

Tunneling and the Casimir effect on a D -dimensional sphere

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Two fundamental signatures of quantum field theory are tunneling and the Casimir effect. We examine the ground state energetic properties of a scalar field confined on a D -dimensional sphere and subjected to these two effects. We focus on $D = 2$ and $D = 3$, with a negative coupling of a massless scalar field to curvature, providing a radius-dependent effective mass that triggers symmetry breaking. This scenario allows tunneling to be more important than the Casimir effect, in a certain regime of parameters, and potential implications in early cosmology are discussed for the case $D = 3$, which could avoid a cosmological singularity.

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Introduction. The Casimir effect [1] is the manifestation of the effects of boundary conditions on the vacuum fluctuations of a quantum field. In its original observational configuration [2], two parallel conducting plates are placed in the presence of the electromagnetic field, and an attractive force between them is observed. This force originates from a reduced vacuum energy between the boundaries due to the quantisation of vacuum fluctuations, an effect that also leads to the violation of the null energy condition (NEC—see [3,4] for reviews). Since its initial discovery, the Casimir effect has been calculated for a variety of different fields and boundaries, where it is found to be highly sensitive to their geometry and topology, in some cases even changing sign and thus satisfying the NEC (see [5] for a review). Such an effect may even be present in the complete absence of boundaries since the identification conditions of a compact topology play the same role as boundary conditions. The latter effect is of interest within early universe cosmology since it may provide a source of NEC violation, required for features such as a cosmological bounce (see [6] for a review). We describe in the paragraph *discussions* at the end of this paper the potential applications of this work within this field.

As detailed further below, another source of NEC violation due to the presence of boundary/identification conditions has been under recent study and originates from a massive scalar field tunneling between degenerate vacua, with symmetry breaking triggered by a negative mass squared [7–13]. The convexity of the effective potential from tunneling results in a negative vacuum energy, leading to NEC violation. Unlike the Casimir effect, tunneling

effects always violate the NEC since the vacuum energy may only be lowered. The effect is exponentially suppressed by the spatial volume, whereas the Casimir effect is at most exponentially suppressed by the length scale and therefore dominates over tunneling.

In this paper, we instead focus on the effects of a massless scalar field tunneling on a D -dimensional sphere of radius a , with symmetry breaking triggered instead by a negative nonminimal scalar/curvature coupling, $\xi < 0$, and find a regime where tunneling effects are dominant. In particular, with the inclusion of tunneling effects, we find the following two modifications to the parameter space where the NEC is violated on a D -dimensional sphere versus the Casimir effect alone:

- (i) On a two-dimensional sphere, NEC violation no longer vanishes at $|\xi| = \xi_c^{(2)} = 1/16$ [cf. Fig. 1(a)];
- (ii) On a three-dimensional sphere, the NEC may now be violated at $|\xi| = \xi_c^{(3)} = 1/12$ [cf. Fig. 1(b)].

We note that the value of $\xi_c^{(D)}$ corresponds to the conformal coupling in the vacuum of the bare theory and depends on the dimension D .

Ground state from degenerate vacua. Consider a symmetric double-well potential, with degenerate vacua and typical frequency ω . In quantum mechanics (QM), tunneling between these vacua leads to a symmetric ground state wave function and a vacuum energy that is lower than the vacuum energy $\omega/2$ in the individual wells. The difference in energy is proportional to $\omega \exp(-S_{\text{inst}}^0)$, where S_{inst}^0 is the action of the instanton configuration relating the two wells.¹

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¹The exponential dependence in S_{inst}^0 arises from the resummation over zero modes of the instanton dilute gas, corresponding to the invariance of the total action describing the gas under translation of each individual instanton (see [14] for a review based on the path integral approach).

