

Three-coupled-channel analysis of $Z_c(3900)$ involving $D\bar{D}^*$, $\pi J/\psi$, and $\rho\eta_c$

Kang Yu,^{1,*} Guang-Juan Wang^{2,†}, Jia-Jun Wu^{1,3,‡}, and Zhi Yang^{4,§}

¹*School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*

²*KEK Theory Center, Institute of Particle and Nuclear Studies (IPNS),
High Energy Accelerator Research Organization (KEK), Tsukuba 305-0801, Japan*

³*Southern Center for Nuclear-Science Theory (SCNT), Institute of Modern Physics,
Chinese Academy of Sciences, Huizhou 516000, China*

⁴*School of Physics, University of Electronic Science and Technology of China, Chengdu 610054, China*



(Received 20 September 2024; accepted 25 November 2024; published 20 December 2024)

In this work, we conduct a three-coupled-channel analysis of the $Z_c(3900)$ structure, focusing on the $D\bar{D}^*$, $J/\psi\pi$, and $\rho\eta_c$ channels, based on the one-boson exchange model. Drawing from previous study on the exotic state T_{cc} , we only utilize one more parameter to construct the interactions between the channels. Our model successfully reproduces the experimental line shapes of the invariant mass distribution at $\sqrt{s} = 4.23$ and 4.26 GeV for the three channels. Additionally, the finite-volume energy levels in our model show agreement with current lattice QCD conclusion. Detailed analysis suggests that the $Z_c(3900)$ peaks in the $\pi J/\psi$ and $\rho\eta_c$ distributions primarily arise from the triangle loop involving the $D_1\bar{D}D^*$ intermediate system. In the $D\bar{D}^*$ distribution, the threshold peak is generated by the cascade decay mechanism enhanced by a triangle diagram. Moreover, we find a virtual pole located far from the threshold, indicating that $Z_c(3900)$ peaks are not associated with a physical pole. We conclude that the $Z_c(3900)$ peaks are predominantly caused by the threshold cusp.

DOI: [10.1103/PhysRevD.110.114029](https://doi.org/10.1103/PhysRevD.110.114029)

I. INTRODUCTION

The discovery of the $X(3872)$ near the $D\bar{D}^*$ threshold in 2003 marked a new era in hadron physics [1]. Since then, dozens of exotic hadrons, collectively referred to as the XYZ structures, have been discovered in the heavy quarkonium region. These states significantly challenge conventional quark models, as their masses or quantum numbers cannot be adequately described by traditional two-quark ($q\bar{q}$) or three-quark (qqq) configurations. The charged state $Z_c(3900)^\pm$ was almost simultaneously observed by BESIII [2] and Belle [3] in the $\pi^\pm J/\psi$ invariant mass spectrum from the process $e^+e^- \rightarrow \pi^+\pi^- J/\psi$ at $\sqrt{s} = 4.26$ GeV. CLEO-c further confirmed the existence of the charged structure in the same channel at a slightly lower e^+e^- collision energy $\sqrt{s} = 4.17$ GeV.

They also reported observation of the neutral partner, $Z_c(3900)^0$, in the process $e^+e^- \rightarrow \pi^0\pi^0 J/\psi$ [4].

Subsequently, BESIII studied the open-charm mode $e^+e^- \rightarrow \pi^\pm(D\bar{D}^*)^\mp$ at both $\sqrt{s} = 4.23$ and 4.26 GeV, initially using a single-tag technique [5] and later refining the analysis with a double-tag technique [6]. In both studies, they observed a near-threshold peak structure at approximately 3.85 GeV in the $(D\bar{D}^*)^\pm$ invariant mass distribution, identified as $Z_c(3885)^\pm$. The Breit-Wigner mass and width for $Z_c(3900)$ and $Z_c(3885)$, as measured by BESIII, are $(m, \Gamma) = (3894.5 \pm 6.6 \pm 4.5, 63 \pm 24 \pm 26)$ MeV and $(m, \Gamma) = (3882.2 \pm 1.1 \pm 1.5, 26.5 \pm 1.7 \pm 2.1)$ MeV, respectively. Due to their similar widths, $Z_c(3885)^\pm$ and $Z_c(3900)^\pm$ are generally considered to be the same entity, despite the slight difference in their masses.¹ A partial-wave analysis suggests that the spin-parity quantum number J^P is likely 1^+ [7]. Recently, a clear signal with a statistical significance exceeding 4σ was observed in the process $e^+e^- \rightarrow \pi^+\pi^0\pi^-\eta_c$ at $\sqrt{s} = 4.23$ GeV. This was achieved by constraining the invariant masses of the $\pi^\pm\pi^0$ and η_c candidates to fall within the $\rho^\pm$ -signal and η_c -signal regions, respectively [8]. So far, the $Z_c(3900)$ has only been observed in e^+e^- production. No clear evidence of

*Contact author: yukang21@mails.ucas.ac.cn

†Contact author: wgj@post.kek.jp

‡Contact author: wujiajun@ucas.ac.cn

§Contact author: zhiyang@uestc.edu.cn

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

¹For simplicity, we will refer to them collectively as Z_c unless a distinction is necessary.

$Z_c(3900)$ has been found in B -decay processes, such as $\bar{B}^0 \rightarrow (J/\psi\pi^+)K^-$ in Belle [9], $B^0 \rightarrow (J/\psi\pi^+)\pi^-$ and $B^0 \rightarrow (\psi(nS)\pi^+)K^-$ in LHCb [10,11].

Since the discovery of Z_c , extensive theoretical research has been conducted to elucidate its nature. The proposed explanations for the Z_c structure include compact tetraquark states [12–16], $D\bar{D}^*$ resonances or virtual molecular states [17–29], cusp effects [30], and nonresonant mechanisms [31,32]. Despite this extensive research, the exact nature of Z_c is still uncertain. In experiments, Z_c is suggested to be a typical resonance based on Breit-Wigner parametrizations. However, some theoretical studies suggest that the observed peak might not necessarily indicate a true resonance. The peak could instead be a virtual state or even a cusp effect, rather than a resonance, depending on the parametrization strategy used.

To reach a solid conclusion, it is essential to impose further constraints on the model and parameters, either through experimental data or underlying symmetries. Therefore, it is essential for a comprehensive model to be capable of explaining as much experimental data as possible. However, capturing the dynamics of the three coupled channels, $\pi J/\psi$, $D\bar{D}^*$, and $\rho\eta_c$, which predominantly govern the nature of $Z_c(3900)$, remains a challenge. There is a pressing need for a model that can consistently describe the $Z_c(3900)$ peaks from different final channels.

The nature of the Z_c structure has also been investigated by several lattice QCD (LQCD) collaborations. The first LQCD study of Z_c was conducted in Ref. [33] using $\pi J/\psi$ and $D\bar{D}^*$ meson-meson interpolators at $m_\pi \approx 266$ MeV. A subsequent study [34] included additional meson-meson and diquark-antidiquark interpolators were included in the simulation. In Ref. [35], MILC Lattice Computation (MILC) claims that they fail to observe an candidate for $Z_c(3900)$ even though they find an candidate for $X(3872)$ on highly improved staggered quark (HISQ) lattices. In Refs. [36,37], the Hadron Spectrum collaboration found that compact tetraquark operators have minimal impact on finite volume spectra at $m_\pi \approx 391$ MeV and no sign for $Z_c(3900)$ was seen. Similarly, the Chinese LQCD collaboration conducted simulations using meson-meson interpolators at three different m_π values [38–40]. However, none of these studies identified a clear candidate for the Z_c state among the energy levels extracted from their lattice configurations. In Refs. [41,42], the HALQCD collaboration extracted the $\pi J/\psi - D\bar{D}^* - \rho\eta_c$ coupled-channel effective potential using their formalism and successfully reproduced the experimental line shape. They concluded that the Z_c state is better understood as a threshold cusp rather than a conventional resonance [41,42]. Besides, it was emphasized that the off-diagonal channel-channel interactions play an important role on Z_c therefore the couple-channel analysis is necessary.

As previously discussed, a significant limitation of prior studies is the absence of a model that integrates the

$D\bar{D}^*(+\bar{D}D^*)$,² $J/\psi\pi$, and $\eta_c\rho$ three coupled channels to construct a comprehensive picture of Z_c . In this work, we employ the Hamiltonian effective field theory (HEFT) to investigate the Z_c state from both phenomenological and lattice perspectives, focusing on the $D\bar{D}^* - \pi J/\psi - \rho\eta_c$ three coupled channels, where Z_c signals have been observed. The one-boson-exchange (OBE) interaction between $D\bar{D}^*$ channels, which respects heavy quark spin symmetry, is utilized here. Building on previous research on the double-charmed exotic state T_{cc}^+ [43], the interaction between $D\bar{D}^*$ and $D\bar{D}^*$ is determined, leaving only one free coupling parameter for the vertex $J/\psi - D\bar{D}^*$ and $\eta_c - D\bar{D}^*$ to construct the full channel-channel interaction. This parameter is calibrated by simultaneously fitting experimental data across the three channels. The vertices like $Y \rightarrow \pi D\bar{D}^*$, $Y \rightarrow D_1\bar{D}$ are also more constrained in the couple-channel analysis compared to the single channel one.

