

Structure constants in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and separation of variables

Carlos Bercini

*Instituto de Física Teórica, UNESP, ICTP South American Institute for Fundamental Research,
Rua Doutor Bento Teobaldo Ferraz 271, 01140-070 São Paulo, Brazil*

Alexandre Homrich 

*Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada
and Perimeter Institute for Theoretical Physics,
31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada*

Pedro Vieira

*Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada
and Instituto de Física Teórica, UNESP, ICTP South American Institute for Fundamental Research,
Rua Dr. Bento Teobaldo Ferraz 271, 01140-070 São Paulo, Brazil*



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We propose a new framework for computing three-point functions in planar $\mathcal{N} = 4$ super Yang-Mills theory where these correlators take the form of multiple integrals of separation of variables type. We test this formalism at weak coupling at leading and next-to-leading orders in a noncompact $SL(2)$ sector of the theory and all the way to next-to-next-to-leading orders for a compact $SU(2)$ sector. We find evidence that wrapping effects can also be incorporated.

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Introduction. Solving planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) in a satisfactory way would mean efficiently computing both the spectrum as well as higher point correlation functions—starting with three points—at any value of the ‘t Hooft coupling.

A formalism for computing three-point functions by means of integrability exists. It is the so-called “hexagon approach” [1]. It is conjectured to hold for any coupling indeed, but it is not easy to use, at least not by the remarkable standards of the spectrum quantum spectral curve approach [2]. With hexagons one needs to go over infinitely many sums and integrals to produce such correlators. At weak coupling perturbation theory, these sums and integrals truncate [3–5] and we can use hexagons to produce a wealth of data to test any putative new framework. We will do it all the time below.

Here we suggest a new approach for correlation functions in $\mathcal{N} = 4$ SYM based on the Baxter Q -functions. The final representations are of so-called separation of variables (SoV) type where these Baxter functions are integrated

against simple universal measures to produce the structure constants.

Q -functions play the central role in the quantum spectral curve, the top of the line tech for computing the dimension of any single trace operator in this conformal gauge theory, so it is only natural to look for a similar central role for these objects in the context of other physical observables such as the operator product expansion (OPE) structure constants.

In the most conventional integrable spin chains, Q -functions are polynomials whose roots are the so-called Bethe roots v_k . In SYM these polynomials are present at leading order at weak coupling but at higher coupling they get dressed by quantum nonpolynomial factors. This is expected; as we crank up the coupling we are no longer dealing with a spin chain or with a classical string but something in between, and so these Baxter polynomials get naturally deformed. For the so-called $SL(2)$ sector of the theory which includes all operators of the schematic form $\text{Tr}(D^J Z^L) + \text{permutations}$ we have

$$\mathcal{Q}(u) \equiv \left(\prod_{k=1}^J \frac{u - v_k}{\sqrt{x_k^+ x_k^-}} \right) e^{\frac{J}{2} Q_1^+ H_1^+(u) + \frac{L}{2} Q_1^- H_1^-(u) + O(g^4)} \quad (1)$$

where the charges $Q_1^\pm$ are simple functions of the Zhukovsky variables $x_k^\pm = x^\pm(v_k)$ and $H_k^\pm$ are harmonic functions, see Supplemental Material [6] for notation.

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An operator is given by a $\mathcal{Q}$ -function (1). What we are after is thus a functional eating these functions and spitting out a number, the structure constant.

For the most part we will consider a single non-Bogomol'nyi-Prasad-Sommerfield (BPS) operator and two BPS operators. The geometry of the three-point function depends on the sizes L_i of these operators or—equivalently—on the so-called bridges $\ell_{ij} = (L_i + L_j - L_k)/2$, i.e., total propagators that connect them. We will usually use $L \equiv L_1$ for the length of the non-BPS operator and $\ell \equiv \ell_{12}$ for its so-called left adjacent bridge (the right adjacent bridge will have length $L - \ell$); the bottom bridge we will often denote as $\ell_B \equiv \ell_{23}$.

