

Constraints on the Dirac spectrum from chiral symmetry restoration

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I derive constraints on the Dirac spectrum in the chirally symmetric phase of a gauge theory with two massless fermion flavors. Using only general properties of correlation functions of scalar and pseudoscalar bilinears, I prove that in the chiral limit of vanishing fermion mass m the corresponding susceptibilities and all their derivatives with respect to m^2 must be finite. I then use the resulting spectral constraints to show that effective breaking of the anomalous $U(1)_A$ symmetry is allowed in the $SU(2)_A$ symmetric phase in the chiral limit, and leads to distinctive spectral features: (i) the spectral density must develop a singular $O(m^4)/\lambda$ peak as $m \rightarrow 0$, (ii) the two-point eigenvalue correlator of near-zero modes must be singular, and (iii) near-zero modes cannot be localized. Moreover, in the symmetric phase the topological charge distribution must be indistinguishable from that of an ideal gas of instantons and anti-instantons of vanishing density, to leading order in m .

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The presence of two light fermions (the u and d quarks) among the strongly interacting fundamental constituents of matter has deep consequences, best understood theoretically by studying quantum chromodynamics (QCD) in the chiral limit where u and d are exactly massless. In this limit QCD has an exact $SU(2)_L \times SU(2)_R$ chiral symmetry, spontaneously broken to its diagonal $SU(2)_V$ part at low temperature. At higher temperature hadronic matter turns into a plasma of quarks and gluons, and the broken $SU(2)_A$ part of the symmetry gets restored. However, the nature of the finite-temperature transition and the properties of the high-temperature symmetric phase are still open questions. A particularly contentious point is the fate of the anomalous $U(1)_A$ symmetry and the related role of gauge-field topology. These affect the nature of the transition [1,2]; due to the vicinity of the chiral and physical points, they can also have phenomenological consequences in collider experiments [3–6] and at the cosmological level [7–10].

There are arguments both in favor and against the necessity of effective $U(1)_A$ restoration as a consequence of $SU(2)_A$ restoration. Arguments in favor are based on the consequences of $SU(2)_A$ restoration for the Dirac spectrum, and on how these reflect on the $U(1)_A$ order parameters. Under plausible analyticity assumptions on the mass dependence of

observables in the symmetric phase, a smooth near-zero behavior of the spectral density of Dirac modes implies the absence of $U(1)_A$ -breaking effects in scalar and pseudoscalar susceptibilities [11–13]. Arguments against effective $U(1)_A$ restoration are instead based on assuming commutativity of the chiral and thermodynamic limits, which generally leads one to expect $U(1)_A$ -breaking effects from the topology-related zero modes [13–16]. This requires, however, the presence in the spectral density of an implausible singular delta at the origin at finite quark mass [17], calling into question the full restoration of $SU(2)_A$ symmetry if $U(1)_A$ remains broken [18].

This issue could be settled in principle by numerical investigations on the lattice, but numerical results are also contradictory, with some supporting [19–21], and some disfavoring [22,23] the effective restoration of $U(1)_A$. Moreover, recent investigations (see also [24]) point at the presence of a singular near-zero peak in the spectral density [19,21,25–37], suggesting that neither the smoothness assumption used in Refs. [11–13,17] nor the commutativity assumption used in Refs. [13–17] may be correct. There is currently no agreement on the physical meaning of the peak and on its fate in the chiral limit. Some ascribe it to a new phase of QCD [27–31] while others suggest a topological origin [24,26,32–36]; some expect it to become a Dirac delta at zero in the chiral limit [26,34], while others claim that it disappears already at finite quark mass [19,21]. In this context, a recent calculation in a QCD-inspired random matrix model for topology-related zero and near-zero modes supports $SU(2)_A$ restoration with $U(1)_A$ remaining broken due to a singular spectral peak [38]. While the validity of results obtained in a mixed-action

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setup [26–28,31,35], using a nonchiral action in the sea sector and a chiral one in the valence sector, has been questioned due to the presence of large lattice artifacts [19,21], Refs. [34,37] show how the peak emerges toward the continuum limit even using the same nonchiral action in both sectors. While waiting for more precise numerical results to definitively answer these questions, an analytic first-principles study of the possible consequences of the spectral peak would be useful.

