

Coon unitarity via partial waves

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In this paper, we study tree-level four-point scattering amplitudes and present a novel approach to partial-wave unitarity that bypasses a lot of technical difficulties of previous approaches. In passing, we explicitly demonstrate that our approach provides a very suggestive form for the partial-wave coefficients in a natural way which is our main result. We use the Coon amplitude to exemplify this method and show how it makes important properties such as partial-wave unitarity manifest. A secondary by-product of our analysis is simple form for the partial-wave coefficients of the Coon amplitude as a single sum.

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I. INTRODUCTION

String amplitudes exhibit fascinating properties: tame high-energy behavior, dual resonance, and an infinite spin tower. The epitome of this triplet is the Veneziano amplitude [1]. Questions on its uniqueness were raised shortly after its discovery, at which time Coon suggested a deformation [2–4] exhibiting all of the aforementioned properties, but with logarithmic Regge trajectories instead of linear ones, interpolating between the Veneziano and scalar theory amplitudes.

Recently the Coon amplitude was put forth as an interesting case, since it evades the universal behavior of linear Regge trajectories [5]. Several of its aspects are by now well understood [6–13], including rigorous unitarity bounds on the $\{q, m^2, D\}$ space with q the deformation parameter of the Coon amplitude, m^2 the mass of external particles, and D the number of dimensions. Further, there are qualitative similarities with open-string scattering on D-branes in anti-de Sitter (AdS) [14].

With two explicit examples exhibiting stringy behavior, a reasonable follow-up was to examine if there are more amplitudes with stringy characteristics and more specifically what are the consequences of imposing an appropriately

tuned spectrum, additionally to the hallmark string properties. Naively, by demanding stringy properties and the string theory spectrum, we would expect that the answer would be uniquely fixed, and quite surprisingly recent studies proved that this is not the case [15–17].

These new amplitudes were bootstrapped in a bottom-up way, and hence, we lack the underlying physics theories, if any. Thus, we have to turn to further consistency conditions to derive more robust statements, and this is where unitarity enters. It is plausible that some S-matrix bootstrap solutions are just mathematical answers inconsistent with unitarity.

Partial-wave unitarity is highly nontrivial, even for the simple Veneziano amplitude [18,19].¹ We still miss a satisfactory derivation of the critical dimension of the (super)string directly from an analysis of tree-level scattering; see [19] for recent work and progress. It becomes an even more pressing matter in cases without an underlying theory, as it imposes nontrivial bounds on the free parameters of the amplitudes.

We provide a different approach to partial-wave unitarity.² The keypoint of our analysis resides in packaging crucial information into harmonic numbers and reducing the analysis to a linear algebra problem. We choose the Coon amplitude, to demonstrate that our method works well even

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¹We stress that we use the terminology partial-wave unitarity loosely in order to be consistent with other related papers on the subject. In order to be more precise we should be writing “partial-wave unitarity of the residues.”

²See also the recent [20,21] for interesting approaches based on analytics and numerics.

when dealing with nontrivial deformations, such as the q deformation.

II. THE COON AMPLITUDE

The Coon amplitude is given by [2]

$$\mathcal{A}(s, t) = (q-1)q^{\frac{\log \sigma \log \tau}{\log q \log q}} \prod_{n=0}^{\infty} \frac{(\sigma \tau - q^n)(1 - q^{n+1})}{(\sigma - q^n)(\tau - q^n)}, \quad (1)$$

with the deformation parameter taking values in the range $q \in (0, 1)$, and we have used the abbreviations

$$\sigma = 1 + (s - m^2)(q - 1), \quad \tau = 1 + (t - m^2)(q - 1). \quad (2)$$

In the center-of-mass frame,

$$\begin{aligned} s &= -(p_1 + p_2)^2 = 4(p^2 + m^2), \\ t &= -(p_1 + p_3)^2 = -2p^2(1 - \cos \theta), \\ u &= -(p_1 + p_4)^2 = -2p^2(1 + \cos \theta), \end{aligned} \quad (3)$$

where p is the momentum transfer and θ the scattering angle. The Mandelstam variables satisfy $s + t + u = 4m^2$ and are related to the scattering angle via

$$\cos \theta = 1 + \frac{2t}{s - 4m^2}. \quad (4)$$

The s -channel poles are located at $\sigma = q^n$, or equivalently,

$$s_N = m^2 + [N]_q, \quad (5)$$

where in the above $[N]_q$ is the q -deformed integer. Our definition is

$$[z]_q = \frac{1 - q^z}{1 - q}. \quad (6)$$

Note that we are using the same conventions as [6,7], and hence in comparing against the results of [8] one has to be cautious with some shifts.

