

Collinear limit of the four-point energy correlator in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

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(Received 2 February 2024; accepted 16 October 2024; published 14 November 2024)

We present a compact formula, expressed in terms of classical polylogarithms up to weight three, for the leading order four-point energy correlator in maximally supersymmetric Yang-Mills theory, in the limit where the four detectors are collinear. This formula is derived by combining a simplified, manifestly dual conformal invariant form of the $1 \rightarrow 4$ splitting function obtained from the square of the tree-level five-particle form factor of stress-tensor multiplet operators, with a novel integration-by-parts algorithm operating directly on Feynman parameter integrals. Our results provide valuable data for exploring the structure of physical observables in perturbation theory, and for calculations of jet substructure observables in quantum chromodynamics.

DOI: [10.1103/PhysRevD.110.L091901](https://doi.org/10.1103/PhysRevD.110.L091901)

Introduction. Explicit calculations of observables in quantum field theory have played a crucial role in uncovering hidden simplicity and structure, and in providing theoretical data for the development of new computational approaches. While a large amount of data exists for perturbative scattering amplitudes, less attention has been focused on physical cross section level observables relevant for colliders. Motivated by the remarkable simplicity hidden in scattering amplitudes, we can hope for more exciting surprises in the study of physical observables.

An interesting class of physical observables are N -point correlation functions of energy flux [1–4], which we denote E^N . The E^N are infrared finite [5,6], exhibit an operator product expansion [7–9], and can be directly measured in experiment [10–18], making them ideal candidates for explorations in perturbation theory. Although the E^N was computed at strong coupling [7] in $\mathcal{N} = 4$ super-Yang-Mills (sYM), little is known about its structure at weak coupling. The E^2 was computed at leading-order (LO) in the coupling expansion in quantum chromodynamics (QCD) in the seminal work of [2], and more recently at next-to-leading order (NLO) in QCD [19,20] and NNLO in $\mathcal{N} = 4$ sYM [21,22]. Explorations of higher-point correlators were initiated with the calculation of the E^3 in the

triple-collinear limit [23], and for generic angles [24,25]. These calculations provided important data for the light-ray OPE [26–29], and were crucial for developing new collider physics observables [14,15], motivating the exploration of higher-point correlation functions.

A simplified limit of the E^N is the multicollinear limit, shown in Fig. 1. In this limit, the E^N becomes independent of the source, and reduces to a function of $2(N-2)$ independent angles. This limit is also motivated by jet substructure [30,31], where the N detectors all lie in a single jet [32–35]. At LO the collinear limit of the E^N is given as a phase space integral of the tree-level universal $1 \rightarrow N$ splitting functions [36–39], $\mathcal{P}_{1 \rightarrow N}^{(0)}$, as

$$E^N \text{ coll} \equiv \int_0^1 dx_1 \cdots dx_N \delta\left(1 - \sum_i x_i\right) (x_1 \cdots x_N)^2 \mathcal{P}_{1 \rightarrow N}^{(0)},$$

where x_i is the fraction of energy carried by the particle absorbed by detector i . Remarkably, the N -point correlators define a class of manifestly finite integrals in $(N-1)$ -dimensional projective space, making it desirable to develop methods that directly compute in four dimensions, profiting from the simplifications of finite Feynman loop integrals [40–44]. This structure is similar to the Feynman-parameter representation of loop integrals, allowing the application of recently developed techniques [45–59]. Given this richness in both physical and mathematical contexts, it is imperative to develop efficient techniques for finite integrals, well adapted to the computation of the E^N for arbitrary N .

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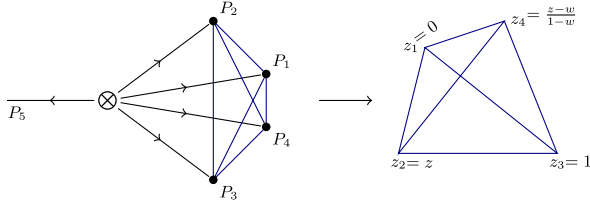


FIG. 1. The collinear limit of the four-point energy correlator parametrized in terms of momenta, see Eq. (1), and complex parameters $(z, \bar{z})$ and $(w, \bar{w})$, see Eq. (10).

In this Letter we obtain a new result for the LO four-point correlator in $\mathcal{N} = 4$ sYM by squaring a $1 \rightarrow 5$ form factor to derive a simplified expression for the $1 \rightarrow 4$ splitting function, which is then integrated using a novel integration-by-parts technique that operates directly in the energy parameter (x_i) space.

Super form factors and splitting functions. We begin by deriving a compact form for the tree-level form factors squared, $|F_{N+1}(\mathcal{O})|^2 = |\langle 0 | \mathcal{O}(q) | p_1, h_1, \dots, p_{N+1}, h_{N+1} \rangle|^2$, for $N = 3, 4$, where $\mathcal{O}$ is a member of the stress-tensor multiplet in planar $\mathcal{N} = 4$ sYM. Here (p_i, h_i) denote massless on-shell momenta and helicities. The particles in the final state (gluons, gluinos, and scalars) form on-shell supermultiplets. The stress-tensor multiplet includes, amongst others, the protected half-BPS operator $\mathcal{O}_{20'}$, the stress-energy tensor and the Lagrangian $\mathcal{L}$ of the theory. $\mathcal{N} = 4$ supersymmetry relates their form factors squared and summed over all helicities in the on-shell multiplets, in a simple way, e.g., $|F_{N+1}(\mathcal{L})|^2 = (q^2)^2 |F_{N+1}(\mathcal{O}_{20'})|^2$, with $q = \sum_{i=1}^{N+1} p_i$ being the total injected momentum.