Several studies of QCD toward the chiral limit have appeared in the literature in recent years, with particular attention to spectral aspects [17,34–42]. The constant advancement in computational power may soon allow for a direct numerical study of the chiral limit, including with chiral discretizations of the Dirac operator [43–47]. A reassessment of the constraints of $SU(2)_A$ and $U(1)_A$ restoration on the Dirac spectrum, separating the consequences of the assumption of symmetry restoration from those of extra, technical assumptions, is therefore timely.

Restoration of chiral symmetry is signalled by correlation functions of local operators related by chiral transformations becoming equal in the chiral limit. (Here and below, this is always understood to be taken after the thermodynamic limit.) Susceptibilities, i.e., connected correlation functions integrated over spacetime (normalized by the four-volume), are then also equal in the chiral limit, provided that the zero-momentum limit corresponding to spacetime integration commutes with the chiral limit, which is expected to be the case in the high-temperature symmetric phase where the correlation length of the system is finite. (The critical temperature is excluded in the case of continuous transitions, since the correlation length diverges there.) In the scalar and pseudoscalar sector, susceptibilities can be expressed in terms of the Dirac eigenvalues only, leading to necessary conditions required of the Dirac spectrum for $SU(2)_A$ restoration. I now outline their derivation; detailed calculations are reported in the Supplemental Material [48].

I consider finite-temperature $SU(N_c)$ gauge theories with two light fermions of mass m , and possibly further massive fermions, regularized on a lattice. Lattice units are used throughout this paper. I employ a discretized massless Dirac operator D obeying the Ginsparg-Wilson (GW) relation $\{D, \gamma_5\} = 2DR\gamma_5D$ with R local [43]. This operator has an exact lattice chiral symmetry [47] that reduces to the usual one in the continuum limit. Denoting by ψ and $\bar{\psi}$ two sets of Grassmann variables, carrying spacetime, color, and flavor indices, all suppressed in what follows, the relevant integrated densities are

$$\begin{aligned} S &= \bar{\psi}(1 - DR)\psi, & P &= \bar{\psi}(1 - DR)\gamma_5\psi, \\ \vec{P} &= \bar{\psi}(1 - DR)\vec{\sigma}\gamma_5\psi, & \vec{S} &= \bar{\psi}(1 - DR)\vec{\sigma}\psi, \end{aligned} \quad (1)$$

where the Pauli matrices $\vec{\sigma}$ act in flavor space, and summation over suppressed indices is understood.

Restoration of chiral symmetry for scalar and pseudoscalar susceptibilities can be expressed using their generating function $\mathcal{W}(V, W; m) = \lim_{V_4 \rightarrow \infty} \frac{1}{V_4} \ln \mathcal{Z}(V, W; m)$, where $V_4 = V_3/T$ is the lattice four-volume, V_3 its spatial volume and $1/T$ its temporal extension, equal to the inverse temperature, and $\mathcal{Z}(V, W; m)$ is the partition function in the presence of sources $V = (j_S, \vec{j}_P)$ and $W = (j_P, \vec{j}_S)$,

$$\begin{aligned} \mathcal{Z}(V, W; m) &= \int DU e^{-S(U)} F(U; V, W; m), \\ F(U; V, W; m) &= \int D\psi D\bar{\psi} e^{-\bar{\psi} D_m(U) \psi - K(\psi, \bar{\psi}, U; V, W)}, \\ K(\psi, \bar{\psi}, U; V, W) &= j_S S + i\vec{j}_P \cdot \vec{P} + i j_P P - \vec{j}_S \cdot \vec{S}, \end{aligned} \quad (2)$$

