 2G_{5\pi\pi}(\dot{\pi})^2 - 4G_{5X\pi}\ddot{\pi}(\dot{\pi})^2 - 4G_{5\pi}\ddot{\pi} + 8G_{5X}\dot{\pi}\ddot{\pi}(H + H_\phi) + 8G_{5X}HH_\phi(\dot{\pi})^2 \\
& + 4G_{5X}\dot{H}(\dot{\pi})^2 + 4G_{5X}H^2(\dot{\pi})^2 + 4G_{5X\pi}(\dot{\pi})^3(H + H_\phi) + 8G_{5XX}\ddot{\pi}(\dot{\pi})^3(H + H_\phi) + 4G_{5X}\dot{H}_\phi(\dot{\pi})^2 \\
& + 4G_{5X}H_\phi^2(\dot{\pi})^2 - 48G_{6XX\pi}\ddot{\pi}(\dot{\pi})^3(H_\phi + H) + 48G_{6XX\pi}HH_\phi(\dot{\pi})^4 + 96G_{6XXX}HH_\phi\ddot{\pi}(\dot{\pi})^4 \\
& + 48G_{6XX}H_\phi\dot{H}(\dot{\pi})^3 + 48G_{6XX}H_\phi H^2(\dot{\pi})^3 + 48G_{6XX}HH_\phi\dot{H}(\dot{\pi})^3 + 48G_{6XX}HH_\phi^2(\dot{\pi})^3 \\
& - 24G_{6X\pi\pi}(\dot{\pi})^3(H_\phi + H) + 24G_{6X}(HH_\phi\ddot{\pi} + HH_\phi\dot{\pi} + H\ddot{\pi}H_\phi^2 + H_\phi\dot{H}\dot{\pi} + H_\phi\dot{\pi}H^2) \\
& - 24G_{6X\pi}(3(H_\phi + H)\dot{\pi}\ddot{\pi} + 2HH_\phi(\dot{\pi})^2 + \dot{H}_\phi(\dot{\pi})^2 + H_\phi^2(\dot{\pi})^2 + \dot{H}(\dot{\pi})^2 + H^2(\dot{\pi})^2) \\
& - 12G_{6\pi}(\dot{H}_\phi + H_\phi^2 + \dot{H} + H^2 + HH_\phi), \tag{C14}
\end{aligned}$$

$$\begin{aligned}
A_{14} = & F_X + 2F_{XX}(\dot{\pi})^2 - K_{X\pi}(\dot{\pi})^2 - K_\pi + 2K_X\dot{\pi}(H_\phi + 3H) + 2K_{XX}(\dot{\pi})^3(H_\phi + 3H) \\
& - 4G_{4XX\pi}(\dot{\pi})^3(H_\phi + 3H) - 6G_{4X\pi}\ddot{\pi}(H_\phi + 3H) + 24G_{4XXX}HH_\phi(\dot{\pi})^4 \\
& + 24G_{4XXX}H^2(\dot{\pi})^4 + 6G_{4X}H(H + H_\phi) + 48G_{4XX}H(\dot{\pi})^2(H + H_\phi) \\
& - 3G_{5\pi}H(H_\phi + H) - 6G_{5XX\pi}HH_\phi(\dot{\pi})^4 - 6G_{5XX\pi}H^2(\dot{\pi})^4 + 264G_{6XXX}H_\phi H^3(\dot{\pi})^4 \\
& + 14G_{5XX}(\dot{\pi})^3 H^2(3H_\phi + H) + 4G_{5XXX}(\dot{\pi})^5 H^2(3H_\phi + H) + 6G_{5X}H^2\dot{\pi}(3H_\phi + H) \\
& - 96G_{6XX\pi}(\dot{\pi})^3 H^2(3H_\phi + H) - 24G_{6XXX\pi}(\dot{\pi})^5 H^2(3H_\phi + H) + 48G_{6XXX}H_\phi H^3(\dot{\pi})^6 \\
& - 54G_{6X\pi}\dot{\pi}H^2(3H_\phi + H) + 18G_{6X}H_\phi H^3 - 15G_{5X\pi}H(\dot{\pi})^2(H_\phi + H) + 252G_{6XX}H_\phi H^3(\dot{\pi})^2, \tag{C15}
\end{aligned}$$