To compute the splitting function $\mathcal{P}_{1 \rightarrow N}^{(0)}$, we consider the multicollinear limit $1 || 2 \dots || N$ of $|F_{N+1}|^2$, in which the angles between the first N particles become small simultaneously (see Fig. 1). The massless particle momenta p_i^μ , $p_i^2 = 0$, for $i = 1, \dots, N+1$, have the light cone components [23]

$$p_i^+ = \frac{q^0 x_i}{1 + |z_i|^2}, \quad p_i^- = \frac{q^0 x_i |z_i|^2}{1 + |z_i|^2}, \quad p_i^\perp = \frac{q^0 x_i z_i}{1 + |z_i|^2}, \quad (1)$$

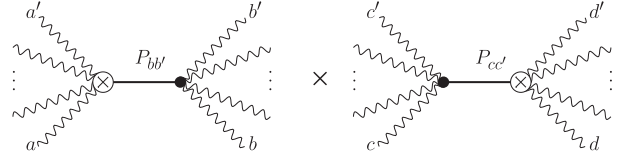


FIG. 2. The product of chiral and antichiral MHV diagrams used to compute the form factor squared.

with $\sum_{i=1}^{N+1} p_i^\mu = q^\mu$. The energy fractions $x_i = 2p_i^0/q^0$ satisfy $\sum_{i=1}^{N+1} x_i = 2$ and the complex parameters z_i are related to the angles between the particle momenta. In the multicollinear limit $z_i \rightarrow 0$ at the same rate, for $i = 1, \dots, N$. In the center-of-mass frame $q^\pm = q^0$, $q^\perp = 0$ momentum conservation implies $z_{N+1} = \infty$ and $x_{N+1} = 1$. The Mandelstam variables simplify to

$$s_{i_1 i_2 \dots i_k N+1} / |z_{N+1}|^2 = x_{i_1} + x_{i_2} + \dots + x_{i_k} \equiv x_{i_1 i_2 \dots i_k},$$

$$s_{i_1 i_2 \dots i_k} = \sum_{i < j} x_i x_j |z_i - z_j|^2, \quad i, j = 1, \dots, N.$$

In $\mathcal{N} = 4$ sYM the supermultiplet of final states is self-conjugate and contains all the helicities $-2, \dots, +2$. Consequently, the five-particle form factors are of the type maximally helicity violating (MHV), next-to-maximally helicity violating (NMHV), and N^2 MHV $\equiv \overline{\text{MHV}}$ [60], so we need only $F_5^{\text{MHV}}(\mathcal{L})$ and $F_5^{\text{NMHV}}(\mathcal{L})$. Tree-level form factors in $\mathcal{N} = 4$ sYM are computed efficiently by the MHV rules [61, 62]. In the planar limit, $N_c \rightarrow \infty$, we specify a single color orientation. The MHV matrix element squared is $\mathbb{F}_{N+1}^{\text{MHV}} = N_c^{N+1} (q^2)^4 / (s_{12} s_{23} \dots s_{1(N+1)})$. The NMHV form factor consists of a sum of graphs shown on the left panel in Fig. 2 (the right panel depicts the conjugate $\overline{\text{NMHV}}$ form factor). They are labeled anticlockwise by $b' > b \in [1, N+1]$ denoting the set of particles coming out of the interaction vertex. The complementary set with $a' = b' + 1$ and $a = b - 1$ consists of particles attached to the nonpolynomial Lagrangian operator denoted by a cross. The momentum $P_{bb'} = \sum_{i=b}^{b'} p_i$ flows from the operator to the vertex. In the multicollinear limit, if the noncollinear leg $N+1$ is on the interaction side, the contribution of the graph to the highest order pole vanishes. Finally, the $(N+1)$ -point matrix element squared $\mathbb{F}_{N+1}^{\overline{\text{MHV}}}$ is obtained by multiplying the sum of all graphs of the same color orientation by its conjugate and summing over the helicities of the final states. We find

$$\frac{\mathbb{F}_{N+1}^{\overline{\text{MHV}}}}{\mathbb{F}_{N+1}^{\text{MHV}}} \stackrel{\text{coll}}{=} \sum_{\{b, b'\}; \{c, c'\}} \left(\sum_{i=b}^{b'} \sum_{j=c}^{c'} \sum_{k=e}^{e'} x_i x_j x_k z_{ik} \bar{z}_{kj} \right)^4 \times \frac{z_{b-1} z_b z_{b'+1}}{s_{b \dots b'} K_{b-1, b'} K_{b, b'} L_{b, b'} L_{b, b'+1}} \frac{\bar{z}_{c-1} \bar{z}_c \bar{z}_{c'+1}}{s_{c \dots c'} \bar{K}_{c-1, c'} \bar{K}_{c, c'} \bar{L}_{c, c'} \bar{L}_{c, c'+1}},$$

$$z_{ij} = z_i - z_j, \quad K_{ab} \equiv \sum_{i=a}^b z_{ai} x_i, \quad L_{ab} \equiv \sum_{i=a}^b z_{bi} x_i, \quad 1 \leq b < b' \leq N, \quad 1 \leq c < c' \leq N, \quad [e, e'] = [b, b'] \cap [c, c']. \quad (2)$$

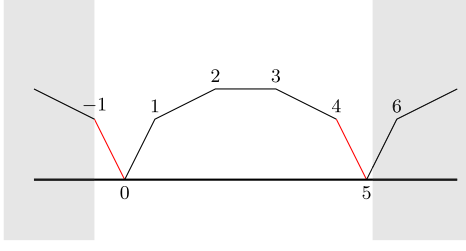


FIG. 3. Dual coordinates for the five-particle form factor. The points lie on an infinite contour with a period equal to the injected momentum q .