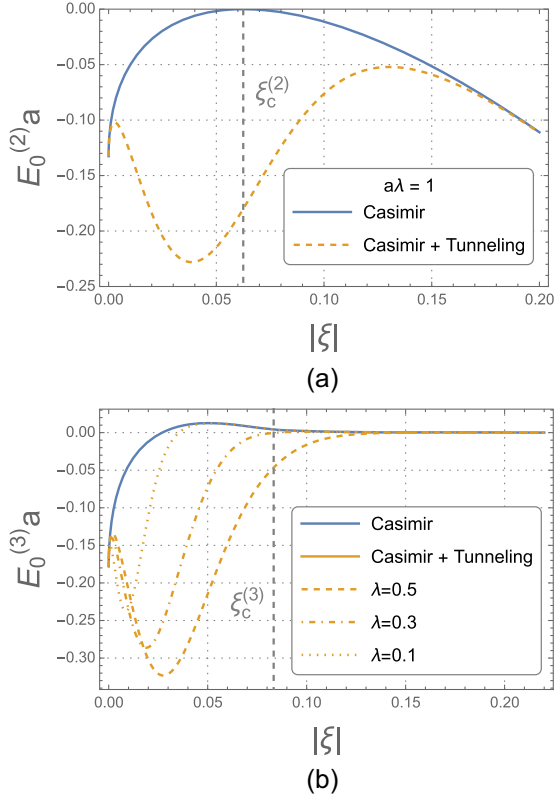


FIG. 1. The vacuum energy $E_0^{(D)}$ times the sphere radius a , as a function of the nonminimal coupling $|\xi|$. (a) 2-dimensions for $\alpha\lambda = 1$. (b) 3-dimensions for $\lambda = 0.5, 0.3, 0.1$.

In quantum field theory (QFT), these features generalize to Euclidean-time-dependent instantons, but the corresponding action S_{inst} is proportional to the volume V of the space, which confines the field. For this reason, tunneling between degenerate vacua in QFT is suppressed for an infinite volume, and spontaneous symmetry breaking occurs instead.² For a finite volume though, tunneling is allowed and, similarly to what happens in QM, symmetry is restored and the ground state energy is lower than the bare minima. This is consistent with convexity of the effective potential [7,17–25], and the construction of a convex one-particle-irreducible effective potential from tunneling is explicitly shown in [8,10].

Taking tunneling into account, the true ground state energy can then be obtained via a semiclassical approximation for the partition function Z in the ground state, based on the dilute instanton gas approximation [14]

$$Z \simeq \sum_{n=0}^{\infty} F_n e^{-nS_{\text{inst}}}, \quad (1)$$

²This situation is different for $O(4)$ -symmetric instantons [15,16], which have a volume-independent action.

where n is the number of instantons and anti-instantons with action S_{inst} , and F_n is the one-loop fluctuation factor. This approximation assumes that the n -instanton configurations are decoupled from each other, they do not overlap in field space, and is valid in the limit of zero temperature ($\beta \rightarrow \infty$) and weak self-coupling ($\lambda \ll 1$).

The vacuum energy is

$$E_0 = \lim_{\beta \rightarrow \infty} \left\{ \frac{-1}{\beta} \ln Z \right\}, \quad (2)$$

where β is the total Euclidean time. If we start from the double-well potential

$$U(\phi) = \frac{\lambda}{24} (\phi^2 - v^2)^2, \quad (3)$$

we obtain [10,12]

$$E_0 \simeq E_{\text{Cas}} - v \sqrt{\frac{\lambda}{\pi}} S_{\text{inst}} \exp(-S_{\text{inst}}). \quad (4)$$

In the above expression, E_{Cas} is the Casimir energy corresponding to a free field oscillating in one potential well with minimum $\phi = \pm v$ and represents the renormalized vacuum energy following the omission of the curvature-independent UV divergence, corresponding to the vacuum energy of flat space [5].

The instanton that connects the bare vacua $\pm v$ is homogeneous and depends on Euclidean time only. It can be decomposed as:

- (i) “Static” parts that asymptotically go to $\pm v$;
- (ii) A Euclidean-time-dependent “jump” that connects the two static parts.

The expression (4) is obtained in the QM approximation, where the space dependence of quantum fluctuations above the jump part of the instanton are ignored. The static parts of the instanton are quantized with spacetime dependence though, involving discrete momenta, and leads to the Casimir energy E_{Cas} .

The full one-loop quantization of the instanton is calculated in [13] in the situation of a three-torus. This calculation is based on the resolvent method [26] and the spectral decomposition of the Green’s function corresponding to the fluctuation operator, which takes into account discrete momenta for fluctuations above the instanton. The result shows that, for a weak self-coupling $\lambda \ll 1$, the QM approximation is very good, and is therefore what we use here.

D-dimensional sphere. We now consider a D -dimensional sphere as a confining space, with a nonminimally coupled massless scalar field. The nonminimal coupling provides an effective mass, which depends on the geometry. As a consequence and as shown below, the tunneling

contribution is not exponentially suppressed when the radius of the sphere increases.