$$\begin{aligned}
A_{15} = & -F_X + K_\pi - K_{X\pi}(\dot{\pi})^2 - 2K_{XX}\ddot{\pi}(\dot{\pi})^2 - 2K_X\ddot{\pi} - 2(H_\phi + 2H)K_X\dot{\pi} + 6G_{4X\pi}\ddot{\pi} \\
& + 6(H_\phi + 2H)G_{4X\pi}\dot{\pi} + 4G_{4XX\pi}\ddot{\pi}(\dot{\pi})^2 - 4(H_\phi + 2H)G_{4XX\pi}(\dot{\pi})^3 - 8(H_\phi + 2H)G_{4XXX}\ddot{\pi}(\dot{\pi})^3 \\
& + 2G_{4X\pi\pi}(\dot{\pi})^2 + (H_\phi + 2H)G_{5X\pi\pi}(\dot{\pi})^3 - 4G_{4XX}(3(2H + H_\phi)\dot{\pi}\ddot{\pi} \\
& + (\dot{\pi})^2(5H^2 + 2\dot{H} + \dot{H}_\phi + H_\phi^2 + 6HH_\phi)) - 2G_{4X}(2\dot{H} + 3H^2 + 2HH_\phi + \dot{H}_\phi + H_\phi^2) \\
& + G_{5\pi}(2\dot{H} + 3H^2 + 2HH_\phi + \dot{H}_\phi + H_\phi^2) + G_{5X\pi}((\dot{\pi})^2(2\dot{H} + 5H^2 + 6HH_\phi + \dot{H}_\phi + H_\phi^2) \\
& + (4H_\phi + 8H)\dot{\pi}\ddot{\pi}) - G_{5XX}(10H(\dot{\pi})^2\ddot{\pi}(2H_\phi + H) + 4(\dot{\pi})^3(12H_\phi H^2 + (H_\phi + H)\dot{H} \\
& + H^3 + HH_\phi + HH_\phi^2)) - 2G_{5X}((2H + 2H_\phi)\dot{H}\dot{\pi} + 2\dot{\pi}H^3 + 5H_\phi\dot{\pi}H^2 + 2H\dot{H}_\phi\dot{\pi} \\
& + 2H\dot{\pi}H_\phi^2 + 2HH_\phi\ddot{\pi} + \ddot{\pi}H^2) + 2G_{5XX\pi}((H_\phi + 2H)\ddot{\pi}(\dot{\pi})^3 - 2HH_\phi(\dot{\pi})^4 - H^2(\dot{\pi})^4) \\
& - 4G_{5XXX}H\ddot{\pi}(\dot{\pi})^4(2H_\phi + H) + 24G_{6XX\pi}(3H\ddot{\pi}(\dot{\pi})^2(2H_\phi + H) \\
& + (\dot{\pi})^3(HH_\phi^2 + 2H_\phi H^2 + (H_\phi + H)\dot{H} + H^3 + HH_\phi)) + 24G_{6XXX}H(\dot{\pi})^4(2H_\phi\ddot{\pi} + \ddot{\pi}H - H_\phi H\dot{\pi}) \\
& - 48G_{6XXX}H_\phi\ddot{\pi}H^2(\dot{\pi})^5 - 24G_{6XXX}H(\dot{\pi})^3(8H_\phi\ddot{\pi}H + 2H_\phi\dot{H}\dot{\pi} + 2H_\phi H^2\dot{\pi} + \dot{H}_\phi H\dot{\pi} + HH_\phi^2\dot{\pi}) \\
& + 12G_{6XX\pi\pi}H(\dot{\pi})^4(2H_\phi + H) - 12G_{6XX}H((\dot{\pi})^2(8H_\phi\dot{H} + 8H_\phi H^2 + 4HH_\phi^2 + 4\dot{H}_\phi H) + 9H_\phi\dot{\pi}\ddot{\pi}H) \\
& + 18G_{6X\pi}(2HH_\phi\ddot{\pi} + \ddot{\pi}H^2 + 2(H + H_\phi)\dot{H}\dot{\pi} + 2\dot{\pi}H^3 + 2H\dot{\pi}H_\phi^2 + 5H_\phi\dot{\pi}H^2 + 2H\dot{H}_\phi\dot{\pi}) \\
& + 6G_{6X\pi\pi}H(\dot{\pi})^2(2H_\phi + H) - 6G_{6X}H(\dot{H}_\phi H + HH_\phi^2 + 2H_\phi\dot{H} + 2H_\phi H^2), \tag{C16}
\end{aligned}$$

