

Real-time scattering in the lattice Schwinger model

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Tensor network methods have demonstrated their suitability for the study of equilibrium properties of lattice gauge theories, even close to the continuum limit. We use them in an out-of-equilibrium scenario, much less explored so far, by simulating the real-time collisions of composite mesons in the lattice Schwinger model. Constructing wave-packets of vector mesons at different incoming momenta, we observe the opening of the inelastic channel in which two heavier mesons are produced and identify the momentum threshold. To detect the products of the collision in the strong coupling regime we propose local quantities that could be measured in current quantum simulation platforms.

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I. INTRODUCTION

Scattering experiments are a well-established tool for probing fundamental physics. In particular, collision experiments allow the production of high energy and rare particles and thereby a study of their interactions. Precise theoretical predictions of such processes, necessary for their interpretation, often involve contributions that can not be extracted from diagrammatic perturbation theory. This is the case, for instance, for hadron collisions, where nonperturbative effects of Quantum Chromodynamics (QCD) may play a significant role [1]. The most powerful tool to address such nonperturbative regimes is lattice gauge theory (LGT), the discrete formulation of gauge field theories [2]. Using advanced numerical methods, like Quantum Monte Carlo [3,4], LGT has allowed the successful exploration of strong coupling phenomena, such as the hadron spectra in QCD, but real-time dynamics represents a challenge. Despite recent progress [5], a precise first-principles calculation of the scattering processes has not yet been possible, one of the reasons that motivates the search for alternative techniques [6].

In recent years quantum methods have unveiled potential alternative ways to explore fundamental physics (see [7–13] for reviews). Their central focus are LGTs, which also appear as effective low-energy descriptions of

strongly correlated condensed matter systems [14]. The Hamiltonian version of LGTs then constitutes a natural object for quantum simulations. While a full simulation of QCD is still beyond reach, current efforts are focused on simplified and low-dimensional LGTs. An important development is the quantum-information based tensor network (TN) methods [15–20] which have been successfully applied to study spectral and thermal equilibrium properties of low dimensional LGTs [6,21–25], in 1 + 1D cases achieving some of the most precise existing extrapolations to the continuum.

Real-time evolution phenomena, such as scattering dynamics, are among the most promising problems for a potential quantum advantage [26], since Monte Carlo methods suffer in this case from a sign problem. Tensor network algorithms, on the other hand, can be used to simulate LGT dynamics—but only for a limited time—before the entanglement in the system becomes too large for an efficient TN description. Yet, out-of-equilibrium simulations of LGT with TN have been used to explore the dynamics of pair production [27] and string breaking dynamics in several LGT models [28–32], whereas, far from the precision achieved by their equilibrium counterparts and not extrapolated to the continuum limit, such simulations overcome the limits of classical Monte Carlo methods and constitute a valuable tool to prepare and analyze the potential quantum simulations.

More recently, scattering experiments in LGT have attracted the attention of TN calculations and quantum simulation proposals [30,33,34]. The basic phenomenology of confinement [35,36] and meson scattering has also been

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explored recently in quantum spin models which share properties of LGT especially in $1 + 1\text{D}$ [37–43].

In this paper we use TN techniques to study elastic and inelastic scattering of composite particles in the lattice Schwinger model, which is the discrete lattice version of quantum electrodynamics in one spatial dimension. Due to its similarities with more complex LGTs, including confinement and chiral symmetry breaking, the Schwinger model is a standard test bench for LGT methods and has been focus of TN simulations, quantum simulation proposals and even experimental realizations (see [7,44] for reviews). Studies of scattering processes in the Schwinger model have been more scarce.

In absence of a background field, the spectrum of the model contains two stable particles, referred to as *vector* and *scalar*. Elastic processes between two vector mesons, below the threshold of production of the scalar, have been simulated in the strong coupling [30] and in the weak and intermediate coupling [33] regimes. In this regime, the particles collide and can bounce back, without the creation of new particles post collision, generating entanglement between the mesons. A first approach to the inelastic regime was explored recently in [34], with a focus on the thermodynamic limit and the nonperturbative regime near the confinement-deconfinement critical point and using a bosonized formulation of the model, aimed at proposing a quantum simulation scheme that could be implemented in circuit-QED.

In this work, we focus instead on the strong coupling regime at vanishing background field and explore the energy threshold required to open the inelastic channel and obtain particle production after the collision. Working in this regime allows us to propose simple initial state preparations and observables that could be amenable to realization on quantum simulators.

The paper is structured as follows: in Sec. II we briefly present the Schwinger model in the continuum limit and we show the lattice discretization we use in this work. In Sec. III we discuss the preparation of the initial state of two meson wave packets, indicate the momentum threshold for particle production and present the method used for the time evolution of the system. In Sec. IV we show the results of meson-meson collisions. We specifically identify the energy threshold for particle production and we compare our results with the predicted threshold obtained with density matrix renormalization group (DMRG) calculations.

II. MODEL

The Schwinger model [45] is quantum electrodynamics in one spatial dimension. It is the simplest gauge theory including dynamical matter and exactly solvable for massless fermions, yet it exhibits common features with those of quantum chromodynamics, like confinement and

chiral symmetry breaking [46]. Its lattice discretization is commonly used as a benchmark for numerical LGT methods [47,48] and has in particular been widely adopted in studies of Tensor Network techniques for lattice gauge theories, which started [49] with the application to the model of the DMRG algorithm.

A. The continuum limit

The continuum Hamiltonian of the Schwinger model in the temporal gauge can be written:

$$\mathcal{H} = - \int dx (i\bar{\Psi}(x)\gamma^1[\partial_1 + igA_1(x)]\Psi(x)) + \int dx \left(m\bar{\Psi}(x)\Psi(x) + \frac{1}{2}E(x)^2 \right), \quad (1)$$

with γ^0 and γ^1 the Dirac matrices in the temporal and spatial dimension, whereas ∂_1 is the partial derivative with respect to x . The fermionic field $\Psi(x)$ has two components and its conjugate is $\bar{\Psi} = \Psi^\dagger\gamma^0$. m is the fermionic mass and g is the coupling constant. In an infinite volume, the only independent parameter of the model is the adimensional ratio m/g . Note that in the temporal gauge $A^0 = 0$. The electric field $E(x)$ is then related to the vector gauge field $A^1(x)$ as $E(x) = -\partial_0 A^1(x)$.