The proposal is that (the square of) the structure constant is given by a ratio of SoV-like scalar products

$$C_{\infty}^2 = \frac{(\mathbb{J}!)^2 \langle \mathcal{Q}, \mathbf{1} \rangle_{\ell}^2}{(2\mathbb{J})! \langle \mathcal{Q}, \mathcal{Q} \rangle_L}. \quad (2)$$

Here $\mathbb{J} = J + g^2 Q_1^+$ and the scalar product

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle_{\ell} \equiv \binom{\mathbb{J}_1 + \mathbb{J}_2 + \ell - 1}{\ell - 1} \int d\mu_{\ell-1} \prod_{i=1}^{\ell-1} \mathcal{Q}_1(u_i) \mathcal{Q}_2(u_i) \quad (3)$$

with a nice factorized measure

$$d\mu_{\ell} = \prod_{i=1}^{\ell} du_i \mu_1(u_i) \prod_{i=1}^{\ell-1} \prod_{j=i+1}^{\ell} \mu_2(u_i, u_j), \quad (4)$$

which is constructed out of the building blocks [using, s_u , c_u , t_u for $\sinh(\pi u)$, $\cosh(\pi u)$, $\tanh(\pi u)$]

$$\mu_1(u) = \frac{\pi}{2c_u^2} (1 + g^2 \pi^2 (3t_u^2 - 1) + \dots) \quad (5)$$

$$\mu_2(u, v) = \frac{\pi(u-v)s_{u-v}}{c_u c_v} \left(1 + g^2 \pi^2 \left((t_u + t_v)^2 - \frac{4}{3} \right) + \dots \right) \quad (6)$$

valid to leading order (LO) and next-to-leading order (NLO); the dots in the above expressions are next-to-next-to-leading order (NNLO) corrections which start at $O(g^4)$. The structure of the result might be corrected at subleading orders as explained below. The contours in (2) are the real axis for all u_i .

We can prove (2) by “exhaustion” by comparing it with hexagon produced data for numerous L 's and ℓ 's and for different operators corresponding to different $\mathcal{Q}$ -functions. We did it; (2) is correct. We can also establish it more honestly as discussed in the next section.

Representations like (2) are the main results of this Letter. In Sec. III we present an $SU(2)$ counterpart of this

representation in (19); we managed to fully test it to LO, NLO, and NNLO. We compare these two rank one sectors in Appendix IV of Supplemental Material [6]. In the Discussion, Sec. IV, we discuss further generalizations such as multiple spinning operators and speculate on all loop structures we expect to find. Many appendixes complement the main text in Supplemental Material.

$SL(2)$. At leading order, that is at tree level, correlation functions are given by Wick contractions. Each operator can be thought of as a spin chain and these Wick contractions are thus given by spin chain scalar products [7–9]. For the $SL(2)$ sector such scalar products can be cast as SoV integrals [10]—see also [11] in the $\mathcal{N} = 4$ context. Once we properly normalize all these scalar products as in [8] to extract the structure constant we precisely end up with (2) for $g = 0$.

What we find remarkable is that this classical $g = 0$ expression seems to have a nice quantum lift as anticipated in the Introduction.

Let us first discuss the simplest possible case where a twist-two operator ($L = 2$) splits evenly into two BPS operators ($\ell = 1$) so that the proposal (2) simply reads

$$C_{\infty}^2 = \frac{\mathbb{J}!^2}{(2\mathbb{J} + 1)!} \left(\int du \mu_1(u) \mathcal{Q}(u)^2 \right)^{-1}. \quad (7)$$

We derived the single-particle measure μ_1 as well as the quantum deformed Baxter functions $\mathcal{Q}$ in two ways. Both are based on looking for a measure such that, for two different twist-two states (i.e., with different spins J and J'), we have an orthogonality relation

$$\int \mu_1(u) \mathcal{Q}_J(u) \mathcal{Q}_{J'}(u) \propto \delta_{JJ'}, \quad (8)$$

a powerful relation which has been extensively exploited by [12–14] in numerous SoV studies, most of which for rational spin chains or for the fishnet reduction [15] of SYM, see most notably [16].

The first uses the fact that the Bethe roots v_k of twist-two operators are given by the zeros of Hahn polynomials, a fact that persists at NLO. We have that $\prod_k (u - v_k)$ is proportional to the Hahn polynomial [17,18] $p_J(u|a, b, b, a)$ where $a - b = 2i\sqrt{2}g$ and $a + b = 1 + 4g^2 H_1(J)$. Hahn polynomials are orthogonal with a simple measure as reviewed in Appendix II A of Supplemental Material [6] which allows us to derive (5) up to some simple tuning due to the mild J dependence in the polynomial parameters a, b , see [6] for details. It is the J dependence that renders the derivation nontrivial and which is responsible for the needed modification of the Baxter functions in (1).