By constraining our model and parameter in this way, we are able to provide a potential explanation for Z_c that is also consistent with previous analyses of T_{cc} . With all parameters determined, we then calculate the finite volume energy levels and compare them with previous LQCD results. It will be shown later that our model provides an interpretation of Z_c that is consistent with both experimental data and lattice results.

This paper is organized as follows. In Sec. II, we outline the formalism used for the fitting process. In Sec. III we present the fitting results, along with a discussion of the observed peak structures. In Sec. IV, we analytically continue the scattering matrix and search for its poles in the complex plane. In Sec. V, we provide a brief introduction to the HEFT and apply it in order to calculate the finite volume energy levels, which are then compared to LQCD results. Finally, in Sec. VI, we make a summary and discuss future prospects.

II. FORMALISM

Since the Z_c structure was discovered in e^+e^- collisions at $\sqrt{s} = 4.23$ and 4.26 GeV, where the exotic state $Y(4230/4260)$ resides, we investigate the invariant mass distribution through the processes $Y(4230/4260) \rightarrow \pi(J/\psi, \rho\eta_c, D\bar{D}^*)$. For convenience, we denote the $[D\bar{D}^*]^\pm$ states, which have definite positive G parity, as follows:

$$[D\bar{D}^*]^- = \frac{1}{\sqrt{2}}(D^0\bar{D}^{*0} + D^-\bar{D}^{*0}), \quad (1)$$

$$[D\bar{D}^*]^+ = \frac{1}{\sqrt{2}}(D^+\bar{D}^{*0} + D^0\bar{D}^{*+}), \quad (2)$$

with the conventions $D \xrightarrow{C} \bar{D}$ and $D^* \xrightarrow{C} -\bar{D}^*$.

²Hereafter simplified as $D\bar{D}^*$.

The schematic Feynman diagrams for the decay of $Y(4230/4260)$ are illustrated in Figs. 1–3 for overview. The black solid box represents the T matrix, which

characterizes the final state interactions (FSIs) and will be discussed in detail later. The diagrams containing bubble loops correspond to the initial-single-pion-emission

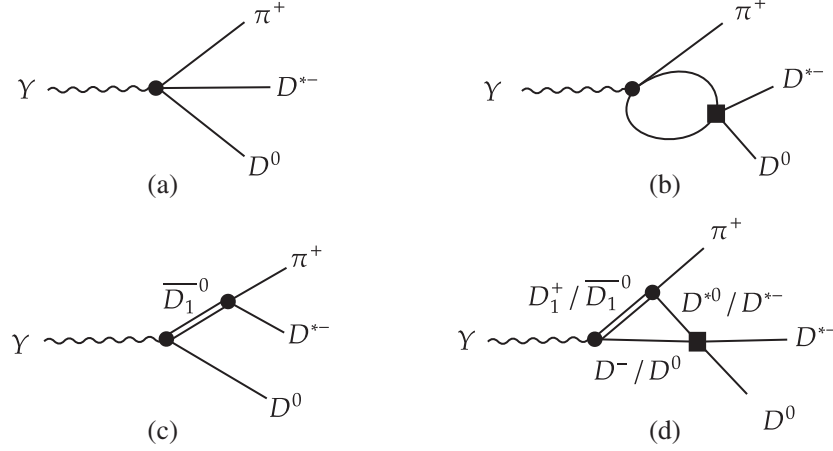


FIG. 1. The schematic Feynman diagrams for the process $Y(4230/4260) \rightarrow \pi^+ D^0 D^{*-}$. The solid circles and boxes denote the interaction vertex and T matrix characterizing the final state interactions. (a), (c) denote the point-like and cascade decay mechanism respectively and (b), (d) include the FSI. The intermediate channels for bubble loops in (b) can include $\pi^- J/\psi$ and $[D\bar{D}^*]^-$.

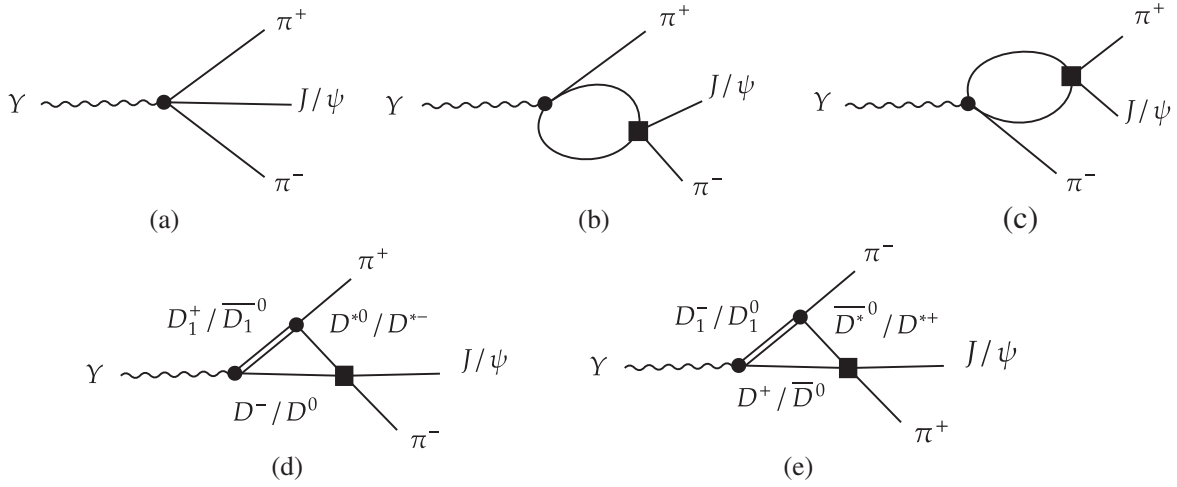


FIG. 2. The schematic Feynman diagrams for the process $Y(4230/4260) \rightarrow \pi^+ \pi^- J/\psi$. The notations are same as in Fig. 1. (a) denotes the point-like mechanism and (b), (c) include the FSI. (d), (e) denote the triangle mechanism. The intermediate channels for bubble loops in (b) and (c) can include $\pi^- J/\psi$ and $[D\bar{D}^*]^-$. π^+ and π^- are treated equivalently due to the isospin symmetry.

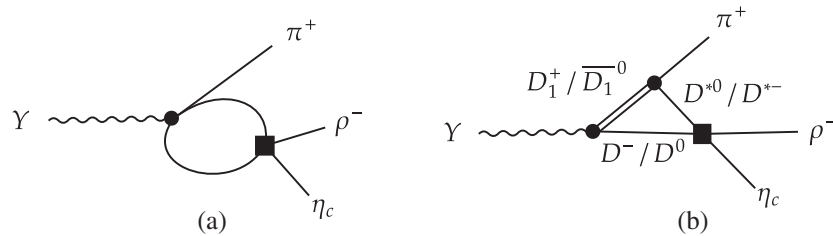


FIG. 3. The schematic Feynman diagrams for the process $Y(4230/4260) \rightarrow \pi^+ \rho^- \eta_c$. The notations are same as in Fig. 2. The intermediate channels for bubble loops in (a) can include $\pi^- J/\psi$ and $[D\bar{D}^*]^-$.

mechanism [44–46]. We have considered the hypothesis that $Y(4230/4260)$ is strongly coupled to a $\bar{D}_1(2420)D$, as suggested in Refs. [17,47].³

A. The Lagrangians

In this subsection, we present the relevant Lagrangians and interaction vertices. The Lagrangians associated with Y field are given as follows [25,31]:

$$\mathcal{L}_{YD_1D} = -g_{YD_1D} Y_i D_1^i \bar{D}, \quad (3)$$

$$\mathcal{L}_{Y\pi J/\psi}^S = c_{\pi J/\psi}^S \pi^+ \pi^- Y_i \psi^i, \quad (4)$$

$$\begin{aligned} \mathcal{L}_{Y\pi\pi J/\psi}^D &= c_{\pi\pi J/\psi}^D Y_i \psi_j \left[\pi^+ \left(\partial^i \partial^j - \frac{1}{3} \vec{\partial}^2 \delta^{ij} \right) \pi^- \right] \\ &+ (\pi^+ \leftrightarrow \pi^-), \end{aligned} \quad (5)$$

$$\mathcal{L}_{Y\pi D\bar{D}^*}^S = c_{D\bar{D}^*}^S Y_i \psi_i (D^0 D^{*-} + D^- D^{*0}) \pi^+, \quad (6)$$

where c_α^L is the coupling constant for the interaction between Y and $\pi + \alpha$ ($\alpha = \pi^- J/\psi, \rho^- \eta_c, D^0 D^-$) with L -wave between π and the α channel denoted by $\pi_L(\alpha)$. Note that for $Y \rightarrow \pi D\bar{D}^*$, we only consider the S -wave $\pi_S(D\bar{D}^*)$ interaction, as the typical momentum of the final states is not sufficiently large to produce significant D -wave contributions. In contrast, for the decay $Y \rightarrow \pi\pi J/\psi$, a D -wave term $\pi_D(\pi J/\psi)$ is included due to the significantly larger phase space. Here, we neglect the Lagrangian for $Y \rightarrow \pi\rho\eta_c$ because experimental data suggest that this vertex is highly suppressed [8]. Notably, we will later demonstrate that the $Y \rightarrow \pi\pi J/\psi$ vertex is also suppressed, which is consistent with the expectations from the heavy quark symmetry.