$$\begin{aligned}
A_{17} = & F_\pi - 2F_{X\pi}(\dot{\pi})^2 + 6G_{4\pi}H^2 + 2(H_\phi + 3H)G_{4\pi\pi}\dot{\pi} + 4(3H + H_\phi)G_{4X\pi\pi}(\dot{\pi})^3 \\
& + 6G_{5X\pi\pi}HH_\phi(\dot{\pi})^4 + 6G_{5X\pi\pi}H^2(\dot{\pi})^4 + 18G_{6\pi\pi}\dot{\pi}H^3 + 54G_{6\pi\pi}H_\phi\dot{\pi}H^2 + K_{\pi\pi}(\dot{\pi})^2 \\
& - 2(H_\phi + 3H)K_{X\pi}(\dot{\pi})^3 + 6G_{4\pi}HH_\phi + 9G_{5\pi\pi}HH_\phi(\dot{\pi})^2 + 9G_{5\pi\pi}H^2(\dot{\pi})^2 \\
& - 24G_{4X\pi}H(\dot{\pi})^2(H + H_\phi) - 24G_{4XX\pi}H(\dot{\pi})^4(H_\phi + H) - 10G_{5X\pi}H^2(\dot{\pi})^3(3H_\phi + H) \\
& - 4G_{5XX\pi}(\dot{\pi})^5 H^2(3H_\phi + H) - 216G_{6XX\pi}H_\phi H^3(\dot{\pi})^4 - 108G_{6X\pi}H_\phi H^3(\dot{\pi})^2 \\
& + 72G_{6X\pi\pi}H^2(\dot{\pi})^3(3H_\phi + H) + 18G_{6\pi}H_\phi H^3 + 24G_{6XX\pi\pi}(\dot{\pi})^5 H^2(3H_\phi + H) - 48G_{6XXX\pi}H_\phi H^3(\dot{\pi})^6, \tag{C17}
\end{aligned}$$

