

Collective coordinate fix in the path integral

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Collective coordinates are frequently employed in path integrals to manage divergences caused by fluctuations around saddle points that align with classical symmetries. These coordinates parametrize a manifold of zero modes and more broadly provide judicious coordinates on the space of fields. However, changing from local coordinates around a saddle point to more global collective coordinates is remarkably subtle. The main complication is that the mapping from local coordinates to collective coordinates is generically multivalued. Consequently one is forced to either restrict the domain of path integral in a delicate way, or otherwise correct for the multivaluedness by dividing the path integral by certain intersection numbers. We provide a careful treatment of how to fix collective coordinates while accounting for these intersection numbers, and then demonstrate the importance of the fix for free theories. We also provide a detailed study of the fix for interacting theories and show that the contributions of higher intersections to the path integral can be nonperturbatively suppressed. Using a variety of examples ranging from single-particle quantum mechanics to quantum field theory, we explain and resolve various pitfalls in the implementation of collective coordinates.

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I. INTRODUCTION

One of the great advantages of the path integral for a quantum theory is that it allows for expansions around saddle points other than the free theory, providing access to nonperturbative physics. For example, the lifetime of the Standard Model can be computed by expanding the path integral around a nontrivial configuration of the Higgs field [1–7]. In many field theories (including the Standard Model) there are families of equivalent nontrivial saddles related by symmetries, such as translations, dilatations, or internal symmetries. Expanding around any one of these saddles leads to a divergent 1-loop saddle point calculation because the zero eigenvalues from the flat directions make the functional determinant vanish. Such infinities are usually handled by introducing collective coordinates along the manifold of zero modes, effectively trading local coordinates around a single saddle for more global coordinates around the entire manifold of zero modes [8–10].

Then the integral over the zero mode manifold can be handled nonperturbatively, giving a factor of the volume of the symmetry group. The remaining path integral can then be evaluated using a saddle-point approximation, and can be regularized to be finite. While this procedure gives sensible answers in some situations where it can be checked (for example, the energy splitting in a symmetric double-well potential can be alternatively computed with the WKB approximation [11–16]), the procedure also happens to give very wrong answers in other situations, such as in free theories.

To see the problem clearly, consider in more detail the decay rate calculation in a quantum field theory with action $S = \int d^4x (\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \lambda \phi^4)$ with $\lambda < 0$. As we will review in Sec. V, the decay rate involves an expansion around a nontrivial instanton background $\phi_b(x)$ normalized by an expansion around the trivial vacuum $\phi(x) = 0$. The decay rate Γ can be written as [5–7]

$$\Gamma = \lim_{T \rightarrow \infty} \frac{1}{T} \text{Im} \frac{\int_{C_b} \mathcal{D}\phi e^{-S[\phi]}}{\int_{C_{\text{FV}}} \mathcal{D}\phi e^{-S_0[\phi]}}, \quad (1.1)$$

where $S_0[\phi] = \int d^4x (\frac{1}{2} \partial_\mu \phi \partial^\mu \phi)$ is the free action. Here C_b is a path of steepest descent in the interacting theory passing through $\phi_b(x)$ and C_{FV} is the path of steepest descent in the free theory passing through the free vacuum $\phi(x) = 0$. To evaluate the numerator in the saddle point approximation, we expand

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$$\phi(x) = \phi_b(x) + \sum_{m=0}^{\infty} a_m \phi_m(x), \quad (1.2)$$

where $\{\phi_m(x)\}$ forms a basis of normal modes around the bounce $\phi_b(x)$. Five of these modes, corresponding to four translations and one dilatation mode, have zero eigenvalue with respect to the operator $L''[\phi_b]$, where L is the Lagrangian, and so a naïve saddle point approach gives infinite contributions to the numerator of Eq. (1.1). A more careful treatment of steepest descent is required to resolve these infinities. To do so, we can change to a collective coordinate basis

$$\phi(x) = \phi_b(x + \xi, R) + \sum_{m \geq 5} \alpha_m \phi_m(x + \xi, R), \quad (1.3)$$

where the five zero-mode coordinates α_m for $m = 0, 1, 2, 3, 4$ are traded for five collective coordinates ξ^μ and R . The integral over ξ^μ then gives a factor of the spacetime volume TV for the numerator of Eq. (1.1), as expected for a decay rate in spacetime. However, a complication arises if we use the same exact basis of Eq. (1.3), which happens to also diagonalize normal modes in the free theory, for the denominator of Eq. (1.1). In particular, the same TV factor apparently arises, which is unexpected. In this paper, we identify the cause of this unexpected factor, and show that the appropriate treatment of collective coordinates yields sensible results for both the numerator and denominator of Eq. (1.1). We more broadly study general coordinates on field space in which similar subtleties arise.

To understand the source of the subtleties, let us recall how δ -functions can be used to change variables in integrals. In a finite-dimensional integral $\int_{\Omega} d^d x A(x)$ over $x \in \Omega \subseteq \mathbb{R}^d$, we can implement the change of variables $x = f(y)$ using the resolution of the identity

$$1 = \int_{f^{-1}(\Omega)} d^d y \left| \frac{\partial f(y)}{\partial y} \right| \delta^d(f(y) - x), \quad (1.4)$$

so that

$$\int_{\Omega} d^d x A(x) = \int_{\Omega} d^d x \int_{f^{-1}(\Omega)} d^d y \left| \frac{\partial f(y)}{\partial y} \right| \delta^d(f(y) - x) A(x) \quad (1.5)$$

$$= \int_{f^{-1}(\Omega)} d^d y \left| \frac{\partial f(y)}{\partial y} \right| A(f(y)). \quad (1.6)$$

The above can be regarded as a trick to find the right Jacobian factor and domain of integration. Crucially, we notice that the above assumes that f is an invertible function on the domain of integration. But what if f multiply covers the domain of integration? A simple example is the 1-dimensional integral $\int_0^\infty dx A(x)$ with the change of

variables $x = y^2$. In this setting the salient resolution of the identity is

$$1 = \frac{1}{2} \int_{-\infty}^{\infty} dy |2y| \delta(y^2 - x). \quad (1.7)$$

Here, we need the factor of $\frac{1}{2}$ out front because the map $f(y) = y^2$ taking $\mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ doubly covers the positive real line. Going back to Eq. (1.4), if $f(y) = x$ has $N_f[x]$ solutions for fixed x , then we need to correct Eq. (1.4) by

$$1 = \frac{1}{N_f[x]} \int_{f^{-1}(\Omega)} d^d y \left| \frac{\partial f(y)}{\partial y} \right| \delta^d(f(y) - x), \quad (1.8)$$

where we call $N_f[x]$ the *intersection number*. A similar subtlety arises in gauge theories where there can be multiple solutions to a local gauge-fixing condition, and so we must introduce a multiplicative factor to correct for the so-called *Gribov copies* [17].

To see how the intersection number appears in a path integral, consider nonrelativistic quantum mechanics where the path integral is over paths $x(t)$. For a time-independent Lagrangian, there will always be a flat direction: $S[x(t)] = S[x(t + t_0)]$ for any path $x(t)$ and any t_0 . To change from orthonormal coordinates on path space to collective coordinates we can employ the resolution of the identity¹

$$1 = \frac{1}{N_X[x]} \int dt_0 \delta[\langle X(t + t_0) | x(t) \rangle] |\langle X(t + t_0) | \dot{x}(t) \rangle|. \quad (1.9)$$

Here $\langle x | y \rangle = \int dt x(t) y(t)$ denotes the L^2 inner product, $x(t)$ is a trajectory in the domain of the path integral, $\dot{x}(t) = \partial_t x(t)$ is the time derivative, $X(t + t_0)$ is the path whose coordinate we want to exchange for t_0 , and $N_X[x]$ is the intersection number. That is, solving $\langle X(t + t_0) | x(t) \rangle = 0$ will replace the time-translation zero mode with the collective coordinate t_0 for any path $x(t)$. Then the intersection number $N_X[x]$ counts the number of values of t_0 for which $\langle X(t + t_0) | x(t) \rangle = 0$ for a given path $x(t)$. The intersection number is a complicated integer-valued functional of the path which must be carefully taken into account.

The textbook narrative is that a collective coordinate can be simply pulled out of the path integral giving a volume factor [18–22]. Unfortunately, this standard treatment does not properly account for the multivaluedness of the coordinate change associated with going from local coordinates to collective coordinates. In this paper, we explore through examples how and when collective coordinates can be used. Our key observations are that (1) the transition to collective coordinates does not involve only an innocuous coordinate change; (2) in interacting theories, non-Gaussian terms in the action are important to suppress high-multiplicity

¹We have used $|\langle \dot{X}(t + t_0) | x(t) \rangle| = |\langle X(t + t_0) | \dot{x}(t) \rangle|$ obtained via integration by parts in t .

intersections; and (3) the Jacobian factor incurred by changing to collective coordinates plays an important role in assuring the consistency of the implementation.

Our paper begins in Sec. II with an overview of how to change from local coordinates to collective coordinates. We then proceed through a set of examples of increasing complexity. In Sec. III, we discuss a free particle with time compactified on a circle. For this system time-translation invariance is exact. We explicitly compute the intersection number associated with the collective coordinates and illustrate the proper treatment of the Jacobian. In Sec. IV, we consider the symmetric double-well potential in quantum mechanics, focusing on the computation of the energy splitting between the two lowest levels. This theory has a nontrivial instanton and a basis of excitations around the instanton whose analytic form is explicitly known. We further show that the intersection number apropos to the collective coordinates counts the number of times a given path $x(t)$ crosses $x = 0$. We then demonstrate that paths with $N_X[x] > 1$ give exponentially small contributions to the path integral in the interacting theory, but give $\mathcal{O}(1)$ contributions in the free theory. In Sec. V we consider $\lambda\phi^4$ quantum field theory and discuss collective coordinates for dilatation and translations. In Sec. VI we conclude with a discussion.

II. COLLECTIVE COORDINATES: GENERAL RESULTS

We begin by discussing how collective coordinates are used in quantum mechanics. In evaluating the path integral, we often expand around some saddle point $x_I(t)$. This saddle can be the free solution $x = 0$, an instanton, or simply some classical path satisfying the boundary conditions of the path integral for the problem of interest. For concreteness, suppose we work in Euclidean signature with a time-independent Lagrangian of the form $L[x] = \frac{1}{2}\dot{x}^2 + V(x)$ with action $S[x] = \int dt L[x]$. Then we can expand a generic path in an orthonormal basis of eigenstates of $\frac{\delta^2 S}{\delta x^2}|_{x=x_I}$. That is, we write

$$x(t) = x_I(t) + a_0 x_0(t) + \sum_{n=1}^{\infty} a_n x_n(t) \quad (2.1)$$

where

$$[-\partial_t^2 + V''(x_I)]x_n = \lambda_n x_n. \quad (2.2)$$

In (2.1) we have separated out the zero mode $x_0(t) \propto \dot{x}_I(t)$. We can see that $\lambda_0 = 0$ using $\frac{\delta S}{\delta x}|_{x=x_I} = 0$ so that

$$0 = \partial_t \frac{\delta S}{\delta x(t)} \Big|_{x=x_I} = \int dt' \frac{\delta^2 S}{\delta x(t) \delta x(t')} \Big|_{x=x_I} \dot{x}_I(t'). \quad (2.3)$$

Then the path integral becomes, to 1-loop order in the saddle-point approximation,

$$\begin{aligned} Z &= \mathcal{N} \int \mathcal{D}x e^{-S} = \mathcal{N} e^{-S[x_I]} \int \frac{da_0}{\sqrt{2\pi}} \prod_{n=1}^{\infty} \int \frac{da_n}{\sqrt{2\pi}} e^{-\lambda_n a_n^2} \\ &= \mathcal{N} e^{-S[x_I]} \sqrt{\frac{1}{\lambda_0}} \prod_{n=1}^{\infty} \sqrt{\frac{1}{\lambda_n}}. \end{aligned} \quad (2.4)$$

Because of the zero eigenvalue, Eq. (2.4) is infinite. The existence of a zero mode is because the action $S[x(t+t_0)]$ is independent of t_0 . We therefore would like to trade a_0 for t_0 and integrate over t_0 giving a factor of the volume of time T .

To trade a_0 for t_0 we transform from coordinates $\{a_0, a_n\}$ to collective coordinates $\{t_0, \alpha_n\}$ of the form

$$x(t) = x_I(t+t_0) + \sum_{n=1}^{\infty} \alpha_n x_n(t+t_0). \quad (2.5)$$

Since $\{\alpha_0, \alpha_n\}$ are global coordinates, for which each path in the integration domain is represented exactly once, the zero mode can be projected out with

$$\langle x(t) - x_I(t+t_0) | x_0(t+t_0) \rangle = 0 \quad (2.6)$$

where $\langle x|y \rangle$ is the inner product under which the x_n are orthonormal. Equation (2.6) can be solved to find a coordinate t_0 for a given path $x(t)$. Importantly, however, there can be multiple solutions and we denote by $N_{x_0}[x]$ the number of solutions for t_0 given x . The Jacobian for the coordinate change is determined by the partial derivatives

$$\begin{aligned} \frac{\partial a_m}{\partial t_0} &= \left\langle x_m(t) | \partial_{t_0} x_I(t+t_0) + \sum_{n=1}^{\infty} \alpha_n \partial_{t_0} x_n(t+t_0) \right\rangle, \\ \frac{\partial a_m}{\partial \alpha_n} &= \langle x_m(t) | x_n(t+t_0) \rangle. \end{aligned} \quad (2.7)$$

After some algebra (see Appendix B of [6]), this Jacobian can be simplified to

$$\mathcal{J} = \langle x_0(t+t_0) | \partial_{t_0} x_I(t+t_0) + \sum_{n=1}^{\infty} \alpha_n \partial_{t_0} x_n(t+t_0) \rangle \quad (2.8)$$

$$= \langle x_0(t) | \dot{x}_I(t) + \sum_{n=1}^{\infty} \alpha_n \dot{x}_n(t) \rangle. \quad (2.9)$$

Alternatively, the functional determinant can be computed as $\sqrt{\det G}$ where G is the Riemannian metric on the space of fields in collective coordinates (see Appendix A). That the Jacobian is independent of t_0 is nontrivial. Thus we have

$$Z = \mathcal{N} e^{-S[x_I]} \int \frac{dt_0}{\sqrt{2\pi}} \int \prod_{m=1}^{\infty} \frac{d\alpha_m}{\sqrt{2\pi}} e^{-\lambda_m \alpha_m^2} \times \frac{1}{N_{x_0}[x]} \left| \langle x_0(t) | \dot{x}_I(t) + \sum_{n=1}^{\infty} \alpha_n \dot{x}_n(t) \rangle \right| \quad (2.10)$$

This is the formulation we should use if we want to implement the $\{t_0, \alpha_n\}$ coordinates as in Eq. (2.5).