The Lagrangian for $D_1 D^* \pi$ vertex reads [25,48]

$$\mathcal{L}_{D_1 D^* \pi}^S = i \frac{2h_S}{\sqrt{3}f_\pi} (D_1^{+i} D_0^{*i\dagger} \partial^0 \pi^-), \quad (7)$$

$$\mathcal{L}_{D_1 D^* \pi}^D = -\frac{2h_D}{\sqrt{3}f_\pi} (D_1^{+i} D_0^{*i\dagger} \vec{\partial}^2 \pi^- - 3D_1^{+i} D_0^{*j\dagger} \partial^i \partial^j \pi^-), \quad (8)$$

with $f_\pi = 132$ MeV, $|h_S| = 0.57$ and $|h_D| = 1.17$ GeV⁻¹ determined by reproducing the decay widths of D_2 and $D_1(2420)$. In fact, the contribution from the D -wave component would shift the peak of the line shape closer to the threshold, as the energy of the pion increases when the invariant mass of the $D^0 D^{*-}$ system approaches the threshold. This is one of the reasons why we have neglected the contribution from the process $e^+ e^- \rightarrow D_0^*(2300) \bar{D}^* \rightarrow (D\pi) \bar{D}^*$ despite the threshold of $D_0^*(2300) \bar{D}^*$ being near

4.23 GeV, since only the S wave is allowed for $D_0^*(2300) \rightarrow D\pi$, and the peak of its line shape is at around 3.95 GeV, which is about 100 MeV away from the experimental peak.

Finally, we present the Lagrangian for the final state interaction between the three coupled channels. For the OBE interaction between $D\bar{D}^*$ and $D^* \bar{D}$, the relevant Lagrangians $\mathcal{L}_{D^{(*)}D^{(*)}M}$ with $M = \pi, \rho, \omega$ are given in Refs. [49,50]. For the values of involved parameters, we follow a recent work [43], where the line shapes of the double-charmed exotic state T_{cc}^+ are successfully reproduced. The interactions between the hidden-charmed channels, i.e., $\pi J/\psi - \pi J/\psi, \rho\eta_c - \rho\eta_c, \pi J/\psi - \rho\eta_c$, are negligible due to the OZI suppression [51,52], which is also confirmed by the HALQCD calculation [41].

For the $D\bar{D}^*$ coupled with the hidden-charmed channels, we adopt the effective Lagrangian respecting the heavy quark spin and SU(3) flavor symmetry that reads [53,54],

$$\mathcal{L}_{HHR} = g' \langle R^{(c\bar{c})} (\bar{H}_a^{(\bar{c})} \overleftrightarrow{\partial} \bar{H}_a^{(c)}) \rangle + \text{H.c.}, \quad (9)$$

with the superfields

$$\bar{H}_a^{(c)} = (P_a^{*\mu\dagger} \gamma_\mu + P_a^\dagger \gamma_5) \frac{1 + \not{v}}{2}, \quad (10)$$

$$\bar{H}_a^{(\bar{c})} = \frac{1 - \not{v}}{2} (\tilde{P}_{a\mu}^\dagger \gamma^\mu + \tilde{P}_a^\dagger \gamma_5), \quad (11)$$

$$R^{c\bar{c}} = \frac{1 + \not{v}}{2} (\psi^\mu \gamma_\mu - \eta_c \gamma_5) \frac{1 - \not{v}}{2}, \quad (12)$$

where $P = (D^0, D^+, D_s^+)$, $P^* = (D^{*0}, D^{*+}, D_s^{*+})$, $\tilde{P} = (\bar{D}^0, D^-, D_s^-)$, $\tilde{P}^* = (\bar{D}^{*0}, D^{*-}, D_s^{*-})$ are SU(3) triplets of the fields that annihilates the corresponding meson. ψ^μ and η_c are the singlets of field that annihilates corresponding charmonium. v is the typical four velocity of the heavy quark. In the heavy quark limit, $v^\mu = (1, \vec{0})$. $\langle \cdots \rangle$ denotes the trace in spinor space and $\overleftrightarrow{\partial} \equiv \vec{\partial} - \vec{\partial}$. After expansion, Eq. (9) can be divided into

$$\mathcal{L}_{\psi D D^*} = 2ig' \epsilon^{\mu\nu\alpha\beta} v_\alpha \psi_\beta (\tilde{P}_\nu^{*\dagger} \overleftrightarrow{\partial}_\mu P^\dagger + P_\nu^{*\dagger} \overleftrightarrow{\partial}_\mu \tilde{P}^\dagger), \quad (13)$$

$$\mathcal{L}_{\psi DD} = 2g' \psi^\mu P^\dagger \overleftrightarrow{\partial}_\mu \tilde{P}^\dagger, \quad (14)$$

$$\mathcal{L}_{\psi D^* D^*} = 2g' (\psi^\nu \tilde{P}_\nu^{*\dagger} \overleftrightarrow{\partial}^\mu P_\mu^{*\dagger} - \psi^\nu P_\nu^{*\dagger} \overleftrightarrow{\partial}^\mu \tilde{P}_\mu^{*\dagger} + \psi_\mu P^{*\nu\dagger} \overleftrightarrow{\partial}^\mu \tilde{P}_\nu^{*\dagger), \quad (15)$$

$$\mathcal{L}_{\eta_c D^* D^*} = -2ig' \epsilon^{\mu\nu\alpha\beta} v_\alpha \eta_c (P_\beta^{*\dagger} \overleftrightarrow{\partial}_\mu \tilde{P}_\nu^{*\dagger}), \quad (16)$$

$$\mathcal{L}_{\eta_c DD^*} = 2g' \eta_c (P_\mu^{*\dagger} \overleftrightarrow{\partial}^\mu \tilde{P}^\dagger + P^\dagger \overleftrightarrow{\partial}^\mu \tilde{P}_\mu^{*\dagger}). \quad (17)$$

The Hermitian conjugate terms are omitted for simplicity.

³For simplicity, we will refer to $D_1(2420)$ and $Y(4230/4260)$ as D_1 and Y , respectively, unless otherwise specified.

B. The amplitudes

In this section we present the squared and initial spin-averaged amplitude. We denote $s_\alpha \equiv (p_{\alpha_1} + p_{\alpha_2})^2$ as the invariant mass of the α channel from now on. First we discuss the T matrix describing the FSI. For the present work, all three couple channels are of pseudoscalar-vector type so the T matrix is given by the following couple-channels Lippmann-Schwinger equation (LSE),

$$\begin{aligned} T_{\alpha\beta}(\mathbf{p}, \sigma_{\alpha_1}, \mathbf{k}, \sigma_{\beta_1}; E) &= V_{\alpha\beta}(\mathbf{p}, \sigma_{\alpha_1}, \mathbf{k}, \sigma_{\beta_1}) \\ &+ \sum_{\sigma_{\gamma_1}, \gamma} \int d^3q V_{\alpha\gamma}(\mathbf{p}, \sigma_{\alpha_1}, \mathbf{q}, \sigma_{\gamma_1}) \\ &\times G_\gamma(\mathbf{q}; E) T_{\gamma\beta}(\mathbf{q}, \sigma_{\gamma_1}, \mathbf{k}, \sigma_{\beta_1}; E), \end{aligned} \quad (18)$$

where σ_{α_1} is the polarization indices of vector meson in the α channel and $G_\alpha(\mathbf{q}, E)$ is the propagator for the intermediate α channel given by⁴

$$\begin{aligned} G_\alpha(\mathbf{q}, E) &= \frac{1}{(2\pi)^3 4\omega_{\alpha_1}(|\mathbf{q}|)\omega_{\alpha_2}(|\mathbf{q}|)} \\ &\times \frac{1}{(E - \omega_{\alpha_1}(|\mathbf{q}|) - \omega_{\alpha_2}(|\mathbf{q}|) + i0^+)}, \end{aligned} \quad (19)$$

where $\omega_{\alpha_i}(q) = \sqrt{q^2 + m_{\alpha_i}^2}$. Practically, LSE is solved in the partial-wave representation where the integral equation is reduced to a one-dimensional form,

$$\begin{aligned} T_{l_\beta, l_\alpha}^J(p, k; E) &= V_{l_\beta, l_\alpha}^J(p, k) + \sum_{l_\gamma} \int q^2 dq V_{l_\beta, l_\gamma}^J(p, q) \\ &\times G_\gamma(q; E) T_{l_\gamma, l_\alpha}^J(q, k; E), \end{aligned} \quad (20)$$

with

$$\begin{aligned} V_{l', l}^J(p, k) &= \sum_{\sigma\sigma'} \int d\Omega_p \sqrt{\frac{4\pi(2l+1)}{2J+1}} C_{l'1}^J(J\sigma; \sigma - \sigma', \sigma') \\ &\times C_{l1}(J\sigma, 0\sigma) Y_{l', \sigma - \sigma'}(\hat{\mathbf{p}}) V(\mathbf{p}, \sigma', k\hat{\mathbf{e}}_z, \sigma), \end{aligned}$$

where $C_{j_1 j_2}(JM; m_1 m_2)$ and Y_{lm} are the Clebsch-Gordon coefficients and spherical harmonics, respectively. Based on the partial wave analysis by BESIII [7], we only focus on $J = 1$ and $l = l' = 0$ in the present paper. The numerical method to solve the partial-wave LSE is given in Appendix A. The kernel of LSE, V , is related to the tree-level Lorentz-invariant amplitude $i\mathcal{M}$ by

⁴The width of ρ meson are neglected here. Since the potential $V_{D\bar{D}^* - \rho\eta_c}$ vanishes at the $\rho\eta_c$ threshold, it will not produce a cusp at the $\rho\eta_c$ threshold which is not observed. The finite width of ρ hardly affect the result of this work.

$$V = -\mathcal{M}. \quad (21)$$

For specific expressions we refer to Appendix C.