$$\begin{aligned}
A_{18} = & -6F_X\dot{\pi} - 6K_{X\pi}(\dot{\pi})^3 + 6K_{\pi}\ddot{\pi} - 12K_{XX}\ddot{\pi}(\dot{\pi})^3 - 12K_X\dot{\pi}\ddot{\pi} - 12(3H + H_\phi)K_X(\dot{\pi})^2 \\
& + 6(4H + H_\phi)G_{4\pi} + 12G_{4X\pi\pi}(\dot{\pi})^3 - 48(H_\phi + 2H)G_{4XXX}\ddot{\pi}(\dot{\pi})^4 + 6(H_\phi + 2H)G_{5X\pi\pi}(\dot{\pi})^4 \\
& + 3(H_\phi + 2H)G_{5\pi\pi}(\dot{\pi})^2 - 12G_{4X}(\dot{H}_\phi\dot{\pi} + \dot{\pi}H_\phi^2 + 2\dot{H}\dot{\pi} + 9\dot{\pi}H^2 + 8HH_\phi\dot{\pi} + (2H + H_\phi)\ddot{\pi}) \\
& + 12G_{4X\pi}((6H + 2H_\phi)(\dot{\pi})^2 + 3\dot{\pi}\ddot{\pi}) - 24G_{4XX}(9H^2(\dot{\pi})^3 + 8HH_\phi(\dot{\pi})^3 + 4(H_\phi + 2H)\ddot{\pi}(\dot{\pi})^2 \\
& + 2\dot{H}(\dot{\pi})^3 + \dot{H}_\phi(\dot{\pi})^3 + H_\phi^2(\dot{\pi})^3) + 24G_{4XX\pi}(\ddot{\pi}(\dot{\pi})^3 - (H_\phi + 2H)(\dot{\pi})^4) \\
& + 6G_{5X\pi}((\dot{\pi})^3(7H^2 + 2\dot{H} + \dot{H}_\phi + H_\phi^2 + 4HH_\phi) + 5(H_\phi + 2H)\ddot{\pi}(\dot{\pi})^2) \\
& + 6G_{5\pi}(\dot{\pi}(9H^2 + 2\dot{H} + \dot{H}_\phi + H_\phi^2 + 8HH_\phi) + (H_\phi + 2H)\ddot{\pi}) \\
& - 36G_{5X}((\dot{\pi})^2((H + H_\phi)\dot{H} + 2H^3 + 5H_\phi H^2 + HH_\phi^2 + HH_\phi\dot{H}) + \dot{\pi}\ddot{H}(H + 2H_\phi)) \\
& + 12G_{5XX\pi}(H_\phi + 2H)\ddot{\pi}(\dot{\pi})^4 - 12G_{5XX}(7\ddot{\pi}H^2(\dot{\pi})^3 + 10H_\phi H^2(\dot{\pi})^4 + 14HH_\phi\ddot{\pi}(\dot{\pi})^3 \\
& + 2(H + H_\phi)\dot{H}(\dot{\pi})^4 + 4H^3(\dot{\pi})^4 + 2HH_\phi\dot{H}(\dot{\pi})^4 + 2HH_\phi^2(\dot{\pi})^4) - 12G_{5XX\pi}(\dot{\pi})^5 H(H + 2H_\phi) \\
& - 24G_{5XXX}(\dot{\pi})^5 \ddot{\pi}H(H + 2H_\phi) + 144G_{6XXX\pi}(\dot{\pi})^5 \ddot{\pi}H(2H_\phi + H) \\
& + 72G_{6XX\pi}(16HH_\phi\ddot{\pi}(\dot{\pi})^3 + 8\ddot{\pi}H^2(\dot{\pi})^3 + 5H_\phi H^2(\dot{\pi})^4 + 2HH_\phi^2(\dot{\pi})^4 + 2HH_\phi\dot{H}(\dot{\pi})^4 \\
& + 2(H_\phi + H)\dot{H}(\dot{\pi})^4 + 4H^3(\dot{\pi})^4) - 144G_{6XXX\pi}H_\phi H^2(\dot{\pi})^6 - 288G_{6XXX}H_\phi\ddot{\pi}H^2(\dot{\pi})^6 \\
& - 1584G_{6XXX}H_\phi\ddot{\pi}H^2(\dot{\pi})^4 - 144G_{6XXX}(\dot{\pi})^5 H(2H_\phi\dot{H} + 4H_\phi H^2 + \dot{H}_\phi H + HH_\phi^2) \\
& - 216G_{6XX}H(\dot{\pi})^2(8H_\phi H^3\dot{\pi} + 7H_\phi\ddot{\pi}H^2 + 2H^2H_\phi^2\dot{\pi} + 2\dot{H}_\phi H^2\dot{\pi} + 4HH_\phi\dot{H}\dot{\pi}) \\
& - 108G_{6X}(H_\phi\ddot{\pi}H^2 + \dot{H}_\phi\dot{\pi}H^2 + \dot{\pi}H^2H_\phi^2 + 2HH_\phi\dot{H}\dot{\pi} + 4H_\phi\dot{\pi}H^3) \\
& + 36G_{6X\pi}\ddot{\pi}(18HH_\phi\ddot{\pi} + 9\ddot{\pi}H^2 + 8HH_\phi^2\dot{\pi} + 39H_\phi H^2\dot{\pi} + 8H\dot{H}_\phi\dot{\pi} + 8(H_\phi + H)\dot{H}\dot{\pi} + 16H^3\dot{\pi}) \\
& + 72G_{6XX\pi\pi}(\dot{\pi})^5 H(2H_\phi + H) + 108G_{6X\pi\pi}H(\dot{\pi})^3(2H_\phi + H) \\
& + 18G_{6\pi}(2H\dot{H}_\phi + 2HH_\phi^2 + 2(H + H_\phi)\dot{H} + 2H^3 + 9H_\phi H^2), \tag{C18}
\end{aligned}$$

