

Nambu-Goto string as a higher-derivative Liouville theory

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I propose a generalization of the Liouville action which corresponds to the Nambu-Goto string like the usual Liouville action corresponds to the Polyakov string. The two differ by higher-derivative terms which are negligible classically but revive quantumly. An equivalence with the four-derivative action suggests that the Nambu-Goto string in four dimensions can be described by the (4,3) minimal model analogously to the critical Ising model on a dynamical lattice. While critical indices are the same as in the usual Liouville theory, the domain of applicability becomes broader.

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Introduction. Strings are with us! A string is generically a one-dimensional object whose propagation in time forms a two-dimensional surface embedded in d -dimensional space-time (we live in $d = 4$). The origin of modern string theory goes back to early 1970s when it was recognized that the dual resonance models of strong interaction are described by strings. Thus relativistic quantum strings do exist at the distances of the order of one Fermi. There are vast applications of strings and two-dimensional surfaces in physics: biological membranes, cosmic strings, Abrikosov and Nielsen-Olesen vortices, etc.

The beauty of bosonic string theory is a simplicity of its action—the area spanned by a string propagation—as proposed by Y. Nambu with T. Goto and also by H. B. Nielsen at the border of 1960s and 1970s. It looks very simple, but this is an illusion. Area is a highly nonlinear functional of d target-space coordinates X^μ which is invariant under diffeomorphism transformations of two coordinates parametrizing the string world sheet. To quantize such a system the symmetry has to be constrained by fixing a gauge, like it happens in quantum electrodynamics. The string quantization of the 1970s has resulted in a very beautiful theory enjoying conformal symmetry which becomes infinite-dimensional in two dimensions and whose generators obey the Virasoro algebra. However, the canonical quantization was consistent only in $d = 26$.

In early 1980s A. M. Polyakov recognized that the reason for this was an additional degree of freedom—one of the components of the metric tensor at the world sheet—which

does not decouple if $d \neq 26$. Its dynamics is governed for the Polyakov string [1] by the Liouville action (the field is accordingly called the Liouville field). Quantization of the Polyakov string is more easy than that of the Nambu-Goto string because the action is quadratic in X^μ and enjoys Weyl’s invariance which makes conformal symmetry manifest. The equivalence of the Nambu-Goto and Polyakov string formulations was shown in classical theory and at the one-loop order [2] of the perturbative expansion in the inverse string tension $2\pi\alpha'$. An exact solution of the quantum Liouville theory was found by Knizhnik-Polyakov-Zamolodchikov (KPZ) [3] and David-Distler-Kawai (DDK) [4,5] using the methods of conformal field theory (CFT). It allows one to compute the so-called string susceptibility index γ_h which determines the large-area behavior of the number of surfaces of genus h . The result for closed Polyakov’s string reads

$$\gamma_0 = 1 - \frac{(d_+ + d_-)/2 - d + \sqrt{(d_+ - d)(d_- - d)}}{12}, \quad (1)$$

with $d_+ = 25$, $d_- = 1$. It is real for $d \leq 1$ describing a vast amount of models in statistical mechanics; in particular, $d = 1/2$ describes the susceptibility of the critical Ising model on a random lattice [6], but does not apply for $d > 1$ where (1) is not real. A pessimistic viewpoint (shared by some of my colleagues in the 1990s) is that $d = 1$ is a barrier for the existence of bosonic string which does not exist nonperturbatively if $1 < d < 25$ including $d = 4$ or $d = 3$. A more optimistic view supported by the recent studies [7–9] of the spectrum of “effective strings” is that the problem may exist only for the Polyakov string rather than for the Nambu-Goto string. Anyway the challenging problem of existence of nonperturbative strings is inherited from the previous millennium along with confinement and quantum gravity.

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I argue in this Letter that Eq. (1) may still hold for the Nambu-Goto string in $d = 4$ with the KPZ barriers shifted to $d_{\pm} = 15 \pm 4\sqrt{6}$. Then $\gamma_0 = -1/3$ like for the critical Ising model on a dynamical lattice [6] which for the Polyakov string was described by $d = 1/2$. It works now in $d = 4$ because $d_- \approx 5.2 > 4$, linking conformal symmetry of the Nambu-Goto string to the (4,3) unitary minimal model. The arguments are based on an equivalence with the four-derivative Liouville action exactly solved [10] previously. Both theories are conformal invariant in spite of the presence of mass parameters.