The second derivation follows [13] closely (the novelty being the extension to g^2 corrections) and makes use of the Baxter equation [19]

$$\mathcal{B} \circ \mathbb{Q} = T(u) \mathbb{Q}(u) \quad (9)$$

where $\mathbb{Q}(u)$ are the Baxter polynomials [i.e., just the parentheses in (1)] and the Baxter operator

$$\mathcal{B} \equiv (x^+)^L \left(1 - \frac{g^2}{x^-(u)} (Q_1^+ - iQ_1^-) \right) e^{i\partial_u} + \text{c.c.} \quad (10)$$

Note that at $g = 0$ we have $x^+ = u + i/2$ and, most important, $\mathcal{B}$ becomes a simple linear operator but as we turn on g corrections this is no longer true since the charges $Q_1^\pm$ depend on $\mathbb{Q}$. Consider first $g = 0$ and $L = 2$ so that the transfer matrix is $T(u) = 2u^2 + c_J$. Multiplying (9) by the sought after measure μ_1 and by another Baxter polynomial with a different spin, subtracting that to the same thing with the spin swapped and integrating yields

$$(c_J - c_{J'}) \int \mu_1 \mathbb{Q}_J \mathbb{Q}_{J'} = \int \mu_1 (\mathbb{Q}_{J'} (\mathcal{B} \circ \mathbb{Q}_J) - (\mathcal{B} \circ \mathbb{Q}_{J'}) \mathbb{Q}_J). \quad (11)$$

If we manage to make $\mathcal{B}$ self-adjoint we will thus have the required orthogonality. An i -periodic μ_1 would do the job since under shifts of contour the two terms of Baxter would swap and cancel in the right-hand side. In detail, $\mathbb{Q}_{J'}(u)(u \pm \frac{i}{2})^2 \mathbb{Q}_J(u \pm i) \mu_1(u)$ becomes $\mathbb{Q}_{J'}(u \mp i)(u \mp \frac{i}{2})^2 \mathbb{Q}_J(u) \mu_1(u \mp i)$ under a shift of contour by $\mp i$ leading to an interchange (and thus cancellation) of the two terms in (11) once we use that the measure is periodic. To make this manipulation kosher we need to make sure no singularities are picked when deforming the contour and to make the measure acceptable we need to make sure it decays fast enough at infinity so that it can be integrated against polynomials of arbitrary degree. Both these problems are solved at once with

$$\mu_1(u) = \frac{\pi/2}{\cosh^2(\pi u)} + O(g^2). \quad (12)$$

The function decays faster than any polynomial and the double poles at $\pm i/2$ precisely cancel the double zeroes in the potential terms $(u \mp i/2)^L$ when $L = 2$ so that they lead to no extra contribution when deforming the contours. A periodic function without these double poles would not decay fast enough and a function with more than double poles would lead to extra contributions when deforming the contours; (12) is the sweet spot.

Turning on g corrections is not complicated. The redefinition (1) brings the Baxter operator to a linear operator again but introduces some further poles at $\pm i/2$ in the (no longer polynomial) Baxter $\mathbb{Q}$ functions so that the measure now needs some extra poles to cancel the contribution of these when deforming the contour. This explains (1) as well as (5); more details in Appendix II B of Supplemental Material [6].

Having derived the measure μ_1 and the Baxter polynomial dressing it remains to fix the overall normalization of the structure constant to be sure everything is in order. Evaluating the SoV integral using the loop corrected Bethe roots for various spins immediately leads to the cute result

$$\left(\int du \mu_1(u) \mathbb{Q}(u)^2 \right)^{-1} = \frac{1 + g^2(4H_2(J) + \frac{8H_1(J)}{2J+1}) + \dots}{2J+1} \quad (13)$$

which is indeed almost the correct loop level structure constant computed in [20]. The prefactor in (7) with J deformed into $\mathbb{J}$ in a *reciprocity* reminiscent fashion [21] neatly combines with this expression to give the full NLO structure constants for twist-two operators.