We consider the nonminimal coupling $\xi R\phi^2/2$, together with a self-interaction term, such that the bare action is

$$S = \int d\tau \int d\mathbf{x} \sqrt{g} \left(\frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{\lambda}{24} \left(\phi^2 + 6 \frac{\xi R}{\lambda} \right)^2 \right). \quad (5)$$

If we choose a negative coupling ξ , the latter action provides an effective double-well potential for the massless scalar field. This action involves a term proportional to R^2 , whose significance is left for the discussion. For time-dependent fields only, which is the case of the relevant saddle points in this problem, we have then

$$S = V \int d\tau \left(\frac{1}{2} (\dot{\phi})^2 + \frac{\lambda}{24} \left(\phi^2 - \frac{6|\xi|R}{\lambda} \right)^2 \right). \quad (6)$$

A negative coupling ξ therefore leads to fluctuations above the two bare minima, with effective mass squared $2|\xi|R$, corresponding to the effective nonminimal coupling to gravity $\xi^{\text{eff}} = 2|\xi|$.

The single-instanton configurations are solutions of the equation of movement, they minimize the action, and therefore have a dominant contribution in the path integral. They are of the form

$$\phi_s(\tau) = \pm \sqrt{6 \frac{|\xi|R}{\lambda}} \tanh \left((\tau - \tau_0) \sqrt{\frac{|\xi|R}{2}} \right), \quad (7)$$

where the center of the instanton τ_0 is not fixed and is at the origin of the translational invariance, or equivalently the zero mode of the fluctuation operator. The action for one instanton is then

$$S_{\text{inst}} = S[\phi_s] = \frac{2\sqrt{2}}{\lambda} (|\xi|R)^{3/2} V. \quad (8)$$

Two-dimensional sphere. We now consider a two-dimensional sphere of radius a as a confining space, for which the Casimir energy satisfies [27]

$$E_{\text{Cas}}^{(2)} a = \begin{cases} \mu^{3/2} \left(-\frac{1}{3} + \int_0^1 \frac{2t\sqrt{1-t^2}}{e^{2\pi\mu t} + 1} dt \right), & \mu > 0 \\ 0, & \mu = 0 \\ -|\mu|^{3/2} \int_0^1 t \sqrt{1-t^2} \tan(\pi|\mu|t) dt, & \mu < 0 \end{cases}$$

where $\mu \equiv 4|\xi| - 1/4$.

The volume is $V = 4\pi a^2$, and the curvature scalar is $R = 2/a^2$, such that the action for one instanton is

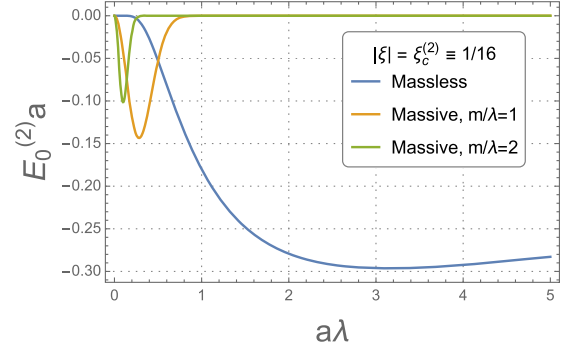


FIG. 2. The vacuum energy $E_0^{(2)}$ times the sphere radius a —see (10)—for $|\xi| = \xi_c^{(2)} = 1/16$, as a function of the dimensionless quantity $a\lambda$ and in comparison to the case of a minimally coupled massive scalar field (4). In the situation of a massless field coupled nonminimally, tunneling is not suppressed exponentially with the radius of the sphere.

$$S_{\text{inst}}^{(2)} = \frac{32\pi|\xi|^{3/2}}{\lambda a}. \quad (9)$$

The vacuum energy (4) satisfies then

$$E_0^{(2)} a \simeq E_{\text{Cas}}^{(2)} a - \frac{16\sqrt{3}|\xi|^{5/4}}{\sqrt{\lambda a}} \exp \left(-\frac{32\pi|\xi|^{3/2}}{\lambda a} \right), \quad (10)$$

which is plotted in Fig. 1(a).

As mentioned above, the tunneling contribution is not exponentially suppressed when the radius of the two-sphere increases or for weak coupling, unlike the case of a massive scalar field where the instanton action is proportional to $a^2 m^3/\lambda$. A comparison between the massless and massive cases is displayed in Fig. 2. In addition, the Casimir energy vanishes for $|\xi| = \xi_c^{(2)} = 1/16$, corresponding to a conformal coupling in the vacuum of the bare theory. The tunneling contribution remains, however, such that the null energy condition is now violated for all values of $|\xi|$, as shown in Fig. 1(a).