The $N \rightarrow \infty$ limit reveals an accumulation point:

$$s_* = m^2 + \frac{1}{1 - q}. \quad (7)$$

Denoting by $\mathcal{R}(t, q, N)$ the residue of Eq. (1) at the s -channel poles we have

$$\mathcal{R}(t, q, N) = \frac{q^N}{(q^{-N}; q)_N} \sum_{n=0}^N \frac{(q^{-N}; q)_n}{(q; q)_n} \tau^{N-n}, \quad (8)$$

where $(x; q)_n$ is the q -Pochhammer symbol defined via Eq. (9):

$$(x; q)_n \equiv \prod_{i=0}^{n-1} (1 - xq^i). \quad (9)$$

We proceed by expanding Eq. (8) in a basis of Gegenbauer polynomials:

$$\mathcal{R}(t, q, N) = \sum_{\ell=0}^N c_{N,\ell} C_{\ell}^{(\alpha)} \left(1 + \frac{2t}{s_N - 4m^2} \right), \quad (10)$$

where $\alpha = \frac{D-3}{2}$, D is the spacetime dimensions, and we will use the shorthand $\mathcal{N} = s_N - 4m^2 = [N]_q - 3m^2$.

The necessary condition for unitarity is

$$c_{N,\ell} \geq 0, \quad (11)$$

such that the spectrum is free of negative-norm states.

To proceed, we exploit the fact that the Gegenbauer polynomials form a complete and orthogonal basis,

$$\int_{-1}^{+1} dx C_{\ell}^{(\alpha)}(x) C_{\ell'}^{(\alpha)}(x) (1 - x^2)^{\alpha-1/2} = 2\mathcal{K}(\ell, \alpha) \delta_{\ell\ell'}, \quad (12)$$

where $\mathcal{K}(\ell, \alpha)$ is the normalization factor

$$\mathcal{K}(\ell, \alpha) = \frac{\pi \Gamma(\ell + 2\alpha)}{2^{2\alpha} \ell! (\ell + \alpha) \Gamma^2(\alpha)}. \quad (13)$$

This allows us to obtain an integral representation formula of the partial-wave coefficients; note that we shift $n \rightarrow N - r$ in Eq. (8):

$$c_{N,\ell} = \frac{4^{\alpha-1/2}}{\mathcal{N}^{2\alpha}} \frac{1}{\mathcal{K}(\ell, \alpha)} \frac{q^N}{(q^{-N}; q)_N} \sum_{r=0}^N \frac{(q^{-N}; q)_{N-r}}{(q; q)_{N-r}} \int_{-\mathcal{N}}^0 dt C_{\ell}^{(\alpha)} \left(1 + \frac{2t}{\mathcal{N}} \right) [(1 - q)(m^2 - t) + 1]^r [t(-t - \mathcal{N})]^{\alpha-1/2}. \quad (14)$$

III. GENERATING FUNCTION FOR THE INTEGRAL

We are going to use a well-established approach to derive the expression for the partial-wave coefficients. We briefly describe the basic steps, and the interested reader can find thorough explanations of this method in [7,18,22]. We proceed by making use of the generating function of the Gegenbauer polynomials (the expression can be found in the Supplemental Material [23]) and the binomial expansion in order to express the $[t(-t-\mathcal{N})]^{\alpha-\frac{1}{2}}$ term as powers of t . This results in the following expression:

$$\begin{aligned} \sum_{j=0}^{\infty} \mathcal{K}(j, \alpha) c_{N,j} h^j &= \frac{4^{\alpha-\frac{1}{2}}}{\mathcal{N}^{2\alpha}} \frac{q^N}{(q^{-N}; q)_N} \\ &\times \sum_{r=0}^N \frac{(q^{-N}; q)_{N-r}}{(q; q)_{N-r}} \\ &\times \sum_{p=0}^{\alpha-\frac{1}{2}} \binom{\alpha-\frac{1}{2}}{p} (-\mathcal{N})^{\alpha-p-\frac{1}{2}} \mathcal{G}, \end{aligned} \quad (15)$$