It is worth mentioning several subtle issues here. First, we note that all the heavy-meson and charmonium fields in Eqs. (7)–(17) are not normalized as in the relativistic field theory, where the particles are normalized as $\langle \vec{p}' | \vec{p} \rangle = (2\pi)^3 2E_{\vec{p}} \delta^3(\vec{p}' - \vec{p})$. They differ from a factor $\sqrt{m}$ at the leading order. Therefore, to obtain $i\mathcal{M}$, extra factors $\sqrt{m}$ with m the mass of heavy-meson or charmonium should be multiplied when evaluating the Feynman diagrams with the Lagrangian above. Second, since a factor $m_c v^\mu$ has been already subtracted when effective Lagrangians in Eqs. (7)–(17) are constructed, the differential operator acting on the heavy-meson field gives the residual momentum $k^\mu := p^\mu - m_c v^\mu$ where p is the four momentum. Third, in the LSE or rescattering equation in the time-ordered-perturbation-theory formalism, only the conservaeach vertex. Therefore, there are at least two schemes to evaluate the transferred momentum q in $\mathcal{M}$ that make differences unless the energy-conservation conditions are imposed:

Scheme 1 [55]:

$$V_{\beta\alpha} = -\mathcal{M} \left(q_0 = \frac{1}{2}(\omega_{\alpha_1} - \omega_{\beta_1} + \omega_{\beta_2} - \omega_{\alpha_2}) \right). \quad (22)$$

Scheme 2 [56]:

$$\begin{aligned} V_{\beta\alpha} &= -\frac{1}{2}(\mathcal{M}(q_0 = \omega_{\alpha_1} - \omega_{\beta_1}) \\ &+ \mathcal{M}(q_0 = \omega_{\beta_2} - \omega_{\alpha_2})). \end{aligned} \quad (23)$$

For scheme 1, the potential would degenerate to a widely used form $V \propto (q^2 + m_{\text{eff}}^2 + i\epsilon)^{-1}$ if $\omega_{\alpha(\beta)_i} \approx m_{\alpha(\beta)_i}$ is further applied. Both schemes are used later and the scheme dependence of the result is investigated. Lastly, to eliminate the ultraviolet divergence of the loop integral in the LSE that arising from the short distance physics, a nonlocal dipole form factor is introduced:

$$V_{\beta\alpha}(\mathbf{p}, \mathbf{k}) \rightarrow V_{\beta\alpha}(\mathbf{p}, \mathbf{k}) \left(\frac{\Lambda_\alpha^2}{\Lambda_\alpha^2 + \mathbf{k}^2} \right)^2 \left(\frac{\Lambda_\beta^2}{\Lambda_\beta^2 + \mathbf{p}^2} \right)^2. \quad (24)$$

The $\Lambda_{[D\bar{D}^*]}$ is fixed at 1 GeV following Ref. [43], and it has been shown that its variation within a reasonable range can be effectively absorbed into coupling constant. Given that $\frac{m_{J/\psi}}{m_D} \approx \frac{m_{\eta_c}}{m_D} \approx 1.5$, the cutoff $\Lambda_{\pi J/\psi} = \Lambda_{\rho\eta_c}$ is chosen to be fixed at 1.5 GeV. Similarly, as will be shown later, within a reasonable range of $\Lambda_{\rho\eta_c, \pi J/\psi}$, its influence can be effectively absorbed into the coupling constant g' .

Using the T matrix, we can now explicitly express $|\mathcal{M}_\alpha|^2 = |\mathcal{M}_\alpha^S|^2 + |\mathcal{M}_\alpha^D|^2$ as follows:

$$|\mathcal{M}_{D^0 D^{*-}}^S|^2 = \left| c_{D\bar{D}^*}^S + \sum_{\alpha=\pi J/\psi, [D\bar{D}^*]} c_\alpha^S \mathcal{I}_{\alpha \rightarrow [D\bar{D}^*]} + g_{YD_1 D} h'_S \omega_\pi \left(\frac{1}{s_{D^* \pi} - m_{D_1}^2 + im_{D_1} \Gamma_{D_1}} + \mathcal{Q}_{D\bar{D}^*} \right) \right|^2, \quad (25)$$

$$|\mathcal{M}_{D^0 D^{*-}}^D|^2 = \frac{2}{9} p_\pi^4 \left| c_{\pi J/\psi}^D \mathcal{I}_{\pi J/\psi \rightarrow [D\bar{D}^*]} + g_{YD_1 D} h'_D \left(\frac{1}{s_{D^* \pi} - m_{D_1}^2 + im_{D_1} \Gamma_{D_1}} + \mathcal{Q}_{D\bar{D}^*} \right) \right|^2, \quad (26)$$

$$|\mathcal{M}_{\pi^- J/\psi}^S|^2 = \left| c_{\pi J/\psi}^S + \sum_{\alpha=\pi J/\psi, [D\bar{D}^*]} x_\alpha c_\alpha^S (\mathcal{I}_{\alpha \rightarrow \pi^+ J/\psi} + \mathcal{I}_{\alpha \rightarrow \pi^- J/\psi}) + \mathcal{F}^+ + \mathcal{F}^- \right|^2, \quad (27)$$

$$|\mathcal{M}_{\pi^- J/\psi}^D|^2 = \frac{2}{9} p_{\pi^+}^4 |\mathcal{A}^+|^2 + \frac{2}{9} p_{\pi^-}^4 |\mathcal{A}^-|^2 + \frac{2}{9} p_{\pi^+}^2 p_{\pi^-}^2 (3 \cos^2 \theta_{\pi^+ \pi^-} - 1) \text{Re}(\mathcal{A}^{+*} \mathcal{A}^-), \quad (28)$$

$$|\mathcal{M}_{\rho \eta_c}^S|^2 = \left| \sum_{\alpha=\pi J/\psi, [D\bar{D}^*]} x_\alpha c_\alpha^S \mathcal{I}_{\alpha \rightarrow \rho \eta_c} + \sqrt{2} g_{YD_1 D} h'_S \mathcal{Q}_{\rho \eta_c} \right|^2, \quad (29)$$

$$\mathcal{I}_{\alpha \rightarrow \beta} = \int \frac{q^2 dq}{(2\pi)^3 4\omega_{\alpha_1}(q) \omega_{\alpha_2}(q) \sqrt{s_\beta} - \omega_{\alpha_1}(q) - \omega_{\alpha_2}(q) + i0^+} \frac{T_{0,0;\alpha \rightarrow \beta}^{J=1}(\bar{p}_\beta, q; \sqrt{s_\beta})}{\sqrt{s_\beta} - \omega_{\alpha_1}(q) - \omega_{\alpha_2}(q) + i0^+}, \quad (36)$$

$$|\mathcal{M}_{\rho \eta_c}^D|^2 = \frac{2}{9} p_\pi^4 \left| c_{\pi J/\psi}^D \mathcal{I}_{\alpha \rightarrow \rho \eta_c} + \sqrt{2} g_{YD_1 D} h'_D \mathcal{Q}_{\rho \eta_c} \right|^2, \quad (30)$$

where

$$\mathcal{A}^\pm = c_{\pi J/\psi}^D + c_{\pi J/\psi}^D \mathcal{I}_{\pi^\mp J/\psi \rightarrow \pi^\mp J/\psi} + \sqrt{2} g_{YD_1 D} h'_D \mathcal{Q}_{\pi^\mp J/\psi}, \quad (31)$$

$$\mathcal{F}^\pm = \sqrt{2} g_{YD_1 D} h'_S \omega_\pi(p_{\pi^\pm}) \mathcal{Q}_{\pi^\mp J/\psi}, \quad (32)$$

and $h'_S = \frac{\sqrt{4m_{D_1} m_{D^*}}}{\sqrt{3}f_\pi} h_S$, $h'_D = -\frac{2\sqrt{3m_{D_1} m_{D^*}}}{f_\pi} h_D$, $x_{\pi J/\psi}([D\bar{D}^*]) = 1(\sqrt{2})$. Additionally, $\omega_\pi = \sqrt{p_\pi^2 + m_\pi^2}$ and $p_\pi = \lambda^{\frac{1}{2}}(\sqrt{s}, \sqrt{s_\alpha}, m_\pi)$ with $\lambda(a, b, c)$ being the modified Källén triangle function,

$$\lambda(a, b, c) \equiv \frac{(a^2 - (b+c)^2)(a^2 - (b-c)^2)}{4a^2}. \quad (33)$$