From Eq. (2.10) we can integrate in α_0 by including $\int d\alpha_0 \delta(\alpha_0)$. If we then relabel $\alpha_0 \rightarrow a_0$ and $\alpha_n \rightarrow a_n$, the resulting expression would be exactly the same as if we had expanded

$$Z = \mathcal{N} \int dt_0 \left\{ \int \mathcal{D}x e^{-S[x]} \frac{1}{N_{x_0}[x]} \times \delta[\langle x_0(t) | x(t) \rangle] |\langle x_0(t) | \dot{x}(t) \rangle| \right\} \quad (2.11)$$

using Eq. (2.1). In the above equation we can no longer use t_0 as a coordinate: it is a constant from the point of view of the path integral (the part of the expression in the curly brackets). That is, we cannot evaluate the path integral expression using the coordinates in Eq. (2.5). If we want to use t_0 as a coordinate, we should instead use Eq. (2.10).

There was nothing essential about $x_I(t)$ or $x_0(t)$ in our derivation. Indeed, $x_I(t)$ does not even appear in Eq. (2.11). We can replace $x_0(t)$ by any path $X(t)$ leading to

$$Z = \mathcal{N} \int dt_0 \left\{ \int \mathcal{D}x e^{-S[x]} \frac{1}{N_X[x]} \times \delta[\langle X(t) | x(t) \rangle] |\langle X(t) | \dot{x}(t) \rangle| \right\} \quad (2.12)$$

where $N_X[x]$ counts the number of times $\langle X(t+t_0) | x(t) \rangle$ vanishes over the integration range of t_0 for fixed $x(t)$. Note that the above equation can also be derived inserting the identity of Eq. (1.9) inside the path integral and using $S[x(t+t_0)] = S[x(t)]$.

To summarize, the key new element in our analysis is the intersection number $N_X[x]$. If we write a generic path $x(t)$ in an orthonormal basis, then $N_X[x]$ is the number of solutions to

$$\langle X(t+t_0) | x(t) \rangle = 0 \quad (2.13)$$

over the range of t_0 . Equation (2.13) allows us to solve for t_0 in terms of the coordinates labeling an orthonormal basis, and there may be multiple solutions. When the collective coordinate t_0 is one of the coordinates, Eq. (2.13) cannot be used. Instead, in collective coordinates, $N_X[x]$ counts the number of ways of constructing the same path $x(t)$ using different values of t_0 . That is, the same function would be overcounted in the path integral in the collective coordinates if it can be described using multiple values of t_0 , and so we

must divide by a $N_X[x]$ factor to compensate. We will see examples of the use of $N_X[x]$ in the following.

III. FREE THEORY

As a first example, we consider a free nonrelativistic quantum theory for a particle with mass $m=1$ and Lagrangian $L = \frac{1}{2} \dot{x}^2$. The propagator going from a to b in time T is

$$D(a, b) = \langle b | e^{-HT} | a \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(b-a)} e^{-\frac{p^2}{2}T} = \sqrt{\frac{1}{2\pi T}} e^{-\frac{(b-a)^2}{2T}} \quad (3.1)$$

and the partition function is

$$Z_0 = \text{Tr}(e^{-HT}) = \int_{-\mathcal{V}/2}^{\mathcal{V}/2} dx_0 D(x_0, x_0) = \sqrt{\frac{1}{2\pi T}} \mathcal{V}, \quad (3.2)$$

where $\mathcal{V}$ is the volume of space used to regulate this integral. To compute the partition function with the path integral, we parametrize paths with periodic boundary conditions in time as

$$x(t) = x_0 + \sum_{n \text{ even}} [a_n s_n(t) + b_n c_n(t)], \quad (3.3)$$

where

$$s_n(t) = \sqrt{\frac{2}{T}} \sin\left(\frac{n\pi(t+\frac{T}{2})}{T}\right), \quad c_n(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{n\pi(t+\frac{T}{2})}{T}\right). \quad (3.4)$$

Then the partition function is given by

$$Z_0 = \mathcal{N} \int \mathcal{D}x e^{-S[x]} = \mathcal{N} \int_{-\mathcal{V}/2}^{\mathcal{V}/2} dx_0 \prod_{n \text{ even}} \left[\int_{-\infty}^{\infty} \frac{da_n}{\sqrt{2\pi}} e^{-\frac{1}{2} \lambda_n a_n^2} \int_{-\infty}^{\infty} \frac{db_n}{\sqrt{2\pi}} e^{-\frac{1}{2} \lambda_n b_n^2} \right] \quad (3.5)$$

$$= \mathcal{N} \mathcal{V} \left[\prod_{n \text{ even}} \sqrt{\frac{1}{\lambda_n}} \right]^2 \quad (3.6)$$

where

$$\lambda_n = \left(\frac{n\pi}{T} \right)^2 \quad (3.7)$$

are the eigenvalues of the fluctuations under $\frac{\delta^2 S}{\delta x(t) \delta x(t')} \big|_{x=0} = -\delta(t-t') \partial_t^2$. We then match on to Eq. (3.2) with

$$\mathcal{N} = \sqrt{\frac{1}{2\pi T}} \left[\prod_{n \text{ even}} \frac{T}{n\pi} \right]^{-2} \quad (3.8)$$

which fixes the normalization of the path integral. Note that $\mathcal{V}$ is only needed to regulate the x_0 integral: we can take the integrals over a_n and b_n to go to from $-\infty$ to ∞ because the path integral is exponentially suppressed at large a_n, b_n .

Now we want to reproduce the partition function Z_0 with collective coordinates, using Eq. (2.12). We first need to choose which path $X(t)$ to project onto, i.e. which coordinate to swap for the collective coordinate t_0 . The coordinate to swap should be one from a parametrization of paths where time is shifted by t_0 , i.e. from

$$x(t) = x_0 + \sum_{n \text{ even}} (\alpha_n s_n(t + t_0) + \beta_n c_n(t + t_0)). \quad (3.9)$$

In the simplest case, we would swap a single coordinate, say β_m , for t_0 . That would mean choosing $X(t) = c_m(t)$. To change to coordinates involving t_0 we have to start from coordinates not involving t_0 , such as the original coordinates in Eq. (3.3). The condition $\beta_m = 0$ is $\langle c_m(t + t_0) | x(t) \rangle = 0$ as in Eq. (2.13), which in the original coordinates amounts to

$$\tan\left(\frac{m\pi t_0}{T}\right) = -\frac{b_m}{a_m}. \quad (3.10)$$

This equation allows us to solve for t_0 and to transform from the $\{a_n, b_n\}$ coordinates to the $\{t_0, \alpha_n, \beta_n\}$ coordinates with no β_m component. Note that over the range $0 < t_0 \leq T$, Eq. (3.10) has m possible solutions so that² $N_X[x] = m$.

The Jacobian arising from Eq. (2.12) is

$$\mathcal{J} = \langle x(t) | \dot{c}_m(t) \rangle = \sqrt{\lambda_m} a_m, \quad (3.11)$$

and thus in collective coordinates the partition function is

$$\begin{aligned} Z_{\text{coll}} = \mathcal{N} \frac{1}{m} \int_0^T dt_0 \int_0^{\mathcal{V}} dx_0 \prod_{\substack{n \text{ even} \\ n \neq m}} \left(\int_{-\infty}^{\infty} \frac{da_n}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda_n a_n^2} \right. \\ \left. \times \int_{-\infty}^{\infty} \frac{db_n}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda_n b_n^2} \right) \int \frac{da_m}{\sqrt{2\pi}} \sqrt{\lambda_m} |a_m| e^{-\frac{1}{2}\lambda_m a_m^2}. \end{aligned} \quad (3.12)$$

Using Eq. (3.8) the above agrees precisely with Z_0 in Eq. (3.2).

²In this case, we can alternatively restrict the range $0 < t_0 \leq \frac{T}{m}$ and take $N_X[x] = 1$, but restricting the range is untenable in more complicated examples.

The next simplest case is to choose $X(t)$ to be a sum of two modes instead of one mode. Suppose we choose $X(t) = A c_m(t) + B c_k(t)$ which is a sum of cosine modes. Then the coordinate t_0 would be determined from the coordinates in Eq. (3.3) using the condition

$$\begin{aligned} \langle X(t + t_0) | x(t) \rangle = A b_m c_m(t_0) + A a_m s_m(t_0) + B b_k c_k(t_0) \\ + B a_k s_k(t_0) = 0. \end{aligned} \quad (3.13)$$

The number of solutions $N_X[x]$ for this equation is a complicated function of a_m, b_m, a_k , and b_k . It turns out to be somewhat easier to use polar coordinates. Let us write an arbitrary linear combination of the m and k sines and cosine modes as

$$X(t) = R_m s_m(t + \phi_m) + R_k s_k(t + \phi_k). \quad (3.14)$$

Changing to the polar coordinates $a_n = \rho_n s_n(\theta_n)$ and $b_n = \rho_n c_n(\theta_n)$ for convenience, we find the overlap

$$\begin{aligned} \langle X(t + t_0) | x(t) \rangle = R_m \rho_m s_m(t_0 + \theta_m + \phi_m) \\ + R_k \rho_k s_k(t_0 + \theta_k + \phi_k) \end{aligned} \quad (3.15)$$

which only involves two frequencies, as expected.

We now want to replace the a_m, b_m, a_k, b_k integrals from the original partition function, which gave rise to a $\frac{1}{\lambda_m} \frac{1}{\lambda_k}$ term, with new integrals in our polar coordinates. As a cross check, we see that before introducing collective coordinates we have

$$\begin{aligned} Z_{\text{polar}} = \left(\frac{\lambda_m \lambda_k}{2\pi 2\pi} Z_0 \right) \frac{m\pi k\pi}{T T} \int_0^{\infty} d\rho_m \rho_m e^{-\frac{1}{2}\lambda_m \rho_m^2} \\ \times \int_0^{\infty} d\rho_k \rho_k e^{-\frac{1}{2}\lambda_k \rho_k^2} \int_0^{\frac{2\pi}{m}} d\theta_m \int_0^{\frac{2\pi}{k}} d\theta_k = Z_0, \end{aligned} \quad (3.16)$$

as expected. Here $\frac{\lambda_m \lambda_k}{2\pi 2\pi} Z_0$ represents the partition function where we have performed all integrals except for those over a_m, b_m, a_k, b_k . The $\frac{m\pi k\pi}{T T}$ factors arise from the Jacobian induced by our polar coordinates.

Now we move to collective coordinates by inserting into the partially integrated partition function our Eq. (2.12) which includes an integral over t_0 , a δ -function, and a Jacobian. Doing so and then shifting $\theta_m \rightarrow \theta_m - \phi_m$, $\theta_k \rightarrow \theta_k - \phi_k$, rescaling $\rho_m \rightarrow \frac{\rho_m}{R_m}$, $\rho_k \rightarrow \frac{\rho_k}{R_k}$, taking $\rho_k \rightarrow \rho_k \rho_m$, and performing the ρ_m integral leads to

$$\begin{aligned}
Z_{\text{coll}} = Z_0 &\times \left(\frac{\lambda_m \lambda_k}{2\pi 2\pi} \right) \frac{m\pi}{T} \frac{k\pi}{T} \int_0^\infty d\rho_k \frac{2\rho_k R_k^2 R_m^2}{(R_k^2 \lambda_m + R_m^2 \lambda_k \rho_k^2)^2} \\
&\times \int_0^{\frac{2T}{m}} d\theta_m \int_0^{\frac{2T}{k}} d\theta_k \frac{1}{N_X(\rho_k, \theta_k, \theta_m)} \\
&\times \int_0^T dt_0 \delta[s_m(\theta_m) + \rho_k s_k(\theta_k)] \\
&\times \left| \frac{m\pi}{T} c_m(\theta_m) + \rho_k \frac{k\pi}{T} c_k(\theta_k) \right|. \quad (3.17)
\end{aligned}$$

One can check that without the second line, we still have $Z_{\text{coll}} = Z_0$. The intersection number N_X counts the number of times the δ -function fires for a given path. We can compute the intersection number by counting the roots of $\sin(\pi m(t_0 + \theta_m)/T) + \rho_k \sin(\pi k(t_0 + \theta_k)/T)$ on the interval $0 < t_0 < T$. Due to periodicity, the intersection number depends only on $\theta_k - \theta_m$. The function for $m = 26$ and $k = 4$ with three choices of $\theta_k - \theta_m$ is shown in Fig. 1.

To proceed, one can first observe that the integral is periodic in θ_m and θ_k , so we can extend their limits of integration to $0 < \theta_m, \theta_k < T$ and pull out an overall factor of $\frac{4}{mk}$. Then we change variables from (θ_m, θ_k) to $\theta_+ = \frac{1}{2}(\theta_m + \theta_k)$ and $\theta_- = \theta_m - \theta_k$ and integrate over θ_+ . In this integral, the δ -function fires the number of times given by the number of solutions of $s_m(\theta_+ + \frac{1}{2}\theta_-) + \rho_k s_k(\theta_+ - \frac{1}{2}\theta_-)$ for θ_+ . But this number is identical to $N_X(\rho_k, \theta_-)$, so that again $Z_{\text{coll}} = Z_0$ as expected. One can of course integrate over θ_m or θ_k first (rather than θ_+) to get the same result, but the integrals are much more difficult.

In the general case where $X(t)$ has components of arbitrary frequency, the intersection number $N_X(\rho_n, \theta_n)$ would be a function of the entire configuration space. Thus the intersection number is generally a horrible mess, but accounting for the intersection number in the path integral is nonetheless required for collective coordinates to work. Luckily, for some choices of $X(t)$ the intersection number

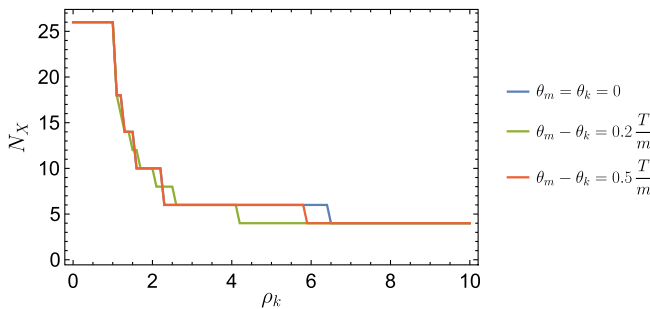


FIG. 1. For the free quantum mechanical theory where the coefficient of a linear combination of two normal modes is traded for a collective coordinate, the intersection number N_X cannot be neglected. N_X is integer-valued. After integrating over the modulus ρ_m , we find that N_X depends on the modulus ρ_k and the relative angle $\theta_m - \theta_k$. Shown are the value of N_X for $m = 26$ and $k = 4$ for a few choices of $\theta_m - \theta_k$.

has a more physical interpretation and can be managed either analytically or approximately. A representative example is the double-well potential in quantum mechanics to which we now turn.