with $D_m = D + m(1 - DR)$. The dependence of $\mathcal{W}$ on T , and of $\mathcal{Z}$ on T and V_3 , is understood. Here U and DU denote collectively the gauge links and the corresponding Haar integration measure, and e^{-S} includes the contributions of the discretized Yang-Mills action and of the massive-fermion determinants. Usual (anti)periodic boundary conditions are understood. Expectation values of gauge-field functionals are defined as $\langle \mathcal{O} \rangle = \int DU e^{-S} F|_0 \mathcal{O} / \mathcal{Z}|_0$, where $|_0$ denotes setting sources to zero; their connected part is denoted $\langle \cdot \rangle_c$. Susceptibilities are the derivatives of $\mathcal{W}$ at zero sources. For present purposes, $\mathcal{W}$ (in its various guises discussed below) is a formal power series that can always be truncated to an arbitrarily large but finite order, and so it can be treated in practice as an analytic function of the sources; in particular, derivation with respect to the sources can be freely exchanged with the chiral limit.

The multiplets $(S, i\vec{P})$ and $(iP, -\vec{S})$ are irreducible under chiral transformations $\mathcal{U}$, whose effect on K is simply $K(\psi, \bar{\psi}, U; V, W) \rightarrow K(\psi, \bar{\psi}, U; R_{\mathcal{U}}V, R_{\mathcal{U}}W)$, with suitable $R_{\mathcal{U}} \in SO(4)$. Thanks to $SU(2)_V$ symmetry, which is exact also at $m \neq 0$, $\mathcal{W}$ depends on $\vec{j}_P$ and $\vec{j}_S$ only through $\vec{j}_P^2$, $\vec{j}_S^2$, and $\vec{j}_P \cdot \vec{j}_S$, or equivalently $Y = (Y_1, Y_2, Y_3) = (V^2, W^2, V \cdot W)$,

$$\mathcal{W}(V, W; m) = \bar{\mathcal{W}}(j_S, j_P; Y; m). \quad (3)$$

Notice that $SU(2)_V$ cannot break spontaneously for GW fermions at any nonzero m [49]. The axial part $SU(2)_A$ is instead explicitly broken at nonzero m .

Full symmetry restoration in the chiral limit for the quantities under investigation requires

$$\lim_{m \rightarrow 0} [\mathcal{W}(R_{\mathcal{U}}V, R_{\mathcal{U}}W; m) - \mathcal{W}(V, W; m)] = 0. \quad (4)$$

Since Eq. (4) must hold for arbitrary transformations, in the chiral limit $\mathcal{W}$ must not depend explicitly on $j_{S,P}$,

$$\lim_{m \rightarrow 0} \partial_{j_{S,P}} \bar{\mathcal{W}}(j_S, j_P; Y; m) = 0, \quad (5)$$

where $\partial_x = \partial/\partial x$. As $\mathcal{W}(V, W; m)$ depends on j_S and m only through the combination $j_S + m$, it is identical to $\mathcal{W}_0(\tilde{V}, W) = \lim_{V_4 \rightarrow \infty} \frac{1}{V_4} \ln \mathcal{Z}(\tilde{V}, W; 0)$, where $\tilde{V} = (j_S + m, \vec{j}_P)$ is a modified, mass-dependent source. Since D and the integration measure are invariant under (nonsinglet) chiral transformations, $\mathcal{W}_0(\tilde{V}, W)$ depends only on $\text{SO}(4)$ invariant combinations of $\tilde{V}$ and W ,

$$\begin{aligned} \mathcal{W}(V, W; m) &= \mathcal{W}_0(\tilde{V}, W) = \hat{\mathcal{W}}(\tilde{V}^2, W^2, 2\tilde{V} \cdot W) \\ &= \hat{\mathcal{W}}(m^2 + u, w, \tilde{u}), \end{aligned} \quad (6)$$