The continuum Schwinger model is exactly solvable for the case $m/g = 0$ and $m/g \rightarrow \infty$ [45,50]. For $m/g = 0$, the Schwinger model can be solved using bosonization and corresponds to a free massive boson, the so-called Schwinger boson. For nonzero m/g , this boson becomes interacting.

We denote the ground state energy as E_{GS} . The first stable particle of the theory is the vector meson, a bound state (Schwinger boson) with mass M_V . The second stable particle is the scalar meson, with mass M_S , which can be considered a bound state of two Schwinger bosons [46,51].

These two bound states are stable particles with parity and charge conjugation being well-defined quantum numbers (CP symmetry) [46]. The bound state of two Schwinger bosons belongs in the same sector as the ground state (scalar sector), whereas the Schwinger boson belongs to the vector sector.

In open boundary conditions, integrating Gauss law fixes the electric field up to a constant, which can be interpreted as an external background field $g\theta/2\pi$, with θ the so-called vacuum angle [46,52]. In the strong coupling (small mass) regime we target in this study, it is possible to estimate the particle masses perturbatively. Specifically, the mass of the mesons M_V in the strong coupling regime can be obtained from second-order perturbation theory as [53,54]

$$\begin{aligned}
 M_V^2 &\equiv \mu_2^2 \\
 &= \mu_0^2 \left(1 + 3.5621 \frac{m}{\mu_0} \cos(\theta) \right. \\
 &\quad \left. + 5.4807 \frac{m^2}{\mu_0^2} - 2.0933 \frac{m^2}{\mu_0^2} \cos(2\theta) \right) \quad (2)
 \end{aligned}$$

with $\mu_0^2 = \frac{g^2}{\pi}$ the result from zero-order perturbation theory. The mass of the second excitation of the theory, the bound state M_S , is given in second-order perturbation theory by [51]

$$M_S^2 = 4M_V^2 - \Delta \quad (3)$$

with $\Delta = \frac{4\pi^4 m^2 \Sigma^2 \cos^2(\theta)}{M_V^2}$. $\Sigma = \frac{g}{2\pi} \mu_0$ is the fermion condensate with γ the Euler constant [53].

The form of the wave function of the bound states in the strong coupling regime of the continuum limit was previously studied first by Ref. [55] up to and including four-body states and then by Ref. [56] up to and including six-body states with the use of the six-body light-front Tamm-Dancoff approximation. In these works it was found that the composition of the bound states depends on the fermionic mass m/g . They specifically consist of a two-body component, a four-body component and a negligible six-body component. The contribution of the four-body component increases as the fermionic mass m/g is getting smaller.

B. The lattice

This section presents the lattice Schwinger model, the scattering processes of which will be studied in this paper. Here we are using the Kogut-Susskind staggered fermion formulation for the model [57]

$$\begin{aligned}
 W &= \frac{g^2 a}{2} \sum_n L_n^2 + m \sum_n (-1)^n \Phi_n^\dagger \Phi_n, \\
 &\quad - \frac{i}{2a} \sum_n (\Phi_n^\dagger e^{i\theta_n} \Phi_{n+1} - \text{H.c.}), \quad (4)
 \end{aligned}$$

with the lattice spacing being ga in coupling units. Specifically, we focus on the strong coupling regime, which means $m/g \ll 1$.

The fermionic annihilation (creation) operators Φ_n ($\Phi_n^\dagger$) reside on the lattice sites n . As single-component fermionic operators, they satisfy canonical anticommutation relations $\{\Phi_n^\dagger, \Phi_m\} = \delta_{nm}$ and $\{\Phi_n, \Phi_m\} = 0$. The gauge fields θ_n and L_n are the lattice equivalent to the vector potential and the electric field in the continuum. They occupy the links between two nearest neighboring sites (n and $n+1$). They are canonically conjugate operators, satisfying the commutation relation $[\theta_n, L_m] = i\delta_{nm}$.

In addition, the lattice version of the Gauss law reads [48]

$$L_n - L_{n-1} = \Phi_n^\dagger \Phi_n - \frac{1}{2} [1 - (-1)^n] \quad (5)$$

and needs to be satisfied by the physical states of the system.

By a Jordan-Wigner transformation, the Hamiltonian (4) can be mapped onto a spin model. The Gauss law, which now reads

$$L_n - L_{n-1} = \frac{1}{2} ((-1)^n + (\sigma_n^z)), \quad (6)$$

can be integrated out if we consider open boundary conditions, resulting in a spin Hamiltonian with long range interactions [48,58]. After rescaling it with a global $2/(ag^2)$ factor, one is left with the adimensional Hamiltonian

$$\begin{aligned}
 H &= \frac{2}{g^2 a} W = H_x + H_\mu + H_l \\
 &= x \sum_{n=0}^{N-2} [\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+] + \frac{\mu}{2} \sum_{n=0}^{N-1} [1 + (-1)^n \sigma_n^z] \\
 &\quad + \sum_{n=0}^{N-2} \left[l + \frac{1}{2} \sum_{k=0}^n ((-1)^k + \sigma_k^z) \right]^2, \quad (7)
 \end{aligned}$$

with $x = \frac{1}{g^2 a^2}$ and $\mu = \frac{2m}{g^2 a}$ being adimensional parameters and $\sigma^\pm = \frac{1}{2} (\sigma^x \pm i\sigma^y)$. l is the background electric field and is related to the vacuum angle as following: $l = \theta/2\pi$. It is set to zero in this work, $l = 0$.

The lattice model with open boundary conditions breaks most of the continuum symmetries. In particular, charge conjugation is no longer a good quantum number. However, the symmetry is restored in the thermodynamic limit and, if the system is large enough, it is possible to define an approximate charge conjugation operator whose expectation value can be used to distinguish the scalar and vector sectors [59].

1. The strong coupling regime

In this paper we study the spin formulation of the lattice Schwinger model solely focusing on the strong coupling regime. Although the strong-coupling regime that we consider is far from the continuum limit—where we aim to study physical phenomena without lattice artifacts—it remains an important regime. This is mainly because the system size and lattice spacing parameters chosen in this work are close to the parameters of the systems accessible by quantum simulators. Since here we use the rescaled Hamiltonian of Eq. (7) with the adimensional parameters μ and x , being in the strong coupling regime for a finite lattice means $x \ll 1$. In this regime H_x can be considered the

perturbation, with $H_\mu + H_l$ being the unperturbed part of the Hamiltonian. The strong coupling regime offers the advantage of having a simple ansatz for the initial mesons of the collision, as well as a clear energy threshold for producing the scalar mesons of interest. We note that a truncated Hilbert space approach, which has been employed for spin chains [41], would not work here.