The unphysical poles cancel out in the sum of all terms. As mentioned above, the helicity decomposition of the matrix elements squared is $\mathbb{F}\bar{\mathbb{F}}_5 = 2\mathbb{F}\bar{\mathbb{F}}_5^{\text{MHV}} + 2\mathbb{F}\bar{\mathbb{F}}_5^{\text{NMHV}}$. The splitting function is obtained from the multicollinear limit of $\mathbb{F}\bar{\mathbb{F}}_{N+1}$ by symmetrizing over all permutations of the collinear particles, and by dividing by the two-particle matrix element squared to cancel the dependence on the operator $\mathcal{L}$,

$$\mathcal{P}_{1 \rightarrow N}^{(0)} = \frac{N_c^{N-1}}{(x_1 \dots x_N)^2 |z_{12} \dots z_{N-1N}|^2} \mathcal{G}_N + \text{perm}(1, 2, \dots, N)$$

$$\text{where } \mathcal{G}_N := \lim_{1||2 \dots ||N} \frac{\mathbb{F}\bar{\mathbb{F}}_{N+1}}{\mathbb{F}\bar{\mathbb{F}}_{N+1}^{\text{MHV}}}. \quad (3)$$

Our general result in Eq. (2) reproduces that for $N = 3$ in [23], but in a way which makes one of the remarkable features of amplitudes [60,63,64] and form factors [65–69] in $\mathcal{N} = 4$ sYM manifest, namely the dual conformal symmetry acting linearly on the particle momenta. To this end we introduce dual coordinates y_i by the rule $p_i = y_i - y_{i-1}$. In the amplitude case with vanishing total momentum $q = 0$ the dual points form a closed contour. For the form factor $q \neq 0$, and the dual points lie on an infinite contour with period q , such that $y_{ij} = y_{i+N+1, j+N+1}$ and $y_{i+N+1, i} \equiv q$, where $y_{i,j} \equiv y_i - y_j$. This is illustrated in Fig. 3 for the relevant case of the five-particle form factor. To reproduce the result of [23], we choose the points $y_{-1}, \dots, y_4$ on the contour and write the result in terms of dual conformal cross ratios $(a, b, c, d) = y_{ab}^2 y_{cd}^2 / (y_{ac}^2 y_{bd}^2)$,

$$\begin{aligned} \frac{\mathbb{F}\bar{\mathbb{F}}_4^{\text{NMHV}}}{\mathbb{F}\bar{\mathbb{F}}_4^{\text{MHV}}} &\stackrel{\text{coll}}{=} (-1, 1, 2, 4) + (-1, 3, 2, 0) + (3, 1, 0, 4), \\ &= \frac{s_{34}s_{14}}{s_{234}s_{124}} + \frac{s_{12}s_{1234}}{s_{123}s_{124}} + \frac{s_{23}s_{1234}}{s_{234}s_{123}}. \end{aligned}$$

The dual coordinates outside of the main period are marked in red. A new result reported here is the splitting function for $N = 4$ in $\mathcal{N} = 4$ sYM. It is given by a very compact dual conformal function of the points y_i with $i = -1, 0, \dots, 5$:

$$\begin{aligned} \frac{\mathbb{F}\bar{\mathbb{F}}_5^{\text{NMHV}}}{\mathbb{F}\bar{\mathbb{F}}_5^{\text{MHV}}} &\stackrel{\text{coll}}{=} -1 + (-1, 1, 2, 5) + (-1, 2, 3, 5) + (-1, 4, 3, 0) + (4, 1, 0, 5) + (-1, 3, 2, 0) + (4, 2, 1, 5) + (0, 4, 3, 1) \\ &+ (0, 4, 3, 1)(-1, 1, 3, 5) + (-1, 4, 3, 1)(3, 1, 0, 5) + (-1, 4, 2, 0)(0, 2, 3, 5) \\ &+ (-1, 1, 2, 4)(4, 2, 0, 5) + (-1, 3, 2, 0)(4, 2, 0, 5) + (-1, 4, 2, 0)(4, 2, 1, 5) + (-1, 4, 3, 0)(-1, 1, 2, 4) \\ &+ (4, 1, 0, 5)(0, 2, 3, 5) + (-1, 4, 3, 1)(-1, 4, 2, 0) + (3, 1, 0, 5)(4, 2, 0, 5) \\ &+ (-1, 4, 3, 1)(-1, 1, 2, 5) + (3, 1, 0, 5)(-1, 2, 3, 5) + (-1, 1, 2, 4)(-1, 1, 3, 5) + (0, 2, 3, 5)(-1, 1, 3, 5). \end{aligned}$$

Phase-space integration from integration-by-parts. To compute the E^{NC} in the collinear limit we invent an automated method exploiting simplifications from inter-dimensional integration-by-parts (IBPs) identities. Further developing algorithms from [41], we show how to find a basis of master integrals which is both free from divergences and avoids higher-degree poles, and illustrate this algorithm for the specific case of the $E^4\text{C}$.