where $u = 2mj_S + V^2$, $w = W^2$, and $\tilde{u} = 2(mj_P + V \cdot W)$. (This relation holds exactly at any finite m , even if $\text{SU}(2)_A$ breaks down spontaneously in the chiral limit, as it follows from the symmetry properties of the massless partition function in a finite volume in the presence of nonzero sources.) Then, $\partial_{j_S} \bar{\mathcal{W}} = 2m\partial_{V^2} \bar{\mathcal{W}}$ and $\partial_{j_P} \bar{\mathcal{W}} = m\partial_{V \cdot W} \bar{\mathcal{W}}$, and chiral symmetry will be restored if the derivatives of $\bar{\mathcal{W}}$ with respect to V^2 , W^2 , and $V \cdot W$ at zero sources are finite in the chiral limit. The converse is also true: since

$$\begin{aligned} \partial_m \bar{\mathcal{W}}(j_S, j_P; Y; m) &= \partial_m \mathcal{W}(V, W; m) = \partial_{j_S} \mathcal{W}(V, W; m) \\ &= (\partial_{j_S} + 2j_S \partial_{V^2} + j_P \partial_{V \cdot W}) \bar{\mathcal{W}}(j_S, j_P; Y; m), \end{aligned} \quad (7)$$

taking further derivatives with respect to Y_i (that commute with the differential operator appearing on the second line), setting sources to zero, and taking the chiral limit, Eq. (5) implies

$$\begin{aligned} \lim_{m \rightarrow 0} \partial_m \left(\prod_i \partial_{Y_i}^{n_i} \right) \bar{\mathcal{W}}(j_S, j_P; Y; m)|_0 \\ = \lim_{m \rightarrow 0} \partial_{j_S} \left(\prod_i \partial_{Y_i}^{n_i} \right) \bar{\mathcal{W}}(j_S, j_P; Y; m)|_0 = 0, \end{aligned} \quad (8)$$

and so $\mathcal{A}_{n_1, n_2, n_3}(m) = (\prod_i \partial_{Y_i}^{n_i}) \bar{\mathcal{W}}(j_S, j_P, Y, m)|_0$ are finite (meaning nondivergent, here and below) in the chiral limit. Taking now repeated derivatives of $\bar{\mathcal{W}}$ with respect to m and Y_i and using Eqs. (6) and (7) one finds

$$\partial_m^n \mathcal{A}_{n_1, n_2, n_3}(m) = \sum_{a=0}^{\lfloor \frac{n}{2} \rfloor} C_a^{(n)} (2m)^{n-2a} \mathcal{A}_{n_1+n-a, n_2, n_3}(m), \quad (9)$$

with mass-independent coefficients $C_a^{(n)}$. By the result above, odd derivatives vanish in the chiral limit, and even derivatives are finite. Chiral symmetry restoration requires then all $\mathcal{A}_{n_1, n_2, n_3}$ (including the free energy $\mathcal{A}_{0,0,0}$) to be infinitely differentiable in m^2 at zero (“ m^2 -differentiable,” for short), i.e., to be power series in m^2 (possibly with zero radius of convergence, but this does not affect the following arguments) up to terms vanishing faster than any power. This implies that susceptibilities are m^2 -differentiable, or m times a m^2 -differentiable function, depending on whether

they involve an even or odd number of S and P [see Eq. (6)]. This condition, which is commonly assumed, is here proved to be necessary for $\text{SU}(2)_A$ restoration, using only general properties of correlators in the symmetric phase.

By a simple generalization of the argument, also connected correlation functions of integrated scalar and pseudoscalar densities and (mass-independent) local fields built out of gauge links, and their spacetime integrals, must be m^2 -differentiable for chiral symmetry to be restored. Since gauge fields are unaffected by chiral transformations, one expects this result to extend to any functional of the gauge fields; however, for nonlocal functionals (such as the spectral density of the Dirac operator) this will be treated as an additional assumption, that I will refer to as “nonlocal restoration.”