In the infinite coupling limit, $x = 0$, the ground state of the unperturbed Hamiltonian $H_\mu + H_l$ is the state with all the sites empty

$$|0\rangle = |\downarrow\uparrow\downarrow\uparrow\cdots\downarrow\uparrow\rangle \quad (8)$$

The sites are numbered from left to right $n = 0, 1, \dots, N - 1$. Due to the fact that we are using the staggered fermions formulation, the odd sites correspond to the fermions and the even sites to the antifermions. Specifically, the occupied sites are $|\downarrow\rangle_O$ for the odd sites and $|\uparrow\rangle_E$ for the even sites ($|\uparrow\rangle_O$ and $|\downarrow\rangle_E$ are the empty odd and even sites correspondingly). Therefore, $|\downarrow\rangle_O$ are the particles (electrons) and $|\uparrow\rangle_E$ are the antiparticles (positrons).

In the limit of infinite coupling, $x = 0$, the first excitation of the theory $|1_V\rangle$ (the vector meson), with energy E_V , can be obtained simply as

$$|1_V\rangle = \frac{1}{\sqrt{N}} \sum_n (\sigma_n^+ \sigma_{n+1}^- - \sigma_{n+1}^+ \sigma_n^-) |0\rangle. \quad (9)$$

For small but nonvanishing values of x , the strong coupling expansion (SCE) [48], a perturbative expansion in x , can be used to obtain very precise estimates of the ground state energy and the mass spectrum, for small bare fermion mass [60].

Our goal is to create states of the mesons with well-defined momenta. Although we do not have the explicit solution for the operator that creates these particles, in the strong coupling regime we can use an ansatz that resembles the first order of SCE (10) but where the strong coupling vacuum has been substituted by the true ground state at finite x , which we denote $|\Omega\rangle$ and compute numerically following the method in [59]. Therefore, our ansatz for the state of the mesons can be written as

$$|1_V\rangle = \frac{1}{\sqrt{N}} \sum_n (\sigma_n^+ \sigma_{n+1}^- - \sigma_{n+1}^+ \sigma_n^-) |\Omega\rangle. \quad (10)$$

III. METHODS

A. Preparing the initial state

The first step for studying the elastic and inelastic meson-meson collisions is creating two meson wave packets in the initial state and giving them opposite momentum so that they eventually collide. We prepare two wave packets for

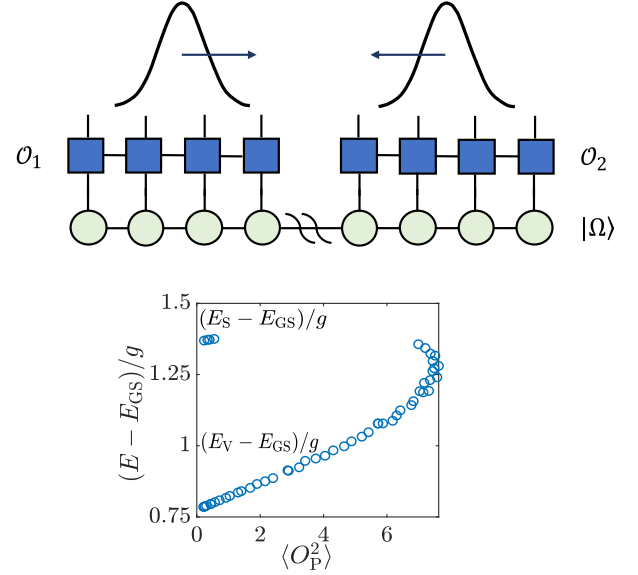


FIG. 1. Diagram above shows the TN structure for the initial state preparation following Eq. (12). The lower panel shows the dispersion relation of the adimensional Hamiltonian of Eq. (7) as $E = f(\langle O_P^2 \rangle)$, with $O_P = -ix \sum_n (\sigma_n^- \sigma_{n+1}^z \sigma_{n+2}^+ - \text{H.c.})$ the dimensionless momentum operator for the fermion field (see [59]). The lower branch corresponds to the vector meson and the upper one to the scalar, for which only the lowest momentum excitations are shown. The ground state energy, E_{GS}/g , is subtracted from all the energies plotted. We consider a finite system with lattice size $N = 100$ and physical volume $N/\sqrt{x} = 100$. The mass is $m/g = 10^{-5}$. The energies were calculated with variational MPS as in [59] with small bond dimension, $D = 50$.

the mesons as Gaussian superpositions of the operator in Eq. (10) acting on different positions, namely

$$\begin{aligned} |\varphi_i\rangle &= \mathcal{N}_i \sum_{n=\alpha_i^{\text{beg}}}^{\alpha_i^{\text{end}}} e^{-(n-c_i)^2/(2\sigma^2)} e^{-ink_i} (\sigma_n^+ \sigma_{n+1}^- - \sigma_{n+1}^+ \sigma_n^-) |\Omega\rangle \\ &= \mathcal{O}_i |\Omega\rangle \end{aligned} \quad (11)$$

with indices $i = 1, 2$ corresponding to the two mesons. We choose the wave packets to have centers c_i and common width σ . Initially, the wave packet of the first (second) meson is spread in the left (right) half of the system (see Fig. 1). We therefore have $\alpha_1^{\text{beg}} = 0, \alpha_1^{\text{end}} = N/2 - 1$ for the first meson and $\alpha_2^{\text{beg}} = N/2$ and $\alpha_2^{\text{end}} = N - 1$ for the second one. We consider the mesons to have opposite momenta with the same magnitude, $k_1 = -k_2 = k$. $\mathcal{N}_i$ are the necessary normalization factors. The total initial state is constructed as the product

$$|\varphi\rangle = (\mathcal{O}_1 \mathcal{O}_2) |\Omega\rangle. \quad (12)$$

Notice that this approach is made possible by working in the strong coupling regime, as this provides us with the explicit approximate description of the vector meson state shown in Eq. (10), which can be directly used for the creation of the wave packets.