Generalized conformal anomaly. The standard representation of the Nambu-Goto string via auxiliary fields which I learned from [11] is

$$S_{\text{NG}} = \int \sqrt{\det(\partial_a X \cdot \partial_b X)} \\ = \int \left[\sqrt{g} + \frac{1}{2} \lambda^{ab} (\partial_a X \cdot \partial_b X - g_{ab}) \right], \quad (2)$$

where the (imaginary) Lagrange multiplier λ^{ab} is a tensor density and g_{ab} is an independent metric tensor. I use the units where the bare string tension is set to 1.

The action (2) becomes quadratic in X^μ that makes it easy to path-integrate X^μ out. For a closed string this results in an effective action

$$S[g_{ab}, \lambda^{ab}] = \int \sqrt{g} - \frac{1}{2} \int \lambda^{ab} g_{ab} + S_X[g_{ab}, \lambda^{ab}], \quad (3)$$

where

$$S_X[g_{ab}, \lambda^{ab}] = \frac{d}{96\pi} \int \left[-\frac{12\sqrt{g}}{\tau \sqrt{\det \lambda^{ab}}} + \sqrt{g} R \frac{1}{\Delta} R \right. \\ \left. - \left(\beta \lambda^{ab} g_{ab} R + 2 \lambda^{ab} \nabla_a \partial_b \frac{1}{\Delta} R \right) \right] \quad (4)$$

and terms of higher orders in Schwinger's proper-time ultraviolet cutoff τ are dropped like they are dropped in the derivation of the Liouville action from the Polyakov string. As argued in [12] the higher-derivative terms do not change the results in that case. Equation (4) has been derived [12] from the DeWitt-Seeley expansion of the operator $(\sqrt{g})^{-1} \partial_a \lambda^{ab} \partial_b$ which becomes the Laplacian for

$$\lambda^{ab} = \bar{\lambda} \sqrt{g} g^{ab} \quad (5)$$

with constant $\bar{\lambda}$, reproducing the usual Liouville action derived for the Polyakov string. Thus Eq. (4) generalizes the usual conformal anomaly. One has $\beta = 1$ for the Nambu-Goto string but I keep β arbitrary for generality.

The action (4) is nonlocal just as in the case of the Polyakov string. It becomes local in the conformal gauge

$$g_{ab} = \hat{g}_{ab} e^\varphi, \quad (6)$$

where $\hat{g}_{ab}$ is the background (or fiducial) metric tensor and the Liouville field φ is a dynamical variable. Fixing the gauge produces the usual ghosts and their usual contribution to the effective action after path-integrating over the ghosts. A subtlety which will be crucial in what follows is that the curvature acquires the shift

$$\sqrt{g} R = \sqrt{\hat{g}} (\hat{R} - \hat{\Delta} \varphi), \quad (7)$$

where $\hat{\Delta}$ is the Laplacian for the metric tensor $\hat{g}_{ab}$. It vanishes only if the background curvature $\hat{R}$ vanishes. This produces an additional nonminimal interaction of φ with background gravity.

As always in Euclidean CFT we use conformal coordinates z and $\bar{z}$ in a flat background when $g_{zz} = g_{\bar{z}\bar{z}} = 0$, $g_{z\bar{z}} = g_{\bar{z}z} = 1/2$. Then the action (3) plus the ghost contribution takes the form

$$\mathcal{S}[\varphi, \lambda^{ab}] = \int e^\varphi (1 - \lambda^{z\bar{z}}) + \frac{1}{24\pi} \int \left[-\frac{3e^\varphi}{\tau} \left(\frac{d}{\sqrt{\det \lambda^{ab}}} - 2 \right) \right. \\ \left. + (d-26) \varphi \partial \bar{\partial} \varphi + d\kappa (2(1+\beta) \lambda^{z\bar{z}} \partial \bar{\partial} \varphi + \lambda^{z\bar{z}} \nabla \partial \varphi \right. \\ \left. + \lambda^{\bar{z}z} \bar{\nabla} \bar{\partial} \varphi) \right], \quad (8)$$

where $\kappa = 1$ as follows from (4), but it may be renormalized. In the action (8) $\nabla = \partial - \partial \varphi$ is the covariant derivative in the conformal gauge and it describes a theory with interaction. The representation of the R^2 case by an auxiliary field [13] is reproduced as $\beta \rightarrow \infty$.