I now specialize to γ_5 -Hermitian D obeying the GW relation with $R = \frac{1}{2}$ [44,45], whose spectrum consists of $N_{\pm}$ zero modes of definite chirality ± 1 , N_2 modes with eigenvalue 2, and N complex-conjugate pairs of eigenvalues $\lambda_n^2/2 \pm i\lambda_n \sqrt{1 - \lambda_n^2/4}$, $\lambda_n \in (0, 2)$. I denote by $N_0 = N_+ + N_-$ and $Q = N_+ - N_-$ the total number of zero modes and the topological charge, respectively, and by $n_0 = \lim_{V_4 \rightarrow \infty} \langle N_0 \rangle / V_4$ and $\chi_t = \lim_{V_4 \rightarrow \infty} \langle Q^2 \rangle / V_4$ the zero-mode density and the topological susceptibility. Due to CP symmetry, $\mathcal{W}$ must be invariant under $j_P, \vec{j}_P \rightarrow -j_P, -\vec{j}_P$, so one can express it as a power series in u , w , and $\tilde{u}^2$,

$$\begin{aligned} \mathcal{W} &= \mathcal{C}_0 + u\mathcal{C}_u + w\mathcal{C}_w + \tilde{u}^2\mathcal{C}_{\tilde{u}^2} \\ &+ \frac{1}{2}(u^2\mathcal{C}_{uu} + 2uw\mathcal{C}_{uw} + w^2\mathcal{C}_{ww} \\ &+ 2u\tilde{u}^2\mathcal{C}_{u\tilde{u}^2} + 2w\tilde{u}^2\mathcal{C}_{w\tilde{u}^2} + \tilde{u}^4\mathcal{C}_{\tilde{u}^2\tilde{u}^2}) + \dots \end{aligned} \quad (10)$$

Chiral symmetry restoration requires all the coefficients to be finite in the chiral limit. To first order one finds

$$\begin{aligned} \mathcal{C}_u &= \frac{\chi_\pi}{2} = \frac{n_0}{m^2} + 2I^{(1)}[f], \\ \mathcal{C}_w &= \frac{\chi_\delta}{2} = -\frac{n_0}{m^2} + 2I^{(1)}[\tilde{f}], \\ \mathcal{C}_{\tilde{u}^2} &= \frac{\chi_\pi - \chi_\delta}{8m^2} - \frac{\chi_t}{2m^4} = \frac{1}{m^2} \left(\frac{n_0 - \chi_t}{2m^2} + m^2 I^{(1)}[f^2] \right), \end{aligned} \quad (11)$$

where $\rho(\lambda; m) = \lim_{V_4 \rightarrow \infty} \langle \rho_U(\lambda) \rangle$ is the spectral density,

$$\begin{aligned} \rho_U(\lambda) &= \frac{1}{V_4} \sum_{n=1}^N \delta(\lambda - \lambda_n), \\ I^{(1)}[g] &= \int_0^2 d\lambda g(\lambda) \rho(\lambda; m), \\ f(\lambda; m) &= \frac{h(\lambda)}{\lambda^2 + m^2 h(\lambda)}, \quad h(\lambda) = 1 - \frac{\lambda^2}{4}, \end{aligned} \quad (12)$$

and $\tilde{f} = f - 2m^2 f^2$. The dependence of ρ and other spectral quantities on T is understood. Since both terms in $\mathcal{C}_u$ are positive, they must be separately finite,

$$\lim_{m \rightarrow 0} \frac{n_0}{m^2} < \infty, \quad \lim_{m \rightarrow 0} I^{(1)}[f] < \infty. \quad (13)$$

This automatically implies the vanishing of the chiral condensate $\Sigma = -m\chi_\pi$ in the chiral limit. Since $m^2 f^2 \leq f$ for $\lambda \in [0, 2]$, χ_δ is automatically finite if χ_π is. Finiteness of $\mathcal{C}_{\tilde{u}^2}$ requires that the term in brackets be $O(m^2)$, and so that the $U(1)_A$ -breaking parameter $\Delta = \lim_{m \rightarrow 0} \frac{\chi_\pi - \chi_\delta}{4}$ equal

$$\Delta = \lim_{m \rightarrow 0} \left(\frac{n_0}{m^2} + 2m^2 I^{(1)}[f^2] \right) = \lim_{m \rightarrow 0} \frac{\chi_t}{m^2}, \quad (14)$$

and $\chi_t = O(m^2)$. Without further assumptions on n_0 and ρ , $SU(2)_A$ restoration is compatible with $U(1)_A$ breaking.