IV. DOUBLE WELL POTENTIAL

For the next example, we consider the symmetric double well potential in quantum mechanics whose Euclidean action is

$$S[x] = \int_{-T/2}^{T/2} dt \left[\frac{m}{2} \dot{x}^2 + V(x) \right], \quad V(x) = \frac{m\omega^2}{8a^2} (x^2 - a^2)^2. \quad (4.1)$$

There are two dimensionless scales in the problem: $ma^2\omega/\hbar$ which indicates the size of the barrier and $\omega T/\hbar$ which indicates time. We will set $m = \hbar = 1$ but keep a, ω and T to assist with the physical interpretation of various limits.

The main quantity of interest in this theory to compute with path integral methods is the splitting of the two lowest energy eigenstates. This splitting, and the entire spectrum, have been studied in great detail both analytically and numerically [13,23–29]. It is nevertheless difficult to find a rigorous analysis of the path integral computation of the energy splitting. Typically treatments are not only cavalier with regard to the introduction of collective coordinates, but also with regard to the associated Jacobian and finite-time regulation. We will attempt to clarify all of these issues in this section.

Since the potential $V(x)$ is symmetric, the parity operator P which takes $x \rightarrow -x$ commutes with the Hamiltonian. Thus energy eigenstates are parity even or parity odd. In the limit $a^2\omega \rightarrow \infty$, the barrier becomes enormous and the spectrum is that of two harmonic oscillators with frequency ω . That is, $E_0^+ = E_0^- = \frac{\omega}{2}$ and the higher modes are spaced by ω . Expanding the potential around a minimum and working perturbatively in the limit of large but finite $a^2\omega$, the energies receive corrections but the even and odd modes remain degenerate to all orders. The WKB approximation can be used to compute the energy splitting with the leading order result [13,14,29,30]

$$\Delta_n^{\text{WKB}} = E_n^- - E_n^+ = \omega \frac{2}{\sqrt{2\pi}} (8\omega a^2)^{n+1/2} e^{-\frac{2\omega a^2}{3}}. \quad (4.2)$$

In particular, the ground state splitting can be written as

$$\Delta_0^{\text{WKB}} = 4\omega \sqrt{\frac{\omega a^2}{\pi}} e^{-\frac{2\omega a^2}{3}}. \quad (4.3)$$

In fact, the energies can be determined for this system by solving the Riccati equation iteratively though the

Exact WKB method relatively easily to arbitrary order [12,28,31–34]. However, our focus here is not on precision but on the role of collective coordinates in the path integral version of the calculation.

To compute the energy splitting with the path integral, it is helpful to set up the problem so that the leading result as $T \rightarrow \infty$ is directly proportional to Δ_0 . Defining $E_n^\pm = E_n \mp \frac{\Delta_n}{2}$ the partition function is

$$Z = \text{Tr}[e^{-HT}] = \sum_n (e^{-E_n^+ T} + e^{-E_n^- T}), \quad (4.4)$$

and the twisted partition function is

$$Z_P = \text{Tr}[Pe^{-HT}] = \sum_n (e^{-E_n^+ T} - e^{-E_n^- T}). \quad (4.5)$$

Z_P is useful for extracting the energy since $E_n^+ = E_n^-$ to all orders in perturbation theory; that is, the splittings are nonperturbative. Since $Z - Z_P$ involves only odd modes and $Z + Z_P$ involves only even modes we see that

$$\lim_{T \rightarrow \infty} -\frac{1}{T} \ln \left(\frac{Z - Z_P}{Z + Z_P} \right) = E_0^- - E_0^+ = \Delta_0. \quad (4.6)$$

The above formula makes no assumption about Δ_0 being small, but is useful only if Z and Z_P are known non-perturbatively in Δ_0 . If we only know Z_P in the small Δ_n limit, then we can use

$$\lim_{T \rightarrow \infty} \frac{2}{T} \Delta_n \lim_{\Delta_n \rightarrow 0} \frac{1}{\Delta_n} \frac{Z_P}{Z} = \lim_{T \rightarrow \infty} \frac{2}{T} \frac{\sum e^{-E_n^+ T} (\Delta_n T)}{\sum e^{-E_n^+ T}} = \Delta_0. \quad (4.7)$$

This formula will allow us to compute the splitting of the lowest energy states directly with the path integral: if we compute Z_P and Z in the saddle-point approximation and find at small splitting and large T that $Z_P \sim ZAT$, then the energy splitting is $\Delta_0 = 2A$.

The twisted partition function can be computed with the path integral using twisted boundary conditions

$$\begin{aligned} Z_P &= \text{Tr}(Pe^{-HT}) = \mathcal{N} \int_{-\infty}^{\infty} db \langle -b | e^{-HT} | b \rangle \\ &= \mathcal{N} \int_{-\infty}^{\infty} db \int_{x(-\frac{T}{2})=-b}^{x(\frac{T}{2})=b} \mathcal{D}x e^{-S[x]}. \end{aligned} \quad (4.8)$$

All paths involved go from $-b$ to b in time T and we integrate over b to take the trace. For the normalization $\mathcal{N}$, we can match to the partition function of the harmonic oscillator.

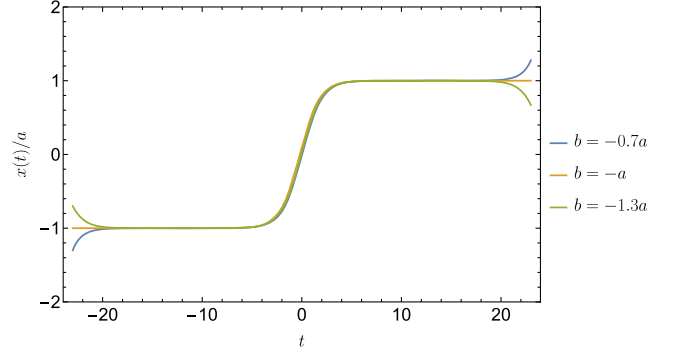


FIG. 2. The twisted partition function sums over paths starting and ending at any b . At finite T , the double well admits exact instanton solutions for any b . For any b and large values of T , these solutions spend most of the time in one of the two wells.

For any $b > 0$, the classical path from $-b$ to b for a large time T is exponentially close to the $T = \infty$ instanton

$$x_I(t) = a \tanh \frac{\omega t}{2}. \quad (4.9)$$

For $b < 0$, the classical path from $-b$ to b for a large time T is closer to the anti-instanton solution $-x_I(t)$. Both the instanton and anti-instanton have an action given by

$$\begin{aligned} S_I \equiv S[x_I] &= a^2 \omega \tanh \frac{\omega T}{4} - \frac{a^2 \omega}{3} \tanh^3 \frac{T \omega}{4} \\ &= \frac{2a^2 \omega}{3} + \mathcal{O}(e^{-T \omega}). \end{aligned} \quad (4.10)$$

In particular for $b \neq a$ these (anti-)instanton paths start at $-b$, shoot up to the top of hill of the inverted potential, stay there for almost the entire time, roll across, and then stay at the other hill, ending at b after a total time T . Examples (found numerically with the shooting method) that have b slightly lower or larger than a are shown in Fig. 2. In particular, the part of the motion between $-b$ and $-a$ is given by a solution to the Euclidean equations of motion

$$\ddot{x}_b(t) = -a \coth \frac{\omega t}{2}, \quad (4.11)$$

and provides the corresponding action

$$S[\tilde{x}_b(t)] = \int_a^b dx \sqrt{2V(x)} = \frac{\omega}{6a} (a-b)^2 (2a+b). \quad (4.12)$$

Thus in the large T limit, we find that the classical path from $-b$ to b ($b > 0$) has action

$$S[x_b(t)] = S[x_I] + 2S[\tilde{x}_b(t)] = S_I + \frac{\omega}{3a}(a-b)^2(2a+b). \quad (4.13)$$

Note that when $|\frac{b}{a}| \gg 1$ then $S[x] \approx \frac{2a^2\omega}{3} + \frac{\omega|b|^3}{3a}$ and so the part of the action arising from the path approaching the hills is significant. The effect is that the contributions to the twisted partition function with $|b| \gg a$ will be small, however we still need to include the b dependence to get the numerical prefactor correct.

For the fluctuations, the tail regions before and after $|x| \approx a$ give exponentially small contributions, so we can expand paths as $x(t) \rightarrow x_I(t) + x(t)$. Then the twisted partition function in the leading saddle point approximation becomes

$$Z_P \approx 2\mathcal{N} \int_0^\infty db e^{-S_I} e^{-2S[\tilde{x}_b(t)]} \int_{x(-\frac{T}{2})=0}^{x(\frac{T}{2})=0} \mathcal{D}x \times \exp \left[-\frac{1}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt x(t) [-\partial_t^2 + V''(x_I)] x(t) \right] \quad (4.14)$$

$$\approx \mathcal{N} \sqrt{\frac{4\pi}{\omega}} e^{-S_I} \int_{x(-\frac{T}{2})=0}^{x(\frac{T}{2})=0} \mathcal{D}x \times \exp \left[-\frac{1}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt x(t) [-\partial_t^2 + V''(x_I)] x(t) \right] \quad (4.15)$$

where we have performed the integral over b in the large S_I limit. The factor of 2 in front accounts for the anti-instanton solutions, which go from $-b$ to b where $b < 0$, and give the same contribution as the instanton solutions. Note that the instantons saturate the twisted boundary conditions and the fluctuations are Dirichlet. Next, we want to compute the path integral in the leading saddle-point approximation, carefully treating the collective coordinate and Gaussian integration over the fluctuations.

A. Normal modes at finite T

Fluctuations around the instanton are described by normal modes satisfying

$$[-\partial_t^2 + V''(x_I)]x_n = \lambda_n x_n \quad (4.16)$$

where $V''(x_I) = \omega^2(1 - \frac{3}{2}\text{sech}^2 \frac{\omega t}{2})$. Upon rescaling $t \rightarrow t/2$, we recognize that Eq. (4.16) is equivalent to the time-independent Schrödinger equation with a Pöschl-Teller potential [35] having $\lambda = 2$. The spectrum of the differential operator in Eq. (4.16) has bound states and a continuum. The continuum becomes discrete at finite T and we can write a trajectory as

$$x(t) = x_I(t) + a_0 x_0(t) + a_1 x_1(t) + \sum_k (a_k^+ x_k^+(t) + a_k^- x_k^-(t)) \quad (4.17)$$

with modes defined as follows.

The first bound state is³

$$x_0(t) = \sqrt{\frac{3\omega}{8}} \frac{1}{\cosh^2 \frac{\omega t}{2}} = \frac{\sqrt{6}}{2} \dot{x}_I(t), \quad \lambda_0 \approx 0. \quad (4.18)$$

Note that $x_0(t)$ is proportional to the velocity of the instanton.

The second bound state has odd parity and finite eigenvalue

$$x_1(t) = \frac{\sqrt{3\omega}}{2} \frac{\sinh \frac{\omega t}{2}}{\cosh^2 \frac{\omega t}{2}}, \quad \lambda_1 = \frac{3}{4} \omega^2. \quad (4.19)$$

For $T \rightarrow \infty$ the remaining modes form a continuum. However, if we consistently maintain boundary conditions at finite T , these remaining modes are discrete. The eigenvalue equation is solved by

$$x_k(t) = \frac{1}{4} \left(1 + 4 \frac{k^2}{\omega^2} + 6i \frac{k}{\omega} \tanh \frac{\omega t}{2} - 3 \tanh^2 \frac{\omega t}{2} \right) e^{ikt}, \quad \lambda_k = \omega^2 + k^2. \quad (4.20)$$

Paths $x_k(-t)$ are also eigenstates with the same eigenvalues. Since the Pöschl-Teller potential is symmetric, it is sensible to use parity eigenstates. Explicitly, these modes are

$$x_k^+(t) = N_k [x_k(t) + x_k(-t)] = \frac{N_k}{2} \left[\left(1 + \frac{4k^2}{\omega^2} \right) \cos(kt) - \frac{6k}{\omega} \sin(kt) \tanh \left(\frac{\omega t}{2} \right) - 3 \cos(kt) \tanh^2 \left(\frac{\omega t}{2} \right) \right] \quad (4.21)$$

with $N_k = \sqrt{\frac{2\omega^4}{T(4k^4 + 5k^2\omega^2 + \omega^4)}}$, and

³This function as written has eigenvalue exactly 0, but does not satisfy the Dirichlet boundary conditions. If we impose the Dirichlet boundary conditions at $t = \pm \frac{T}{2}$, the eigenvalue gets lifted to $\lambda_0 \approx e^{-\omega T}$ (see [[24], Appendix E] or [[30], Eq. (23.55b)]). While an exponentially small eigenvalue renders finite the functional determinant in the path integral, it does not produce the correct answer for the twisted partition function because higher order terms in fluctuations in the x_0 direction are important. Collective coordinates allow for inclusion of fluctuations due to time translations to all orders.