- (1) Family definitions: The splitting function exhibits a uniform scaling weight in the energy fractions, allowing them to be treated as projective coordinates $[x_1 : \dots : x_N] \in \mathbb{P}_{N-1}(\mathbb{R}_+)$, parametrizing the N -body

collinear phase space. This yields a master formula for the E^{NC} in the collinear limit,

$$\begin{aligned} E^{\text{NC}} &\stackrel{\text{coll}}{=} \frac{1}{|z_{12} \dots z_{N-1N}|^2} \int \frac{d^N x}{\text{GL}(1)} (x_1 + \dots + x_N)^{-N} \mathcal{G}_N \\ &+ \text{perm}(z_1, \dots, z_N), \end{aligned} \quad (4)$$

where $\mathcal{G}_N$ is defined in Eq. (3) and we introduced the short-hand notation

$$\int \frac{d^N x}{\text{GL}(1)} \equiv \int_0^\infty dx_1 \cdots dx_N \delta\left(1 - \sum_i x_i\right).$$

Investigating the integrand topologies, we observe that the integrals for the E^4C belong to a family $A_{1,1,1,1,1,1,0,0,0}$ defined as

$$A_{a_1, \dots, a_{10}} \equiv \int \frac{d^4 x}{\text{GL}(1)} \frac{1}{\prod_i D_i^{a_i}},$$

$$D_1 = s_{1234}, \quad D_2 = s_{123}, \quad D_3 = s_{234},$$

$$D_4 = x_{1234}, \quad D_5 = x_{234}, \quad D_6 = x_{123}, \quad D_7 = x_{34},$$

$$D_8 = s_{12}, \quad D_9 = s_{23}, \quad D_{10} = s_{34}, \quad (5)$$

with multiparticle poles only in D_{1-7} . The integrals for the E^3C are given by the simpler family $B_{1,1,1,1,0,0}$,

$$B_{a_1, \dots, a_6} \equiv \int \frac{d^3 x}{\text{GL}(1)} \frac{s_{12}^{-a_5} s_{23}^{-a_6}}{s_{123}^{a_1} x_{12}^{a_2} x_{23}^{a_3} x_{123}^{a_4}}. \quad (6)$$

When computing the energy correlators, we wish to work with integrands satisfying the following conditions:

- (a) Homogeneity: The integrand has an overall scaling dimension $(-N)$ under the transformation $(x_1, \dots, x_N) \rightarrow (\kappa x_1, \dots, \kappa x_N)$.
- (b) Finiteness: The integral is free from IR divergences as any subset of energy variables go to zero. Given condition (a), condition (b) is equivalent to the UV power-counting behavior $\prod_i D_i^{-a_i} \sim O(\kappa^{-1-|S|})$ as $S \rightarrow \kappa S, \kappa \rightarrow \infty, \forall S \subset \{x_1, \dots, x_N\}$.

Such integrals fulfill the same criteria as the “admissible integrals” in [41], for which four-dimensional IBP methods are particularly well suited. One can reduce the integrals with poles of degree up to N onto a basis of master integrals of lower complexity. The latter can be evaluated with the Maple program HyperInt [70].

Our IBP algorithm operates directly on the projective coordinates $[x_1 : \dots : x_N]$. Integrals defined without a dimensional regulator exhibit several distinctive features. First, there exist integrand-level partial fractioning identities among the integrals with different propagator indices. It is advantageous to set up partial-fractioning identities among the (finite) integrands

$$A_{a_1, \dots, a_7; q_1, \dots, q_4} \equiv \int \frac{d^4 x}{\text{GL}(1)} \frac{x_1^{-q_1} x_2^{-q_2} x_3^{-q_3} x_4^{-q_4}}{D_1^{a_1} D_2^{a_2} \cdots D_7^{a_7}}, \quad (7)$$

where we demand that $a_i \geq 0, q_i \leq 0, \forall i$, and place an upper bound on the total numerator degree. This allows us to

set up a simple basis of finite integrands with x_i monomials in the numerator. Second, boundary terms are generated in the IBP relations on the surface of the integration domain: $x_i = 0, i \in \{1, 2, 3, 4\}$. These boundary terms are given by lower-point finite integrals over coordinates on the codimension-1 boundary $[x_j : x_k : x_l] \equiv [x_1 : \dots : \hat{x}_i : \dots : x_4]$,

$$B_{a_1, a_2, a_3, a_4; q_1, q_2, q_3}^{[jkl]} \equiv \int \frac{d^3 x}{\text{GL}(1)} \frac{x_j^{-q_1} x_k^{-q_2} x_l^{-q_3}}{s_{jkl}^{a_1} x_{jk}^{a_2} x_{kl}^{a_3} x_{jkl}^{a_4}}, \quad (8)$$

which are equivalent to the three-point energy integral family in Eq. (6) upon setting $(j, k, l) = (1, 2, 3)$. Hence our algorithm must proceed iteratively, by generating the relevant lower-point integrals and recycling them for the reduction of the higher-point integral families.