B. Momentum threshold k_{thr}

In order to open an inelastic channel (i.e. to produce a final state allowed by the symmetries of the problem), the momentum of the incoming wave packets needs to be high enough to produce the new particles satisfying the kinematic constraints imposed by energy and momentum conservation.

We consider collisions in the center of mass of the incoming mesons, where the initial wave packets have opposite spatial momenta $p_1 = -p_2 \equiv p$. Momentum conservation imposes that the outgoing particles, two identical scalar mesons, have the same momenta with opposite directions, q and $-q$. Conservation of energy then requires

$$2\sqrt{M_V^2 + p^2} = 2\sqrt{M_S^2 + q^2}. \quad (13)$$

The final state with the smallest possible energy would correspond to creating the pair of scalars at rest, $q = 0$. This sets the momentum threshold for the incoming vector mesons at

$$p^2 = p_{\text{thr}}^2 = M_S^2 - M_V^2. \quad (14)$$

Notice that the relations above are written in terms of physical (continuum) momenta. Our initial state (11) is however written in terms of adimensional lattice quantities. In particular, the lattice momentum k is related to the physical one as $k = p/(g\sqrt{\chi})$.

Because we work on a finite lattice with open boundary conditions, the scalar mesons cannot be produced at rest but will carry at least the smallest lattice momentum $k_{\text{min}} = O(1/N)$, which will correct the threshold.

In the continuum limit, one could calculate the momentum threshold with the use of perturbation theory and Eqs. (2) and (3). Since in our parameter regime $N = 100$, $\chi = 1$ and $\mu = 2 \times 10^{-5}$, we are far away from that limit; we instead calculate the masses using a variational optimization of matrix product state (MPS), with the algorithm in Ref. [59] for which we obtain the masses $M_V/g = 0.785$, $M_S/g = 1.370$. With these values we estimate the lattice momentum threshold at

$$k_{\text{thr}}^{\text{MPS}} \approx 1.123. \quad (15)$$

C. Time evolution

In our work, we use TN methods to prepare the initial state and simulate the time evolution. First, we find a matrix

product state [16] approximation to the interacting vacuum $|\Omega\rangle$ by optimizing variationally the MPS ansatz that minimizes the energy, using the method in [59]. We truncate the bond dimension of the ground state to $D_{\text{GS}} = 40$. The operators $\mathcal{O}_1$ and $\mathcal{O}_2$ of Eq. (11) can be written as exact matrix product operators (MPOs) [61] with bond dimension $D = 4$. They can be then applied to the ground state, $|\Omega\rangle$ to create the meson wave packets of Eq. (11). We truncate the bond dimension of the wave packets to $D_V = 50$, which, when compared to the exact initial state of meson wave packets (with bond dimension $4D_{\text{GS}}$), has fidelity very close to 1 (with negligible corrections).

After creating the initial state with the two meson wave packets, we perform time evolution with standard tensor network techniques [15,16]. Specifically, we use a second order Suzuki-Trotter expansion of the evolution operator [62,63] with time step δ , using the splitting of the Hamiltonian described in [64], namely,

$$e^{-i\tau H} \approx (e^{-i\frac{\delta}{2}H_x^e} e^{-i\frac{\delta}{2}H_x^o} e^{-i\delta H_z} e^{-i\frac{\delta}{2}H_x^o} e^{-i\frac{\delta}{2}H_x^e})^{\tau/\delta}. \quad (16)$$

H_x^e and H_x^o are the hopping terms (part of H_x) acting on the two-body terms of even-odd and odd-even pairs of sites correspondingly. H_z is the mass term of the Hamiltonian together with the electric field term, $H_z = H_\mu + H_l$.

The exponentials $e^{-i\frac{\delta}{2}H_x^e}$ and $e^{-i\frac{\delta}{2}H_x^o}$ can be written as exact MPOs with bond dimension 4. The exponential of the term H_z can also be written as an exact MPO, diagonal in the z basis, with bond dimension that scales linearly with the system size as $\chi = N + 1$ [64].

Specifically, the elements of the MPO are

$$(M_n^{ij})_{L_{n-1}L_n} = e^{-i\delta h_n(i)} \delta_{ij} \delta_{L_n - L_{n-1}, G_n(i)}, \quad (17)$$

where the virtual indices label the electric flux on the links $L_n \in [-N/2, N/2]$ and the second delta ensures the fulfillment of Gauss law (6),

$$G_n(i) = \frac{1}{2} [(-1)^n + (-1)^i], \quad i = 0, 1 \quad (18)$$

and $h_n(i) = \mu(-1)^n G_n(i) + L_n^2$ for $n < N - 1$, with $h_{N-1}(i) = \mu(-1)^{N-1} G_{N-1}(i)$.

To be able to deal with large system sizes (of the order of 100 or more sites) it is convenient to truncate the bond dimension of this MPO to a constant value χ_{tr} , which can be related to a cutoff in the absolute value of the electric flux L_{max} , as $\chi_{\text{tr}} = 2L_{\text{max}} + 1$. This truncation, which was successfully used for thermal equilibrium states in [64,65], will be also valid in our case as far as the dynamics does not generate states with large occupations of the electric flux. The cutoff in the electric flux is restricted to $L_{\text{max}} = 8$ in our study. Applying the evolution operator increases the bond dimension of the time-evolved state $|\Psi(t)\rangle$ at each

time step. In order to control the computational cost, we truncate the bond dimension of the evolved state to a maximum value of $D = 50$. Even though we use relatively small bond dimensions for the ground state, the initial mesons and the evolved state, as well as a relatively small cutoff of the electric flux, we check that our qualitative observations are stable under changes of the truncation parameters, and this level of precision suffices to observe the effects of inelastic scattering in the cases under study (see the Appendix for details). Specifically, knowing that the total energy has to be conserved in our system, we control the errors by measuring the (relative) change of the total energy in comparison to the initial total energy of the system. We find that for the times of interest (up to $gt_{\text{phys}} = 140$) the relative difference is less than 6%.

IV. RESULTS

In this section we present the results of elastic and inelastic meson-meson scattering processes. Specifically, in Sec. IV A we observe the first signs of particle production, when having energies above the momentum threshold of Eq. (15) (inelastic channel). In Sec. IV B we calculate quantities that can be directly measured in experiments, indicating the signal of particle production for the inelastic channel. In Sec. IV C we study the four-body projector.