It is tempting to path-integrate over λ^{ab} expanding about the value $\bar{\lambda}^{ab} = \bar{\lambda} \delta^{ab}$ minimizing (8) with constant $\bar{\lambda} < \bar{\lambda}_{\text{cl}} = 1$ in the world sheet coordinates which is associated with a quantum ground state (determined by vanishing of μ^2 in Eq. (11) below). However, nothing is expected to depend on $\bar{\lambda}$ because of the background independence. I often keep the same notation λ^{ab} for the fluctuations $\delta \lambda^{ab}$ about $\bar{\lambda}^{ab}$ when there is no confusion. The path integral over λ^{ab} has a saddle point at

$$\delta \lambda^{ab} = \sqrt{g} \tau \left(g^{ac} g^{bd} \nabla_c \partial_d \varphi + \frac{(\beta-1)}{4} g^{ab} \Delta \varphi \right) \frac{\kappa}{3} + \mathcal{O}(\tau^2) \quad (9)$$

justified by the smallness of τ . Naively $\delta \lambda^{z\bar{z}}$ is not $\sim \tau$ from (8), but we should not forget the linear $\delta \lambda^{z\bar{z}}$ term entering also the classical part of the action (8), which causes the renormalization of the bare string tension in the scaling regime [14].

Thus in the saddle-point approximation we arrive at the four-derivative action

$$\mathcal{S}[\varphi] = \frac{1}{16\pi b_0^2} \int \sqrt{\hat{g}} \left[\hat{g}^{ab} \partial_a \varphi \partial_b \varphi + 2\mu^2 e^\varphi \right. \\ \left. + \epsilon e^{-\varphi} \hat{\Delta} \varphi (\hat{\Delta} \varphi - G \hat{g}^{ab} \partial_a \varphi \partial_b \varphi) \right], \quad (10)$$

where

$$b_0^2 = \frac{6}{26-d}, \quad \mu^2 = -b_0^2 \left[8\pi(\bar{\lambda} - 1) + \frac{1}{\tau} \left(\frac{d}{\bar{\lambda}} - 2 \right) \right],$$

$$G = -\frac{1}{1 + (1 + \beta)^2/2}, \quad \varepsilon = -\frac{db_0^2 \kappa^2 \bar{\lambda}^3}{9G} \tau. \quad (11)$$

It is precisely the action exactly solved in [10].

It is clear from Eq. (9) that the presence of the dimensionful parameter τ was crucial in the passage from (8) to (10) where it becomes ε . I refer to each of these actions as “massive” CFT because its energy-momentum tensor (EMT) will be conserved and traceless in spite of the presence of the mass parameters.

Equation (9) for the saddle-point values of λ^{ab} is not the end of the story because of the next orders in τ coming from the expansion of $1/\sqrt{\det \lambda^{ab}}$. But these terms are at least quartic in φ and thus are expected not to change the one-loop results while they may contribute to the next orders. I shall now apply a more sophisticated technique of CFT to go toward proving the equivalence of the actions (4) and (10).

Improved energy-momentum tensor. The central role in CFT is played by the traceless EMT. It is derived by applying the variational derivative $\delta/\delta \hat{g}_{ab}$ to the action in a curved background which produces terms additional to the part associated to minimal interaction with gravity. It was called [15] “improved” to be distinguished from the minimal one and used by KPZ-DDK in solving the Liouville theory. A specifics of the “improvement” in two dimensions is outlined in [16].

A very nice feature of the improved EMT is that it is always traceless thanks to the classical equation of motion for φ . This is a general property because in the conformal gauge (6) we have

$$T_a^a \equiv \hat{g}^{ab} \frac{\delta \mathcal{S}}{\delta \hat{g}^{ab}} = -\frac{\delta \mathcal{S}}{\delta \varphi}, \quad (12)$$

where the left-hand side represents the trace of the improved EMT while the right-hand side represents the classical equation of motion for φ .