- (2) IBP algorithm: A generic IBP relation in projective space has the form

$$\int \frac{d^N x}{\text{GL}(1)} O_i \circ f = - \int \frac{d^{N-1} x}{\text{GL}(1)} v \circ f|_{x_i=0},$$

$$v = \prod_k x_k^{-q_k}, \quad f = \frac{1}{\prod_j D_j^{a_j}}, \quad (9)$$

where $O_i = \frac{\partial}{\partial x_i} v, i = 1, \dots, N$, are differential operators acting on projective coordinates, and we demand $a_j \geq 0, q_k \leq 0, \forall j, k$. The validity of the IBP relation requires conditions (a) and (b) both fulfilled on the left-hand side of Eq. (9). Hence we use individual scaling dimensions to impose a well-defined power counting on the IBP differential operator O_i in each integration region. This means that under the scaling of any subset of integration variables $S \rightarrow \kappa S$,

$$O_i \rightarrow \kappa^{\beta_S} O_i, \quad \forall S \subset \{x_1, x_2, x_3, x_4\},$$

where the scaling dimension is given by $\beta_S = -\sum_{k \in S} q_k - \delta_{i \in S}$. Given a system of operators graded by individual scaling dimensions, we derive finite IBPs in the following steps:

- (i) Seeding: The seeding process is optimized by an integrand reduction procedure. In a given sector we pair potential vectors v with an integrand f , and impose conditions (a) and (b) on $O_i \circ f$. We generate partial fractioning identities on finite fields for $v \circ f$, organized in block triangular form [71, 72], and solve for an integrand basis for IBP generation.
- (ii) IBP generation: Applying the differential operators to the integrand, we generate IBP relations $\mathcal{M}_N^a$ in the form of Eq. (9), where the left-hand side is expanded over integrals in the N -point family and the right-hand side defines the

boundary integral families. Each integral that appears in $\mathcal{M}_N^a$ is finite by construction.

- (iii) Integrand reduction: $\mathcal{M}_N^a$ encloses the complete set of N -point finite integrals. Among them we generate all partial fractioning identities $\mathcal{M}_N^b$ following the sector-by-sector procedure described in step (i).
- (iv) Reducing boundary integrals: When possible, we integrate out the boundary integrals generated on the right-hand side of $\mathcal{M}_N^a$. Otherwise, we repeat our procedure on the $(N-1)$ -point integral family.
- (v) Integral reduction for the full family: We carry out the reduction of $\mathcal{M}_N^a$ together with $\mathcal{M}_N^b$ iteratively, starting from $N=2$.

In the case of the E⁴C, our algorithm generates 3×10^3 finite integrals in the form of Eq. (7) with total numerator degree $p < 7$. To reduce the target integrals in Eq. (4), the reduction was carried out for 30^4 iterations with different kinematic values over finite fields, allowing functional reconstruction powered by [73,74].

- (3) Master integrals and symbol alphabets: For the four-point integral family in Eq. (5), after performing the integrand reduction, and discarding empty topologies that contain no finite integrals, the four-point integrals are mapped onto six subtopologies, plus their images under a reflection symmetry that flips the detector orientation: $1 \leftrightarrow 4, 2 \leftrightarrow 3$. The six topologies are divided into the following three categories based on the presence of multiparticle invariants s_{123} , s_{234} , and s_{1234} on the denominators:

Type I (one 3-particle cut): (2,4,5,6) (2,4,6,7),

Type II (4- and 3-particle cut): (1,2,4,5) (1, 2,4,7),

Type III (two 3-particle cuts): (2, 3,4,5) (2, 3,5,6),

where each sub-topology is specified by the location of its maximal cuts: (i, j, k, l) stands for integrals in Eq. (5) with $a_i, a_j, a_k, a_l \in \mathbb{Z}_{>=0}$ and the remaining indices = 0.

After IBP reduction, we find 28 four-point master integrals A_{1-28} , 14 three-point boundary integrals B_{1-14} and one constant function C_0 . Using the parametrization (see Fig. 1)

$$(z_1, z_2, z_3, z_4) = \left(0, z, 1, \frac{z-w}{1-w}\right), \quad (10)$$

the master integrals evaluate to polylogarithmic functions up to weight three in $(z, \bar{z})$ and $(w, \bar{w})$.

The boundary integrals B_{1-14} , C_0 all arise from three-point topologies $B_{1,0,1,1}^{[ijk]}$ with different detector orientations. For $(i, j, k) = (1, 2, 3)$, the master integrals define two dilogarithmic and two logarithmic functions, whose symbol alphabets read

$$\mathcal{A}_{123} = \{z, \bar{z}, 1-z, 1-\bar{z}, 1-|z|^2\}.$$

The type-I and -II topologies contain 18 four-point master integrals A_{1-18} , which couple to four distinct boundary integral families: $B_{1,0,1,1}^{[123]}$, $B_{1,0,1,1}^{[124]}$, $B_{1,0,1,1}^{[134]}$, and $B_{1,0,1,1}^{[234]}$. Their symbol alphabets are drawn from the set

$$\bar{\mathcal{A}}_{123} \cup \bar{\mathcal{A}}_{124} \cup \bar{\mathcal{A}}_{134} \cup \bar{\mathcal{A}}_{234} \cup \mathcal{A}_{\text{Quad}},$$

where $\bar{\mathcal{A}}_{123} \equiv \mathcal{A}_{123} \cup \{z - \bar{z}\}$ and

$$\mathcal{A}_{\text{Quad}} \equiv \{w\bar{z} - \bar{w}z, w\bar{z} - \bar{w}, z\bar{w} - w, 1 - w - \bar{w} + z\bar{w}, 1 - w - \bar{w} + w\bar{z}\}.$$

$\bar{\mathcal{A}}_{124}, \bar{\mathcal{A}}_{134}, \bar{\mathcal{A}}_{234}$ can be generated from $\bar{\mathcal{A}}_{123}$ through an S_4 symmetry transformation.