The results shown in this section correspond to a fixed fermion mass $m/g = 10^{-5}$ and physical volume $N/\sqrt{x} = 100$ (notice that we express all magnitudes in units of the coupling g).

These magnitudes are related to their corresponding lattice analogs through the lattice spacing $ga = 1/\sqrt{x}$. In particular, the time parameter τ appearing in the evolution operator $e^{-iH\tau}$ (16) is the adimensional lattice time. It is related to the physical time gt_{phys} as

$$\tau = \frac{gt_{\text{phys}}}{2\sqrt{x}} \quad (19)$$

The lattice size was fixed to $N = 100$ sites and the inverse lattice spacing to the value $x = 1$ in all the cases below.

A. Entropy

We first calculate the entanglement entropy $S(n, t)$ corresponding to a bipartition where one subsystem includes two neighboring sites n and $n + 1$, with $n = 0, \dots, N - 2$, and the other subsystem consists of the remaining sites

$$S(n, t) = -\text{Tr}[\rho_{n,n+1}(t) \log_2 \rho_{n,n+1}(t)], \quad (20)$$

with $\rho_{n,n+1}$ being the reduced density matrix of one of the subsystems. $S(n, t)$ is interpreted as the entanglement between the sites $\{n, n + 1\}$ and the rest of the system. Since the two-site reduced density matrix can in principle be recovered from local measurements, as recently realized in optical lattices in [66], this quantity could be more easily accessible in an experiment than the half-chain

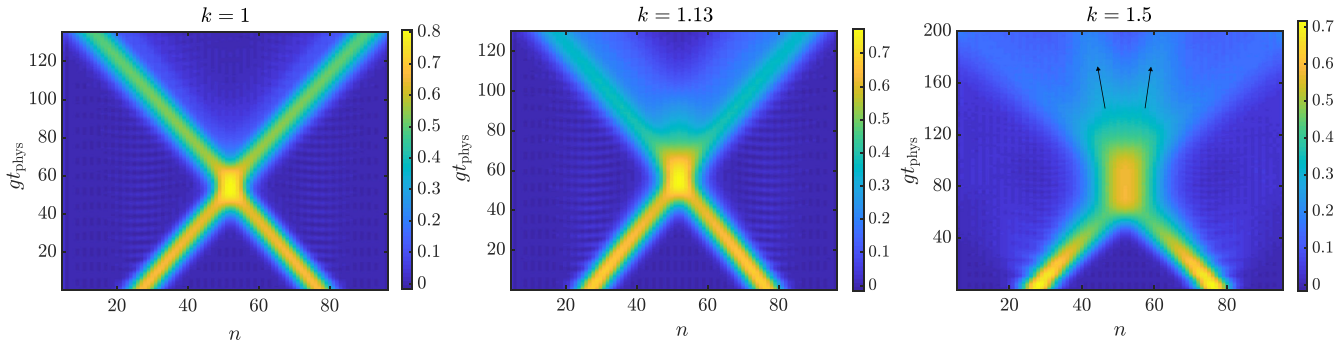


FIG. 2. Plots show the two-site entanglement entropy $S(n, t)$, defined in the main text (20), minus the entanglement entropy of the vacuum. The fermion mass is fixed to $m/g = 10^{-5}$. We consider a finite system with lattice size $N = 100$ and physical volume $N/\sqrt{x} = 100$. The standard deviation of the wave packets is $\sigma = 4$. In the left plot, the momentum magnitude of the incoming mesons is $k = 1$ and only the elastic channel is present. The elastic channel corresponds to two mesons exiting the collision (yellow color) with the same momentum as the momentum of the incoming mesons. In the middle plot, the momentum of the incoming mesons is $k = 1.13$, slightly above the threshold predicted by the variational MPS calculation $k_{\text{thr}}^{\text{MPS}}$ (15). At this point, the inelastic channel opens, with two bound states produced with velocities close to 0. The signal of the two bound states that were created after the collision is visible a bit below $gt_{\text{phys}} \sim 80$, with the two outgoing mesons coming from the elastic channel also being visible. The momentum of the incoming mesons for the plot on the right is $k = 1.5$, also above the momentum threshold. Both the elastic and inelastic channels are open. The two bound states created from the collision (inelastic channel) have non-zero velocities with opposite signs and less magnitude than the momenta of the outgoing mesons of the elastic channel. The elastic channel is represented by the light blue cone-like shape, after the collision, whereas the velocities of the bound states are shown with the two arrows. The parameters used in the numerical simulation were bond dimension $D = 50$, Trotter step $\delta = 0.1$ and maximum electric flux $L_{\text{trunc}} = 8$.

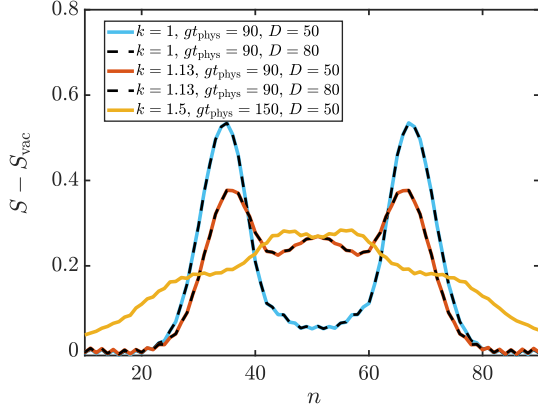


FIG. 3. Plot shows the two-site entanglement entropy $S(n, t)$ (20) minus the entanglement entropy of the vacuum, S_{vac} , on a lattice with size $N = 100$ and physical volume $N/\sqrt{x} = 100$, for the cases where the initial particles have momenta below, slightly above and above the momentum threshold, at $k = 1$, $k = 1.13$, and $k = 1.5$. The mass is $m/g = 10^{-5}$, and the standard deviation of the wave packets $\sigma = 4$. As in Fig. 2, the parameters used in the numerical simulation were bond dimension $D = 50$, Trotter step $\delta = 0.1$ and maximum electric flux $L_{\text{trunc}} = 8$. To demonstrate the numerical convergence, for the cases $k = 1$ and $k = 1.13$ we show also the results for $D = 80$ (black dash lines) while keeping all the other parameters the same.

entanglement that has been analyzed in other scattering simulations [30,33,34].