Representing $\lambda^{ab} = \sqrt{g} g^{ac} \alpha_c^b$ with the mixed tensor α_c^b being dimensionless as required by conformal invariance and varying with respect to $\hat{g}^{ab}$, we find the symmetric minimal EMT

$$T_{zz}^{(\min)} = \frac{(d-26)}{24} (\partial\varphi)^2 + \frac{d\kappa}{24} \left[2(1+\beta) \partial \lambda^{z\bar{z}} \partial \varphi + \bar{\partial} \lambda^{z\bar{z}} \partial \varphi - \partial \lambda^{z\bar{z}} \bar{\partial} \varphi - 2 \lambda^{z\bar{z}} \bar{\partial} \partial \varphi + 2 \lambda^{z\bar{z}} \partial \varphi \bar{\partial} \varphi \right], \quad (13a)$$

$$T_{z\bar{z}}^{(\min)} = 4\pi e^\varphi (1 - \lambda^{z\bar{z}}) - \frac{de^\varphi}{2\tau \sqrt{\det \lambda^{ab}}} + \frac{1}{\tau} e^\varphi + \frac{d\kappa}{24} \left[\bar{\partial} \lambda^{z\bar{z}} \bar{\partial} \varphi + \lambda^{z\bar{z}} \bar{\partial}^2 \varphi + \partial \lambda^{z\bar{z}} \partial \varphi + \lambda^{z\bar{z}} \partial^2 \varphi \right]. \quad (13b)$$

It is conserved obeying $\bar{\partial} T_{zz}^{(\min)} + \partial T_{z\bar{z}}^{(\min)} = 0$ but not traceless.

The improved EMT is the sum $T_{ab} = T_{ab}^{(\min)} + T_{ab}^{(\text{add})}$ of the minimal EMT and an addition, coming from the nonminimal interaction, with the component

$$T_{zz}^{(\text{add})} = -\frac{(d-26)}{12} \partial^2 \varphi - \frac{d\kappa}{24} \left[2(1+\beta) \partial^2 \lambda^{z\bar{z}} + \partial \bar{\partial} \lambda^{z\bar{z}} + \partial (\lambda^{z\bar{z}} \bar{\partial} \varphi) \right] + T_{zz}^{(\text{NL})}, \quad (14a)$$

$$T_{z\bar{z}}^{(\text{NL})} = -\frac{d\kappa}{24} \left[\frac{1}{\bar{\partial}} (\partial^3 \lambda^{z\bar{z}} + \partial^2 (\lambda^{z\bar{z}} \partial \varphi)) \right], \quad (14b)$$

where explicitly

$$\frac{1}{\bar{\partial}} f(\omega) = \int d^2 z \frac{f(z)}{\pi(\omega - z)}. \quad (15)$$

In covariant notations the improved EMT reads

$$T_{zz} = \frac{1}{4b_0^2} \left[(\partial\varphi)^2 + 2\nabla \partial \varphi - 2(1+\beta) \nabla^2 \lambda^{z\bar{z}} - \nabla \bar{\nabla} \lambda^{z\bar{z}} - \nabla \lambda^{z\bar{z}} \bar{\partial} \varphi - 2 \lambda^{z\bar{z}} \bar{\partial} \partial \varphi - 4 \nabla^2 \frac{1}{\Delta} \nabla^2 \lambda^{z\bar{z}} \right], \quad (16)$$

where we set $\kappa = 6/db_0^2$ to simplify the formulas. It obeys $\bar{\partial} T_{zz} = 0$, $T_{z\bar{z}} = 0$ thanks to the classical equations of motion. The improved EMT is thus conserved and traceless as expected in spite of the massive parameter τ ! A price for that is the nonlocal term (14b). This is just as was discovered in [17] for the action (10).

The conservation of the improved EMT (16) at the classical level follows from

$$\frac{1}{\pi} \bar{\partial} T_{zz} = \partial \varphi \frac{\delta \mathcal{S}}{\delta \varphi} - \partial \frac{\delta \mathcal{S}}{\delta \varphi} - \lambda^{z\bar{z}} \partial \frac{\delta \mathcal{S}}{\delta \lambda^{z\bar{z}}} + \partial \lambda^{z\bar{z}} \frac{\delta \mathcal{S}}{\delta \lambda^{z\bar{z}}} + \partial \left(\lambda^{z\bar{z}} \frac{\delta \mathcal{S}}{\delta \lambda^{z\bar{z}}} \right) + \partial \lambda^{z\bar{z}} \frac{\delta \mathcal{S}}{\delta \lambda^{z\bar{z}}}. \quad (17)$$

In quantum theory the variations of $\mathcal{S}$ are no longer zeros but are substituted by the variational derivatives with respect to the corresponding fields in the path integral.