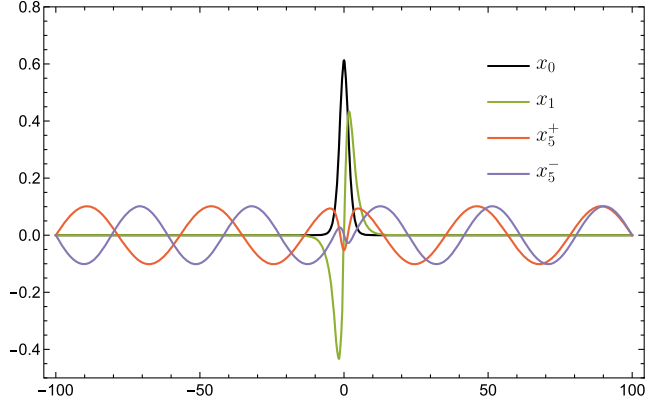


FIG. 3. Normal modes of the double well at finite T (plotted with $T = 200$), with Dirichlet boundary conditions at $t = \pm \frac{T}{2}$. Modes have even or odd parity. The two bound state modes x_0 and x_1 are shown, as are two of the continuum modes $x_k^\pm$ with $k = 5$. These continuum modes are similar to sine functions except near $t = 0$.

$$\begin{aligned} x_k^-(t) &= -iN_k[x_k(t) - x_k(-t)] \\ &= \frac{N_k}{2} \left[\left(1 + \frac{4k^2}{\omega^2}\right) \sin(kt) + \frac{6k}{\omega} \cos(kt) \tanh\left(\frac{\omega t}{2}\right) \right. \\ &\quad \left. - 3 \sin(kt) \tanh^2\left(\frac{\omega t}{2}\right) \right]. \end{aligned} \quad (4.22)$$

Note that the modes $x_k^\pm(t)$ are manifestly real. As written, the modes are normalized at large T to $\langle x_k^\pm | x_k^\pm \rangle = 1 + \mathcal{O}(\frac{1}{T})$. The eigenvalues are quantized by the boundary conditions $x_k^\pm(\frac{T}{2}) = 0$ which leads at large T to

$$k^-(n) \approx \frac{2n\pi}{T}, \quad k^+(n) \approx \frac{(2n-1)\pi}{T}, \quad n = 1, 2, 3, \dots \quad (4.23)$$

As $T \rightarrow \infty$, the modes approach

$$x_k^\pm(t) \rightarrow s_n(t), \quad \lambda_n \approx \omega^2 + \left(\frac{n\pi}{T}\right)^2, \quad n = 1, 2, 3, \dots \quad (4.24)$$

with s_n as in Eq. (3.4). As $T \rightarrow \infty$ the x_k^+ become the even- n modes and x_k^- become the odd- n modes. The modes $x_k^\pm$ and s_n differ only close to $t = 0$, as shown in Fig. 3. For small t , it is enough to know the values of the modes and their derivatives at $t = 0$

$$\begin{aligned} x_k^+(0) &= \frac{1}{\sqrt{2T}}, & x_k^-(0) &= 0, \\ \dot{x}_k^+(0) &= 0, & \dot{x}_k^-(0) &= \frac{4\sqrt{2}n\pi}{T^{3/2}}. \end{aligned} \quad (4.25)$$

When t is not small, the s_n are excellent approximations to the $x_k^\pm$.

B. Collective coordinates

Now we would like to change to collective coordinates. To do so we need to consider the intersection number $N_X[x]$. This is the number of times a given path $x(t)$ has vanishing projection against $x_0(t + t_0)$ over the integration range of t_0 . Parametrizing $x(t)$ as in Eq. (4.17), we need to find the number of zeros of

$$\begin{aligned} \langle x_0(t + t_0) | x(t) \rangle &= \langle x_0(t + t_0) | x_I(t) \rangle + a_0 \langle x_0(t + t_0) | x_0(t) \rangle \\ &\quad + a_1 \langle x_0(t + t_0) | x_1(t) \rangle \\ &\quad + \sum_k [a_k^+ \langle x_0(t + t_0) | x_k^+(t) \rangle \\ &\quad + a_k^- \langle x_0(t + t_0) | x_k^-(t) \rangle] \end{aligned} \quad (4.26)$$

for fixed values of the coordinates $\{a_0, a_1, a_k^\pm\}$. To evaluate the above expression we note that there are three relevant timescales: the overall time T , the collective coordinate time t_0 , and the inverse frequency ω^{-1} associated with the curvature of the potential. The two bound states x_0 and x_1 have support of order ω^{-1} while the continuum modes have support over all t . For $t_0 \ll \omega^{-1} \ll T$, we have

$$\langle x_0(t + t_0) | x(t) \rangle = -\sqrt{\frac{2a^2\omega}{3}} t_0 + a_0 - \frac{3\pi}{16\sqrt{2}} a_1 \omega t_0 + \dots \quad (4.27)$$

If we are permitted to drop the a_1 term and the additional terms thereafter, the above reduces to $t_0 \approx a_0 S_I^{-1/2}$. This fiducial trade-off between a_0 and t_0 is the change to collective coordinates one normally anticipates: motion along the direction of the zero mode x_0 is exchanged for motion along t_0 . However, the relationship between a_0 and t_0 only holds for $t_0 \ll \omega^{-1} \ll T$. For larger t_0 the other terms in Eq. (4.27) are important.

In the regime $\omega^{-1} \ll t_0 \ll T$, the detailed structure of $x_0(t + t_0)$ of order ω^{-1} cannot be resolved. In fact, as $\omega t_0 \rightarrow \infty$, the mode $x_0(t - t_0) \sim \frac{1}{\cosh^2 \frac{\omega(t-t_0)}{2}}$ approximates a δ -function

$$x_0(t - t_0) \approx \sqrt{\frac{6}{\omega}} \delta(t - t_0), \quad \omega^{-1} \ll t_0 \ll T, \quad (4.28)$$

so that

$$\langle x_0(t - t_0) | x(t) \rangle \approx \sqrt{\frac{6}{\omega}} \int dt \delta(t - t_0) x(t) = \sqrt{\frac{6}{\omega}} x(t_0). \quad (4.29)$$

The equation above leads to a nice interpretation of $N_{x_0}[x]$: it is the number of times $x(t_0) = 0$, i.e. the number of times a given path $x(t)$ crosses $x = 0$ on its way from $-b$ to b .

Similarly, in the same limit the Jacobian in Eq. (2.12) $\mathcal{J} = \langle \dot{x}_0(t) | x(t) \rangle \approx -\dot{x}(0)$ becomes the velocity of the path at t_0 , i.e. as the path crosses the origin.

C. Energy splitting

Now let us use the collective coordinates to compute the energy splitting. To do so, it is easiest to parametrize paths by

$$x(t) = x_I(t + t_0) + \alpha_1 x_1(t + t_0) + \sum_k [\alpha_k^+ x_k^+(t + t_0) + \alpha_k^- x_k^-(t + t_0)] \quad (4.30)$$

and to use Eq. (2.10). The Jacobian evaluates to

$$\mathcal{J} = \langle \dot{x}_0(t) | x_I(t) + \alpha_1 x_1(t) + \sum_k [\alpha_k^+ x_k^+(t) + \alpha_k^- x_k^-(t)] \rangle \quad (4.31)$$

$$\approx -\sqrt{\frac{2\omega a^2}{3}} - \frac{3\pi\omega}{16\sqrt{2}}\alpha_1 + \sum_k \alpha_k^+ \frac{4\sqrt{2}n\pi}{T^{3/2}} \quad (4.32)$$

where n refers to the approximate wave number from Eq. (4.23).

The action of a path $x(t)$ parametrized in collective coordinates is

$$S[x] = \frac{1}{2}\alpha_1^2\lambda_1 + \frac{1}{2}\sum_k [(\alpha_k^+)^2\lambda_k^+ + (\alpha_k^-)^2\lambda_k^-]. \quad (4.33)$$

Since $\lambda_1 = \frac{3}{4}\omega^2$ and $\lambda_k \approx \omega^2 + (\frac{n\pi}{T})^2$, the above action implies that the path integral is dominated by the regime $\alpha_1 \lesssim \omega^{-1}$ and $\alpha_k^\pm \lesssim \omega^{-1}$ (and even for $n > T\omega$ the scaling is $\alpha_k^\pm \lesssim \frac{T}{n} \lesssim \omega^{-1}$ as well). Thus, for purposes of computing the energy splitting where we are interested in the limit of large $a^2\omega$ and large T , the Jacobian factor in Eq. (4.32) reduces to

$$\mathcal{J} \approx -\sqrt{\frac{2\omega a^2}{3}} = -\sqrt{S_I}. \quad (4.34)$$

This is the result usually found in the literature.

In collective coordinate, the twisted partition function is then

$$Z_P = \mathcal{N} \sqrt{\frac{4\pi}{\omega}} \sqrt{S_I} e^{-S_I} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_0 \int_{-\infty}^{\infty} \frac{d\alpha_1}{\sqrt{2\pi}} \prod_k \int_{-\infty}^{\infty} \frac{d\alpha_k^\pm}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda_1\alpha_1^2 - \frac{1}{2}\lambda_k(\alpha_k^\pm)^2} \frac{1}{N_{x_0}[x]} \quad (4.35)$$

where, as discussed, $N_{x_0}[x]$ counts the number of times a path $x(t)$ crosses $x = 0$. Although $N_{x_0}[x]$ is an incredibly

complicated functional of paths, it can be simplified for the calculation of the twisted partition function in the large T limit.

We reiterate that according to Eq. (4.29), $N_{x_0}[x]$ counts the number of times the path crosses the origin. Since all paths integrated over in the twisted partition function go from $-b$ to b they must all cross the origin at least once—this is essential, otherwise the change to collective coordinate t_0 would not be valid. Because the potential $V(x)$ is largest at $x = 0$, larger values of $N_{x_0}[x]$ will give exponentially suppressed contributions to the path integral compared with the contributions at $N_{x_0}[x] = 1$. Consequently $N_{x_0}[x] \geq 1$ and paths with $N_{x_0}[x] > 1$ are negligible. Thus we can approximate $N_{x_0}[x] = 1$. So we have

$$Z_P \approx \mathcal{N} \sqrt{\frac{4\pi}{\omega}} T \sqrt{\frac{S_I}{2\pi}} e^{-S_I} \int_{-\infty}^{\infty} \frac{d\alpha_1}{\sqrt{2\pi}} \times \prod_k \int_{-\infty}^{\infty} \frac{d\alpha_k^\pm}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda_1\alpha_1^2 - \frac{1}{2}\lambda_k(\alpha_k^\pm)^2} \quad (4.36)$$

$$= \mathcal{N} \sqrt{\frac{4\pi}{\omega}} T \sqrt{\frac{S_I}{2\pi}} e^{-S_I} \sqrt{\frac{1}{\lambda_1}} \prod_k \sqrt{\frac{1}{\lambda_k^\pm}} \quad (4.37)$$

$$= \mathcal{N} \sqrt{\frac{4\pi}{\omega}} T \sqrt{\frac{S_I}{2\pi}} e^{-S_I} [\det'(-\partial_t^2 + V''(x_I))]^{-1/2}. \quad (4.38)$$

As such we have reduced our calculation to the textbook treatment.

We can also understand the $N_{x_0}[x] \approx 1$ replacement from a different perspective, which allows a more straightforward generalization to the QFT case in Sec. V. If we solve the condition $\langle x_0(t + t_0) | x(t) \rangle = 0$ for t_0 using Eq. (4.27) and expand at large S_I , we find

$$t_0 = \frac{a_0}{\sqrt{S_I}} - \frac{3\pi\omega}{16\sqrt{2}S_I} a_0 a_1 + \dots \quad (4.39)$$

Here $a_n \lesssim \omega^{-1}$ just as $\alpha_n \lesssim \omega^{-1}$. However, the mode with zero eigenvalue with coordinate a_0 is unsuppressed. Considering only the zero mode, the condition determining t_0 is

$$0 = \langle x_0(t + t_0) | x_I(t) \rangle + a_0 \langle x_0(t + t_0) | x_0(t) \rangle \quad (4.40)$$

$$= \frac{3}{\omega} \sqrt{S_I} \frac{(\omega t_0 - \sinh(\omega t_0))}{\cosh \omega t_0 - 1} + a_0 \frac{3(-2 + \omega t_0 \coth \frac{\omega t_0}{2})}{\cosh \omega t_0 - 1}. \quad (4.41)$$

The first function on the second line is a monotonically decreasing odd function of t_0 and the second function is a positive monotonically decreasing function of $|t_0|$ with a maximum at $t_0 = 0$; these conditions imply that there

is exactly one solution to (4.40) for a given a_0 . There are more solutions for t_0 if we turn on some of the other a_n 's, analogous to the nonlinear terms in Eq. (4.39). Such additional solutions for t_0 require large values of $a_n \gtrsim \sqrt{S_I}$. However, when $a_n \gtrsim \omega^{-1}$, the path integral becomes exponentially suppressed. Thus any path with $N_{x_0}[x] > 1$ is exponentially suppressed by the contributions of large a_n and so we can simply set $N_{x_0}[x] = 1$.

The above argument can be stated more physically. We expand around a particular instanton which crosses the origin at $t_0 = 0$. We can shift the path using the zero mode, changing the time the path crosses the origin. But to get an additional crossing, the quantum corrections must be enormous: as large as the classical contribution. Such large corrections are exponentially suppressed.

There is a close analogy between this calculation, where the energy splitting is given by paths that cross $x = 0$ exactly once, and the calculation of the tunneling rate in an asymmetric double well. In the latter, one way to derive the formula for the decay rate is to restrict to paths that cross the origin only once [5,6]. This restriction can be understood physically: if the paths do not cross the origin at all, they do not mediate transitions from the false vacuum to the true one; to find a well-defined decay rate one must be in a regime where the back-scattering off the far end of the potential is irrelevant, which can be enforced by demanding at most one crossing. Thus the decay rate is determined by paths that cross the origin exactly once, akin to the energy splitting in the symmetric double well.

For completeness, we finish the energy-splitting calculation. The product over continuum eigenvalues is nearly the same as for a simple harmonic oscillator. For the harmonic oscillator, one expands around the classical solution $x_{cl}(t) = b \cosh(\omega t) / \cosh(\omega T/2)$ where the fluctuations have eigenvalues $\lambda_n^{\text{SHO}} = \omega^2 + (n\pi/T)^2$, and so the partition function is

$$Z_{\text{SHO}} = \mathcal{N} \int_{-\infty}^{\infty} db e^{-\omega b^2} \prod_n \int_{-\infty}^{\infty} \frac{da_n}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda_n a_n^2} \quad (4.42)$$

$$= \mathcal{N} \sqrt{\frac{\pi}{\omega}} \prod_n \sqrt{\frac{1}{\lambda_n^{\text{SHO}}}} = \mathcal{N} \sqrt{\frac{\pi}{\omega}} \det^{-1/2}(-\partial_t^2 + \omega^2) \quad (4.43)$$

which defines for us the normalization $\mathcal{N}$ in terms of Z_{SHO} . For the double well, we can then plug $\mathcal{N}$ into Eq. (4.38), obtaining

$$Z_P = Z_{\text{SHO}} 2T \sqrt{\frac{S_I}{2\pi}} e^{-S_I} \left(\frac{\det(-\partial_t^2 + \omega^2)}{\det'(-\partial_t^2 + V''(x_I))} \right)^{1/2}. \quad (4.44)$$

The ratio of determinants can be computed using the Gelfand-Yanglom method [36,37] and yields $12\omega^2$. Using Eq. (4.7), we find

$$\Delta_0 = 2\sqrt{\frac{6S_I}{\pi}} e^{-S_I} \omega \quad (4.45)$$

Recalling $S_I = \frac{2\omega a^2}{3}$, the above exactly matches the WKB result in Eq. (4.3). We have used that $Z \approx 2Z_{\text{SHO}}$ in the small-energy-splitting limit.