where in the above we have defined a “pseudogenerating function” to be given by

$$\mathcal{G} = \int_{-\mathcal{N}}^0 dt \frac{(-(1-q)t + 1 + m^2(1-q))^r t^{\alpha+p+\frac{1}{2}}}{\left((h-1)^2 - \frac{4ht}{\mathcal{N}}\right)^\alpha}. \quad (16)$$

The integral in Eq. (16) can be performed analytically in terms of the Appell hypergeometric function and yields

$$\begin{aligned} \mathcal{G} &= -\frac{2(-\mathcal{N})^{\alpha+p+\frac{1}{2}}(1+m^2(1-q))}{1+2\alpha+2p} \frac{(h-1)^{2\alpha}}{(h-1)^{2\alpha}} \\ &\times {}_2F_1\left(\alpha+p+\frac{1}{2}; -r, \alpha; \alpha+p+\frac{3}{2}; -\frac{\mathcal{N}}{s_*}, -\frac{4h}{(h-1)^2}\right). \end{aligned} \quad (17)$$

We proceed further by using the definition of the Appell function as a formal power series (see the Supplemental Material [23]) and the binomial expansion for the h -dependent term in order to derive

$$\begin{aligned} \mathcal{G} &= -2^{r+1}(-\mathcal{N})^{\alpha+p+\frac{1}{2}}(1+m^2(1-q))^r \\ &\times \sum_{w=0}^r \frac{(\alpha)_w}{w_2!} \frac{(-4)^w (-1)^{-2w-2\alpha}}{1+2\alpha+2p+2w} \\ &\times {}_2F_1\left(-r, \alpha+p+w+\frac{1}{2}; \alpha+p+w+\frac{3}{2}; -\frac{\mathcal{N}}{s_*}\right) \\ &\times h^w \sum_{\epsilon} \frac{(2w+2\alpha)_\epsilon}{\epsilon!} h^\epsilon. \end{aligned} \quad (18)$$

Substituting Eq. (18) into Eq. (15) we have expressed both sides as powers of h , and we simply wish to extract the h^ℓ term from both sides which gives the $c_{N,\ell}$ coefficients. The final result is

$$\begin{aligned} c_{N,\ell} &= \frac{4^\alpha}{\mathcal{K}(\ell, \alpha)} \frac{q^N}{(q^{-N}; q)_N} \\ &\times \sum_{r=0}^N \sum_{p=0}^{\alpha-\frac{1}{2}} \sum_{w=0}^{\ell} \frac{(q^{-N}; q)_{N-r}}{(q; q)_{N-r}} (1+m^2(1-q))^r \\ &\times \binom{\alpha-\frac{1}{2}}{p} (-1)^p \frac{(-4)^w (\alpha)_w (2w+2\alpha)_{\ell-w}}{w!(1+2\alpha+2p+2w)(\ell-w)!} \\ &\times {}_2F_1\left(-r, \alpha+p+w+\frac{1}{2}; \alpha+p+w+\frac{3}{2}; -\frac{\mathcal{N}}{s_*}\right). \end{aligned} \quad (19)$$

We have, thus, derived a representation of all partial-wave coefficients in terms of triple truncated sums. While it might appear a bit more cumbersome than the original integral representation, Eq. (14), it is algorithmically simpler to consider. However, we proceed to demonstrate an even simpler representation of the partial-wave coefficients.

IV. DISCUSSING OTHER APPROACHES

We would like to comment on Eq. (19) and related results in the literature. In [16] [Eq. (67)] the authors presented a formula for all partial-wave coefficients of the Coon amplitude as a nested fourfold sum. Compared to that way of writing the partial-wave coefficients Eq. (19) appears to be simpler. Also, in [8] the authors used the contour integral representation, that was originally developed in [19] for the Veneziano amplitude, to solve for the partial-wave coefficients. However, an explicit expression for all spins and mass levels was not derived in closed form. The authors in [7] derived the expressions for the leading Regge trajectory and the general partial-wave coefficients but only for the $D=4$ case. Hence, Eq. (19) presents itself as the simpler and most general result so far. As we shall see, we will be able to derive a much simpler expression even compared to Eq. (19) below.