For the generator of the (infinitesimal) conformal transformation $\delta z = \xi(z)$ this yields¹

$$\hat{\delta}_\xi = \frac{1}{\pi} \int \xi \bar{\partial} T_{zz} = \int \left[(\xi' + \xi \partial \varphi) \frac{\delta}{\delta \varphi} + \xi \partial \lambda^{z\bar{z}} \frac{\delta}{\delta \lambda^{z\bar{z}}} + (\xi' \lambda^{z\bar{z}} + \xi \partial \lambda^{z\bar{z}}) \frac{\delta}{\delta \lambda^{z\bar{z}}} + (-\xi' \lambda^{zz} + \xi \partial \lambda^{zz}) \frac{\delta}{\delta \lambda^{zz}} \right]. \quad (18)$$

Classically it produces the right transformation laws of φ and λ^{ab} whose components $\lambda^{z\bar{z}}$, $\lambda^{z\bar{z}}$, λ^{zz} have conformal weights 1, 0, -1, respectively.

While the improved EMT (16) classically generates the usual conformal transformation, a short calculation gives

$$\delta_\xi T_{zz} = \xi''' \frac{1}{2b_0^2} + 2\xi' T_{zz} + \xi \partial T_{zz} - \frac{1}{4b_0^2} \xi'' \frac{1}{\partial} \partial \nabla \lambda^{zz} \quad (19)$$

for the transformation of the improved EMT itself, where the last nonlocal term is additional to the standard transformation of the conserved primary² tensor of the conformal weight 2. Nevertheless I do not see any contradiction with Belavin-Polyakov-Zamolodchikov (BPZ) [18] because of the nonlocality. Equation (19) may simply show that the nonlocal term (14b) makes the improved EMT to be nonprimary.

The emergence of the additional nonlocal term in (19) does not mean *a priori* a violation of the Virasoro algebra. The commutator of two generators (18) reads

$$\langle (\hat{\delta}_\eta \hat{\delta}_\xi - \hat{\delta}_\xi \hat{\delta}_\eta) X \rangle = \langle \hat{\delta}_\zeta X \rangle + \frac{1}{24} \oint_{C_1} \frac{dz}{2\pi i} \left[\xi'''(z) \eta(z) - \xi(z) \eta'''(z) \right] \langle \hat{\partial} X \rangle, \quad (20)$$

where $\zeta = \xi\eta' - \xi'\eta$ in the first term on the right-hand side is as it should be. The contour C_1 encircles the singularities of $\xi(z)$ and $\eta(z)$, leaving outside possible singularities of X . As usual, $\hat{\partial}$ in Eq. (20) is linked to the central extension of the Virasoro algebra. It is determined by the second variation of the action with respect to the fields as it emerges from the commutator on the left-hand side like in [19]. It is the usual c-number for the quadratic action or at one loop, but it will be field-dependent for the action (4) involving the cubic part. It is my argument in favor of the intelligent one-loop approximation described in the next section because $\hat{\partial}$ in Eq. (20) would be operator-valued otherwise.

“Massive” versus massless CFT. The one-loop computation of the central charge and conformal weights can be performed by the propagators

¹Note that $\delta_\xi \lambda^{ab} = -(\partial_c \xi^a) \lambda^{bc} - (\partial_c \xi^b) \lambda^{ac} + (\partial_c \xi^c) \lambda^{ab} + \xi^c \partial_c \lambda^{ab}$ under diffeomorphism transformations.