D. Free twisted partition function

In the interacting theory, integration over the collective coordinate t_0 gave a factor of T in the twisted partition function, as expected for the computation of the energy splitting. The free theory is also time-translation invariant, and so using collective coordinates would seem to give a factor of T . However, the free twisted partition function is just $Z_P = \frac{1}{2}$ (as we review shortly). So what is the fate of the putative factor of T ? To find out, we need to properly compute the twisted partition function with action $S_0[x] = \int dt \frac{1}{2} \dot{x}^2$ using collective coordinates.

1. Expected results

First we recall the expected result. The twisted partition function in the free theory can be computed as the zero frequency limit of the twisted partition function for the harmonic oscillator, namely

$$Z_P^0 = \text{Tr}(P e^{-HT}) = \lim_{\omega \rightarrow 0} \sum_{n=0}^{\infty} e^{-\omega(n+\frac{1}{2})T} (-1)^n = \frac{1}{2}. \quad (4.46)$$

Alternatively, we can recall that the position-space propagator in quantum mechanics is

$$\begin{aligned} D(a, b) &= \langle b | e^{-HT} | a \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle b | e^{-\frac{1}{2}p^2 T} | p \rangle \langle p | a \rangle \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(b-a)} e^{-\frac{p^2}{2}T} = \sqrt{\frac{1}{2\pi T}} e^{-\frac{(b-a)^2}{2T}}. \end{aligned} \quad (4.47)$$

So the twisted partition function is

$$Z_P^0 = \int_{-\infty}^{\infty} db D(b, -b) = \int_{-\infty}^{\infty} db \sqrt{\frac{1}{2\pi T}} e^{-\frac{b^2}{T}} = \frac{1}{2}. \quad (4.48)$$

In the path integral approach, we compute

$$Z_P^0 = \mathcal{N} \int_{-\infty}^{\infty} db \int_{x(-\frac{T}{2})=-b}^{x(\frac{T}{2})=b} \mathcal{D}x e^{-S[x]}. \quad (4.49)$$

Here the classical path going from $-b$ to b in time T is $x_c(t) = \frac{2}{T}b$ with action $S_0[x_c] = \frac{2b^2}{T}$, as expected from the exponent of $D(b, -b)$. A general path is then parametrized by

$$x(t) = x_c(t) + c_n s_n(t), \quad n = 1, 2, 3, \dots, \quad (4.50)$$

with $s_n(t)$ as in Eq. (3.4). These $s_n(t)$ modes satisfy $\partial_t^2 s_n = -\lambda_n s_n$ with $\lambda_n = (\frac{n\pi}{T})^2$. Then

$$\begin{aligned}
Z_P^0 &= \mathcal{N} \int_{-\infty}^{\infty} db e^{-\frac{2b^2}{T}} \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda_n c_n^2} \\
&= \mathcal{N} \frac{\sqrt{2\pi T}}{2} \prod_n \frac{T}{n\pi}
\end{aligned} \tag{4.51}$$

so that to have $Z_P^0 = \frac{1}{2}$ we need

$$\mathcal{N} = \frac{1}{\sqrt{2\pi T}} \left[\prod_n \frac{T}{n\pi} \right]^{-1}. \tag{4.52}$$

2. Collective coordinates

Now we attempt to compute Z_P^0 using collective coordinates in the double-well basis. Noting that the formula in

Eq. (2.12) is basis-independent, the only feature from the double well basis we carry over is that the function $X(t) = x_0(t)$ is the one used to project out the collective coordinate. Indeed, if we had tried to use the parametrization in Eq. (4.17) for the paths in the free theory case we would have run into problems because $x_I(t)$ is not a saddle point of the free action (in fact, it has much larger action than the classical path; for paths from $-a$ to a , $S_0[x_I] = \frac{a^2\omega}{3} \gg S_0[x_c] = \frac{2a^2}{T}$). Since $x_I(t)$ is not a saddle point, in attempting to use Eq. (4.17) in the path integral, we would find linear fluctuation terms which would require completing the square in the action, effectively reducing the computation to expanding around the classical path. The result would be the same as using the parametrization in Eq. (4.50) to begin with.

Marching forward, we want to evaluate

$$Z_P^{\text{coll},0} = \mathcal{N} \int_{-\infty}^{\infty} db e^{-\frac{2b^2}{T}} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_0 \prod_n \int_{-\infty}^{\infty} \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda_n c_n^2} \frac{1}{N_{x_0}[x]} \delta[\langle x_0(t)|x(t)\rangle] |\langle x_0(t)|\dot{x}(t)\rangle|. \tag{4.53}$$

As discussed in the double-well case, $\delta[\langle x_0(t)|x(t)\rangle]$ forces the path to cross $x = 0$ at $t = 0$ and $N_{x_0}[x]$ counts the number of times a given path crosses $x = 0$. To compute the argument of the δ function we need the projections

$$\begin{aligned}
\langle x_0(t)|s_n(t)\rangle &= \sqrt{\frac{2}{T}} \int_{-\infty}^{\infty} dt x_0(t) \sin \frac{n\pi(t + \frac{T}{2})}{T} \\
&= \sqrt{\frac{12}{T\omega}} \sin \frac{n\pi}{2} Q_n
\end{aligned} \tag{4.54}$$

$$= (-1)^{\frac{n-1}{2}} \sqrt{\frac{12}{T\omega}} \delta_{n,\text{odd}} Q_n \tag{4.55}$$

which we calculate in the $T \rightarrow \infty$ limit, where

$$Q_n = \frac{\frac{n\pi^2}{T\omega}}{\sinh \frac{n\pi^2}{T\omega}} \approx 1. \tag{4.56}$$

Thus the δ function involves only the odd modes, and sets $c_1 - c_3 + c_4 - c_5 + \dots = 0$ up to corrections in $\frac{1}{T}$. The Jacobian is

$$\begin{aligned}
\langle x_0(t)|\dot{x}(t)\rangle &= \langle x_0(t)|\dot{x}_c(t)\rangle + \sum_n c_n \langle x_0(t)|\dot{s}_n(t)\rangle \\
&= \frac{2b}{T} \sqrt{\frac{6}{\omega}} + \sum_n (-1)^{\frac{n}{2}} \sqrt{\frac{12}{\omega T}} \sqrt{\lambda_n} c_n Q_n \delta_{n,\text{even}}
\end{aligned} \tag{4.57}$$

which involves only the even modes. Since the δ function only involves the odd modes and the Jacobian only involves the even modes, the integration over the even and odd modes factorizes.

With the factor of $N_{x_0}[x]$, even though the path integral factorizes, the integrals are challenging. To show that $N_{x_0}[x]$ is necessary, it suffices to ignore it by setting it to 1 and show that the wrong answer results. Let us do so and behold the consequences of such ignorance.

The formulas

$$\int d^N c e^{-\pi \vec{c} \cdot A \cdot \vec{c}} |\vec{w} \cdot \vec{c}| = \frac{1}{\pi} \frac{\sqrt{\vec{w} \cdot A^{-1} \cdot \vec{w}}}{\sqrt{\det(A)}} \tag{4.58}$$

and

$$\int d^N c e^{-\pi \vec{c}^2} \delta(\vec{w} \cdot \vec{c}) = \frac{1}{\sqrt{\vec{w}^2}} \tag{4.59}$$

are helpful. For the odd modes we find

$$\begin{aligned}
I_{\text{odd}} &= \prod_{n \text{ odd}} \int \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2}c_n^2 \lambda_n} \delta\left(c_n \sqrt{\frac{12}{\omega T}} \sin \frac{n\pi}{2} Q_n\right) \\
&= \left(\prod_{n \text{ odd}} \sqrt{\frac{1}{\lambda_n}} \right) \sqrt{\frac{\omega}{3\pi T}}
\end{aligned} \tag{4.60}$$

and for the even modes we have

$$\begin{aligned}
I_{\text{even}} &= \int db e^{-\frac{2b^2}{T}} \prod_{n \text{ even}} \int \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2}c_n^2 \lambda_n} \left| \frac{2b}{T} \sqrt{\frac{6}{\omega}} \right. \\
&\quad \left. + \sum_n (-1)^{\frac{n}{2}} \sqrt{\frac{12}{\omega T}} \sqrt{\lambda_n} c_n Q_n \delta_{n,\text{even}} \right|
\end{aligned} \tag{4.61}$$

$$= \left(\prod_{n \text{ even}} \sqrt{\frac{1}{\lambda_n}} \right) \left(\sqrt{T} + \frac{3}{\omega\sqrt{T}} + \dots \right). \quad (4.62)$$

So all together we obtain

$$Z_P^{\text{coll},0} = \frac{1}{\sqrt{2\pi T}} \left[\prod_n \sqrt{\frac{1}{\lambda_n}} \right]^{-1} \left(\prod_{n \text{ odd}} \sqrt{\frac{1}{\lambda_n}} \right) \times \sqrt{\frac{\omega}{3\pi T}} \left(\prod_{n \text{ even}} \sqrt{\frac{1}{\lambda_n}} \right) \sqrt{T} \int_{-T/2}^{T/2} dt_0 \quad (4.63)$$

$$= \sqrt{\frac{\omega T}{6\pi^2}}. \quad (4.64)$$

This does not agree with the correct answer $Z_P^0 = \frac{1}{2}$, as in Eq. (4.48). Indeed Eq. (4.64) is too large by a factor of $\sqrt{\omega T}$. Thus we confirm that using collective coordinates without the factor of $N_{x_0}[x]$ is incorrect.

To see that the right answer does emerge if $N_{x_0}[x]$ is included is challenging because of the extreme complexity of $N_{x_0}[x]$. However, incorporating $N_{x_0}[x]$ is essentially the same challenging problem discussed in Sec. III. It does not seem that working in the double-well basis rather than a Fourier basis provides any additional insight, so we simply refer the reader back to Sec. III and move on to quantum field theory.

V. QUANTUM FIELD THEORY

For our final example, we consider the $\lambda\phi^4$ quantum field theory in 4 Euclidean dimensions. The Lagrangian density is $\mathcal{L}[\phi] = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)$ with $V(\phi) = \frac{1}{4}\lambda\phi^4$. For $\lambda < 0$ this theory has a 5-parameter family of Fubini-Lipatov instanton saddle points [38,39] called “bounces” given by

$$\phi_b^{R,\xi}(x) = \sqrt{\frac{8}{-\lambda R^2 + (x^\mu + \xi^\mu)^2}} R, \quad (5.1)$$

with action

$$S_b = S[\phi_b] = -\frac{8\pi^2}{3\lambda}. \quad (5.2)$$

We will generally pick a fiducial member of this family to expand around and denote it by $\phi_b(r)$ with R fixed and $\xi = 0$. Here $r = \sqrt{\sum_\mu x_\mu^2}$ is the Euclidean radial coordinate.

With $\lambda < 0$ the vacuum at $\phi = 0$ is unstable. The corresponding decay rate can be computed perturbatively as

$$\Gamma = \lim_{T \rightarrow \infty} \frac{1}{T} \text{Im} \frac{\int_{C_b} \mathcal{D}\phi e^{-S[\phi_b] + \frac{1}{2}S''[\phi_b]\phi^2 + \dots}}{\int_{C_{\text{FV}}} \mathcal{D}\phi e^{-\frac{1}{2}\int d^4x (\partial_\mu\phi)^2}} \quad (5.3)$$

where C_b is a path of steepest descent passing through the bounce ϕ_b and C_{FV} is the expansion around the false vacuum $\phi = 0$. A derivation of this formula can be found in [7]. The numerator of this expression is expected to have a factor of $T\mathcal{V}$ from the translation invariance of the action (the ξ^μ moduli). Such a factor makes $\Gamma \propto \mathcal{V}$, i.e. the decay is a rate per unit volume, as expected in a translationally invariant theory. The denominator, which comprises an expansion around the free theory, is not expected to have a factor of $T\mathcal{V}$. The goal of this section is to understand why the numerator is proportional to $T\mathcal{V}$ but the denominator is not, despite the fact that both actions are translation and scale invariant.

A. Basis and cross check

Remarkably, the eigenstates for expanding around the bounce are known in closed form [7]. Even more remarkably, these same eigenstates also diagonalize fluctuations around the false vacuum. The modes are

$$\hat{\phi}_{n\ell m}(r, \alpha, \theta, \phi) = \frac{\sqrt{2n+3}}{2\pi} \frac{1}{r} P_{n+1}^{-s-1} \left(\frac{R^2 - r^2}{R^2 + r^2} \right) \times \sqrt{\frac{\pi}{2}} Y^{s\ell m}(\alpha, \theta, \phi) \quad (5.4)$$

where $P_\ell^m(x)$ are associated Legendre polynomials and $Y^{s\ell m}(\alpha, \theta, \phi)$ are 3D spherical harmonics (which are also Legendre polynomials). We will use the “box” inner product

$$\langle \phi_i | \phi_j \rangle = - \int d^4x \phi_i(x) \square \phi_j(x) \quad (5.5)$$

under which the modes are orthonormal, namely

$$\langle \hat{\phi}_{n\ell m} | \hat{\phi}_{n'\ell'm'} \rangle = - \int d^4x \hat{\phi}_{n\ell m} \square \hat{\phi}_{n'\ell'm'} = \delta_{nn'} \delta_{ss'} \delta_{\ell\ell'} \delta_{mm'}. \quad (5.6)$$

The modes satisfy

$$\square \hat{\phi}_{n\ell m} = \lambda_n V''[\phi_b] \hat{\phi}_{n\ell m}, \quad \lambda_n = \frac{(n+1)(n+2)}{6} = \frac{1}{3}, 1, 2, \frac{10}{3}, \dots, \quad (5.7)$$

and there are $d_n = \frac{(n+1)(n+2)(2n+3)}{6} = 1, 5, 14, \dots$ modes with eigenvalue λ_n .