The results of the calculation of the entanglement entropy can be seen in Figs. 2 and 3. We study three different cases depending on the magnitude of the momenta of the incoming mesons. The three cases consider momenta below, slightly above and above the momentum threshold $k_{\text{thr}}^{\text{MPS}}$. The first case is the case where the momenta are below the momentum threshold, as seen in the left plot of Fig. 2. We specifically have lattice momenta $k_1 = -k_2 = k = 1$. Here, the incoming mesons do not have enough energy to create new particles. Therefore, only the elastic channel is present and the outgoing particles are two vector mesons with the same momenta as the incoming ones. Figure 3 shows the excess entanglement entropy with respect to the vacuum after the collision, for physical time $g t_{\text{phys}} = 90$ (blue line). The two peaks represent the wave packets of the two outgoing mesons. Notice the small increase in the entanglement between the two wave packets in comparison to the entanglement entropy of the vacuum. Because the local entropy we compute is not directly measuring entanglement between the outgoing mesons, we attribute this increment to the fluctuations imprinted onto the vacuum by the collision.

The second case is the middle plot of Fig. 2, where the incoming particles have enough energy to produce two scalar mesons of mass M_S with zero velocity. Specifically, as mentioned in Sec. III A in Eq. (15), the threshold for particle production is at $k_{\text{thr}}^{\text{MPS}} \approx 1.12$. As the initial momenta of the vector mesons is $k = 1.13$, we expect

producing scalar mesons that are almost at rest after the collision. Notice here that it is not possible to create particles with momentum exactly 0, because as mentioned in Sec. III B, the scalar mesons have to have at least momentum $k_{\text{min}} = O(1/N)$. One could observe that after the collision ($t_{\text{phys}} \sim 80$) there is a sign of particle production in the middle of the system, where the entanglement entropy is greater than the entanglement entropy between the outgoing particles of the elastic channel. This signal can be more clearly observed in Fig. 3 (orange line). The peak in the middle of the system corresponds to the two bound states produced with almost zero velocities. The higher peaks on the left and right of this signal correspond to the outgoing mesons of the elastic channel.

Finally, the right plot of Fig. 2 represents the case where the momentum of the incoming particles is considerably larger than the threshold, such that they have enough energy to produce two scalar particles of mass M_S , and the excess is transformed in kinetic energy of the outgoing mesons. In this case, we can notice both the elastic and the inelastic channels. Since the scalar mesons are produced with small velocities, their signal is found near the middle of the system, as also seen by Fig 3. One could also observe the signal of the elastic channel, near the edges of the same plot.

We notice here that the momentum threshold for particle production calculated from the variational MPS spectrum calculations was $k_{\text{thr}}^{\text{MPS}} \approx 1.12$. The simulated collisions behave as expected, with the first particle production observed at $k = 1.13$ and no signs of particle production at $k = 1$. We could also compare with the momentum threshold predicted by perturbation theory in the continuum limit, $p_{\text{thr}}^{\text{continuum}}/g \approx 0.98$, which was done with the use of Eqs. (2), (3), and (14).

B. Correlators

In this subsection we plot the electric flux correlator,

$$\mathcal{C}^{mn}(t) = \langle L^m L^n \rangle - \langle L^m \rangle \langle L^n \rangle, \quad (21)$$

with $L^n = \frac{1}{2} \sum_{k=0}^n ((-1)^k + \sigma_k^z)$. In principle, we can expect that this quantity can be measured in experiments, showing the signals of particle production.

The plots in this section show the electric flux correlators, $\mathcal{C}^{mn}(t)$, specifically for times after the collision. Figure 4 shows $\mathcal{C}^{mn}(t)$ for $g t_{\text{phys}} = 100$ for the case in which only the elastic channel appears, with the incoming mesons having momentum $k = 1$. As seen in the left plot of Fig. 2, the outgoing mesons appear to be occupying the lattice sites 20–40 and 60–80. This is verified by Fig. 4, where the two oval shapes around the diagonal correspond to the two outgoing mesons of the elastic channel and are concentrated around the same positions.

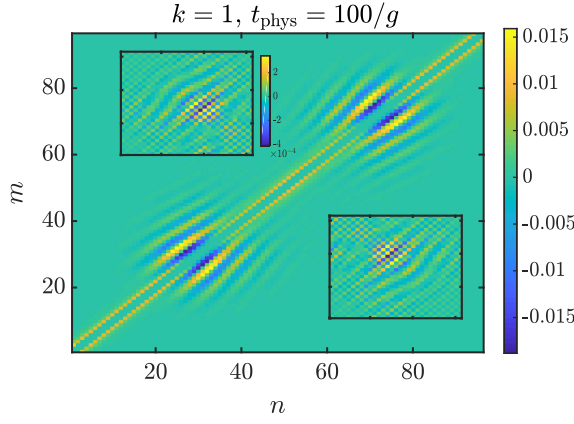


FIG. 4. Plot shows the electric flux correlator, C_l^{mn} , of a system size $N = 100$, corresponding to physical volume $N/\sqrt{x} = 100$ at lattice spacing $x = 1$, for times after the meson-meson collision, $gt_{\text{phys}} = 100$. The initial momentum of the mesons, $k = 1$, is below the momentum threshold. The fermion mass is $m/g = 10^{-5}$, and the standard deviation of the initial wave packets $\sigma = 4$. We set $C^{mn} = 0$ for $|m - n| \leq 1$ so that $L^{m,n}$ do not overlap. The insets show the enclosed area rescaled by a factor, to highlight the signal of the correlations between the two outgoing meson wave packets of the elastic channel. The bond dimension used for the calculations is $D = 50$ and the maximum electric flux $L_{\text{trunc}} = 8$.

In Fig. 5 we plot the correlator for $gt_{\text{phys}} = 180$ and initial meson momenta above the momentum threshold, at $k = 1.5$. In this case both the elastic and inelastic channel are present. The inelastic channel appears as the yellow signal around the main diagonal, with the two bound states visible around sites 40–50 and 60–70.