The relative angle $\cos \theta_{\pi^+ \pi^-}$ is given by

$$\cos \theta_{\pi^+ \pi^-} = \frac{s_{\pi^+ \pi^-} - 2m_\pi^2 - 2\omega_\pi \omega_{\pi^+}}{2p_{\pi^+} p_{\pi^-}}, \quad (34)$$

$$s_{\pi^+ \pi^-} = s + 2m_\pi^2 + m_{J/\psi}^2 - s_{\pi^+ J/\psi} - s_{\pi^- J/\psi}. \quad (35)$$

$\mathcal{I}_\alpha$ and $\mathcal{Q}_\alpha$ come from the FSI in the bubble diagram and triangle diagram, respectively. The explicit expressions are given by

and

$$\mathcal{Q}_\alpha = \int \frac{d^3 \mathbf{q}}{(2\pi)^3 4\pi 8\omega_{D_1}(|\mathbf{p}_\pi + \mathbf{q}|) \omega_{D^*}(q) \omega_D(|\mathbf{p}_\pi + \mathbf{q}|)} \frac{T_{0,0;\alpha \rightarrow \alpha}^{J=1}(q^*, \bar{p}_\alpha; \sqrt{s_\alpha})}{1} \times \frac{1}{(\sqrt{s} - \omega_{D_1}(|\mathbf{p}_\pi + \mathbf{q}|) - \omega_D(|\mathbf{p}_\pi + \mathbf{q}|) + i\frac{\Gamma_{D_1}}{2})} \times \frac{1}{(\sqrt{s} - \omega_\pi(p_\pi) - \omega_{D^*}(q) - \omega_D(|\mathbf{p}_\pi + \mathbf{q}|) + i0^+)}, \quad (37)$$

where $\bar{p}_\beta \equiv \lambda^{\frac{1}{2}}(\sqrt{s_\beta}, m_{\beta_1}, m_{\beta_2})$ is the on-shell momentum in the rest frame of β channel. The integration variable in Eq. (37) is the momentum of D^* in the rest frame of Y while q^* is that in the rest frame of $D\bar{D}^*$ and hence given by $q^* = \sqrt{(\gamma \sqrt{q^2 + m_{D^*}^2} + \gamma \beta q \cos \theta)^2 - m_{D^*}^2}$ with $\gamma = \sqrt{\frac{p_\pi^2 + s_\alpha}{s_\alpha}}$ and $\gamma\beta = -\frac{p_\pi}{\sqrt{s_\alpha}}$. Note that the finite width of $\Gamma_{D_1(2420)} = 31.3$ MeV has been taken into account. The numerical method for the calculation of $\mathcal{Q}_\alpha$ is discussed in Appendix B.

As the final result, the invariant mass distribution of $Y \rightarrow \pi\alpha$ reads

$$\frac{d\Gamma_{Y \rightarrow \pi^+ \alpha}}{d\sqrt{s_\alpha}} = \mathcal{N}_\alpha \int_{t_\alpha^-(s_\alpha)}^{t_\alpha^+(s_\alpha)} \frac{|\mathcal{M}_\alpha|^2}{(2\pi)^3 16\sqrt{s^3}} \sqrt{s_\alpha} dt_\alpha + \mathcal{B}_\alpha, \quad (38)$$

with $t_{\pi^- J/\psi} = s_{\pi^+ J/\psi}$, $t_{\rho^- \eta_c} = s_{\rho^- \pi^+}$ and $t_{D^0 D^{*-}} = s_{\pi^+ D^{*-}}$. $t_\alpha^\pm$ is the upper and lower limit of Mandelstam variables [57]. $\mathcal{N}_\alpha$ are scale factors. Here, we introduce two polynomial distribution functions $\mathcal{B}_\alpha$ incoherently for $\alpha = \pi J/\psi$ and $D^0 D^{*-}$, which is defined by

$$\mathcal{B}_\alpha = b_0^\alpha (\sqrt{s_\alpha} - m_\alpha^-)^{b_1^\alpha} (m^+ - \sqrt{s_\alpha})^{b_2^\alpha}, \quad (39)$$

with $m_\alpha^- = m_{\alpha_1} + m_{\alpha_2}$ and $m^+ = \sqrt{s} - m_\pi$. For $D\bar{D}^*$, it is introduced to mimic the possible contributions from processes such as $e^+e^- \rightarrow Y(4160) \rightarrow \pi D\bar{D}^*$, which may be important to reproduce the tail of line shape [32]. For $\pi J/\psi$, it is introduced to approximate the $\pi\pi$ FSI, which are not explicitly included here as in Ref. [25]. In a recent work [26], the $\pi\pi$ (and even $K\bar{K}$) FSI is carefully treated using dispersion relation to accurately calculate the pole position of Z_c . Nevertheless, the result do not differ significantly from their previous result in Ref. [25], where the $\pi\pi$ FSI is also mimicked by a polynomial function. Therefore, we believe that the treatment here will not significantly impact the conclusions. Later, we will see that the contribution from polynomial function has little interference with the $|\mathcal{M}_\alpha|^2$ term, which justifies their incoherent addition. Besides, to reduce the number of fitting parameters, we set $b_1^{\pi J/\psi} = b_2^{\pi J/\psi}$.

C. Parameters in the formalism

In this subsection, we summarize the free parameters in our formalism. For the T matrix, there is only one free parameter g' as defined in Eq. (9). For the vertices involving a Y particle, the free parameters are g_{YD_1D} and $c_\alpha^{S,D}$. For the final invariant mass distribution, the free parameters are N_α and $b_{0,1,2}^\alpha$. With the exception of g' , all parameters are $\sqrt{s}$ dependent, which means they may vary when fitting experimental data at $\sqrt{s} = 4.23$ and 4.26 GeV. In a word, our model incorporates a total of $21 + 1 = 22$ free parameters to fit over 250 experimental data points at $\sqrt{s} = 4.23$ and 4.26 GeV. The number of parameters is somewhat reduced compared to the previous work.

III. FITTING RESULTS AND DISCUSSIONS

We are now ready to fit the invariant mass distributions of $\rho^-\eta_c$ at $\sqrt{s} = 4.23$ GeV and $\pi^-J/\psi, D^0D^{*-}$ at $\sqrt{s} = 4.23$ as well as 4.26 GeV. The results for g' across two schemes at several fixed cutoffs are presented in Table I, with values from other references included for comparison. The results for some other parameters are given in Appendix E. The experimental line shapes are successfully reproduced in both schemes, yielding a reduced chi-squared $\hat{\chi}^2 \sim 1.6$. For illustration, we present the results for scheme 1 at $\Lambda = 1.5$ GeV in Figs. 4–6. Specific contributions are identified and depicted with dashed lines for further analysis.

A. $D\bar{D}^*$ invariant mass distribution

For the $D\bar{D}^*$ invariant mass distribution, the cascade decay $Y \rightarrow \bar{D}_1D \rightarrow D\bar{D}^*\pi$ Fig. 1(c) is found to be the dominant contribution. This can be explained by noting that, because the OBE isovector couple-channel

TABLE I. Fitting results for g' at different fixed cutoff $\Lambda_{\pi J/\psi(\rho\eta_c)}$ for two schemes of OBE potential. The fitting quality for these values are nearly identical, indicating the cutoff independence of the result. The value of g' in other references are also shown for comparison. The difference can be attributed to the different renormalization scheme.

	Scheme	$\Lambda_{\pi J/\psi(\rho\eta_c)}$ (fixed)	
		[GeV]	$g' (10^{-5} \text{ MeV}^{-\frac{3}{2}})$
This work	1	1.3	7.506 ± 0.120
		1.5	6.143 ± 0.350
		1.7	5.129 ± 0.070
	2	1.5	5.584 ± 0.470
		1.7	4.69 ± 0.280
Other references	...	...	3.68 [21]
	...	...	3.85 [54]

interactions are weak, the contributions from the two rescattering diagrams shown in Figs. 1(b) and 1(d) are much smaller compared to the corresponding tree diagrams Figs. 1(a) and 1(c). Nonetheless, the interference between the triangle diagram Fig. 1(d) and the cascade decay Fig. 1(c) is crucial in bridging the gap between the cascade decay and the experimental data at threshold, as illustrated by the purple dot-dashed lines in Fig. 4. The incoherent polynomial function contributes significantly only at the tail, as anticipated. The pointlike tree diagram, depicted in Fig. 1(a) and can only produce a smooth line shape, contributes very little here and hence not shown in Fig. 4. In fact, if the $Y \rightarrow \pi_D(D\bar{D}^*)$ interaction were introduced, the contribution of that pointlike diagram could be large enough to reproduce the peak even without the cascade decay mechanism. However, this interpretation is implausible, as it would be unnatural for the contribution from $Y \rightarrow \pi_D(D\bar{D}^*)$ to exceed that from $Y \rightarrow \pi_S(D\bar{D}^*)$ near the threshold. Further details can be referred to Appendix D.