The $n = 0$ mode with multiplicity $d_0 = 1$ is proportional to the bounce

$$\phi_{0000} = \frac{\sqrt{3}}{2\pi} \frac{R}{r^2 + R^2} = \frac{1}{2\pi} \sqrt{\frac{-3\lambda}{8}} \phi_b = \frac{1}{2\sqrt{S_b}} \phi_b. \quad (5.8)$$

Normalizing with respect to our box inner product, the mode is independent of λ . Of the five $n = 1$ modes the one with $s = 0$ is the dilatation mode associated with changes in R , namely

$$\phi_d \equiv \phi_{1000} = \frac{\sqrt{5}R}{2\pi} \frac{R^2 - r^2}{(R^2 + r^2)^2} = -\sqrt{\frac{5}{12S_b}} R \partial_R \phi_b. \quad (5.9)$$

The other four $n = 1$ modes, with $s = 1$, correspond to translations. They are linear combinations of the $\partial_\mu \phi_b$.

Field configurations around the bounce can be parametrized as

$$\phi(x) = \phi_b(x) + \sum_n c_n \hat{\phi}_n(x) \quad (5.10)$$

where we are using n to abbreviate all the indices. The quadratic action around the bounce is

$$S[\phi] \approx S[\phi_b] + \sum_{n,m} \frac{1}{2} \int d^4x c_n \hat{\phi}_n [-\square + V''(\phi_b)] c_m \hat{\phi}_m \quad (5.11)$$

$$= S_b + \sum_n \frac{1}{2} c_n^2 \cdot \left(\frac{\lambda_n - 1}{\lambda_n} \right) \quad (5.12)$$

$$= S_b + \frac{1}{2} c_0^2 \cdot (-2) + \frac{1}{2} c_1^2 \cdot (0) + \frac{1}{2} c_2^2 \cdot \left(\frac{1}{2} \right) + \dots \quad (5.13)$$

The unusual factor $(\lambda_n - 1)/\lambda_n$ is a result of the box inner product. If we had used the “ V ” inner product of Ref. [7], namely $\langle \phi_i | \phi_j \rangle_V = -\int d^4x V''[\phi_b] \phi_i \phi_j$, we would have found the action $S = S_b + \frac{1}{2} c_n^2 (\lambda_n - 1) + \dots$. For present purposes, the box inner product is preferable because it is well defined even for $V = 0$ so we can use it in the free theory. Importantly, there is one mode ($n = 0$) with a negative eigenvalue and five modes ($n = 1$) with zero eigenvalue in either inner product.

Writing $\phi(x) = \sum_n b_n \hat{\phi}_n(x)$, the free action in these coordinates is

$$\begin{aligned} S[0] + \frac{1}{2} \int d^4x S''[0] \phi(x)^2 &= \sum_{n,m} \frac{1}{2} \int d^4x (b_n \hat{\phi}_n) [-\square] (b_m \hat{\phi}_m) \\ &= \sum_n \frac{1}{2} b_n^2. \end{aligned} \quad (5.14)$$

The reason that the same basis diagonalizes the expansion around the bounce and around the free theory is that the bounce is itself one of the normal modes (the $n = 0$ one).

Thus we can move from the bounce to the false vacuum by varying a single coordinate.

For a cross check, let us compute the propagator in the free theory. We can write the propagator in terms of the path integral as

$$\begin{aligned} D(x, 0) &= \frac{\int \mathcal{D}\phi e^{-S[\phi]} \phi(x) \phi(0)}{\int \mathcal{D}\phi e^{-S[\phi]}} \\ &= \frac{\int \prod_n db_n e^{-\frac{1}{2} b_n^2} \sum_{k,j} [b_k \phi_k(x)] [b_j \phi_j(0)]}{\int \prod_n dc_n e^{-\frac{1}{2} c_n^2}} \end{aligned} \quad (5.15)$$

$$= \sum_n \phi_n(x) \phi_n(0) \quad (5.16)$$

$$= \frac{1}{4\pi^2 R r} \sum_n (2n + 3) P_{n+1}^{-1} \left(\frac{R^2 - r^2}{R^2 + r^2} \right). \quad (5.17)$$

Using the integral representation

$$P_{n+1}^{-1}(x) = \frac{1}{i\pi(n+1)} \int_0^\pi dt \cos(t) (\sqrt{x^2 - 1} \cos t + x)^{n+1} \quad (5.18)$$

we can perform the sum, leading to

$$D(x, 0) = \frac{1}{4\pi^2 r^2} \quad (5.19)$$

which is the correct answer and as expected is independent of R .

B. Collective coordinates for dilatations

To go to collective coordinates, we consider first the R coordinate associated with dilatations. The $\lambda\phi^4$ theory is classically scale invariant, including if we expand around the bounce or the false vacuum. Ultimately, the divergence associated with an integration over R will be regulated by the scale anomaly, which shows up at higher orders in the saddle point approximation. We are not interested in this regulation here. Rather, we are interested in showing why a divergent integral over the collective coordinate R should show up in the expansion around the bounce but not around the free theory.

Let us denote the dilatation mode as $\phi_d \equiv \phi_{1000}$. This is the mode we denoted by $X(t) = x_0(t)$ in the quantum mechanical case, and is the mode we would like to remove. While R is already incorporated into our basis expansion, let us make the R -dependence explicit by fixing $R = \bar{R}$ and writing

$$\phi(x) = \phi_b(\bar{R}, x) + c_d \phi_d(\bar{R}, x) + \sum_{n \geq 1} c_n \phi_n(\bar{R}, x). \quad (5.20)$$

Then the collective coordinate parametrization where R is a coordinate and ϕ_d is removed is

$$\phi(x) = \phi_b(R, x) + \sum_{n \geq 1} \alpha_n \phi_n(R, x). \quad (5.21)$$

In these collective coordinates, the Jacobian factor from Eq. (2.10) is

$$\mathcal{J} = \langle \phi_d | \partial_R \phi \rangle = -\sqrt{\frac{3}{5}} \frac{1}{R} (2\sqrt{S_b} + \alpha_0) + \frac{8}{\sqrt{35}} \frac{1}{R} \alpha_2 \quad (5.22)$$

where α_2 is the coefficient of ϕ_{2000} . Only the radial modes with $\ell = 0$ (and only two of them, with $n = 0, 2$) contribute. Accordingly the expansion of the path integral around the bounce is

$$e^{-S_b} \int \frac{dR}{R} \int \prod_n d\alpha_n e^{-\frac{1}{2} \alpha_n^2 \frac{2n-1}{\lambda_n}} \delta(\alpha_1) \times \left| \sqrt{\frac{3}{5}} (2\sqrt{S_b} + \alpha_0) - \frac{8}{\sqrt{35}} \alpha_2 \right| \frac{1}{N_d[\phi]} \quad (5.23)$$

where $N_d[\phi]$ counts the number of solutions to $\langle \phi_d(R, x) | \phi(\bar{R}, x) \rangle = 0$. For the free theory, the path integral in the collective coordinate basis is

$$\int \frac{dR}{R} \int \prod_n d\alpha_n e^{-\frac{1}{2} \alpha_n^2} \delta(\alpha_1) \left| \sqrt{\frac{3}{5}} \alpha_0 - \frac{8}{\sqrt{35}} \alpha_2 \right| \frac{1}{N_d[\phi]}. \quad (5.24)$$

The integrands of both Eqs. (5.23) and (5.24) have a dependence on R which enters through $N_d[\phi]$.

In the interacting theory, $S_b \sim \frac{1}{-\lambda} \gg 1$ while $\lambda_n \sim 1$ so that the part of the path integral domain with $\alpha_n \lesssim 1$ dominates. Then we can approximate the Jacobian as $\sqrt{\frac{3}{5}} (2\sqrt{S_b} + \alpha_0) - \frac{8}{\sqrt{35}} \alpha_2 \approx \sqrt{\frac{12S_b}{5}}$. If we can justify ignoring the $N_d[\phi]$ factor, the path integral reduces to the usual expression used to compute the decay rate in $\lambda\phi^4$ QFT.

In the free theory, we also have $\alpha_n \lesssim 1$. There S_b does not appear and we can simply perform the integrals over α_0 and α_2 and compare to what we would have gotten without collective coordinates, where the R integral is replaced by α_1 . The ratio of “collective coordinate integral” and the “noncollective coordinate integral” if we set $N_d[\phi] = 1$ is

$$\frac{\int \frac{dR}{R} \int d\alpha_0 \int d\alpha_2 e^{-\frac{1}{2} \alpha_0^2 - \frac{1}{2} \alpha_2^2} \left| \sqrt{\frac{3}{5}} \alpha_0 - \frac{8}{\sqrt{35}} \alpha_2 \right|}{\int d\alpha_0 d\alpha_1 d\alpha_2 e^{-\frac{1}{2} \alpha_0^2 - \frac{1}{2} \alpha_1^2 - \frac{1}{2} \alpha_2^2}} = \int \frac{dR}{R} \sqrt{\frac{17}{7\pi^2}} \neq 1. \quad (5.25)$$

Since this ratio is not 1, we cannot expect $N_d[\phi] = 1$ to be a good approximation in the free theory. So if the standard computation for the interacting theory is correct, why can we drop $N_d[\phi]$ when expanding around the bounce but not around the false vacuum?

To proceed, we have to understand the intersection number $N_d[\phi]$. By analogy to the nonrelativistic quantum mechanics case, $N_d[\phi]$ counts the number of solutions to $\langle \phi_d(x, R) | \phi \rangle = 0$ for a given ϕ over the range of R . As in the quantum mechanics case, we need to work in a basis other than the collective coordinate one to evaluate $N_d[\phi]$. In the basis with fixed $\bar{R}$ as in Eq. (5.20), we find

$$\langle \phi_d(x, R) | \phi(x) \rangle = 2\sqrt{S_b} f_0(R, \bar{R}) + \sum_n c_n f_n(R, \bar{R}) \quad (5.26)$$

where the functions

$$f_n(R, \bar{R}) = - \int d^4x \phi_d(x, R) \square \phi_n(x, \bar{R}) \quad (5.27)$$

are represented in Fig. 4. For example, with $z = R/\bar{R}$ we have

$$f_0(z) = \frac{\sqrt{15}z(-1 - 9z^2 + 9z^4 + z^6 - 12(z^4 + z^2) \ln z)}{(z^2 - 1)^4} \quad (5.28)$$

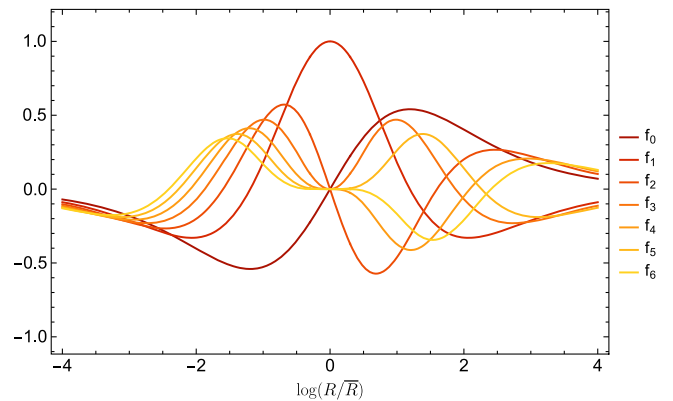


FIG. 4. We depict the functions $f_n(R/\bar{R}) = \langle \phi_d(x, R) | \phi_n(x, \bar{R}) \rangle$. The collective coordinate R is determined by the condition $2\sqrt{S_b} f_0(R/\bar{R}) + c_n f_n(R/\bar{R}) = 0$ which may have multiple solutions.

$$f_1(z) = -\frac{5z(-1 - 46(z^2 - z^6) + z^8 - 12z^2(3 + z^2)(1 + 3z^2)\ln z)}{(z^2 - 1)^5}. \quad (5.29)$$

For the interacting theory, we can examine whether $N_d[\phi] = 1$ is a good approximation using the same approach as for the double-well potential in Sec. IV C. Close to $R = \bar{R}$ we have

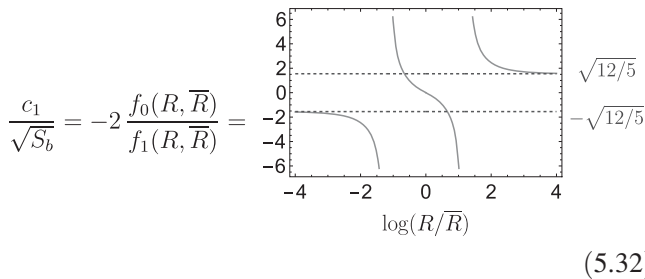
$$\begin{aligned} \langle \phi_d(x, R) | \phi(x) \rangle = & c_1 + \left(\sqrt{\frac{12S_b}{5}} + \sqrt{\frac{3}{5}}c_0 - \frac{8}{\sqrt{35}}c_2 \right) \left(\frac{R - \bar{R}}{\bar{R}} \right) \\ & + \frac{-7\sqrt{15}c_0 - 85c_1 + 8\sqrt{35}c_2 + 40\sqrt{5}c_3 - 14\sqrt{15}S_b}{70} \left(\frac{R - \bar{R}}{\bar{R}} \right)^2 + \dots \end{aligned} \quad (5.30)$$

Since $\sqrt{S_b} \gg 1$ we can solve $\langle \phi_d | \phi \rangle = 0$ as a series in $\frac{1}{\sqrt{S_b}}$. The leading terms are

$$R = \bar{R} \left(1 - \sqrt{\frac{5}{12S_b}}c_1 + \frac{14\sqrt{15}c_0c_1 + 35c_1^2 - 16\sqrt{35}c_1c_2}{168S_b} + \dots \right). \quad (5.31)$$

Since $\lambda_n \approx 1$, the path integral becomes suppressed if any $c_n \gtrsim 1$ on account of the $e^{-c_n^2 \lambda_n}$ terms. The only exception is the zero mode: excursions in the c_1 direction by arbitrary amounts only affect the integrand by an order 1 amount. From Eq. (5.31) we see that there is always at least one solution with nonzero c_1 . To have multiple solutions the nonlinear terms need to become important. If we just consider the c_n contributions, we require $c_n \gtrsim \sqrt{S_b} \gg 1$ and so the path integral is affected by an exponentially small amount (of order $e^{-c_1^2 \lambda_R} \approx e^{-S_b}$). The situation is akin to our quantum mechanical examples: the quantum fluctuations are too weak to significantly alter the classical trajectory. It remains to check the possibility of additional solutions due to large values of c_1 .