It is argued, and well supported by numerical results, that $\langle N_+ N_- \rangle = 0$ [50], implying $\langle N_0^2 \rangle = \langle Q^2 \rangle$ and $n_0 = 0$; Δ is then entirely determined by the density of near-zero modes, since the region $\lambda > \delta$ does not contribute to $m^2 I^{(1)}$ in the chiral limit for arbitrary $\delta > 0$. If $\rho(\lambda; 0)$ is well defined, and $\rho(\lambda; m)$ admits an expansion in positive powers of λ , $\rho(\lambda; m) = \sum_{n=0}^{\infty} \rho_n(m) \lambda^n$, convergent within some mass-independent radius (as assumed in Refs. [12,13]), then the $SU(2)_A$ restoration condition, Eq. (13), and positivity of ρ require only $\rho_0(m) = O(m)$ and $\rho_1(m) = O(1/\ln|m|)$. Effective $U(1)_A$ restoration, i.e., $\Delta = 0$, requires $\rho_0(m) = o(m)$: this is the case if ρ_n are power series in m^2 , as assumed in Refs. [12,13], and follows if restoration is nonlocal. A power-series behavior of ρ as a function of both λ and m^2 leads then to $U(1)_A$ restoration [12,13].

However, such a behavior is called into question by recent results showing a singular near-zero peak in ρ [19,21,24–37]. For a spectral density dominated by a power-law behavior $\rho \simeq C(m) \lambda^{\alpha(m)}$ at small λ , with $|\alpha(m)| < 1$ at small nonzero m and $\alpha(0) \neq 1$, Eq. (13) requires that $C(m) = \frac{\cos(\alpha(m)\pi/2)}{(1-\alpha(0))\pi/2} m^{1-\alpha(0)} (c + o(1))$, resulting in $\Delta = c$, and so in effective $U(1)_A$ breaking if $c \neq 0$. In this case ρ cannot be m^2 -differentiable, unless $\alpha(0) = -1$ with $\alpha(m)$ and $C(m)/m^2$ m^2 -differentiable. This leads to a surprising and highly nontrivial result: effective $U(1)_A$ breaking in the symmetric phase requires a singular peak in ρ tending to C/λ with $C = O(m^4)$ in the chiral limit, if restoration is nonlocal and ρ an ordinary function. (In Ref. [12] a nonanalytic behavior $\rho \sim \lambda^\alpha$ was considered, but only for mass-independent $\alpha > 0$. A behavior $\rho \sim 1/\lambda$ was proposed in Refs. [28–30] in a different context.) This is possible only if the chiral and thermodynamic limits do not commute (otherwise one needs a term $m^2 \Delta \delta(\lambda)$ in ρ) [17]. In this case one also finds $2 \int_0^2 d\lambda C(m) \lambda^{\alpha(m)} = \chi_t + o(m^2)$ for the density of peak modes. This supports the topological origin of the peak [24,26,32–36], and is consistent with the

instanton model of Ref. [38]. Moreover, it shows how strongly suppressed the peak is in the chiral limit, requiring very large volumes of order $\chi_t^{-1} \propto m^{-2}$ to become fully visible. The disappearance of the peak at finite m reported in Refs. [19,21] could then simply be a finite-volume artifact.

Further constraints are obtained requiring finiteness of the second-order terms in Eq. (10). The first two are

$$\frac{b_{N_0^2} - \chi_t}{2m^2} - m^2 \frac{\partial}{\partial m^2} \frac{n_0}{m^2} - 2m^2 I^{(2)}[f, f] = O(m^2),$$

$$I^{(2)}[\hat{f}, \hat{f}] = O(m^0), \quad (15)$$

where $\hat{f} = f - m^2 f^2$, $b_X = \lim_{V_4 \rightarrow \infty} \langle X \rangle_c / V_4$,

$$I^{(2)}[g_1, g_2] = \int_0^2 d\lambda \int_0^2 d\lambda' g_1(\lambda) g_2(\lambda') \rho_c^{(2)}(\lambda, \lambda'; m), \quad (16)$$

and $\rho_c^{(2)}$ is the connected two-point function,

$$\rho_c^{(2)}(\lambda, \lambda'; m) = \lim_{V_4 \rightarrow \infty} (V_4 \langle \rho_U(\lambda) \rho_U(\lambda') \rangle_c - \delta(\lambda - \lambda') \langle \rho_U(\lambda) \rangle). \quad (17)$$