²More accurately, it is a descendant of the unit operator.

$$\langle \varphi(-p) \varphi(p) \rangle = \frac{8\pi b^2}{p^2 + \varepsilon p^4}, \quad (21a)$$

$$\langle \lambda^{z\bar{z}}(-p) \varphi(p) \rangle = \frac{(1+\beta)G}{2} \frac{8\pi b^2 \varepsilon}{1 + \varepsilon p^2}, \quad (21b)$$

$$\langle \lambda^{zz}(-p) \varphi(p) \rangle = 4G \frac{8\pi b^2 \varepsilon p_z^2}{p^2 + \varepsilon p^4}, \quad (21c)$$

where ε and G are given by Eq. (11) and $b^2 = b_0^2 + \mathcal{O}(b_0^4)$ is a renormalization of b_0^2 . We see that λ^{ab} has mass squared ε^{-1} and does *not* propagate to large distances what was the original Polyakov’s argument [20] for the equivalence of the two string formulations. However, like shown in [17] for the action (10), a private life of λ ’s which occurs at the distances of order of the cutoff ε is seen nevertheless at macroscopic distances as a result of doing the uncertainty $\varepsilon \times \varepsilon^{-1}$ where ε^{-1} cuts momentum-space integrals and ε is the coupling of interaction between φ and λ^{ab} . This is like an appearance of anomalies in quantum field theory

It is clear that the local terms involving λ^{ab} in (16) do not contribute to the central charge because of massiveness, except for the nonlocal term (14b) which *does* contribute in a full analogy with the four-derivative Liouville theory [17]. Its computation drastically simplifies when the generator of the conformal transformation is represented by Eq. (18) which accounts for tremendous cancellations in the quantum case, while there are subtleties associated with singular products emerging in the averages like in the definition of the central charge c ,

$$\langle \hat{\delta}_\xi T_{zz}(0) \rangle = \frac{c}{12} \xi'''(0). \quad (22)$$

The normal ordering has to be implemented in T_{zz} .

With or without a little use of *Mathematica* we obtain

$$\langle \hat{\delta}_\xi T_{zz}^{(NL)} \rangle = -\frac{1}{2b^2} \partial^2 \frac{1}{\partial} \int d^2 z \xi'(z) \langle \partial \lambda^{zz}(z) \varphi(0) \rangle \delta^{(2)}(z), \quad (23)$$

where the singular product does not vanish as naively expected, but equals [19]

$$\frac{1}{\partial} \int d^2 z \xi'(z) \langle \partial \lambda^{zz}(z) \varphi(0) \rangle \delta^{(2)}(z) = -Gb^2 \xi'(0). \quad (24)$$

Equation (22) then contributes at one loop the additional $\delta c = 6G$ to the central charge which is the same as for the four-derivative action (10).

It is tempting to repeat the arguments [10] that like for the action (10) the intelligent one loop will give an exact answer in our case as well. I call this way the procedure proposed by DDK [4,5] for solving the usual Liouville theory where $T_{zz}^{(add)}$ is simply multiplied by a parameter Q

describing a renormalization of the nonminimal interaction. The arguments rely on a cancellation of skeleton diagrams which is represented by Eq. (18). We then would obtain $\delta c = 6QG$ for the additional contribution to the central charge at all loops. The vanishing of the total central charge would then require

$$d - 26 + \frac{6Q^2}{b^2} + 1 + 6QG = 0 \quad (25)$$

recovering DDK for $G = 0$.

The second equation that fixes the conformal weight of e^φ to be 1 remains unchanged:

$$1 = Q - b^2, \quad (26)$$

what is easily seen from the propagators (21) when $\varepsilon \rightarrow 0$. Like for the four-derivative action the nonlocal term in T_{zz} does not contribute to the conformal weight. In fact there exists a whole family of primary operators with the weights 1 thanks to Eq. (26), including a renormalized version of $e^\varphi / \sqrt{\det \lambda^{ab}}$.

The difference between massless and “massive” CFTs is explicitly seen in the pure R^2 case where it can be illustrated by the computation of the average (22) at large β . Then only φ and $\lambda^{z\bar{z}}$ are essential and we obtain

$$\begin{aligned} \langle \hat{\delta}_\xi T_{zz} \rangle = & \frac{1}{2b^2} \left\{ Q^2 \xi'''(0) + \int d^2z [\xi'(z) \partial^2 \varphi(z) + \xi(z) \partial^3 \varphi(z) \right. \\ & + (1 + \beta)(\xi''(z) \partial \lambda^{z\bar{z}}(z) + 3\xi'(z) \partial^2 \lambda^{z\bar{z}}(z) \\ & \left. + 2\xi(z) \partial^3 \lambda^{z\bar{z}}(z))] \varphi(0) \delta^{(2)}(z) \right\} \end{aligned} \quad (27)$$