At this point it is straightforward to guess the general structure constant for any $SL(2)$ operators of any twists by simply taking the tree level $g = 0$ result and deforming the new ingredient for higher twists—the two-particle measure μ_2 —by a bunch of hyperbolic tangents following what was derived for twist two. The coefficients of these hyperbolic tangents are then fixed by requiring orthogonality between any two different Baxter solutions. The last line deformation of the Baxter functions in (1)—which was invisible for twist-two operators for which odd charges vanish—is also fixed by imposing orthogonality. In the end, we just need to check that with a minimal reciprocity friendly prefactor as in (2) we precisely agree with perturbative data produced by hexagons. We do.

Even without matching with data, there is a nice self-consistency check of the full construction including the deformed prefactor: The structure should be invariant under swapping left and right bridges $\ell \leftrightarrow L - \ell$; we checked that this is indeed realized by our expressions once the prefactors are deformed as in (2).

We made some progress at higher loops, in particular at NNLO (two loops) and for the smallest possible sizes and bridge lengths. For twist $L = 2$ operators for example we found a Baxter function dressing as well as an orthogonal one-particle measure realizing (8) as

$$\mathbb{Q}(u) \equiv \prod_{k=1}^J \frac{u - v_k}{\sqrt{x_k^+ x_k^-}} \times e^{\frac{1}{2}g^2 Q_1^+ H_1^+ + \frac{1}{8}g^4 Q_1^+ Q_2^+ H_1^+ - \frac{1}{2}g^4 Q_1^+ H_3^+}$$

and

$$\mu_1 = \frac{\pi}{2c_u^2} \exp \left(1 + \pi^2 g^2 (3t_u^2 - 1) + \pi^4 g^4 \left(\frac{5}{6} - 7t_u^2 + \frac{11}{2} t_u^4 \right) - \frac{g^4}{8} H_1^+ (Q_1^+(\mathbf{v}_1) Q_2^+(\mathbf{v}_2) + Q_1^+(\mathbf{v}_2) Q_2^+(\mathbf{v}_1)) \right), \quad (14)$$

where we use the fact that $Q_k^- = 0$ for twist-two operators.

The last line is exotic as it depends now on the charges of the two operators in (8) with Bethe roots $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Note, however, that when considering the pairings $\langle \mathcal{Q}, \mathbf{1} \rangle$ and $\langle \mathcal{Q}, \mathcal{Q} \rangle$ a single Baxter function shows up and thus this mixing term can be absorbed as new factor dressing $\mathcal{Q}$. One would then have different dressings in each pairing, a phenomenon we observe in the next section as well [in the SU(2) sector].

The first line in (14) is also not any random combination of trigonometric functions. Take the tree level measure $1/\cosh^2(\pi u)$ —which is periodic with period i and has poles at all the imaginary half integers—and promote it to a periodic function where all these poles are opened up into small cuts following [16]

$$\hat{\mu}_1 \equiv \oint \frac{dv}{2\pi i} \frac{\pi/2}{\cosh^2(\pi(u-v))} \frac{1}{x(v)}. \quad (15)$$

The integration contour encircles the Zhukovsky cut $v \in [-2g, 2g]$. Evaluating this integral in perturbation theory precisely reproduces the first line in (14) up to an overall normalization constant. It is tempting to conjecture that the finite coupling expression (15) might play an important role in an all loop SoV formulation. We make further comments on NNLO structure constants in the Discussion.

SU(2). Up to NNLO the SU(2) magnon S-matrix does not receive corrections. Quantum effects do correct the propagation of scalar particles on top of the Berenstein-Maldacena-Nastase vacuum. One could imagine mimicking this dressed propagation through appropriate deformations of the background vacuum by the insertions of inhomogeneities θ_i . This was made concrete in [22] where the authors determined NLO (one loop) structure constants in the SU(2) sector through the construction of a differential “ θ -morphism” operator acting on the (θ -deformed) spin chain scalar products defining the LO structure constants [9] and outputting the quantum three-point functions.