Additionally, the bubble diagram $Y \rightarrow \pi(\pi J/\psi) \rightarrow \pi(D\bar{D}^*)$, as shown in Fig. 1(b), could have a sizable contribution if the $Y \rightarrow \pi(\pi J/\psi)$ coupling is sufficiently large so that such contribution is comparable to the cascade decay. However, if this were the case, the line shape in the $\pi J/\psi$ distribution would not be accurately reproduced. This shows that the key advantage of our approach is the ability to achieve a correct description by analyzing both datasets together. We also note that, in Ref. [21], the near-threshold peak appears to be reproduced using the OBE potential without the cascade decay. However, we argue that the formalism in that work may be inconsistent, as in their Eqs. (13) and (14), the tree diagram Fig. 1(a) is artificially suppressed while the rescattering diagram Fig. 1(b) remains unchanged so the FSI is no longer weak relatively.

In conclusion, the sharp near-threshold peak in the $D\bar{D}^*$ contribution is produced by the cascade decay and the

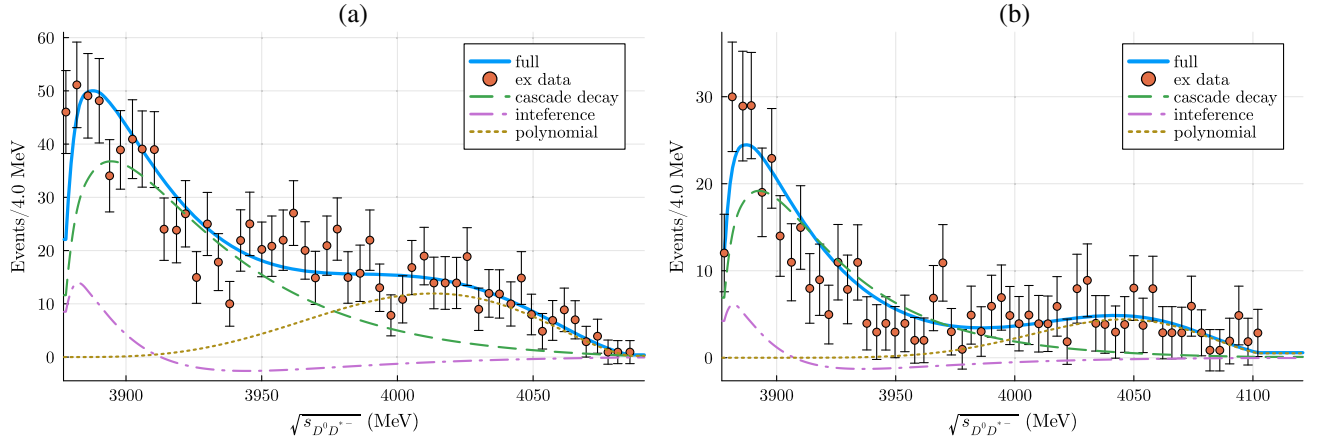


FIG. 4. Experimental data as well as theoretical line shapes for $D^0 D^{*-}$ distribution of $e^+ e^- \rightarrow \pi^+ D^0 D^{*-}$ process at (a) $\sqrt{s} = 4.23$ and (b) $\sqrt{s} = 4.26$ GeV. Orange solid circles are experimental data. Blue solid, green dashed, pink dot-dashed, yellow dotted lines denote the contribution cascade decay, interference between cascade decay and triangle diagram, and the polynomial function, respectively.

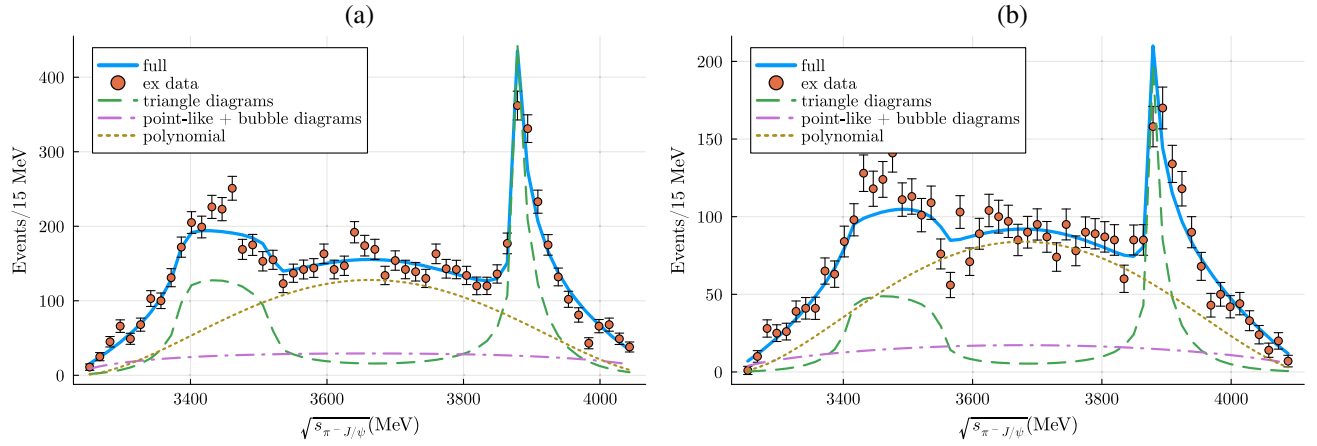


FIG. 5. Experimental data as well as theoretical line shapes for $\pi^- J/\psi$ distribution of $e^+ e^- \rightarrow \pi^+ \pi^- J/\psi$ process at (a) $\sqrt{s} = 4.23$ and (b) $\sqrt{s} = 4.26$ GeV. Orange solid circles are experimental data. Blue solid, green dashed, pink dot-dashed, yellow dotted lines denote the contribution from full amplitude, triangle diagram, pointlike as well as bubble diagrams, and the polynomial function, respectively.

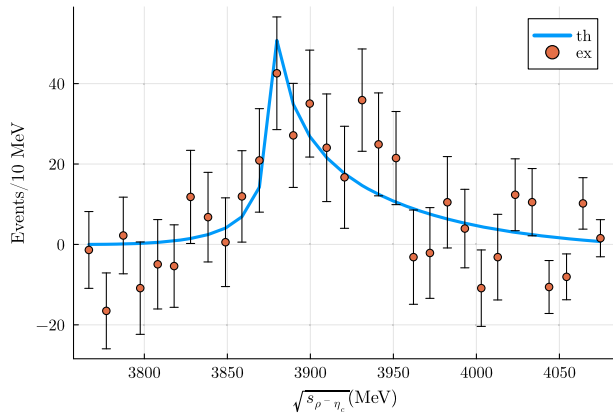


FIG. 6. Experimental data as well as theoretical line shape for $\rho^- \eta_c$ invariant mass distribution of $e^+ e^- \rightarrow \pi^+ \rho^- \eta_c \rightarrow \eta_c \pi^0 \pi^+ \pi^-$ process at $\sqrt{s} = 4.23$ GeV. Orange solid circle and blue solid line denote the experimental data and theoretical line shape, respectively.

triangle diagram in our model, consistent with the Refs. [31,32,58,59].

We now study the triangle diagram Fig. 1(d) in more detail, which enhance the peak structure observed in the $D\bar{D}^*$ distribution through the interference with cascade decay as just discussed. The enhancement may originate from two sources: T matrix in Fig. 1 and the kinematic triangle singularity (TS). In Refs. [17,59,60], it is suggested that the TS could be crucial for the peak structure in the $D\bar{D}^*$ distribution even the system does not fall exactly within the region where the TS occurs, as the $D_1\bar{D}$ threshold is above 4.26 GeV. This raises the question of whether the enhancement is entirely due to TS or partially due to other factors such as the T matrix.

To investigate this, we replace the T matrix with a constant in Eq. (37) and present the line shapes produced by the pure triangle integral at $\sqrt{s} = 4.23$ and 4.26 GeV in Figs. 7(a) and 7(b). While the peak near the threshold is evident, it is not

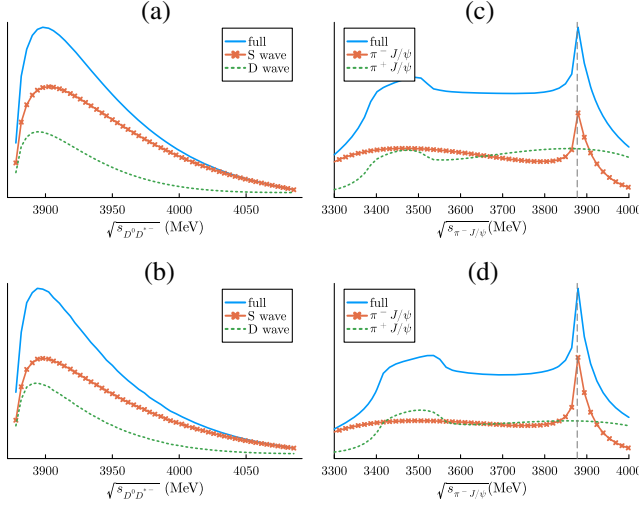


FIG. 7. Line shapes produced by pure triangle loops for (a),(b) $D^0 D^{*-}$ and (c),(d) $\pi^- J/\psi$ distribution at (a),(c) $\sqrt{s} = 4.23$ and (b),(d) $\sqrt{s} = 4.26$ GeV. Blue solid lines, orange lines with markers, green dotted lines correspond to Figs. 2(d) and 2(e), only Fig. 2(d) and only Fig. 2(e), respectively, for (a),(b) and Figs. 1(d) and 1(d) with only S -wave $D_1 \rightarrow D^* \pi$ and Fig. 1(d) with only D wave $D_1 \rightarrow D^* \pi$, respectively, for (c),(d). The gray dashed vertical lines indicate the threshold of $D\bar{D}^*$. The peak at $\sqrt{s} = 4.26$ GeV is slightly narrower since it is closer to the region where TS occurs.

sharp enough to reproduce the purple dot-dashed lines in Fig. 4 when interference with the cascade decay is considered. This indicates that, although the kinematic triangle loop is significant, a T matrix describing the FSI is still necessary in our model. In summary, the threshold peak in the $D\bar{D}^*$ distribution arises from the combination of $Y \rightarrow D\bar{D}_1 \rightarrow D\bar{D}^* \pi$, the T matrix for $D\bar{D}^* \rightarrow D\bar{D}^*$, and the kinematic triangle loop. The peak cannot be reproduced as sharply without all of these elements.