In the type-III topologies we identify 11 four-point master integrals which couple to two boundary integral families, $B_{1,0,1,1}^{[123]}$ and $B_{1,1,0,1}^{[234]}$, related by reflection. The type-III integrands do not factorize linearly in terms of $(z, \bar{z}, w, \bar{w})$. One must further introduce roots of a cubic polynomial, (a, b, c) , defined by

$$abc = -|z|^2|w|^2, \quad a + b + c = 1 - w - \bar{w} - z - \bar{z}, \\ a^{-1} + b^{-1} + c^{-1} = 1 - z^{-1} - \bar{z}^{-1} - w^{-1} - \bar{w}^{-1}. \quad (11)$$

The symbol alphabet of the type-III integrals are drawn from the set

$$\mathcal{A}_{123} \cup \mathcal{A}_{\text{Tri}^2} \cup \{1 \leftrightarrow 4, 2 \leftrightarrow 3\},$$

where

$$\mathcal{A}_{\text{Tri}^2} = \left\{ |z|^2 - |w|^2, z - |w|^2, \bar{z} - |w|^2, \frac{a}{b}, \frac{a + |w|^2}{b + |w|^2}, \frac{a + w\bar{b} + \bar{w}}{b + w\bar{a} + \bar{w}} \right\} \cup \{a \rightarrow b, b \rightarrow c\}, \quad (12)$$

and the reflection maps (z, w) to $(1/w, 1/z)$ and (a, b, c) to $(1/a, 1/b, 1/c)$. The origin of the cubic-root letters can be understood from the maximal-cut condition $s_{123} = s_{234} = x_{1234} = 0$, whose solutions, when projected onto the (x_2, x_3) plane, define a singular cubic curve $(ax_2 - x_3)(bx_2 - x_3) \times (cx_2 - x_3) = 0$, as illustrated in Fig. 4. The structure of the master integrals in this sector are discussed in Appendix A. Despite the cubic-root dependence, the master integrals are single-valued functions whose branch cuts cancel on the Euclidean sheet [75–77]. They all satisfy the expected first-entry condition [78], that the first entry of the symbol is $|z_{ij}|^2$. In Appendix B, we define the full set of master integrals A_{1-28} , B_{1-14} , and C_0 , and provide their transformation into a basis of uniform transcendental weight.

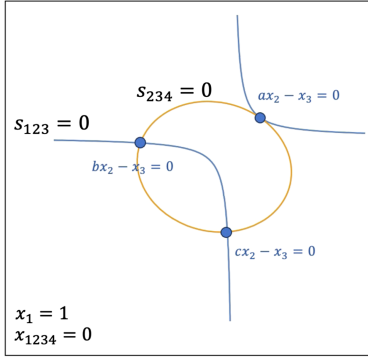


FIG. 4. The intersection between the hyperplane $x_{1234} = 0$ and the hypersurfaces $s_{123} = 0$ and $s_{234} = 0$ projected onto the (x_2, x_3) plane, gives cubic-root letters in the E^4C .

Results. Integrating the squared form factor, we obtain an analytic result for the collinear limit of the E^4C in $\mathcal{N} = 4$ sYM. We find that the LO E^4C function space is comprised of polylogarithms up to weight 3, whose symbol alphabets are drawn from the set

$$\mathcal{A}_4(1, 2, 3, 4) = \bar{\mathcal{A}}_{123} \cup \mathcal{A}_{\text{Quad}} \cup \mathcal{A}_{\text{Tri}^2} \cup \text{perm}(1, 2, 3, 4).$$

Interestingly, the first entry of the symbol only sees five letters: $\{|z|^2, |1-z|^2, |w|^2, |1-w|^2, |z-w|^2\}$. The final answer can be expressed in terms of a basis of single-valued polylogarithmic functions as well as their images under S_4 permutations of the detector orientations,

$$E^4C_{\mathcal{N}=4}^{\text{LO, coll}}(z_{ij}) = \frac{\sum_{I=1}^{51} R_I(z, \omega) F_I(z, \omega)}{|z_{12}|^2 |z_{23}|^2 |z_{34}|^2} + \text{perm}(1, 2, 3, 4).$$

In the ancillary files, we provide the analytic expressions for the basis $\vec{F}$ and algebraic coefficients $\vec{R}$.

We have checked our result through direct numerical integration over the energy fractions. An additional check comes from the investigation of kinematic limits. In particular, we can access the triple-collinear limit by sending z_1, z_2, z_3 to the origin; and the double-collinear limit by taking $|z_{12}|^2, |z_{34}|^2$ to zero simultaneously. In these limits the E^4C factorizes [23]

$$E^4C \xrightarrow{z_1, z_2, z_3 \sim 0} \frac{4}{3|z_4|^2} E^3C(z_1, z_2, z_3),$$

$$E^4C \xrightarrow{z_{12}, z_{34} \sim 0} \frac{2}{3|z_{23}|^2} E^2C(z_1, z_2) \times E^2C(z_3, z_4),$$

into lower-point correlators, E^2C, E^3C , defined in Eq. (4).

Outlook. Our result provides a number of directions for further exploring the perturbative structure of physical observables. First, it is important to study the singularity structure of the result to understand if its alphabet can be predicted *a priori*. There has been recent progress in the study of the Landau equations [79–81], intersection theory [82–84], as well as the structure of integrals directly in Feynman parameter space [47, 85–88], all of which are ideally suited for analyzing the integrals appearing in the $E^N C$. Second, it is interesting to push towards higher-point correlators, and to understand if statements can be made about the function space of the generic $E^N C$. Third, our result suggests to further explore the origin and the role of dual conformal symmetry for physical observables. Finally, it is highly desirable to extend our calculation to real world QCD, where our insight into the $\mathcal{N} = 4$ sYM alphabet should prove invaluable.