Together with the assumption $\langle N_+ N_- \rangle = 0$, these constraints lead to remarkable results. I assume here that ρ and $\rho_c^{(2)}$ are ordinary functions; if delta-like singularities are present, the arguments below apply to their ordinary parts. Under general integrability conditions the region $\lambda, \lambda' > \delta > 0$ does not contribute to $m^2 I^{(2)}$ in the chiral limit, for arbitrarily small δ . If one assumes furthermore that $\rho_c^{(2)}(\lambda, \lambda'; m) = A(m) + B(\lambda, \lambda'; m)$, with B vanishing at the origin and obeying the loose bound $|B(\lambda, \lambda'; m)| \leq b(\lambda^2 + \lambda'^2)^{\beta/2}$ with $0 < \beta < 1$, the first constraint implies

$$-\pi^2 A(0) = \lim_{m \rightarrow 0} \frac{\chi_t - b_{N_0^2}}{m^2} = \lim_{m \rightarrow 0} \lim_{V_4 \rightarrow \infty} \frac{\langle N_0 \rangle^2}{V_4 m^2} = \Delta'. \quad (18)$$

(This corrects a result of Ref. [13] where exact zero modes were not fully accounted for.) On the other hand, the second constraint implies that $(\pi/4)^2 A(0) = \lim_{m \rightarrow 0} m^2 I^{(2)}[\hat{f}, \hat{f}] = 0$, so Δ' vanishes. This is possible only if the measure of the positive random variable $\frac{N_0}{\sqrt{V_4|m|}}$ is concentrated in zero in the thermodynamic and chiral limit, implying in turn $\frac{\chi_t}{m^2} \rightarrow 0$ and $\Delta = 0$. $U(1)_A$ breaking by $\Delta \neq 0$ in the $SU(2)_A$ -restored phase requires then a singular behavior at the origin of the two-point function $\rho_c^{(2)}$ for $m \neq 0$.

The current wisdom is that in the high-temperature phase of QCD, low modes are localized below a “mobility edge,” λ_c , and obey Poisson statistics [51]. For a purely Poisson spectrum of $N_P V_4$ eigenvalues one has $\rho_{Pc}^{(2)}(\lambda, \lambda') = -\rho_P(\lambda) \rho_P(\lambda') / N_P$ [13,52], so one expects $|\rho_c^{(2)}(\lambda, \lambda'; m)| \leq C \rho(\lambda; m) \rho(\lambda'; m)$ for some constant C if $\lambda, \lambda' < \lambda_c$. As localized modes fluctuate independently of each other, one

expects their correlations with modes beyond the mobility edge to be proportional to their density. It is then reasonable to assume $|\rho_c^{(2)}(\lambda, \lambda'; m)| \leq C' \rho(\lambda; m)$ for $\lambda < \lambda_c < \lambda'$. If λ_c remains finite in the chiral limit, under the assumptions above $m^2 I^{(2)}[f, f]$ vanishes in the chiral limit (and the second constraint in Eq. (15) is satisfied), so $\Delta' = 0$. Barring delta-like contributions, the persistence of $U(1)_A$ -breaking topological effects in the symmetric phase requires then that near-zero modes be not localized, either because λ_c vanishes in the chiral limit, or because another mobility edge is present near zero at nonzero m . (A mobility edge exactly at zero, as proposed in Refs. [29–31], is not sufficient, as the zero-measure lines $\lambda = 0$ and $\lambda' = 0$ can be excluded from the integral defining $I^{(2)}$.) The second possibility seems more likely in the presence of a singular spectral peak of topological origin. In any case, $U(1)_A$ breaking requires strong (singular) repulsion of near-zero modes. Numerical evidence for non-Poissonian repulsion toward the chiral limit was found using staggered fermions [34].