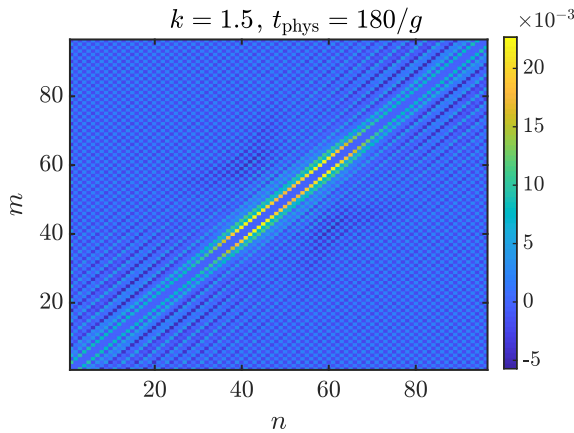


FIG. 5. Plot shows the electric flux correlator, C_l^{mn} , of a system size $N = 100$ and physical volume $N/\sqrt{x} = 100$, for times after the meson-meson collision, $gt_{\text{phys}} = 180$. The initial momenta of the mesons are above the momentum threshold, $k = 1.5$. The mass is $m/g = 10^{-5}$, and the standard deviation of the wave packets $\sigma = 4$. The bond dimension is $D = 50$ and the maximum electric flux $L_{\text{trunc}} = 8$. We set $C^{mn} = 0$ for $m = n$, $m = n - 1$ and $m = n + 1$.

C. Four-body projector

As seen in the subsections above, two different particles are produced when the momenta of the initial mesons are above the momentum threshold $k_{\text{thr}}^{\text{MPS}} \approx 1.12$. These are scalar mesons, which can be interpreted, as discussed in Sec. II A, as bound states of two Schwinger bosons.

Ideally, to unambiguously identify the particles produced as scalar mesons, we would like to measure the corresponding scalar operator, which is not known, in general. However, in the strong coupling regime, $m/g \rightarrow 0$, the continuum theory results ensure that the composition of the scalar meson in that case is dominated by four fermion terms, whereas the vector meson is dominated by two-fermion terms [55,56]. Even though we are far from that limit and both the ground states and excitations are strongly renormalized, we propose probing the appearance of the scalar mesons by calculating the time-dependent expectation value of the four-body projector

$$P_n(t) = \langle \Pi_{\uparrow}^n \Pi_{\downarrow}^{n+1} \Pi_{\uparrow}^{n+2} \Pi_{\downarrow}^{n+3} \rangle |_{n=\text{even}} \quad (22)$$

with $\Pi_{\uparrow}^n$, $\Pi_{\downarrow}^n$ the projectors

$$\begin{aligned} \Pi_{\uparrow}^n |\uparrow\rangle &= |\uparrow\rangle \\ \Pi_{\downarrow}^n |\downarrow\rangle &= |\downarrow\rangle \end{aligned} \quad (23)$$

and $\Pi_{\uparrow}^n |\downarrow\rangle = 0$, $\Pi_{\downarrow}^n |\uparrow\rangle = 0$. This will give a signal that the time-evolved state has a four-body component if and only if all the consecutive sites n , $n + 1$, $n + 2$ and $n + 3$ are occupied. The underlying assumption is that the wave functions are continuously connected to the strong-coupling regime, such that we can identify its signal as the presence of the scalar.

The results shown in Fig. 6 give yet another indication of the presence of the bound states. Specifically, in the left plot of Fig. 6 ($k = 1.13$) one observes the first signal of bound state production (yellow color). This corresponds to two scalar mesons produced almost at rest (approximately zero velocities). The right plot of Fig. 6 shows the case where the incoming mesons have momentum $k = 1.5$. After the collision, one observes signals of two bound states with opposite velocities (yellow color).

V. DISCUSSION

In this work we have used tensor network states (TNS) methods to simulate scattering processes in the lattice version of the Schwinger model. More concretely, we have studied the scattering in the center of mass of two vector mesons (composite particles) at different incident momenta and have been able to observe the threshold at which the inelastic channel opens and other particles of higher mass are produced. We have proposed entanglement and local observables to identify the character of the collision and the

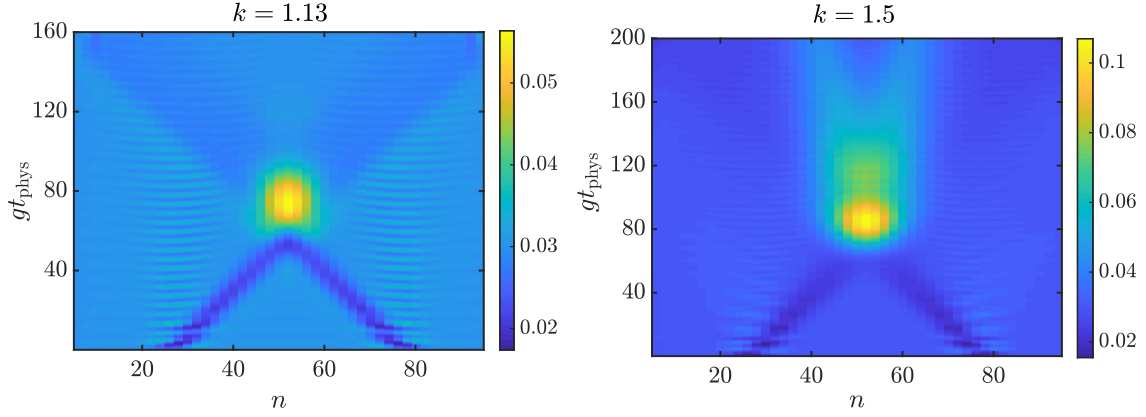


FIG. 6. Plot shows the expectation value of the four-body projector, $P_n(t)$, of a system with lattice size $N = 100$ and physical volume $N/\sqrt{x} = 100$. For the first plot the initial momenta of the mesons are $k = 1.13$ and for the second plot they are $k = 1.5$. The mass is $m/g = 10^{-5}$, and the standard deviation of the wave packets $\sigma = 4$. The bond dimension is $D = 50$ and the maximum electric flux $L_{\text{trunc}} = 8$.

nature of the products. More concretely, we show that the entanglement of two neighboring sites (which would be accessible from measurements on two sites) with the rest of the chain and the correlators of the electric flux provide clear signatures of the collision character. Moreover, a 4-site projector can be used as another measure for observing the presence of the collision products.