V. INTERMEZZO: HARMONIC NUMBERS

Before we proceed, we give some very basic facts about harmonic numbers. This is the set that will serve as the basis in the analysis below.

We define the q -deformed harmonic numbers recursively by

$$Z_{\{i_1, i_2, \dots, i_k\}}^q(N) \equiv \sum_{n=1}^N \frac{q^n}{[n]_q} Z_{\{i_2, i_3, \dots, i_k\}}^q(n-1), \quad (20)$$

where i is the symbol. We also define $Z_{\emptyset}^q(N) = 1$ and $Z_{\{i_1, i_2, \dots, i_k\}}^q(0) = 0$ with $k \geq 1$. These are also called the multiple harmonic q series [24].

In our context, we only encounter the condition that every symbol i is 1. Therefore, we introduce the notation

$$Z_k^q(N) = Z_{\{1, 1, \dots, 1\}}^q(N), \quad (21)$$

where the length of the symbol list is k . More useful properties of q -deform harmonic numbers can be found in the Supplemental Material [23].

VI. SUM RULES AND HARMONIC NUMBERS

Next we provide a different and new approach to determine all partial-wave coefficients. We will bypass many difficulties in the generating function method. This method also offers a novel perspective that allows us to encode information into harmonic numbers and then simply obtain all the coefficients by solving some linear equations.

First of all, we need to rearrange the form of the residues slightly. Following [9], we reorganize Eq. (8) as

$$\mathcal{R}(t, q, N) = q^N \prod_{n=1}^N \left(1 + (t - m^2) \frac{q^n}{[n]_q} \right). \quad (22)$$

The next observation is crucial. Let us consider the series coefficient of t , which leads us to the key identity

$$\prod_{n=1}^N \left(1 + t \frac{q^n}{[n]_q} \right) = \sum_{n=0}^N t^n Z_n^q(N), \quad (23)$$

where $Z_n^q(N)$ is the q -deformed multiple harmonic number; for the definition see Eqs. (20) and (21) in the Supplemental Material [23]. In fact, we treat the residue $\mathcal{R}(t, q, N)$ as a generating function of t with coefficients $Z_n^q(N)$.

By using the binomial theorem, we straightforwardly derive the series expansion of $\mathcal{R}(t, q, N)$ as a series in t :

$$\mathcal{R}(t, q, N) = q^N \sum_{n,k=0}^N \binom{k}{n} (-m^2)^{k-n} t^n Z_k^q(N). \quad (24)$$

On the other hand, let us revisit the decomposition of the Gegenbauer polynomials. By substituting

$$C_{\ell}^{(\alpha)}(x) = \frac{(2\alpha)_{\ell}}{\Gamma(\ell+1)} {}_2F_1 \left(-\ell, \ell+2\alpha; \alpha + \frac{1}{2}; \frac{1-x}{2} \right) \quad (25)$$

into Eq. (10) we get

$$\mathcal{R}(t, q, N) = \sum_{n,\ell=0}^N c_{N,\ell} \mathcal{T}_{n,\ell} \left(-\frac{t}{N} \right)^n, \quad (26)$$

where

$$\mathcal{T}_{n,\ell} = \frac{\Gamma(\ell+2\alpha)(-\ell)_n(\ell+2\alpha)_n}{n!\ell!\Gamma(2\alpha)(\frac{1}{2}+\alpha)_n}. \quad (27)$$

So far we have obtained two expansions of $\mathcal{R}(t, q, N)$: one is in the basis of harmonic numbers, and the other is the decomposition of the partial-wave coefficients. We can identify them, and since they are both series in t we arrive at the sum rules equations,

$$\sum_{\ell=0}^N c_{N,\ell} \mathcal{T}_{n,\ell} \left(-\frac{1}{N} \right)^n = q^N \sum_{k=0}^N \binom{k}{n} (-m^2)^{k-n} Z_k^q(N). \quad (28)$$