Two more constraints obtained at second order are

$$\left| \lim_{m \rightarrow 0} \frac{\partial}{\partial m^2} \frac{\chi_t}{m^2} \right| < \infty, \quad \left| \lim_{m \rightarrow 0} \frac{\partial \mathcal{C}_{\bar{u}^2}}{\partial m^2} \right| < \infty. \quad (19)$$

The first one requires that in the symmetric phase $\chi_t = \Delta m^2 + c_1 m^4 + o(m^4)$. This follows (to all orders in m^2) from the argument under Eq. (9), since Q^2 admits a representation as the integral of a local correlator [53–55]. The second constraint implies that the term in brackets in Eq. (11) is of the form $d_1 m^2 + d_2 m^4 + o(m^4)$; this also follows to all orders from the argument under Eq. (9).

The last two constraints at second order are

$$\lim_{m \rightarrow 0} \frac{8}{m^2} \int_0^2 d\lambda \hat{f}(\lambda; m) \rho_{Q^2 c}(\lambda; m) = \lim_{m \rightarrow 0} (\mathcal{C}_{uu} - \mathcal{C}_{ww}), \quad (20)$$

where $\rho_{Q^2 c}(\lambda; m) = \lim_{V_4 \rightarrow \infty} \langle Q^2 \rho_U(\lambda) \rangle_c$, and

$$\frac{b_{Q^4} - \chi_t}{m^2} = O(m^2). \quad (21)$$

No constraint on ρ is found at second order; no more can be obtained, as higher-order coefficients involve only higher-point eigenvalue correlators. In Eq. (20), $\mathcal{C}_{uu} - \mathcal{C}_{ww} \propto \langle (iP_a)^2 (iP_b)^2 \rangle - \langle S_a^2 S_b^2 \rangle$ ($a \neq b$), so if this

is nonzero one has $U(1)_A$ -breaking effects originating in non-negligible correlations between zero and nonzero modes.

The constraint Eq. (21) requires that in the chiral limit, to leading order in m and to the lowest nontrivial cumulant, the distribution of Q be indistinguishable from that of an ideal (noninteracting, Poisson-distributed) gas of instantons and anti-instantons, with equal and vanishingly small densities $\chi_i/2 = O(m^2)$. In the symmetric phase, the coefficient b_2 in the partition function at finite θ angle [56] must then tend to $-1/12$ in the chiral limit. This leads one to expect an early onset (near the transition temperature T_c) of an ideal instanton gaslike behavior for physical quark masses (though not necessarily that of the usual semiclassical dilute instanton gas [57,58]), as seen in the pure gauge case [56]. On the other hand, the results of Ref. [7] for b_2 show large deviations from an ideal gas up to $2T_c$, which calls for a deeper scrutiny of the size of finite- m corrections. As will be shown elsewhere, chiral symmetry restoration actually requires all cumulants to be those of an ideal gas, a result already obtained in Ref. [16] assuming a free energy analytic in m^2 . (This does not mean that typical gauge configurations must look like an ordinary ideal instanton gas [16].)

Summarizing, $SU(2)_A$ restoration does not prevent effective $U(1)_A$ breaking in the chiral limit, but puts severe constraints on the Dirac spectrum for it to happen: the spectral density must develop a $O(m^4)/\lambda$ peak; the two-point function of near-zero Dirac modes must be singular; near-zero modes cannot obey Poisson statistics and be localized; and the distribution of the topological charge must be the same as that of an ideal instanton gas with density of order m^2 . These predictions of distinctive spectral features, that can lead to a more accurate characterization of the quark-gluon plasma phase of QCD, should be verified by numerical investigations. The necessary and sufficient condition for chiral symmetry restoration in the scalar and pseudoscalar sector can probably be generalized to other sectors and to more massless fermion flavors by suitably adapting the proof.

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