Here we make two observations. First we note that there exists a θ -morphism that promotes XXX_{1/2} inhomogeneous spin chain scalar products,

$$\begin{aligned} \mathcal{A}_\theta &= \sum_{\mathbf{u}=\alpha\cup\bar{\alpha}} (-1)^{|\bar{\alpha}|} \prod_{n=1}^{\ell} \prod_{i \in \bar{\alpha}} \frac{u_i - \theta_n + i/2}{u_i - \theta_n - i/2} \prod_{j \in \alpha} \frac{u_j - u_i + i}{u_j - u_i}, \\ \mathcal{B}_\theta &= \det \left[\partial_{u_i} \log \left(\prod_{n=1}^L \frac{u_j - \theta_n + i/2}{u_j - \theta_n - i/2} \prod_{k \neq j} \frac{u_j - u_k - i}{u_j - u_k + i} \right) \right] \\ &\quad \times \prod_{i < j} \frac{(u_i - u_j)^2}{1 + (u_i - u_j)^2}, \end{aligned} \quad (16)$$

into $\mathcal{N} = 4$ SU(2) structure constants all the way to NNLO,

$$C_{\infty}^2 = \left| \frac{(\mathcal{M} \circ \mathcal{A}_\theta)^2}{\Lambda_B \mathcal{M} \circ \mathcal{B}_\theta} \right|_{\theta=0} + O(g^6), \quad (17)$$

where

$$\begin{aligned} \mathcal{M} &= \exp \left[\sum_{i=1}^L \left(g^2 (\partial_{i,i+1})^2 - \frac{1}{4} g^4 (\partial_{i,i+1})^2 (\partial_{i+1,i+2})^2 \right) \right. \\ &\quad \left. - i g^2 \mathcal{Q}_1^+ (\partial_1 - \partial_L) + g^4 \delta \mathcal{M}_{\text{NNLO-b}} \right], \end{aligned} \quad (18)$$

with $\partial_i = \partial_{\theta_i}$ and $\partial_{i,i+1} = \partial_i - \partial_{i+1}$, see Appendix III A of [6] for $\delta \mathcal{M}_{\text{NNLO-b}}$, and where Λ_B is a simple norm factor, (S15) of [6]. To NLO this is just the construction of [22]. We fix the NNLO part through comparison with the hexagons prediction, see Appendix III A in [6] for details.

The second observation is that given the known integral representation for the scalar products in the inhomogeneous XXX_{1/2} spin chain, derived from Sklyanin’s [23], Baxter’s [12], and hexagon methods [11,24], one can straightforwardly derive the functional SoV representation for the NNLO structure constants simply by acting with the θ -morphism operator on this classical measure.

The result is once again (the square of) a $\ell - 1$ -dimensional integral over an $L - 1$ -dimensional integral,

$$C_{\infty}^2 = \Lambda_\ell(\mathcal{Q}) \times \frac{(J!)^2}{(2J)!} \times \frac{\langle \mathcal{Q}, \mathbf{1} \rangle_{\ell,L}^2}{\langle \mathcal{Q}, \mathcal{Q} \rangle_{L,L}} + O(g^6), \quad (19)$$

with $\langle f, g \rangle_{\ell,L} \equiv \langle f, g \rangle_{\ell,L} / \langle \mathbf{1}, \mathbf{1} \rangle_{\ell,L}$ and

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle_{\ell,L} \equiv \binom{\ell}{J_1 + J_2} \oint_{\gamma} d\mu_{\ell,L} \prod_{i=1}^{\ell-1} \mathcal{Q}_1(u_i) \mathcal{Q}_2(u_i). \quad (20)$$

The contour of integration is a circle wrapping the singularities of the measure (i.e., both Zhukovsky cuts), which once again is factorized,

$$d\mu_{\ell,L} = \prod_{i=1}^{\ell-1} du_i \mu_{\ell,L}^1(u_i) \prod_{j \neq i}^{\ell-1} \mu_{\ell,L}^2(u_i, u_j), \quad (21)$$

with

$$\mu_{\ell,L}^1(u) = \frac{\sinh(2\pi u)}{(x_u^+ x_u^-)^2} e^{\delta_{\ell \neq L} \mathcal{A}_{\ell,L}(u)} \quad (22)$$

$$\mu_{\ell,L}^2(u, v) = \frac{\sinh(2\pi(u-v))(u-v)}{2x_u^+ x_u^- x_v^+ x_v^-} e^{\delta_{\ell \neq L} \mathcal{B}_{\ell,L}(u,v)} \quad (23)$$

$$\mathcal{Q}(u) \equiv \prod_{k=1}^J \frac{u - v_k}{\sqrt{x_k^+ x_k^-}} e^{\delta_{\ell \neq L} \mathcal{B}(u)}. \quad (24)$$