Ergo the problem has reduced to an elementary linear algebra exercise the solution of which is given by

$$c_{N,\ell} = q^N \sum_{n,k=0}^N \binom{n}{\ell} \underbrace{\frac{\sqrt{\pi}}{\mathcal{K}(\ell, \alpha)} \frac{(-1)^{\ell} (\alpha)_{\frac{1}{2}+n}}{(\ell+2\alpha)_{1+n}}}_{\mathcal{T}_{n,\ell}^{-1}} \times \underbrace{\binom{k}{n}}_{\text{external mass}} \underbrace{(-m^2)^{k-n} (3m^2 - [N]_q)^n}_{\text{scattering angle}} Z_k^q(N). \quad (29)$$

The above is the main result of the new approach we developed in this work, and more specifically its structural simplicity. The structure of Eq. (29) suggests a natural explanation of each term. The first part, $\mathcal{T}_{n,\ell}^{-1}$, is just the inverse of the matrix $\mathcal{T}_{n,\ell}$ and is universal. The second is controlled by the external mass and reduces to 1 for massless particles. The third part is related to the scattering angle, and the last part is our q -deformed harmonic number serving as a basis.

We point out that the summation over n in Eq. (29) can be performed analytically. Since the binomial coefficient becomes 0 when $n > k$, we can extend the upper bound of n to infinity. Thus, we finally obtain a single-sum version,

$$c_{N,\ell} = q^N \sum_{k=0}^N 2(-1)^{\ell} \Gamma(2\alpha) (-m^2)^k (\alpha + \ell) Z_k^q(N) \times {}_3\tilde{F}_2 \left(1, \alpha + \frac{1}{2}, -k; 1 - \ell, \ell + 2\alpha + 1; -\frac{N}{m^2} \right), \quad (30)$$

where we have used $p\tilde{F}_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$ to denote the regularized hypergeometric function; for the definition see the Supplemental Material [23].

VII. ANALYSIS

Having pursued a new way to compute the partial-wave coefficients, we make some consistency checks. Firstly, it is

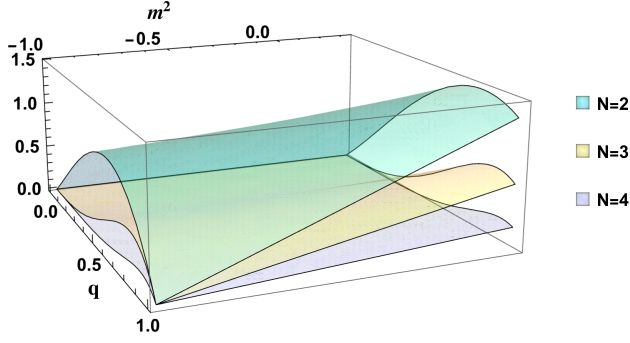


FIG. 1. The sub-leading trajectory is controlled by function $f(q, m^2)$. Let us consider $0 < q < 1$ and $-1 \leq m^2 \leq \frac{1}{3}$. We find that the roots of $f(q, m^2) = 0$ are at $q = 0$ or $q = 1$. We choose $N = 2, 3, 4$ as a vivid illustration.

easy to check that the expressions in Eqs. (14), (19), and (30) agree with one another. Further, it is straightforward to check that for the special case $D = 4$ there is agreement with the leading Regge trajectory and the general formula derived in [7]. We have, also, checked our results against the coefficients [6] [Eq. (42)]; these are explicit checks up to level 3. We provide further tests for the validity of Eqs. (19) and (30) by comparing against the integral representation of the partial-wave coefficients as given by Eq. (14) for some level up to $N = 20$ and $\ell = 0, 1, \dots, 5$ in the Supplemental Material [23].

The next comment pertains to the structure of Eq. (29). Notice how much more suggestive the answer is compared to the integral representation, Eq. (14), and the hypergeometric sum expression, Eq. (19). We proceed to examine some unitarity bounds.

Using the basis of harmonic numbers facilitates the analysis of Regge trajectories. We present the formulas for the leading and subleading trajectories to illustrate. For the former we have the simple

$$c_{N,N} = \left(\frac{q}{4}\right)^N \frac{N!}{(\alpha)_N} \mathcal{N}^N Z_N^q(N). \quad (31)$$

Requiring unitarity yields

$$\mathcal{N} = [N]_q - 3m^2 \geq 0, \quad \forall N \geq 1. \quad (32)$$

This is our first nontrivial bound. When $N = 1$, we easily obtain $m^2 \leq \frac{1}{3}$. It is well known that the Veneziano amplitude is nonunitary when $m^2 < -1$. Taking this into consideration, we set $-1 \leq m^2 \leq \frac{1}{3}$ as our parameter interval.