For large c_1 , meaning $c_1 \gtrsim \sqrt{S_b}$, the expansion in Eq. (5.31) is no longer useful. If we consider only paths of the form $\phi(x) = \phi_b(x, \bar{R}) + c_1 \phi_1(x, \bar{R})$, then a solution to $\langle \phi_d(x, R) | \phi(x) \rangle = 0$ requires



$$\frac{c_1}{\sqrt{S_b}} = -2 \frac{f_0(R, \bar{R})}{f_1(R, \bar{R})} = \quad (5.32)$$

For a given c_1 there is always at least one solution to the above equation. More precisely, since $f_1(R, \bar{R}) = 0$ at $R/\bar{R} = x_c$ and $R/\bar{R} = 1/x_c$ where $x_c \approx 0.303$, there will be one solution with $x_c \bar{R} < R < \frac{1}{x_c} \bar{R}$. However, if

$|c_1/\sqrt{S_b}| > \sqrt{12/5}$, there will be a second solution. So here $1 \leq N_d[\phi] \leq 2$, but we would like to determine whether the second solution is exponentially suppressed or not.

Although the action is exactly independent of the collective coordinate R , it is only independent of the coordinate c_1 to quadratic order. Expanding to higher orders there is a quartic term

$$S[\phi_b + c_1 \phi_1] = S_b - \frac{5}{36S_b} c_1^4. \quad (5.33)$$

The sign is negative since $\lambda < 0$; when expanding around the bounce to quadratic order, there is only one direction in which the potential decreases, but to quartic order it decreases in every direction (except the flat ones)! So we do not want to include terms to quartic order in c_1 . We do, however, want to include terms to quartic order in the collective coordinate R . If we did not include these terms, the action would not be independent of R . Large R leads to a large factor of the volume of the dilatation group, but not an exponentially large factor from fluctuations in a direction of instability.

We are primarily interested in whether a field configuration has large quadratic action in our collective coordinates and not in the coordinates at fixed $\bar{R}$. The α_n coordinates are given by

$$\alpha_n = \langle \phi_b(x, \bar{R}) + c_1 \phi_1(x, \bar{R}) - \phi_b(x, R) | \phi_n(x, R) \rangle \quad (5.34)$$

$$= g_n(R, \bar{R}) + c_1 f_n(R, \bar{R}) - 2\sqrt{S_b} \delta_{n0} \quad (5.35)$$

where $g_n(R, \bar{R}) = \langle \phi_b(x, \bar{R}) | \phi_n(x, R) \rangle$ and we note that $g_1(R, \bar{R}) = f_0(R, \bar{R})$. When $c_1 \ll \sqrt{S_b}$, we have

$c_1 \approx \sqrt{\frac{12S_b}{5}}(R - \bar{R})/\bar{R}$. In this regime R is very close to $\bar{R}$ and all the α_n are small, namely $\alpha_n \ll \sqrt{S_b}$. When $c_1 \gtrsim \sqrt{S_b}$ then all $\alpha_n \gtrsim \sqrt{S_b}$ as well. The corresponding field configurations, which include all of the $N_d[\phi] = 2$ field configurations with only c_1 turned on, are all suppressed. Other field configurations with $N_d[\phi] > 1$ would have various $c_n \gtrsim \sqrt{S_b}$ leading to $\alpha_n \gtrsim \sqrt{S_b}$ as well. Thus we can approximate $N_d[\phi]$ as equal to 1 up to exponentially small corrections in the path integral.

The free field theory behaves very differently, akin to the free quantum mechanics example. In the free field theory, the $\sqrt{\frac{12S_b}{5}}(\frac{R-\bar{R}}{\bar{R}})$ term in Eq. (5.30) is not present. Then the leading solution in the regime $\frac{R-\bar{R}}{\bar{R}} \ll 1$ is

$$R = \bar{R} \left(1 - \frac{7\sqrt{5}c_1}{7\sqrt{3}c_0 - 8\sqrt{7}c_2} + \dots \right). \quad (5.36)$$

There is no small parameter in the free theory and no zero mode. So even with $c_n \lesssim 1$ there are multiple solutions to $\langle \phi_d | \phi \rangle = 0$. That is, field configurations with $N_d[\phi] \gtrsim 1$ are not suppressed and in fact $N_d[\phi]$ can be arbitrarily large. Including these large values of $N_d[\phi]$ must compensate for the unregulated $\int \frac{dR}{R}$ integral, akin to the free quantum mechanics case. Showing the cancellation explicitly seems extraordinarily challenging: collective coordinates around the bounce are very cumbersome to use for the free theory.

An alternative approach to understand the exponential suppression when $N_d[\phi] > 1$ in the interacting theory, but not in the free theory, is to change coordinates to map directly to the double well. This approach is considered in Appendix B.

C. Collective coordinates for translations

In addition to the dilatation zero mode proportional to $\partial_R \phi_b$, the $\lambda \phi^4$ theory also has zero modes associated with translations which are linear combinations of the $\partial_\mu \phi_b$. When introducing collective coordinates for translations, one must also justify why the approximation $N_\delta = 1$ is valid in the interacting theory but not for the free theory. The argument is essentially the same as for the double-well potential and for the dilatation mode: in the interacting theory paths with $N_\delta > 1$ give an exponentially suppressed contribution to the action. Because $\partial_\mu \phi_b$ breaks Lorentz invariance, or rather $SO(4)$ invariance in Euclidean signature, the modes which contribute to the projections $\langle \partial_\mu \phi_b(x) | \phi(x^\mu + x_0^\mu) \rangle$ are not so easy to identify.

To prove that the analysis for the translation modes proceeds as with the dilatation mode, the easiest approach is to make a coordinate transformation to map any particular direction in spacetime to r . This can be done because the theory is scale invariant and the classical theory has an exact $SO(5)$ symmetry as part of a larger

conformal symmetry. To make the map explicit, we employ the conformal mapping with coordinates

$$\eta^\mu = \frac{2x^\mu R}{R^2 + r^2}, \quad \eta^5 = \frac{R^2 - r^2}{R^2 + r^2}. \quad (5.37)$$

These new coordinates satisfy $\sum_{\mu=1}^4 \eta_\mu^2 + \eta_5^2 = 1$, i.e. they parametrize the 4-sphere. The inverse of this conformal mapping is the stereographic projection to Cartesian coordinates.

On the 4-sphere, the integration volume is

$$d^4x = \left(\frac{2R}{R^2 + r^2} \right)^{-4} d\Omega_4. \quad (5.38)$$

If we also rescale the fields as

$$\Phi(\eta) = \frac{r^2 + R^2}{2R} \phi(x) = \frac{R}{1 + \eta_5} \phi(x), \quad (5.39)$$

the action becomes

$$\begin{aligned} & \int d^4x \left[-\frac{1}{2} \phi \square \phi + \frac{\lambda}{4} \phi^4 \right] \\ &= \int d\Omega_4 \left[\frac{1}{2} \Phi (-\bar{L}^2 + 2) \Phi + \frac{\lambda}{4} \Phi^4 \right], \end{aligned} \quad (5.40)$$

where $\bar{L}^2$ is the angular momentum operator. The equations of motion are now easily solved by a mode with $\bar{L}^2 \Phi = 0$, namely $\Phi_b = \sqrt{\frac{2}{-x}}$. Using Eq. (5.39) we find $\Phi_b = \frac{r^2 + R^2}{2R} \phi_b$, as expected.

The normal modes from expanding around Φ_b are spherical harmonics in 5 dimensions, which are identical to the conformally mapped fluctuations in Eq. (5.4). Indeed, it is through this conformal mapping that the exact eigenfunctions were first derived [7,40,41]. In these coordinates there is an exact $SO(5)$ symmetry among the 5 zero modes. We can therefore rotate any of the translation modes to the dilatation mode. Mapping back to flat space, the analysis of the path integral and N_δ proceeds identically to the discussion in Sec. VB.

VI. CONCLUSIONS

In this paper we have provided a resolution to a long-standing puzzle involving collective coordinates. Collective coordinates are a way to factor out an integral over the volume of a symmetry group from the path integral and still exploit a saddle point approximation in the nonsymmetry directions. The puzzle is how collective coordinates can be consistently introduced through a change of coordinates, independent of the action. Doing so appears to produce correct results in interacting theories where the symmetry volume is expected, but not in free theories, where it is

unexpected. The resolution to this puzzle is that the change of coordinates is allowed and valid, but it is multivalued: a given path $x(t)$ in quantum mechanics may be described by multiple values of the collective coordinates t_0 . We quantified this multivaluedness with the intersection number $N_X[x]$: it counts the number of times two paths $X(t)$ and $x(t)$ intersect as one is shifted by the collective coordinate over the symmetry volume. The correct way to introduce a collective coordinate is therefore with a resolution of the identity of the form

$$1 = \int dt_0 \frac{1}{N_X[x]} \delta[\langle X(t+t_0)|x(t) \rangle] |\langle X(t+t_0)|\dot{x}(t) \rangle|. \quad (6.1)$$

Without the factor of $N_X[x]$, the path integral can give the wrong answer.

To call out and rectify the mistreatment of collective coordinates we have explored examples in quantum mechanics and quantum field theory. For the splitting between the two lowest energy levels in the symmetric double well potential in quantum mechanics, the $N_X[x]$ factor counts the number of times a path crosses the potential barrier. Because each crossing gives additional suppression by e^{-S_I} where S_I is the instanton action, we argued that one can approximate $N_X[x] = 1$ up to exponentially small corrections. We made special effort to regulate the theory in finite time and carefully take the limit of a combination of the twisted and untwisted partition function to extract the energy splitting. While the final result is well-known through the WKB approximation, most path integral treatments involve unjustified approximations. We have attempted to provide a complete path integral treatment of the splitting where each step is under explicit control.

In the interacting theories we have explored here, namely the double well in quantum mechanics and interacting scalar quantum field theory, we found that the intersection number could be approximated as 1. In the free theories, the intersection number cannot be approximated as 1 and in fact it can be arbitrarily large, as needed to compensate for the volume factor which arises in collective coordinates.

It would be interesting to explore cases intermediate between these two extremes. An example is the instanton-anti-instanton pair in QCD as a function of separation. This configuration interpolates between the vacuum at zero separation and a saddle point of the action at infinite separation. The separation distance is a quasicollective coordinate [42–44], becoming a flat direction at infinite separation and a unexceptional normal mode fluctuation at zero separation. In the dilute instanton gas approximation, the dependence of the action on the separation distance is neglected, so that the separation is treated as a collective coordinate. Thus the techniques developed here could provide insights into the validity of the dilute instanton gas approximation. Although QCD is a significant step up in complexity from the examples considered here, it is of

direct phenomenological relevance and is certainly worth examining in more detail.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: METRIC APPROACH TO JACOBIANS IN THE PATH INTEGRAL

In this appendix we explain an approach for computing Jacobians arising from changes of coordinates in path integrals. To this end, we review how to use Riemannian geometry to manipulate the path integral measure. For concreteness we consider a quantum mechanical path integral for a particle in d spatial dimensions, although our arguments here readily generalize to field theory.

The path integral measure and its normalization depend on a metric on the space of paths. If K is a positive-definite operator, we can begin by defining an inner product

$$\langle x(t)|y(t) \rangle_K = \int dt x(t) \cdot K \cdot y(t). \quad (A1)$$

It is most common to then choose a basis of functions $x_n(t)$ that are orthonormal with respect to this inner product. If we write $x(t) = \sum_n c_n x_n(t)$ where $\langle x_m|x_n \rangle = \delta_{mn}$ then the path integral measure is

$$\mathcal{D}x = \prod_n dc_n. \quad (A2)$$

Factors of 2π or an overall normalization $\mathcal{N}$ can be optionally absorbed into the measure; these will drop out of physical quantities calculated by ratios of path integrals.

Now we consider a set of coordinates which are not orthonormal. If we have any set of coordinates $\{a_n\}$ which can parametrize a general path $x(t)$ then we can define a distance element between paths as

$$ds^2 = \sum_{m,n} G_{mn}(a) da_m da_n = \sum_{m,n} \left\langle \frac{\partial x}{\partial a_m} \middle| \frac{\partial x}{\partial a_n} \right\rangle da_m da_n. \quad (A3)$$

In this case the path integral measure becomes

$$\mathcal{D}x = \sqrt{\det G} \prod_n da_n, \quad (\text{A4})$$

which is the analog of the measure $\sqrt{\det g} d^d x$ in Riemannian geometry.

When we compute correlation functions, we divide by a partition function normalization computed using the same measure in Eq. (A4). We observe that correlation functions are independent of the choice of positive-definite K in Eq. (A1), since changing K to K' affects the square root determinant multiplicatively by $\sqrt{\frac{\det K'}{\det K}}$ which cancels out the same factor in the partition function normalization.

Let us show how these more abstract ideas manifest in two examples.