The scalar product in the denominator of (19) enters with $\ell = L$, and therefore the exponential in all these expressions should be dropped in that case. For the numerator we need these extra quantum dressing factors. They read

$$\begin{aligned} \mathbb{A}_{\ell,L}(u) &= g^2(q_1^-)^2 + g^4 \left(\frac{1}{2}(q_2^+)^2 + 4\alpha(q_1^-)^2 - 6\pi^2 q_2^+ \delta_{\ell=2} \right), \\ \mathbb{A}_{\ell,L}(u, v) &= g^2 q_1^- \tilde{q}_1^- + g^4 \left(\frac{1}{2} q_2^+ \tilde{q}_2^+ + 2\alpha q_2^+ + 4\alpha q_1^- \tilde{q}_1^- \right), \\ \mathbb{B}(u) &= g^2 q_1^- Q_1^- \\ &\quad + g^4 \left(\frac{1}{2} q_2^+ Q_2^+ + \alpha Q_2^+ + 4\alpha q_1^- Q_1^- - \pi^2 q_1^- Q_1^- \delta_{\ell=2} \right. \\ &\quad + \left(\left(\frac{1}{8} (Q_1^+)^2 - \frac{1}{4} Q_2^+ - Q_1^+ + \frac{3}{8} (Q_1^-)^2 \right) q_2^+ \right. \\ &\quad \left. \left. - \frac{1}{2} q_3^- Q_1^- \right) \delta_{\ell,L-1} \right), \end{aligned} \quad (25)$$

where the charges $q_k^\pm = q_k^\pm(u)$ and $\tilde{q}_k^\pm = q_k^\pm(v)$ are defined in Supplemental Material [6] and $\alpha = \frac{2}{3}\pi^2 - 1$.

Finally $\Lambda_\ell(Q)$ is a simple function of the higher conserved charges also given in [6].

Having expressed the results, a number of remarks are in order. The first is that (19) is an exact result capturing finite-volume effects around the seams adjacent to the non-BPS operator [25]. These effects start at N³LO in the SL(2) sector and therefore were not considered in the previous section, but for SU(2) they are present already at NNLO. Their effect in the SoV representation is encoded in the dressing of the Q -function through the $\delta_{\ell,L-1}$ factor, see Appendix III C in Supplemental Material [6] for a lengthy discussion on these finite-volume effects. Here we emphasize that the structure constant geometry dresses the Q -function: Q depends both on the state and ℓ .

The second remark is the presence of the $\delta_{\ell=2}$ “anomaly” in (25). Its origin is not clear to us. Should it be put on foot with the $\ell = 1$ result (26) in the SL(2) sector at NNLO in which a new integral appears? Is it an indication that we are integrating out a simpler higher-dimensional integral for this short bridge overlap?

To a more basic point, in contrast to the SL(2) sector result (2), different measures and Q -dressings enter the numerator and denominator of (19) already at NLO. From the θ -morphism point of view, the mismatch is due to the boundary terms in the second line of (18) which cancel when acting on the denominator, a symmetric function of the θ_i , but do not on the numerator. It turns out that the denominator’s Gaudin measure follows as in the SL(2) case from an orthogonality principle—see Appendix III B in Supplemental Material [6]. The numerator measure is more complicated and we could not identify an orthogonality principle that fixes it. Is there an alternative principle that generalizes to higher loops and allow us to move forward? We hope so. As usual, one can always rely on hexagons to

compare any new proposal with data, as we did to confirm the correctness of (19).

Discussion. We have initiated here a project for recasting correlators in $\mathcal{N} = 4$ SYM in a language closer to that of the spectrum quantum spectral curve. With all recent SoV related advances [12–14, 16, 26–30] we believe the time is ripe for seeking out for such a new approach. We are probably scratching the tip of an iceberg as far as the full finite coupling structure goes but some elements are clearly emerging.