Three-dimensional sphere. We now consider a three-dimensional sphere of radius a as a confining space, for which the Casimir energy satisfies [27]

$$E_{\text{Cas}}^{(3)} a = \begin{cases} \nu^2 \int_1^\infty \frac{t^2 \sqrt{t^2-1}}{e^{2\pi\nu t} - 1} dt, & \nu > 0 \\ 1/240, & \nu = 0 \\ \nu^2 \int_0^\infty \frac{t^2 \sqrt{t^2+1}}{e^{2\pi\nu t} - 1} dt \\ + \frac{\nu}{2} \int_0^1 t^2 \sqrt{1-t^2} \cot(\pi\nu t) dt, & \nu < 0 \end{cases},$$

where $\nu \equiv 12|\xi| - 1$.

The volume is $V = 2\pi^2 a^3$, and the curvature scalar is $R = 6/a^2$, such that the action for one instanton is

$$S_{\text{inst}}^{(3)} = \frac{48\pi^2 \sqrt{3} |\xi|^{3/2}}{\lambda}. \quad (11)$$

The vacuum energy (4) satisfies then

$$E_0^{(3)} a \simeq E_{\text{Cas}}^{(3)} a - \frac{24\sqrt{2\pi} \times 3^{3/4} |\xi|^{5/4}}{\sqrt{\lambda}} \exp\left(-\frac{48\pi^2 \sqrt{3} |\xi|^{3/2}}{\lambda}\right), \quad (12)$$

and is plotted in Fig. 1(b).

As in the situation of the two-sphere, the tunneling contribution is not exponentially suppressed when the radius increases. In fact, the instanton action is independent of the length scale, such that the total vacuum energy has a simple $1/a$ dependence, like the Casimir energy. The tunneling contribution is, however, suppressed for a weak self-coupling λ , but this may be counteracted with a sufficiently small value for the coupling $|\xi|$.

Discussions. In this work, we consider the inclusion of tunneling on a static D -dimensional sphere, in addition to the well-known Casimir effect. Tunneling is achieved by considering a negative coupling, $\xi < 0$, in the bare theory, as well as a self-interaction for the scalar field, such that the bare theory exhibits two degenerate vacua. Similarly to the Casimir effect, tunneling between degenerate vacua contributes to a nontrivial ground state energy, but tunneling always leads to NEC violation, independently of the parameters. In addition, the scenario studied here allows tunneling to dominate over the Casimir effect, modifying the parameter space where the NEC is violated on a D -dimensional sphere, as exhibited in Figs. 1(a) and 1(b).

Renormalizing the ground state energy is achieved by the replacement of the bare vacuum energy with the Casimir energy. We note that the divergent contribution only coincides with the vacuum energy of flat space in the large a limit, which is the procedure used in [28]. However, the result obtained is identical to that via the more rigorous method of renormalizing the gravitational sector, as was taken in later work by the same authors [27]. Our work relies on the introduction of an R^2 term, for the time-integral of the

instanton Lagrangian to be finite, and thus the instanton action to be proportional to the spatial volume only. Since we do not consider dynamical gravity here, this amounts to the addition of a field independent constant, which we are free to do. For the case of dynamical gravity though, one would need to include an R^2 term in the gravity action and consider a general coefficient to avoid fine-tuning. This term leads to the well-known Starobinsky scenario [29] and would require a more thorough analysis of renormalization for the ground state energy.

A natural extension of the three-sphere example is to study early universe cosmology, where NEC violation could dynamically induce a cosmological bounce, in a scenario where the scale factor decreases. The Casimir energy of a conformally coupled massless scalar field in the Einstein static universe (ESU) with spatial topology of a three-sphere was considered in [30,31], without NEC violation though, and the situation of a dynamical space-time metric was studied in [32–34]. The case of an ESU was then extended in [27,28], with a nonconformally coupled scalar field, and a repulsive Casimir force was found, violating the NEC and potentially sustaining a cosmological bounce. Similar studies involving tunneling have been done in [9,11,13], assuming a three-torus as a confining space. In these works, NEC violation indeed leads to a cosmic bounce, which is induced dynamically when the comoving volume decreases. In the present study though, because of the R^2 contribution, we would need to study the possibility of a bounce with the modified Friedman-Lemaître-Robertson-Walker equations, and these studies are left for future work.

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