Example 1: First, suppose we consider paths $x: [0, T] \rightarrow \mathbb{R}$ with $x(0) = x_i$ and $x(T) = x_f$. The space of such paths can be parametrized by

$$x(t) = x_i + \frac{t}{T}(x_f - x_i) + \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{T}} \sin\left(\frac{\pi n}{T} t\right) \quad (\text{A5})$$

with coordinates $\{c_n\}$. Taking $K = 1$, corresponding to the ordinary “flat” L^2 inner product, the metric is

$$G_{mn} = \left\langle \frac{\partial x}{\partial c_m} \middle| \frac{\partial x}{\partial c_n} \right\rangle = \frac{2}{T} \int_0^T dt \sin\left(\frac{\pi m}{T} t\right) \sin\left(\frac{\pi n}{T} t\right) = \delta_{mn} \quad (\text{A6})$$

and so

$$ds^2 = \sum_n dc_n^2. \quad (\text{A7})$$

For this metric $\sqrt{\det G} = 1$ and the path integral measure is simply

$$\mathcal{D}x = \prod_n dc_n. \quad (\text{A8})$$

Example 2: Now consider coordinates for paths around some instanton $x_I(t)$, namely $x(t) = x_I(t) + \sum_{n=0}^{\infty} a_n x_n(t)$. If $\langle x_m(t) | x_n(t) \rangle = \delta_{mn}$, then, by the same arguments as in the first example above, the path integral measure can be expressed in coordinates $\{a_n\}$ as $\prod_{n=0}^{\infty} da_n$. If $x_0(t)$ is a time translation zero mode of the action of our theory, then it is instead convenient to work with an expansion $x(t) = x_I(t + t_0) + \sum_{n=1}^{\infty} \alpha_n x_n(t + t_0)$ in coordinates $\{t_0, \alpha_{n>0}\}$. In these coordinates, the metric is

$$ds^2 = \langle \dot{x}(t) | \dot{x}(t) \rangle dt_0^2 + 2 \sum_{n=1}^{\infty} \langle \dot{x}(t) | x_n(t + t_0) \rangle dt_0 d\alpha_n + \sum_{n=1}^{\infty} d\alpha_n^2 \quad (\text{A9})$$

and so

$$\sqrt{\det G} = \left(\langle \dot{x}(t) | \dot{x}(t) \rangle - \sum_{n=1}^{\infty} \langle \dot{x}(t) | x_n(t + t_0) \rangle \times \langle x_n(t + t_0) | \dot{x}(t) \rangle \right)^{1/2}. \quad (\text{A10})$$

Since $\sum_{n=0}^{\infty} \langle a(t) | x_n(t + t_0) \rangle \langle x_n(t + t_0) | b(t) \rangle = \langle a(t) | b(t) \rangle$ by completeness of $\{x_{n \geq 0}(t + t_0)\}$ for fixed t_0 , the above simplifies to

$$\sqrt{\det G} = |\langle x_0(t + t_0) | \dot{x}(t) \rangle| = \left| \langle x_0(t) | \dot{x}_I(t) + \sum_{n=1}^{\infty} \alpha_n \dot{x}_n(t) \rangle \right|. \quad (\text{A11})$$

Thus in the $\{t_0, \alpha_{n>0}\}$ coordinates the path integral measure is

$$\mathcal{D}x = dt_0 \prod_{m=1}^{\infty} d\alpha_m \left| \langle x_0(t) | \dot{x}_I(t) \rangle + \sum_{n=1}^{\infty} \alpha_n \langle x_0(t) | \dot{x}_n(t) \rangle \right| \quad (\text{A12})$$

which agrees with Eq. (2.9). Of course, we also have to be careful about the change in the domain of path integration coming from our change in coordinates. The modification of the domain and subtleties with multivalued coordinate changes are more readily understood using the methods in the main text.

APPENDIX B: MAPPING QFT TO THE DOUBLE WELL

In the quantum mechanics case, we saw that N_δ counted the number of times a particular path crossed $x = 0$. This led to a simple physical argument for why paths with $N_\delta > 1$ were exponentially suppressed in the interacting theory where the potential is high at $x = 0$, but unsuppressed in the free theory where $x = 0$ is not special in any sense. In the scalar quantum field theory we justified why $N_\delta = 1$ is appropriate for the interacting theory, but did not provide a physical interpretation of N_δ . We can do so by mapping the theory to quantum mechanics. This mapping provides an alternative way of thinking about the bounces in QFT.

To map QFT to QM we follow [44,45] and use a conformal mapping to turn the dilatation direction into time. Let us take $r = Re^{i\omega\tau}$ with ω an arbitrary frequency,

and consider spherically symmetric field configurations of the form

$$\phi(r) = e^{-\omega\tau} h(\tau). \quad (\text{B1})$$

Then the action of $\lambda\phi^4$ theory evaluated on such field configurations is

$$S[\phi] = \frac{2\pi^2 R^2}{\omega} \int d\tau \left[\frac{1}{2} (\partial_\tau h)^2 + \frac{1}{2} \omega^2 h^2 + R^2 \omega^2 \frac{\lambda}{4} h^4 \right]. \quad (\text{B2})$$

The equations of motion are then $h''(\tau) - V'(h) = 0$ where $V(h) = \frac{1}{2} \omega^2 h^2 + R^2 \omega^2 \frac{\lambda}{4} h^4$. Recalling that $\lambda < 0$, the above is an inverted double-well potential. The bounce is written as

$$h_b(\tau) = \sqrt{\frac{2}{-\lambda R \cosh(\omega\tau)}}. \quad (\text{B3})$$

The above satisfies the equation of motion of a particle in an inverted double well, starting at the metastable local maximum at $h = 0$. The particle rolls slowly off of this hump starting at $\tau = -\infty$, bounces off a wall, and ends up back on top of the hump at $\tau = \infty$. The flat direction is associated with the time at the center of the bounce. The normal modes which had satisfied $\square\phi_n = \lambda_n V''(\phi_b)\phi_n$ now satisfy

$$[\partial_\tau^2 - \omega^2]h_n = -\lambda_n \frac{6\omega^2}{\cosh^2(\omega\tau)} h_n. \quad (\text{B4})$$

However, there is a better basis we can use. Expanding around $h_b(\tau)$ the quadratic fluctuations satisfy

$$\left[\partial_\tau^2 - \omega^2 + \frac{6\omega^2}{\cosh^2(\omega\tau)} \right] \eta_n = \hat{\lambda}_n \eta_n, \quad (\text{B5})$$

which reproduces the Pöschl-Teller potential with $\lambda = 2$. There are thus two bound states and a continuum. The h_n and η_n bases are different, except for the dilatation zero mode

$$h_d(\tau) = \eta_d(\tau) = \frac{\sqrt{5}}{4\pi R} \frac{\sinh(\omega\tau)}{\cosh^2(\omega\tau)} \quad (\text{B6})$$

which has the same form in both bases [it has $\hat{\lambda}_1 = 0$ and $\lambda_1 = 1$ so that $\frac{\lambda_1 - 1}{\lambda_1} = 0$ as in Eq. (5.13)].

The zero condition $\langle \phi_d(x, R) | \phi(x) \rangle = 0$ becomes $\langle \eta_1(\tau - \tau_0) | h(\tau) \rangle = 0$ after the mapping to the double well, and can be studied using η modes quite similarly to the way we studied the double well in Sec. IV. We write

$$h(\tau) = b_0 \eta_0(\tau) + b_1 \eta_1(\tau) + \sum_{n \geq 2} b_n(\tau) \eta_n(\tau) \quad (\text{B7})$$

where η_n are the continuum modes and η_0, η_1 are the bound states. As with the double well, the bound states have support for $\tau \sim \omega^{-1}$ while the continuum modes must be regulated with $-T < \tau < T$, or equivalently $Re^{-\omega T} < r < Re^{\omega T}$. The projection that needs to be studied now involves the collective coordinate τ_0 and is

$$\langle \eta_1(\tau - \tau_0) | h(\tau) \rangle = 0. \quad (\text{B8})$$

In the regime $\tau_0 \ll \omega^{-1}$ we reduce to the case where we can trade τ_0 for b_1 . In the regime $\omega^{-1} \ll \tau_0 \ll T$ the two bound states have support only near $\tau = 0$ and in particular

$$\eta_1(\tau - \tau_0) \approx \delta'(\tau - \tau_0), \quad \omega^{-1} \ll \tau_0 \ll T, \quad (\text{B9})$$

so that the zero overlap condition becomes $h'(\tau_0) = 0$. Thus τ_0 is the time when $h(\tau)$ stops and $N_\delta[h]$ counts the number of times h stops.

Plugging (B1) into the action of the free theory, the potential is $V_0(h) = \frac{1}{2} \omega^2 h^2$ which is just a stable quadratic potential. Thus the h actions for the free and interacting theories are similar, and the action of a given path in the interacting theory is always strictly lower than the action in the free theory. Nevertheless, since the saddle point we expand around has large action in the interacting theory, perturbations around this saddle must be large to slow the particle down so that it stops multiple times. Such large quantum fluctuations are exponentially suppressed. In the free theory, the saddle point is just $h = 0$ and the particle can stop multiple times without much penalty. Hence in the full quantum field theory setting, $N_d[\phi] = 1$ is a good approximation for the interacting theory but not for the free theory.

- [1] P. H. Frampton, Vacuum instability and Higgs scalar mass, *Phys. Rev. Lett.* **37**, 1378 (1976).
- [2] S. R. Coleman, The fate of the false vacuum. 1. Semi-classical theory, *Phys. Rev. D* **15**, 2929 (1977).
- [3] C. G. Callan, Jr. and S. R. Coleman, The fate of the false vacuum. 2. First quantum corrections, *Phys. Rev. D* **16**, 1762 (1977).
- [4] G. Isidori, G. Ridolfi, and A. Strumia, On the metastability of the standard model vacuum, *Nucl. Phys.* **B609**, 387 (2001).
- [5] A. Andreassen, D. Farhi, W. Frost, and M. D. Schwartz, Direct approach to quantum tunneling, *Phys. Rev. Lett.* **117**, 231601 (2016).
- [6] A. Andreassen, D. Farhi, W. Frost, and M. D. Schwartz, Precision decay rate calculations in quantum field theory, *Phys. Rev. D* **95**, 085011 (2017).
- [7] A. Andreassen, W. Frost, and M. D. Schwartz, Scale invariant instantons and the complete lifetime of the standard model, *Phys. Rev. D* **97**, 056006 (2018).
- [8] J.-L. Gervais and B. Sakita, Extended particles in quantum field theories, *Phys. Rev. D* **11**, 2943 (1975).
- [9] C. W. Bernard, Gauge zero modes, instanton determinants, and QCD calculations, *Phys. Rev. D* **19**, 3013 (1979).
- [10] G. 't Hooft, Computation of the quantum effects due to a four-dimensional pseudoparticle, *Phys. Rev. D* **14**, 3432 (1976).
- [11] A. Voros, The return of the quartic oscillator. The complex WKB method, *Ann. l'I. H. P. Phys. Théor.* **39**, 211 (1983).
- [12] E. Delabaere and F. Pham, Unfolding the quartic oscillator, *Ann. Phys. (N.Y.)* **261**, 180 (1997).
- [13] G. Álvarez, Langer–Cherry derivation of the multi-instanton expansion for the symmetric double well, *J. Math. Phys. (N.Y.)* **45**, 3095 (2004).
- [14] G. V. Dunne and M. Unsal, Uniform WKB, multi-instantons, and resurgent trans-series, *Phys. Rev. D* **89**, 105009 (2014).
- [15] G. Rastelli, Semiclassical formula for quantum tunneling in asymmetric double-well potentials, *Phys. Rev. A* **86**, 012106 (2012).
- [16] A. Garg, Tunnel splittings for one-dimensional potential wells revisited, *Am. J. Phys.* **68**, 430 (2000).
- [17] V. N. Gribov, Quantization of non-Abelian gauge theories, *Nucl. Phys.* **B139**, 1 (1978).
- [18] S. Coleman, *Aspects of Symmetry: Selected Erice Lectures* (Cambridge University Press, Cambridge, England, 1985).
- [19] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 5th ed. (World Scientific, Singapore, 2009).
- [20] H. J. W. Müller-Kirsten, *Introduction to Quantum Mechanics: Schrödinger Equation and Path Integral* (World Scientific, Singapore, 2012).
- [21] M. Mariño, *Instantons and Large N: An Introduction to Non-Perturbative Methods in Quantum Field Theory* (Cambridge University Press, Cambridge, England, 2015), 9.
- [22] M. Mariño, *Advanced Topics in Quantum Mechanics* (Cambridge University Press, Cambridge, England, 2021).
- [23] E. Brezin, G. Parisi, and J. Zinn-Justin, Perturbation theory at large orders for potential with degenerate minima, *Phys. Rev. D* **16**, 408 (1977).
- [24] E. Gildener and A. Patrascioiu, Instanton contributions to the energy spectrum of a one-dimensional system, *Phys. Rev. D* **16**, 423 (1977).
- [25] E. B. Bogomolny, Calculation of instanton-anti-instanton contributions in quantum mechanics, *Phys. Lett.* **91B**, 431 (1980).
- [26] J. Zinn-Justin, Multi-instanton contributions in quantum mechanics, *Nucl. Phys.* **B192**, 125 (1981).
- [27] J. Zinn-Justin, Multi-instanton contributions in quantum mechanics. 2, *Nucl. Phys.* **B218**, 333 (1983).
- [28] U. D. Jentschura and J. Zinn-Justin, Higher order corrections to instantons, *J. Phys. A* **34**, L253 (2001).
- [29] J. Zinn-Justin and U. D. Jentschura, Multi-instantons and exact results I: Conjectures, WKB expansions, and instanton interactions, *Ann. Phys. (Amsterdam)* **313**, 197 (2004).
- [30] H. J. W. Müller-Kirsten, *Introduction to Quantum Mechanics: Schrödinger Equation and Path Integral* (World Scientific, Singapore, 2012).
- [31] K. Iwaki and T. Nakanishi, Exact WKB analysis and cluster algebras, *J. Phys. A* **47**, 474009 (2014).
- [32] E. Delabaere, H. Dillinger, and F. Pham, Exact semiclassical expansions for one-dimensional quantum oscillators, *J. Math. Phys. (N.Y.)* **38**, 6126 (1997).
- [33] B. Bucciotti, T. Reis, and M. Serone, An anharmonic alliance: Exact WKB meets EPT, *J. High Energy Phys.* **11** (2023) 124.
- [34] A. van Spaendonck and M. Vonk, Exact instanton transseries for quantum mechanics, *SciPost Phys.* **16**, 103 (2024).
- [35] G. Poschl and E. Teller, Bemerkungen zur Quantenmechanik des anharmonischen Oszillators, *Z. Phys.* **83**, 143 (1933).
- [36] I. M. Gel'fand and A. M. Yaglom, Integration in functional spaces and its applications in quantum physics, *J. Math. Phys. (N.Y.)* **1**, 48 (1960).
- [37] G. V. Dunne, Functional determinants in quantum field theory, *J. Phys. A* **41**, 304006 (2008).
- [38] S. Fubini, A new approach to conformal invariant field theories, *Nuovo Cimento A* **34**, 521 (1976).
- [39] L. N. Lipatov, Divergence of the perturbation theory series and the quasiclassical theory, *Sov. Phys. JETP* **45**, 216 (1977).
- [40] I. Drummond and G. Shore, Dimensional regularisation and instantons: A scalar field theory model, *Ann. Phys. (N.Y.)* **121**, 204 (1979).
- [41] A. J. McKane and D. J. Wallace, Instanton calculations using dimensional regularisation, *J. Phys. A* **11**, 2285 (1978).
- [42] I. I. Balitsky and A. V. Yung, Instanton molecular vacuum in $\mathcal{N} = 1$ supersymmetric quantum mechanics, *Nucl. Phys.* **B274**, 475 (1986).
- [43] I. I. Balitsky and A. V. Yung, Collective-coordinate method for quasizero modes, *Phys. Lett.* **168B**, 113 (1986).
- [44] A. V. Yung, Instanton vacuum in supersymmetric QCD, *Nucl. Phys.* **B297**, 47 (1988).
- [45] V. V. Khoze and A. Ringwald, Valley trajectories in gauge theories, Technical Report, CERN, Geneva, 1991.