Our initial state preparation and detection observables are based on perturbative expansions valid in the strong coupling (small fermion mass and small x parameter) regime, and we performed our numerical simulations at a fixed value of the lattice spacing, far from the continuum limit. Yet, this parameter regime would be closer to the ones possible for current quantum simulation platforms, for example ultracold atoms as recently proposed for a related U(1) quantum link model [67].

Quantum inspired classical simulations with TNS, as well as proposed quantum simulation protocols are tackling progressively challenging regimes of LGT and could in the future study problems that are out of the reach of traditional lattice methods. Out of equilibrium processes, such as collision experiments, are among the most interesting ones, but they also come with their own challenges. In particular, performing a systematic extrapolation of lattice results in the continuum limit for a real-time problem, with a precision close to the one attained for spectral or thermal properties, requires performing real-time simulations on larger lattices far from the strong coupling limit as $x \rightarrow \infty$. It is thus a natural extension of this work to investigate how to prepare suitable initial states and to identify other observables that can detect the products of the collision away from the strong coupling perturbative regime.

Further extensions of our work could involve studying the effect of a background electric field—which is known to induce a phase transition—or considering collisions at much higher energies, with the possibility to open other

inelastic channels. Overall, we hope that our results provide a stepping stone for exploring real-time scattering phenomena in regimes inaccessible by perturbative methods.

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APPENDIX: CONVERGENCE OF NUMERICAL RESULTS

In this appendix we show the robustness of the qualitative results presented in Sec. IV with respect to the various sources of numerical errors in TNS numerics.

The first step in our simulations involves finding an MPS approximation to the ground state of the model, in order to prepare the initial state of the two mesons. With a relatively small bond dimension D_{GS} we can already find a very good approximation (see also [59]), such that the effect of this truncation is negligible in the error of the results, and, for this error analysis, we can consider this approximation to be the exact ground state.

The exact state of the two mesons is prepared by applying a MPO of bond dimension 4 on the ground state.

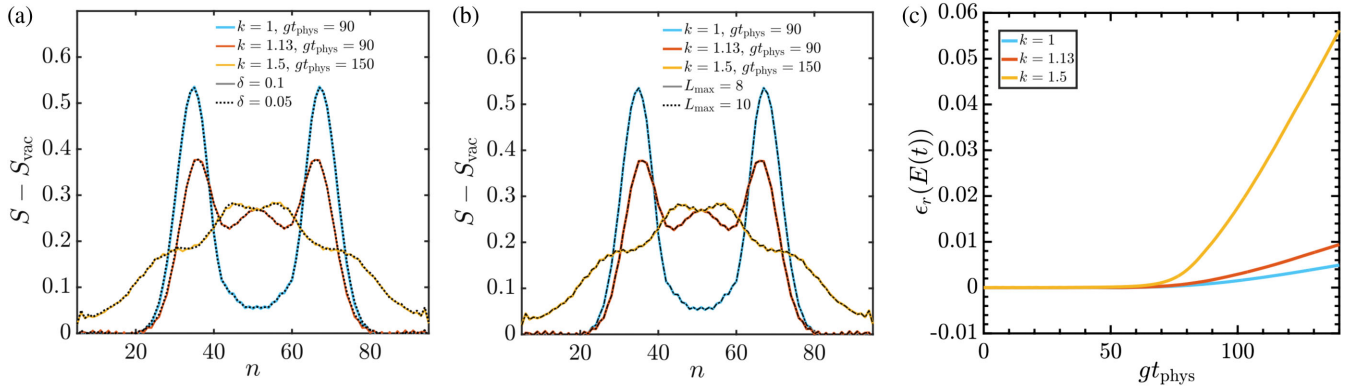


FIG. 7. Convergence of the results with respect to the various sources of numerical errors. The two left panels show the convergence of the two-site entanglement entropy $S(n, t)$ (20) on a lattice with size $N = 100$ and physical volume $N/\sqrt{x} = 100$ for the three momentum cases explored in this paper at a time after the collision (indicated by different colors). In all cases, the mass is $m/g = 10^{-5}$, the standard deviation of the wave packets $\sigma = 4$ and the bond dimension used $D = 50$. (a) shows convergence with respect to Trotter step, by comparing the results at $\delta = 0.1$ (solid colour lines) and $\delta = 0.05$ (black dotted lines), at constant maximum of the electric flux ($L_{\max} = 8$). (b) explores convergence with $L_{\max}$ by comparing results at constant Trotter step $\delta = 0.1$ for $L_{\max} = 8$ (solid color lines) and $L_{\max} = 10$ (black dashed lines). (c) shows the relative deviation of total energy as a function of time in the three cases, with $\delta = 0.1$ and $L_{\max} = 8$.

Thus the initial state has bond dimension upper bounded by $D_{\text{V}}^{\text{exact}} \leq 4D_{\text{GS}}$, but it can in fact be compressed to a much smaller value. Truncating it to $D_{\text{V}} = 50$, we check that the fidelity to the exact application of the MPO is very close to 1, with only small corrections.

The main sources of error in our results come from the time evolution. Specifically, the errors arise from the finite Trotter step δ (used in a second order Suzuki-Trotter expansion), the cutoff of the electric flux to $L_{\max}$ [related to the truncation of the bond dimension when expressing the exponential $e^{-i\frac{1}{2}H_z}$ of Eq. (16) in an MPO form] and the truncated bond dimension D of the evolved MPS state $|\Psi(t)\rangle$. The growing entanglement of the system with time allows TN techniques to be utilized only for short-enough times. The truncation of the bond dimension is expected to play the most significant role, but Fig. 3 shows that the entropy results after the collision are well converged, by comparing data with different values of the bond dimension

for the various initial momenta. Figure 7 shows the convergence with respect to the Trotter step δ [panel (a)] and $L_{\max}$ [panel (b)]. The figure shows, for three different values of the momenta, the two-site entropy at a time after the collision obtained with our chosen numerical parameters, together with the value obtained when a single one of these parameters is improved. The latter (shown as black dashed lines) coincides almost perfectly with our results, demonstrating their robustness.

An alternative way to monitor the accuracy of the simulations is to check for deviations of the total energy from its initial value. Since this should be conserved along the evolution, deviations can only follow from numerical errors. As shown in panel (c) of Fig. 7, in all the cases discussed in our analysis, for the set of parameters we use, the relative deviation of the energy with respect to its initial value is at most $O(10^{-2})$, which further confirms our conclusion about the qualitative robustness of our results.

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