$$\begin{aligned}
A_{20} = & \frac{1}{2}F_{\pi\pi} - K_{X\pi}((\dot{\pi})^2(9H^2 + 3\dot{H} + \dot{H}_\phi + H_\phi^2 + 6HH_\phi) + 2\dot{\pi}\ddot{\pi}(H_\phi + 3H)) \\
& - 6G_{4X\pi}(\dot{\pi}(\dot{H}(2H + H_\phi) + 3H^3 + HH_\phi\dot{H} + HH_\phi^2 + 4H_\phi H^2) + \ddot{\pi}H(H + H_\phi)) \\
& + 2G_{4X\pi\pi}(3\dot{\pi}\ddot{\pi}(H_\phi + 3H) + (\dot{\pi})^2(6HH_\phi + 9H^2 + 6\dot{H} + \dot{H}_\phi + H_\phi^2)) \\
& + 4G_{4XX\pi\pi}(\ddot{\pi}(\dot{\pi})^3(H_\phi + 3H) - 3(\dot{\pi})^4 H(H_\phi + H)) - 24G_{4XXX\pi}H\ddot{\pi}(\dot{\pi})^4(H_\phi + H) \\
& - 12G_{4XX\pi}(4\ddot{\pi}(\dot{\pi})^2 H(H + H_\phi) + (\dot{\pi})^3(4H_\phi H^2 + \dot{H}(H_\phi + 2H) + 3H^3 + HH_\phi\dot{H} + HH_\phi^2)) \\
& + G_{4\pi\pi}(\dot{H}_\phi + H_\phi^2 + 3\dot{H} + 6H^2 + 3HH_\phi) + 3G_{5\pi\pi}(\dot{\pi}(HH_\phi^2 + \dot{H}(2H + H_\phi) + 3H^3 + HH_\phi\dot{H} \\
& + 4H_\phi H^2) + \ddot{\pi}H(H_\phi + H)) + G_{5X\pi\pi}((\dot{\pi})^3(3HH_\phi^2 + \dot{H}(6H + 3H_\phi) + 7H^3 + 6H_\phi H^2 + 3HH_\phi\dot{H} \\
& + 15\ddot{\pi}(\dot{\pi})^2 H(H_\phi + H)) - G_{5XX\pi}(14\ddot{\pi}(\dot{\pi})^3 H^2(3H_\phi + H) + 2(\dot{\pi})^4 H(10H_\phi H^2 + 6H_\phi\dot{H} + 3\dot{H}H \\
& + 3H^3 + 3\dot{H}_\phi H + 3HH_\phi^2)) - 3G_{5X\pi}((\dot{\pi})^2 H(3HH_\phi^2 + 3\dot{H}H + 3H^3 + 10H_\phi H^2 + 3\dot{H}_\phi H \\
& + 6H_\phi\dot{H}) + 2\dot{\pi}\ddot{\pi}H^2(3H_\phi + H)) + \frac{3}{2}G_{5\pi\pi\pi}(\dot{\pi})^2 H(H_\phi + H) + 2G_{5XX\pi\pi}(3\ddot{\pi}(\dot{\pi})^4 H(H_\phi + H) \\
& - (\dot{\pi})^5 H^2(3H_\phi + H)) - 4G_{5XXX\pi}\ddot{\pi}(\dot{\pi})^5 H^2(3H_\phi + H) + 3G_{5X\pi\pi\pi}(\dot{\pi})^4 H(H_\phi + H) \\
& + G_{6XX\pi\pi}(96\ddot{\pi}(\dot{\pi})^3 H^2(3H_\phi + H) + 12(\dot{\pi})^4 H(3HH_\phi^2 + 5H_\phi H^2 + 6H_\phi\dot{H} + 3\dot{H}H + 3H^3 + 3\dot{H}_\phi H)) \\
& + 24G_{6XXX\pi\pi}(\ddot{\pi}(\dot{\pi})^5 H^2(3H_\phi + H) - H_\phi H^3(\dot{\pi})^6) - 24G_{6XXX\pi}(11H_\phi\ddot{\pi}H^3(\dot{\pi})^4 \\
& + (\dot{\pi})^5 H^2(3H_\phi\dot{H} + 3H_\phi H^2 + \dot{H}_\phi H + HH_\phi^2))
\end{aligned}$$