Acknowledgments. We acknowledge enlightening discussions with Nima Arkani-Hamed, Hofie Hannesdottir, Gregory Korchemsky, Andrzej Pokraka, Gang Yang, Xiaoyuan Zhang, and Hua Xing Zhu. This work is supported by the National Natural Science Foundation of China under Grant No. 12357077. K. Y. would like to thank the sponsorship from Yangyang Development Fund. I. M. is supported by Grant No. DOE-HEP-GR120647.

Appendix A. In the type-III topologies, the origin of the cubic-root letters is sector (2,3,4), where the solutions to maximal-cut condition $s_{123} = s_{1234} = x_{1234} = 0$ define three projective lines. Projected onto the (x_2, x_3) plane, the line equations become a singular cubic curve $(ax_2 - x_3)(bx_2 - x_3)(cx_2 - x_3) = 0$, with a, b, c defined in Eq. (11). Examining the leading singularity [89–91] of the master integrals in sector (2,3,4), we observe a subsystem $\{f_1, f_2\} \equiv \frac{|z_{13}z_{34}|^2}{|z_{23}|^4} \{A_{22}, A_{23}\}$ whose leading singularities contain odd structures in the cubic roots,

$$f_1 = \frac{1}{(c-a)(a-b)} \int d\ln x_3 \wedge d\ln \frac{ax_2 - x_3}{bx_2 - x_3} \wedge d\ln \frac{D_2}{D_4} \wedge d\ln \frac{D_3}{D_4}$$

$$- \frac{1}{(b-c)(c-a)} \int d\ln x_3 \wedge d\ln \frac{bx_2 - x_3}{cx_2 - x_3} \wedge d\ln \frac{D_2}{D_4} \wedge d\ln \frac{D_3}{D_4},$$

$$f_2 = \frac{a}{(c-a)(a-b)} \int d\ln x_3 \wedge \ln \frac{ax_2 - x_3}{bx_2 - x_3} \wedge d\ln \frac{D_2}{D_4} \wedge d\ln \frac{D_3}{D_4}$$

$$- \frac{c}{(b-c)(c-a)} \int d\ln x_3 \wedge \ln \frac{bx_2 - x_3}{cx_2 - x_3} \wedge d\ln \frac{D_2}{D_4} \wedge d\ln \frac{D_3}{D_4}. \quad (\text{A1})$$

The above expressions have manifest cyclic symmetry. Thus we introduce pure functions $F_{a,b,c}$ through

$$f_1 \equiv \frac{F_a}{(c-a)(a-b)} + \frac{F_b}{(a-b)(b-c)} + \frac{F_c}{(b-c)(c-a)},$$

$$f_2 \equiv \frac{aF_a}{(c-a)(a-b)} + \frac{bF_b}{(a-b)(b-c)} + \frac{cF_c}{(b-c)(c-a)}.$$

Equation (A1) defines two dlog integrals $\{F_a - F_b, F_b - F_c\}$, whose symbol alphabets contain the odd cubic-root letters given in the last line of Eq. (12). More specifically for F_a , we read off the following cubic-root letters in its symbol:

$$a, \quad a + |z|^2, \quad a + |w|^2, \quad u_a \equiv (a+w)(a+\bar{w}),$$

$$v_a \equiv (a+z)(a+\bar{z}), \quad \frac{a+w}{a+\bar{w}}, \quad \frac{a+z}{a+\bar{z}}.$$

The identities $u_a = \frac{|w|^2(a+|w|^2)(b+|z|^2)(c+|z|^2)}{bc(|w|^2-|z|^2)}$, $v_a = \frac{a(a+|z|^2)(b+|w|^2)(c+|w|^2)}{|w|^2(|w|^2-|z|^2)}$, together with the shuffle algebra [92], allow us to show in the first and second entry of $\mathcal{S}[F_a]$, the cubic-root alphabets appear as cyclic products: $abc, (a+|z|^2)(b+|z|^2)(c+|z|^2)$ and $(a+|w|^2)(b+|w|^2)(c+|w|^2)$. Therefore $\mathcal{S}[F_a]$ contains only rational letters in the first and second entry, and in particular the first-entry condition for single valuedness is satisfied. Moreover, in the cyclic combination $\{F_a + F_b + F_c\}$, the dependence on cubic roots fully cancels, and its symbol letters are rational in all entries.

The top sector (2,3,4,5) contains two additional finite dlog integrals f_3, f_4 , which are linear combinations of $A_{23}, A_{24}, A_{25}, A_{26}$. They can be cast into the following dlog form,

$$f_3 = \int d\ln x_3 \wedge d\ln \frac{wx_2 + x_3}{\bar{w}x_2 + x_3} \wedge d\ln \frac{D_2}{D_4} \wedge d\ln \frac{D_3}{D_5},$$

$$f_4 = \frac{1}{6} \int d\ln x_3 \wedge d\ln \frac{[(ax_2 - x_3)(bx_2 - x_3)(cx_2 - x_3)]^2}{[(wx_2 + x_3)(\bar{w}x_2 + x_3)]^3} \wedge d\ln \frac{D_2}{D_4} \wedge d\ln \frac{D_3}{D_4}$$

$$+ \frac{1}{2} \int d\ln x_3 \wedge d\ln [(wx_2 + x_3)(\bar{w}x_2 + x_3)] \wedge d\ln \frac{D_2}{D_4} \wedge d\ln \frac{D_5}{D_4},$$

which suggests that f_3, f_4 do not depend on the cubic roots. While the first piece of f_4 contains $F_a + F_b + F_c$, its dependence on the cubic roots is spurious.