One is the central importance of SoV quantum corrected measures $\mu(u_1, \dots, u_p)$ which can be found by combining a myriad of different approaches from orthogonality of quantum corrected Baxter equations, matching SoV integrals with hexagon combinatorics, converting hexagons sums into contour integrals which one then massages into SoV-like integrals, θ -morphism operations on inhomogeneous SoV integrals, resumming wrapping corrections, extending orthogonality relations of known classical polynomials, Zhukovsky upgrades of trigonometric functions, etc. (This “etc.” often includes a good deal of “guesswork.”)

These measures are then used to build scalar-product-like integrals which couple Baxter functions $Q(u)$ for the various involved operators. The Baxter functions also are corrected away from their tree level polynomial forms.

The most obvious question is then how to fix the measures and the Baxter Q -functions. Do these measures obey any sort of all loop bootstrap set of axioms? They must. It is crucial to work it out. Are these Q -functions some of the finite coupling solutions to the quantum spectral curve [31]? We hope so.

Is the number of SoV integrals going to remain constant at higher loops or at least grow in a controllable way? Take for example the SL(2) correlation function involving a single non-BPS operator with twist $L = 2$ and adjacent bridges $\ell = 1$. With $\ell = 1$ the numerator $\langle Q, \mathbf{1} \rangle_1 = 1$ to LO and NLO since the scalar product in (3) is an $\ell - 1$ dimensional integral. This is of course consistent with the hexagon approach where this numerator is given by the sum over partitions of magnons $\mathcal{A}_1$ weighted by the hexagon form factors—see Sec. I B of [6]—and this sum over partitions normalized as in this Letter indeed evaluated to $1 + O(g^4)$. At NNLO $O(g^4)$, however, it is no longer unity. Instead it is given by a cute expression involving Bernoulli numbers which we summarize in Appendix II C of Supplemental Material [6]. Also, at this loop order, if the bottom bridge $\ell_{23} = \ell_B = 1$ there is a new wrapping correction to the structure constant [1]. Both these new effects can be compactly incorporated by simply modifying the trivial scalar product

$$\begin{aligned} \langle Q, \mathbf{1} \rangle_1 &\rightarrow 1 + \eta g^4 \int du \nu(u) \frac{\mathbb{Q}(u)}{\mathbb{Q}(i/2)} \\ &\quad + \text{simple contact terms}_\eta + O(g^6), \end{aligned} \quad (26)$$

with the various ingredients spelled out in [6] and where $\eta = 1$ for $\ell_B > 1$ (no wrapping effects) and $\eta = 2$ for $\ell_B = 1$ (wrapping effects included). It is very encouraging to see wrapping and asymptotic effects cast in such unified way. We see it as evidence that the number of integrals in this SoV approach should increase as we go to higher and higher loops but that increase is not directly related to wrapping effects which can be automatically incorporated [as seen in this bottom wrapping example and also in the SU(2) adjacent wrapping example, see Appendix III C of [6]]. This growth of the number of SoV integrals is consistent with the picture that at finite coupling we are dealing with three quantum strings with infinitely many degrees of freedom and in the SoV approach the number of integrals is related to the number of such degrees of freedom [16].

As we go to higher loops, multiple wrapping effects come in at once and we should make contact with [33] where $PSU(2, 2|4)$ transfer matrices are shown to appear. Again, our hope is that, even in those cases, wrapping and no wrapping are all cast in the same unified way.

Another limit worth exploring is large spin. This is a particular limit where SoV should be quite powerful since the number of integrals does not grow with spin. Very interesting structures seem to emerge which might also shed light on Wilson loop/correlator dualities [34,35] along the lines of the recent works [36,37].

It would also be very interesting to think about how all this fits into a higher-point function picture [38]. What is

the SoV description of a four-point function of BPS operators? Does it involve integrating over intermediate Baxter functions? Will the measures derived here play a role in this integration? A starting point for these explorations could be recasting Coronado's all loop octagon correlator [39,40] in this language, perhaps using its determinant representations [41]. The octagon is a large R-charge correlator though so at the same time we should probably figure out what sort of simplifications take place on the SoV side when we consider large operators. (Another motivation for studying this limit.)

We explored the SU(2) and SL(2) sectors of $\mathcal{N} = 4$ SYM. There is life beyond it. We should look for it.

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