$$\begin{aligned}
& + 12G_{6XX\pi\pi\pi}(\dot{\pi})^5 H^2(3H_\phi + H) - 36G_{6XX\pi}(7H_\phi \ddot{\pi} H^3(\dot{\pi})^2 \\
& + (\dot{\pi})^3(2H^3 H_\phi^2 + 6H_\phi \dot{H} H^2 + 6H_\phi H^4 + 2\dot{H}_\phi H^3)) + G_{6X\pi\pi}(54\dot{\pi} \ddot{\pi} H^2(3H_\phi + H) \\
& + 18(\dot{\pi})^2(4\dot{H}_\phi H^2 + 4H^2 H_\phi^2 + 144HH_\phi \dot{H} + 13H_\phi H^3 + 4\dot{H} H^2 + 4H^4)) \\
& + 18G_{6X\pi\pi\pi}(\dot{\pi})^3 H^2(3H_\phi + H) - 18G_{6X\pi} H^2(\dot{\pi}(3H_\phi \dot{H} + 3H_\phi H^2 + HH_\phi^2 + \dot{H}_\phi H) \\
& + H_\phi H \ddot{\pi}) + 9G_{6\pi\pi} H(\dot{H}_\phi H + HH_\phi^2 + \dot{H} H + 2H_\phi \dot{H} + H^3 + 3H_\phi H^2) \\
& - F_{X\pi\pi}(\dot{\pi})^2 + \frac{1}{2}K_{\pi\pi\pi}(\dot{\pi})^2 - 2F_{XX\pi}\ddot{\pi}(\dot{\pi})^2 - F_{X\pi}(\ddot{\pi} + \dot{\pi}(H_\phi + 3H)) \\
& + K_{\pi\pi}(\ddot{\pi} + \dot{\pi}(H_\phi + 3H)) + K_{X\pi\pi}(\ddot{\pi}(\dot{\pi})^2 - (\dot{\pi})^3(H_\phi + 3H)) + 2G_{4X\pi\pi\pi}(\dot{\pi})^3(H_\phi + 3H) \\
& - 2K_{XX\pi}\ddot{\pi}(\dot{\pi})^3(H_\phi + 3H) - 48G_{6XXX\pi}H_\phi \ddot{\pi} H^3(\dot{\pi})^6, \tag{C19}
\end{aligned}$$

$$B_1 = \frac{1}{2}G_4 - G_{4X}(\dot{\pi})^2 + \frac{1}{4}G_{5\pi}(\dot{\pi})^2 - G_{5X}H(\dot{\pi})^3 - 6G_{6XX}H^2(\dot{\pi})^4 + 6G_{6X\pi}H(\dot{\pi})^3 - 6G_{6X}H^2(\dot{\pi})^2 + \frac{3}{2}G_6H^2 + 3G_{6\pi}H\dot{\pi}, \tag{C20}$$

$$\begin{aligned}
B_2 = & -G_4(3H_\phi + H) + 2G_{4X}((\dot{\pi})^2(3H_\phi + H) + \dot{\pi} \ddot{\pi}) + G_{5X\pi}(\dot{\pi})^3(2H\dot{\pi} - \ddot{\pi}) - \frac{1}{2}G_{5\pi}(2\dot{\pi} \ddot{\pi} + (\dot{\pi})^2(H + 3H_\phi)) \\
& + 2G_{5X}(3H\ddot{\pi}(\dot{\pi})^2 + (\dot{\pi})^3(3HH_\phi + \dot{H} + H^2)) + 12G_{6XX}(\dot{\pi})^3 H(6\ddot{\pi}H + \dot{\pi}(2\dot{H} + H^2 + 3H_\phi H)) \\
& + 12G_{6X\pi}H(\dot{\pi})^4(H\dot{\pi} - 2\ddot{\pi}) + 6G_{6X}\dot{\pi}H(3\ddot{\pi}H + 2\dot{\pi}(2\dot{H} + H^2 + 3H_\phi H)) - 12G_{6X\pi}(\dot{\pi})^2(4H\ddot{\pi} + \dot{\pi}(3HH_\phi + \dot{H})) \\
& - 3G_6H(2\dot{H} + H^2 + 3H_\phi H) - 3G_{6\pi}(\dot{\pi}(3H^2 + 2\dot{H} + 6HH_\phi) + 2H\ddot{\pi}) - G_{4\pi}\ddot{\pi} + 2G_{4X\pi}(\dot{\pi})^3 - \frac{1}{2}G_{5\pi\pi}(\dot{\pi})^3 \\
& + 4G_{4XX}\ddot{\pi}(\dot{\pi})^3 + 4G_{5XX}H\ddot{\pi}(\dot{\pi})^4 + 24G_{6XX}\ddot{\pi}H^2(\dot{\pi})^5 - 12G_{6X\pi\pi}H(\dot{\pi})^4 - 6G_{6\pi\pi}H(\dot{\pi})^2, \tag{C21}
\end{aligned}$$