Appendix B. We provide the definition of the master integrals in the type-I, -II, and -III subtopologies. The integrands for the four-point master integrals A_{1-28} are

$$A_{1,2}: \frac{x_3|z_{12}|^2}{s_{123}x_{34}^2x_{1234}}, \quad \frac{|z_{12}|^2}{s_{123}x_{34}x_{1234}},$$

$$A_3: \frac{|z_{12}|^2}{s_{123}x_{234}x_{1234}},$$

$$A_{4,5}: \frac{x_3|z_{23}|^2}{s_{123}x_{123}x_{34}x_{1234}}, \quad \frac{x_2|z_{23}|^2}{s_{123}x_{123}x_{34}x_{1234}},$$

$$A_{6,7}: \frac{x_3|z_{23}|^2}{s_{123}x_{123}x_{234}x_{1234}}, \quad \frac{x_1|z_{23}|^2}{s_{123}x_{123}x_{234}x_{1234}},$$

$$A_8: \frac{x_3|z_{12}|^4}{s_{1234}^2x_{34}}, \quad A_9: \frac{|z_{12}|^2}{s_{1234}x_{34}x_{1234}},$$

$$A_{10}: \frac{|z_{12}|^2}{s_{1234}x_{234}x_{1234}}, \quad A_{11}: \frac{x_3|z_{12}|^4}{s_{1234}x_{123}x_{34}},$$

$$A_{12,13}: \frac{x_2|z_{12}|^4}{s_{1234}x_{123}x_{234}}, \quad \frac{x_3|z_{12}|^4}{s_{1234}x_{123}x_{234}},$$

$$A_{14-16}: \frac{x_1|z_{12}|^4}{s_{1234}x_{123}x_{1234}}, \quad \frac{x_2|z_{12}|^4}{s_{1234}x_{123}x_{1234}}, \quad \frac{x_3|z_{12}|^4}{s_{1234}x_{123}x_{1234}},$$

$$A_{17}: \frac{x_1x_2|z_{12}|^4}{s_{1234}x_{123}x_{34}x_{1234}}, \quad A_{18}: \frac{x_2^2|z_{12}|^4}{s_{1234}x_{123}x_{234}x_{1234}},$$

$$A_{19-23}: \frac{x_1x_2|z_{23}|^4}{s_{123}x_{234}x_{1234}^2}, \quad \frac{x_1x_3|z_{23}|^4}{s_{123}x_{234}x_{1234}^2}, \quad \frac{x_2x_4|z_{23}|^4}{s_{123}x_{234}x_{1234}^2},$$

$$\frac{x_2|z_{23}|^4}{s_{123}x_{234}x_{1234}}, \quad \frac{x_3|z_{23}|^4}{s_{123}x_{234}x_{1234}},$$

$$A_{24,25}: \frac{x_2^2|z_{23}|^4}{s_{123}x_{234}x_{234}x_{1234}}, \quad \frac{x_2x_3|z_{23}|^4}{s_{123}x_{234}x_{234}x_{1234}},$$

$$A_{26-28}: \frac{x_2x_3|z_{23}z_{14}|^2}{s_{123}x_{234}x_{123}x_{234}}, \quad \frac{x_2^2|z_{23}z_{14}|^2}{s_{123}x_{234}x_{123}x_{234}},$$

$$\frac{x_3^2|z_{23}z_{14}|^2}{s_{123}x_{234}x_{123}x_{234}}.$$

The integrands for the three-point boundary integrals B_{1-14} are

$$\begin{aligned}
B_{1-3} &: \frac{x_1 |z_{12}|^2}{s_{123} x_{123}^2}, \quad \frac{x_2 |z_{12}|^2}{s_{123} x_{123}^2}, \quad \frac{|z_{12}|^2}{s_{123} x_{123}}, \\
B_4 &: \frac{x_2 |z_{12}|^2}{s_{123} x_{23} x_{123}}, \\
B_{5,6} &: \frac{x_1 |z_{12}|^2}{s_{124} x_{124}^2}, \quad \frac{|z_{12}|^2}{s_{124} x_{124}}, \quad B_7: \frac{x_2 |z_{12}|^2}{s_{124} x_{24} x_{124}}, \\
B_8 &: \frac{|z_{34}|^2}{s_{134} x_{134}}, \quad B_9: \frac{x_3 |z_{34}|^2}{s_{134} x_{24} x_{134}}, \\
B_{10-12} &: \frac{x_2 |z_{23}|^2}{s_{234} x_{234}^2}, \quad \frac{x_3 |z_{23}|^2}{s_{234} x_{234}^2}, \quad \frac{|z_{23}|^2}{s_{234} x_{234}}, \\
B_{13} &: \frac{x_3 |z_{23}|^2}{s_{234} x_{34} x_{234}}, \\
B_{14} &: \frac{x_3 |z_{23}|^2}{s_{234} x_{23} x_{234}}.
\end{aligned}$$

The two-point boundary integrals reduce to a constant function, which we define as $C_0 = 1$. The complete basis of master integrals $\{\vec{A}, \vec{B}, C_0\}$ can be brought into a uniform transcendental basis $\{\vec{I}_A, \vec{I}_B, C_0\}$ by a linear transformation. We provide the 43×43 transformation matrix T in the Supplemental Material [93].

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