Next let us focus on the subleading trajectory. From Eq. (29) we have

$$c_{N,N-1} \propto \underbrace{2Z_{N-1}^q(N) - N(2m^2 + \mathcal{N})Z_N^q(N)}_{f(q, m^2)}. \quad (33)$$

Ignoring an overall positive factor, and using the explicit form of harmonic numbers (the expression can be found in the Supplemental Material [23]), we find that the locus of zeroes is on the boundary of the q region; see Fig. 1. By taking the special limit $q \rightarrow 1$, we notice

$$f(1, m^2) = \frac{m^2 + 1}{\Gamma(N)} \geq 0. \quad (34)$$

Hence we prove that the subleading trajectory is *always manifestly positive*.

The use of harmonic numbers, also, provides a significant benefit in low-spin analysis. We choose $D = 4$ and $\ell = 0$ as an example to support this claim. The partial-wave coefficient, Eq. (30), becomes

$$c_{N,0} = \frac{1}{\mathcal{N}} \frac{q^N}{(q^{-N}; q)_N} \sum_{j=0}^N \frac{(q^{-N}; q)_{N-j}}{(q; q)_{N-j}} \frac{1}{(1-q)(1+j)} \times [(m^2 + \mathcal{N})(1-q) + 1]^{j+1} - (m^2(1-q) + 1)^{j+1}. \quad (35)$$

The first line not always positive; however, we know³ that the low-spin states control the unitarity analysis.⁴ The second line of Eq. (35) is *manifestly positive* due to $(m^2 + \mathcal{N})(1-q) + 1 \geq m^2(1-q) + 1 \geq 0$. Using this and some mild numerical analysis, we observe that the partial-wave coefficient $c_{N,0}$ is positive when $D = 4$.

In some special limits, we can analyze the *full* unitarity behavior in a new integral representation. We consider the limit $q \rightarrow 1$ and $m^2 = 0$, which actually is the super-string case. Combining Eq. (29) with the definition of harmonic numbers from the package harmonic polylogarithms (HPL) [27], see Eq. (7) of the Supplemental Material [23], the partial-wave coefficients are

³We wish to clarify, at this point, that there is no rigorous proof that the scalar states determine the positivity bounds. This is merely an observation based on empirical evidence known from the past and also pointed out recently; see, e.g., [6,25].

⁴We stress that what we mean here is not related to the so-called “low-spin dominance” that was introduced in [26]. Indeed, the original meaning of “low-spin dominance” is that the Wilson coefficients of physical theories occupy much smaller regions than the bounds from $2 \rightarrow 2$ scattering considerations. This is depicted most clearly in [[26], Fig. 10]. Interestingly, the Coon amplitude satisfies a weaker version of “low-spin dominance” than that of [26] as was pointed out in [6]. In the case of the Coon amplitude not only the spin-0 states dominate, but also the spin-1 state. The same holds true for the limit that the Coon becomes the Veneziano amplitude.

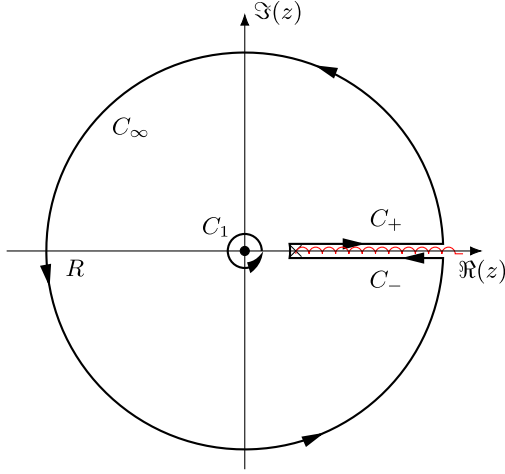


FIG. 2. The complex plane. The original contour is C_1 , and the deformed contour is $C_\infty + C_- + C_+$. We use red bumps to label the branch cut.