$$C_1 = -12B_1, \tag{C22}$$

$$C_2 = (H_\phi - H)C_1,$$

$$\begin{aligned}
C_4 = & -2K_X(\dot{\pi})^2 + 4G_{4X\pi}(\dot{\pi})^2 + 2G_{4\pi} - 24G_{4XX}H(\dot{\pi})^3 - 12G_{4X}H\dot{\pi} + 6G_{5X\pi}H(\dot{\pi})^3 \\
& + 6G_{5\pi}H\dot{\pi} - 18G_{5X}H^2(\dot{\pi})^2 - 12G_{5XX}H^2(\dot{\pi})^4 - 144G_{6XX}H^3(\dot{\pi})^3 + 72G_{6XX\pi}H^2(\dot{\pi})^4 \\
& - 48G_{6XXX}H^3(\dot{\pi})^5 - 36G_{6X}\dot{\pi}H^3 + 144G_{6X\pi}H^2(\dot{\pi})^2 + 18G_{6\pi}H^2, \tag{C23}
\end{aligned}$$

$$\begin{aligned}
C_5 = & -2F_X\dot{\pi} + 2K_\pi\dot{\pi} - 4K_{XX}\ddot{\pi}(\dot{\pi})^3 - 2K_{X\pi}(\dot{\pi})^3 + 4G_{4X\pi\pi}(\dot{\pi})^3 - 48G_{4XXX}H\ddot{\pi}(\dot{\pi})^4 \\
& + 2G_{4\pi}(2H_\phi + 3H) - 4K_X((\dot{\pi})^2(H_\phi + 3H) + \dot{\pi} \ddot{\pi}) - 12G_{4X}(\dot{\pi}(4H^2 + 2HH_\phi + \dot{H}) \\
& + H\ddot{\pi}) + 4G_{4X\pi}(2(\dot{\pi})^2(H_\phi + 3H) + 3\dot{\pi} \ddot{\pi}) + 8G_{4XX\pi}(\dot{\pi})^3(\ddot{\pi} - 3H\dot{\pi}) \\
& - 24G_{4XX}((\dot{\pi})^3(2HH_\phi + 4H^2 + \dot{H}) + 4H\ddot{\pi}(\dot{\pi})^2) + 6G_{5X\pi}(\dot{\pi})^2(\dot{\pi}(2H^2 + 2HH_\phi + \dot{H}) \\
& + 5H\ddot{\pi}) + 6G_{5\pi}(\dot{\pi}(4H^2 + 2HH_\phi + \dot{H}) + H\ddot{\pi}) - 4G_{5XX}H(\dot{\pi})^3(2\dot{\pi}(5H^2 + 3H_\phi H + 3\dot{H}) \\
& + 21\ddot{\pi}H) - 12G_{5X}\dot{\pi}H(\dot{\pi}(5H^2 + 3H_\phi H + 3\dot{H}) + 3\ddot{\pi}H) + 12G_{5XX\pi}H(\dot{\pi})^4(\ddot{\pi} - H\dot{\pi}) \\
& + 24G_{6XX\pi}H(\dot{\pi})^3(\dot{\pi}(5H^2 + 6H_\phi H + 6\dot{H}) + 24\ddot{\pi}H) - 72G_{6XX}H^2(\dot{\pi})^2(2\dot{\pi}(2H_\phi H \\
& + 3\dot{H} + 3H^2) + 7\ddot{\pi}H) - 48G_{6XXX}H^2(\dot{\pi})^4(11\ddot{\pi}H + \dot{\pi}(2H_\phi H + 3\dot{H} + 3H^2)) \\
& + 48G_{6XXX\pi}H^2(\dot{\pi})^5(3\ddot{\pi} - H\dot{\pi}) + 36G_{6X\pi}\dot{\pi}H(\dot{\pi}(13H^2 + 8H_\phi H + 8\dot{H}) + 9\ddot{\pi}H) \\
& - 36G_{6X}H^2(\dot{\pi}(2H_\phi H + 3\dot{H} + 3H^2) + \ddot{\pi}H) + 18G_{6\pi}H(2H_\phi H + 2\dot{H} + 3H^2) \\
& + 3G_{5\pi\pi}H(\dot{\pi})^2 + 6G_{5X\pi\pi}H(\dot{\pi})^4 - 24G_{5XXX}\ddot{\pi}H^2(\dot{\pi})^5 + 72G_{6XX\pi\pi}H^2(\dot{\pi})^5 \\
& - 96G_{6XXX}\ddot{\pi}H^3(\dot{\pi})^6 + 108G_{6X\pi\pi}(\dot{\pi})^3 H^2 \tag{C24}
\end{aligned}$$