Parenthetically, an asymmetry parameter $\mathcal{A} = \frac{n_{>0.5} - n_{<0.5}}{n_{>0.5} + n_{<0.5}}$, where $n_{>0.5}(<0.5)$ represents the number of events with $|\cos \theta_{\pi D}| > 0.5 (< 0.5)$, is defined to study the role of $Y \rightarrow \bar{D}_1 D \rightarrow D\bar{D}^* \pi$ in the context of Z_c on the experimental side [5]. In Ref. [5], $\mathcal{A}_{\text{MC}} = 0.43 \pm 0.04$ and $\mathcal{A}_{\text{data}} = 0.12 \pm 0.06$ were obtained using Monte Carlo simulation and experimental data analysis, respectively. This led to the conclusion that $D_1 \bar{D}$ contributes only minimally. However, in Ref. [31], the authors calculate the asymmetry parameter $\mathcal{A}_{\text{th}}$ for the cascade-decay-dominant mechanism of Z_c , finding $\mathcal{A}_{\text{th}} \approx 0.13$, which is consistent with the experimental value $\mathcal{A}_{\text{data}}$. Therefore, the interpretation here cannot be ruled out based on the asymmetry parameter $\mathcal{A}$.

B. $\pi^- J/\psi$ and $\eta_c \rho$ invariant mass distribution

We now turn to the discussion of the fitting results for the $\pi^- J/\psi$ distribution. First, the polynomial function produces a line shape similar to those generated by the $\pi\pi$ FSI in

Ref. [7], as expected. Second, the peak is primarily contributed by the triangle diagram. To explain why the schematic Feynman diagrams in Figs. 2(a)–2(c) are suppressed, we point out that they can only produce a peak if the $Y \rightarrow \pi(D\bar{D}^*)$ interaction is sufficiently strong, such that the bubble diagram $Y \rightarrow \pi(D\bar{D}^*) \rightarrow \pi(\pi J/\psi)$ becomes comparable with the tree diagram $Y \rightarrow \pi\pi J/\psi$. However, this level of strength of $Y \rightarrow \pi D\bar{D}^*$ is not supported by the experimental data of the $D\bar{D}^*$ distribution. The discussions on both the $\pi J/\psi$ and $D\bar{D}^*$ distributions demonstrate that the model becomes more constrained when experimental data from different channels are considered.

To investigate the origin of the peak from the triangle diagram, we first present the line shape produced by a pure triangle loop in Figs. 7(c) and 7(d), as we did earlier. There is a cusp exactly at the threshold, which arises from a branch cut that begins at the threshold $m_D + m_{D^*}$. Specifically, when $\sqrt{s}_a \geq m_D + m_{D^*}$ the factor $\sqrt{s}_a - \omega_\pi - \omega_{D^*} - \omega_D$ in the denominator can vanish over some interval of the integration variable. In Appendix B, more details are provided. Given that the ratio of the height of the cusp to the smooth platform is much smaller than that in Fig. 5, a T matrix is still necessary, similar to the $D\bar{D}^*$ case.

Based on the above discussion, the contribution from the T matrix plays a crucial role here so it deserves further investigation. In Fig. 8, we illustrate how the magnitude of the half-on-shell T matrix, $|T_{\alpha \rightarrow \pi J/\psi}(k, \bar{p} \pi J/\psi; \sqrt{s} \pi J/\psi)|$, evolves as a function of $\sqrt{s} \pi J/\psi$, with the loop momentum k fixed at several values. Due to the one-pion-exchange interaction in the $D\bar{D}^* \rightarrow D\bar{D}^*$ process, a noticeable cusp appears at the $D\bar{D}^*$ threshold, stemming from the unitary branch cut. It is important to note that this peak originates from a cusp, not from a resonance or (virtual) bound state, which will be discussed later. In this context, we assert that the Z_c structure in the $\pi J/\psi$ distribution is a $D\bar{D}^*$ threshold cusp, driven by the $T_{D\bar{D}^* \rightarrow \pi J/\psi}$ in the triangle loop involving the D_1 particle. This conclusion aligns with the findings of the HALQCD collaboration.

Lastly, regarding the $\eta_c \rho$ channel, although its peak signal in the current experimental data is not as prominent as the others, it can also be interpreted as a threshold cusp within our model.

It is interesting that the major contributions to the peaks in the $\pi J/\psi$ and $D\bar{D}^*$ distributions are different in our model. This disparity may help explain the small deviation in the central values of the Breit-Wigner parameters for $Z_c(3885)$ and $Z_c(3900)$. Furthermore, our model may also provide a possible explanation for why the $Z_c(3900)/Z_c(3885)$ has been observed only in $e^+ e^-$ production but not in B decays thus far.

From the above discussion, it is clear that the quality of experimental data, particularly those near the $D\bar{D}^*$ threshold in each channel, is crucial for a deeper and more accurate understanding of the Z_c states. Therefore,

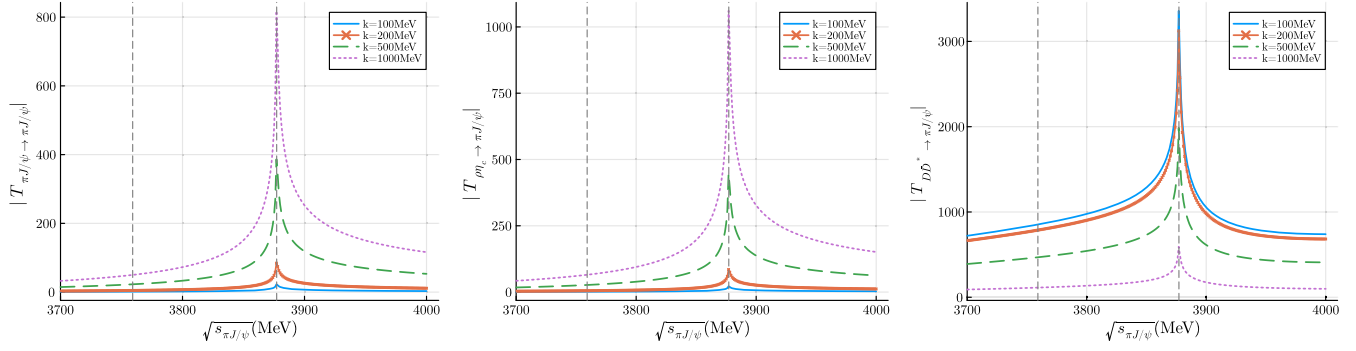


FIG. 8. The half-on-shell $|T_{\alpha \rightarrow \pi J/\psi}(k, \bar{p}_{\pi J/\psi}; \sqrt{s_{\pi J/\psi}})|$ as a function of $\sqrt{s_{\pi J/\psi}}$ with fixed off-shell momentum k . From left to right $\alpha = \pi^- J/\psi, \rho^- \eta_c$ and $[D\bar{D}^*]^-$. The gray dashed vertical lines denote the threshold of $\rho\eta_c$ and $D\bar{D}^*$. A clear cusp appears at the $D\bar{D}^*$ threshold due to the one-pion-exchange interaction between $D\bar{D}^*$ and $D\bar{D}^*$.

high-statistics data from the future Super Tau-Charm Facility(STCF) experiments [61] will be essential.

IV. POLE POSITION

In this study, three coupled channels result in a total of eight Riemann sheets when the T matrix is analytically continued into the complex plane. The branch cuts of the Green function G_α , which extend from the threshold of the α channel to infinity along the positive real energy axis, connect different Riemann sheets. For the purpose of analytical continuation, two types of Green functions are defined as follows:

$$G_\alpha^+(z) = G_\alpha(z), \quad (40)$$

$$G_\alpha^-(z) = G_\alpha(z) - \frac{i}{16\pi^2 z} \bar{p}_\alpha(z), \quad \text{Im} \bar{p}_\alpha < 0. \quad (41)$$

The Riemann sheets adjacent to the physical region are RS^{+++} , RS^{++-} , RS^{+-+} , and RS^{---} , where the superscripts indicate the sign of $G^\pm$ for the $\pi J/\psi$, $\rho\eta_c$, and $D\bar{D}^*$ channels, respectively. Only one pole is found in RS^{---} , and we examine its dependence on the cutoff and the regularization scheme, as presented in Table II.