$$c_{N-1,\ell} = \frac{1}{2\pi i} \oint \frac{2(-1)^\ell}{z^N(1-z)} (\ell + \alpha) \Gamma(2\alpha) \times {}_2\tilde{F}_2\left(1, \alpha + \frac{1}{2}; 1 - \ell, \ell + 2\alpha + 1; N \log(1-z)\right), \quad (36)$$

there is a shift in N for convenience.⁵ The integral is to extract the coefficient before z^{N-1} . The contour C_1 around $z = 0$ can be deformed into $C_\infty + C_- + C_+$; see Fig. 2. The contribution from C_∞ goes to 0 when $R \rightarrow \infty$. There is also a branch cut from $\log(1-z)$; hence we convert Eq. (36) into the integral of its disc. For example, we choose $D = 6$ and $\ell = 0$,⁶

$$c_{N-1,0} = \int_0^\infty dt \frac{12(g(t) + \pi^2(t^N + 1))}{tN^3(t+1)^N(\log^2(t) + \pi^2)^3}. \quad (37)$$

Here $t = z - 1$ and

$$g(t) = \log(t)[N(t^N - 1)(\log^2(t) + \pi^2) - 3(t^N + 1)\log(t)]. \quad (38)$$

The integrand of Eq. (37) is always positive when $t \geq 0$. The only subtlety is the function $g(t)$. However, notice that $g(t)$ has only one root, at $t = 1$, and $g \rightarrow \infty$ when $t \rightarrow 0$ and $t \rightarrow \infty$. We conclude that $g(t) \geq 0$ in the region of

⁵This result matches in [[19], Table 1] up to an overall factor of N . We shift $N \rightarrow N + 1$ in the scattering angle of Eq. (29) and then take the limit $q \rightarrow 1$ and $m^2 \rightarrow 0$.

⁶It is well known that superstring partial-wave coefficient $c_{N-1,\ell} = 0$ when $N - 1 + \ell$ is odd. So we focus on $N = 1, 3, 5, \dots$ here.

integration. A more in-depth analysis with more examples is presented in [25].

Let us comment on the critical q . This is a value for q below which the Coon amplitude is unitary for any choice of D and m^2 . Using the large- N behavior of the harmonic numbers we derive

$$\lim_{N \rightarrow \infty} c_{N,0} \propto \lim_{N \rightarrow \infty} \left[q^N - \frac{N}{2} q^N (-3m^2 + [N]_q) Z_1^q(N) \right] \propto 2q^{-1} - m^2(q-1) - 3. \quad (39)$$

Setting Eq. (39) to zero, we conclude that the critical point for q is

$$q_\infty(m^2) = \frac{m^2 - 3 + \sqrt{9 + 2m^2 + m^4}}{2m^2}, \quad (40)$$

which matches exactly $q_\infty(m^2)$ from [6].

VIII. DISCUSSION

Motivated by the current status in the S-matrix bootstrap and the importance of partial-wave unitarity in these analyses we have suggested a novel method for obtaining an analytic expression for all partial-wave coefficients.

One advantage of this method is the naturally structured and very suggestive form of the answer. Another is that it provides a way to solve for the coefficients by considering a linear algebra problem, much simpler than the original. A third upshot is the use of harmonic numbers and deformations thereof, which allows one to analyze non-trivially deformed amplitudes in a straightforward manner. Furthermore, owing to the properties of the harmonic numbers the simplified cases of the deformations follow by definition. Moreover, this new approach is very powerful in the low-spin trajectories. And finally, we can exploit the well-established asymptotic behavior of harmonic numbers to obtain critical values for the various deformation parameters; here it was the q deformation.

Our approach is easily generalizable to other cases, owing to its versatility. It is interesting to examine the implications and computational power on other amplitudes in order to carve out unitarity regions in the relevant parameter spaces. In [25] one of us demonstrates the applicability of the method developed here in a more elaborate deformation, namely the hypergeometric Coon [15]. As another future avenue it would be very interesting to understand if the basis of the harmonic numbers used in the partial-wave analysis, can be utilized in order to provide information about other observables or other setups, e.g., the Wilson coefficients [28] or in studies in AdS backgrounds [29,30].

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