$$C_7 = -\frac{1}{3}C_1, \quad (C25)$$

$$C_8 = \frac{1}{3}C_1(H - H_\phi), \quad (C26)$$

$$\begin{aligned} C_9 = & -2K_X(\dot{\pi})^3 + 6G_4H - 24G_{4X}H(\dot{\pi})^2 + 4G_{4XX}(\dot{\pi})^3 + 2G_{4\pi}\dot{\pi} - 24G_{4XX}H(\dot{\pi})^4 - 30G_{5X}H^2(\dot{\pi})^3 + 9G_{5\pi}H(\dot{\pi})^2 \\ & + 6G_{5XX}H(\dot{\pi})^4 - 12G_{5XX}H^2(\dot{\pi})^5 - 216G_{6XX}H^3(\dot{\pi})^4 - 48G_{6XXX}H^3(\dot{\pi})^6 + 72G_{6XX\pi}H^2(\dot{\pi})^5 - 108G_{6X}H^3(\dot{\pi})^2 \\ & + 216G_{6X\pi}H^2(\dot{\pi})^3 + 18G_6H^3 + 54G_{6\pi}\dot{\pi}H^2 \end{aligned} \quad (C27)$$

$$\begin{aligned} C_{10} = & -2G_{4\pi} + 4G_{4X}(2H\dot{\pi} + \ddot{\pi}) + 4G_{4XX}(\dot{\pi})^2 + 8G_{4XX}\ddot{\pi}(\dot{\pi})^2 - G_{5\pi\pi}(\dot{\pi})^2 + 8G_{5XX}H\ddot{\pi}(\dot{\pi})^3 - 2G_{5\pi}(2H\dot{\pi} + \ddot{\pi}) \\ & + G_{5X}((\dot{\pi})^2(8H^2 + 4\dot{H}) + 8H\dot{\pi}\ddot{\pi}) + 2G_{5XX}(\dot{\pi})^2(2H\dot{\pi} - \ddot{\pi}) + 48G_{6XX}H(\dot{\pi})^2(2\ddot{\pi}H + \dot{\pi}(\dot{H} + H^2)) \\ & - 24G_{6X\pi\pi}H(\dot{\pi})^3 + 24G_{6XX\pi}H(\dot{\pi})^3(H\dot{\pi} - 2\ddot{\pi}) + 12G_{6X}H(\ddot{\pi}H + 2\dot{\pi}(\dot{H} + H^2)) \\ & - 24G_{6X\pi}\dot{\pi}(\dot{H} + 2H^2) + 3H\ddot{\pi} - 6G_{6\pi}(2\dot{H} + 3H^2) + 48G_{6XXX}\ddot{\pi}H^2(\dot{\pi})^4 \end{aligned} \quad (C28)$$

$$\begin{aligned} C_{11} = & -4G_4 + 24G_{6X}(H\dot{\pi}\ddot{\pi} + (\dot{\pi})^2(H^2 + \dot{H})) + 24G_{6XX}(\dot{\pi})^2(-\ddot{\pi} + H(\dot{\pi})) - 12G_6(\dot{H} + H^2) - 12G_{6\pi}(H\dot{\pi} + \ddot{\pi}) \\ & + (\dot{\pi})^2(2G_{5\pi} - 12G_{6\pi\pi}) + 4G_{5X}\ddot{\pi}(\dot{\pi})^2 + 48G_{6XX}H\ddot{\pi}(\dot{\pi})^3, \end{aligned} \quad (C29)$$

$$C_{12} = \frac{1}{3}C_1. \quad (C30)$$

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