with the singular products equal to

$$\int d^2z \xi(z) \langle \partial^n \varphi(z) \varphi(0) \rangle \delta^{(2)}(z) = G b^2 (-1)^n \partial^n \xi(0) S_n, \quad (28a)$$

$$\begin{aligned} & \int d^2z \xi(z) \langle \partial^n \lambda^{z\bar{z}}(z) \varphi(0) \rangle \delta^{(2)}(z) \\ & = \frac{1}{2} G b^2 (1 + \beta) (-1)^n \partial^n \xi(0) T_n. \end{aligned} \quad (28b)$$

In the free massless case of $\varepsilon = \infty$ like in [13] we have

$$S_n = 0, \quad T_n = \frac{2}{n(n+1)} \quad \boxed{\text{massless}}, \quad (29)$$

while in the massive case

$$S_n = \frac{2}{n(n+1)}, \quad T_n = \frac{2}{(n+1)} \quad \boxed{\text{massive}}. \quad (30)$$

The derivation is as in [19].

Substituting into Eq. (27), we see in the massless case the appearance in the central charge c of 2 rather than 1 as for the usual Liouville action due to the presence of two orthogonal scalars $\varphi - (1 + \beta)\lambda^{z\bar{z}}$ and $\lambda^{z\bar{z}}$ [13]. However, in our case of $\varepsilon \rightarrow 0$ only the first combination remains massless and contributes 1 to c , thus demonstrating the R^2 -term does not change the usual results. In “massive” CFT conformal symmetry holds for all distances, not only for the distances $\gg \sqrt{\varepsilon}$ where the standard CFT technique of BPZ [18] applies. In contrast to CFT without diffeomorphism invariance, the value of ε can now be compensated by a shift of φ . The solution to Eqs. (25) and (26) will be immediately described.

Relation to minimal models. For the Nambu-Goto string we have $G = -1/3$ from (11). Keeping in mind applications of the four-derivative Liouville action (10) in other cases, let us consider arbitrary G . From Eqs. (25), (26) b_0^{-2} renormalizes to

$$b^{-2} = \frac{13 - d - 6G + \sqrt{(d_+ - d)(d_- - d)}}{12}, \quad (31a)$$

$$d_\pm = 13 - 6G \pm 12\sqrt{1 + G} \quad (31b)$$

and $\gamma_0 = 1 - b^{-2}$ ($\gamma_h = 1 + h + (h - 1)b^{-2}$ for surfaces of genus h) is as in Eq. (1) with the KPZ barriers shifted to $d_\pm$ given by (31b).

The values of $d_\pm$ depend on G which has to lie in the interval $[-1, 0]$ for the action (10) to be stable. Then b^{-2} is real for $d < d_-$ which increases from 1 at $G = 0$ to 19 at $G = -1$. For $G = -1/3$ we have $d_- = 15 - 4\sqrt{6} \approx 5.2 > 4$ as is already announced in the Introduction, so γ_0 is real in $d = 4$. Remarkably, the value $G = -1/3$ is associated in $d = 4$ with the $p = 3$, $q = p + 1 = 4$ unitary minimal model as it will be momentarily discussed.

To find the relation to minimal models we note that the operators

$$V_\alpha = e^{\alpha\varphi}, \quad \alpha = \frac{1-n}{2} + \frac{1-m}{2b^2} \quad (32)$$

are the BPZ null-vectors for integers n and m like in the usual Liouville theory [21]. Their conformal weights

$$\Delta_\alpha = \alpha + (\alpha - \alpha^2)b^2 \quad (33)$$

are derived like (26) and reproduce Kac’s spectrum of CFT with the central charge

$$c = 1 + 6(b + b^{-1})^2, \quad (34)$$

where b is given by Eq. (31). This c is the central charge of the Virasoro algebra and not to be confused with the central charge $c^{(\varphi)} = 26 - d$ of φ (or φ plus λ ’s).

The minimal models are obtained by choosing

$$c = 25 + 6 \frac{(p-q)^2}{pq} \quad (35)$$

with coprime $q > p \geq 2$. Then Eqs. (31) and (34) fix

$$G = \frac{\left(1 - d - 6 \frac{(p-q)^2}{pq}\right)q}{6(q+p)}, \quad 1 - 6 \frac{(p-q)^2}{pq} \leq d \leq 19 - 6 \frac{p}{q}. \quad (36)$$

In the usual Liouville theory where $G = 0$ Eq. (36) implies $d = 26 - c = 1 - 6 \frac{(p-q)^2}{pq}$ for the central charge of matter, but d is a free parameter for $G \neq 0$. Contrary to the usual Liouville theory now $c \neq 26 - d$. The inequalities in (36) guarantee $0 \geq G \geq -1$ as is necessary for stability.

Given (36) we finally find from Eq. (31)

$$b^{-2} = \begin{cases} \frac{q}{p} & \text{perturbative branch} \\ -1 + \frac{(25-d)p}{6(q+p)} & \text{the other branch} \end{cases} \quad (37)$$

that applies for $d > 25 - 6 \frac{(p+q)^2}{p^2} \geq 2$. The perturbative branch is as in the usual Liouville theory, but the second branch is no longer $p \leftrightarrow q$ interchangeable with it.

There are no obstacles against $d = 4$ for the unitary case [22] $q = p + 1$! The barriers $d_{\pm}$ coincide (both equal 19) for $d = d_c = 13 - \frac{6}{p}$ ($d_c \geq 10$ for $p \geq 2$). For d from the interval $1 \leq d < d_c$ we have $d \leq d_-$ and γ_0 is *real*.

Final remarks. While our critical indices are as for the perturbative branch in the usual Liouville theory, the domain of applicability is now broader complementing applications of the Liouville action in condensed matter theory inspired by [23]. A hint that conformal symmetry of the Nambu-Goto string and the critical Ising model on a dynamical lattice may both be described by the (4,3) minimal model sounds fascinating and deserves future studies.

Equation (4) generically represents a “massive” generalization of the Liouville action. I expect it may have wide applications and help to gain insight into quantum gravity as the usual Liouville action did [24,25].

The reason why in Eq. (4) I have dropped the terms of higher order in τ , which has resulted in the four-derivative action (10), is twofold. First, this is a simplest extension



FIG. 1. Diagrams (a)–(d) representing the average of (38) over X^μ . The solid line represents X^μ , and the wavy lines represent λ^{ab} .

of the Liouville action, demonstrating how the higher-derivative terms revive. Second, there could be a kind of universality of the higher terms like it happens for the Polyakov string where they are marginal and do not change [12] the results of the usual Liouville action.

I can illustrate this by showing how the nonlocal term (14b) emerges for the Nambu-Goto string in flat space where it precisely comes from averaging over X^μ of EMT for the Nambu-Goto action (2)

$$T_{zz} = 2\pi \langle \lambda^{z\bar{z}} \partial X \cdot \partial X + \lambda^{\bar{z}z} \partial X \cdot \bar{\partial} X \rangle \quad (38)$$

thanks to the interaction $\lambda^{z\bar{z}} \partial X \cdot \partial X$ in (2). The emerging T_{zz} is represented by the diagrams depicted in Fig. 1. For the diagram in Fig. 1(b) we find precisely the linear part of the nonlocal term (14b). For the diagram in Fig. 1(c) nonlocality emerges only when the two λ 's are $\lambda^{z\bar{z}}$ and $\delta\lambda^{z\bar{z}}$. We obtain analogously to (14b)

$$\text{Fig. 1(c)} = -\frac{d\kappa}{12} \frac{1}{\partial} \partial^3 (\lambda^{z\bar{z}} \delta\lambda^{z\bar{z}}), \quad (39)$$

which may be thought of as obtained from a term equivalent to a six-derivative addition to Eq. (10). Nonlocal terms emerging from the diagrams in Fig. 1(d) will not contribute at one loop. The expectation is that the sum of the contributions of (14b) and (39) to $\langle \hat{\delta}_\xi T_{zz} \rangle$ with the singular products computed for the six-derivative action may give the same result as (23) computed for the four-derivative action, like it was the origin of the universality of higher terms for the Polyakov string [12]. I hope to return to this issue elsewhere.

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