The identified pole is located far below $D\bar{D}^*$ threshold on RS^{---} . For comparison, we list the results from other references that report either virtual or resonant poles in Table II. Since the pole found here is far from the threshold, the Z_c structure near the threshold is primarily due to the threshold cusp effect rather than a resonance or virtual bound state in our model. In Refs. [19,25], a pole located about 80 MeV below the threshold, which is similar to the one found here, is observed when an energy-independent interaction is adopted. Additionally, the HALQCD collaboration identifies three poles significantly below the $D\bar{D}^*$ threshold on RS^{---} and suggests that Z_c is not a conventional resonance but rather a cusp effect, based on their lattice setup [41]. Although the value of m_π used in LQCD calculations is unphysical and further investigations

are necessary, all current lattice data from different collaborations have so far shown no direct evidence of a Z_c resonance [33,34,36,38,41].

V. FINITE VOLUME ENERGY LEVELS

In this section, we calculate the finite volume energy levels using the HEFT and compare them with the spectra extracted from LQCD. HEFT is a formalism that links finite energy levels to infinite volume scattering amplitudes and is equivalent [63] to the standard Lüscher formula [64–66].

We begin with a brief introduction. When the system is confined to a finite volume with imposed boundary conditions, the momentum becomes discretized. The symmetry group is reduced from the rotational group $O(3)$ to either the octahedral group O_h or $C_{2/3/4v}$ depending on the total momentum [67–69]. Consequently, the potential matrix

TABLE II. The pole found on RS^{---} for different schemes and cutoffs, far away from the observed peak and therefore has minimal impact. For comparison, the position and type of the pole reported in other references are also shown.

	Pole position	Type	Scheme ($\Lambda_{\pi J/\psi}$)
This work	$3798.72 - 1.10i$	Virtual	1(1.3 GeV)
	$3798.46 - 1.71i$		1(1.5 GeV)
	$3798.12 - 2.26i$		1(1.7 GeV)
	$3798.27 - 2.02i$		2(1.5 GeV)
	$3797.80 - 2.64i$		2(1.7 GeV)
Reference [25]	3798^{+25}_{-31}	Virtual	
	$3902(6) - 38(9)i$	Resonance	
Reference [19]	3831^{+27}_{-38}	Virtual	
	$3894(6) - 30(13)i$	Resonance	
Reference [21]	3870	Virtual	
Reference [20]	3879	Virtual	-
Reference [22]	3872	Virtual	
Reference [26]	$3880(3) - 13(1)i$	Resonance	
Reference [32]	$3884 - 22i$	Resonance	
Reference [27]	3840	Virtual	
Reference [62]	3839(11)	Virtual	

must be defined in the discretized momentum basis. Using the O_h symmetry, the Hamiltonian matrix can be block diagonalized, allowing the extraction of finite volume energy levels corresponding to each irreducible representation (irrep) of O_h . These energy levels can then be compared with the lattice spectrum. This formalism has been successfully applied to describe various systems, such as $\Delta \rightarrow N\pi$ [70], $N^*(1440)$ [71,72], $N^*(1535)/N^*(1650)$ [73,74], and the positive parity D_s states [75] on the lattice.

Given that the spin-parity J^P of Z_c is reported to be 1^+ by BES III, it is sufficient to calculate the finite volume energy levels for the T_1^+ irrep of O_h , which is the only representation that couples to 1^+ [76]. The potential matrix for T_1^+ is given below. The general derivation and details will be provided in a paper that is currently in preparation.

$$\hat{V}_{L;\alpha\beta}^{T_1^+} = \frac{(2\pi)^3}{L^3} \sum_{N,N'=0}^{N_{\text{cut}}} \sum_{F=1}^{\text{oc}(T_1^+,N)} \sum_{F'=1}^{\text{oc}(T_1^+,N')} \sum_{\mathbf{n}} \sum_{\mathbf{n}'} \sum_{\mathbf{n}''=N}^{\mathbf{n}^2=N'} \frac{V_{\alpha\beta}(\mathbf{p}_{\mathbf{n}'}\sigma', \mathbf{p}_{\mathbf{n}}\sigma)}{(2\pi)^3 4\sqrt{\omega_{\alpha_1}(\mathbf{p}_{\mathbf{n}})\omega_{\alpha_2}(\mathbf{p}_{\mathbf{n}'}\sigma')\omega_{\beta_1}(\mathbf{p}_{\mathbf{n}})\omega_{\beta_2}(\mathbf{p}_{\mathbf{n}'}\sigma')}} \times X_F^{T_1^+}(\mathbf{n}\sigma) X_{F'}^{T_1^+}(\mathbf{n}'\sigma') |N; T_1^+, F'\rangle \langle N; T_1^+, F|, \quad (42)$$

where $\mathbf{p}_{\mathbf{n}} \equiv \frac{2\pi}{L}\mathbf{n}$ and $V_{\alpha\beta}$ is the potential in infinite volume. The vector $|N; T_1^+, F\rangle$ is orthonormalized and furnishes T_1^+ -irrep in the invariant subspace $\text{span}\{|\mathbf{p}_{\mathbf{n}}, \sigma\rangle | \mathbf{n}^2 = N\}$, where the occurrence of T_1^+ is denoted by $\text{oc}(T_1^+, N)$. To construct these vectors, one first gets $\text{oc}(T_1^+, N)$ linearly independent states using the representation matrix $D^{T_1^+}$,

$$\sum_{R \in O_h} D_{1\mu}^{T_1^+}(R)^* R |\mathbf{p}_{\mathbf{n}_0}, \sigma\rangle, \quad \mathbf{n}_0^2 = N, \quad (43)$$

by selecting different $\mu, \mathbf{n}_0$, which is followed by orthonormalization via, for example, the Gram-Schmidt process.

The function $X_F^{T_1^+}(\mathbf{n}, \sigma)$ represents the inner product $\langle N; T_1^+, F | \mathbf{p}_{\mathbf{n}_0}, \sigma \rangle$ and depends on the orthonormalization. Note that the Hamiltonian eigenvalues are independent of the orthonormalization since there are equivalent up to a unitary transformation. While the ρ meson is unstable at physical pion mass and requires careful treatment on the lattice, it suffices us to treat it as stable here to qualitatively justify our interpretation of Z_c .

The eigenvalues for various box sizes L are shown in the Fig. 9. As seen, all energy levels are close to the free energies of three channels, which indicates that the interactions between them are weak. This agrees with the findings of several LQCD collaboration studies [33,34,36], despite differences in m_π . Therefore we assert that our model is compatible with the current lattice analyses.

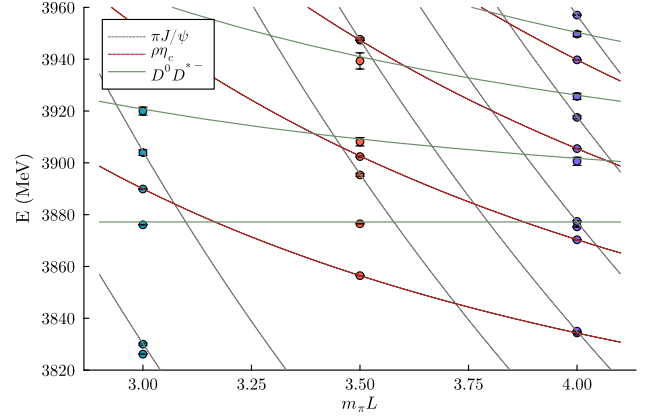


FIG. 9. Finite volume energy spectra at $m_\pi L = 3, 3.5, 4$ with $m_\pi = 139$ MeV calculated in our model. The circles with error bars represent the energy levels calculated by HEFT. The lines denote the free energy levels of $\pi J/\psi$, $\rho\eta_c$, and $D\bar{D}^*$. (ρ mesons are treated as stable particles here.) The energy levels are pretty close to the free energies, indicating the weak interactions.

VI. SUMMARY AND PROSPECT

In this work, a comprehensive coupled-channel model is employed for the first time to analyze the Z_c structure in the $e^+e^- \rightarrow \pi^+(\pi^- J/\psi, D^0 D^{*-}, \rho^- \eta_c)$ processes at $\sqrt{s} = 4.23$ and 4.26 GeV, as observed by BESIII [2,5–8]. The OBE interaction that respects heavy quark spin symmetry is adopted to describe the $D\bar{D}^*$ interactions. Additionally, the rescattering T matrix includes a new parameter, g' which governs the coupling vertex between $D\bar{D}^*$ and charmonium $J/\psi, \eta_c$. Except for g' , all other parameters used for the T matrix were determined by previous works, which successfully reproduced the line shape of the double-charmed exotic state T_{cc} [43]. Assuming that e^+e^- collisions at $\sqrt{s} = 4.23$ and 4.26 GeV are dominated by $e^+e^- \rightarrow Y(4230/4260)$ followed by subsequent decays, the line shapes of the $\pi^- J/\psi$, $D^0 D^{*-}$, and $\rho^- \eta_c$ invariant mass distributions are successfully reproduced in our work.

Upon investigating the different contributions, we find that the nature of the Z_c structures in the $D\bar{D}^*$ and $\pi J/\psi$ ($\rho\eta_c$) distributions are not totally the same, which may explain the small deviation between $Z_c(3885)$ and $Z_c(3900)$. In the $D^0 D^{*-}$ distribution, the near-threshold peak primarily arises from the tree-level cascade decay $Y(4230/4260) \rightarrow \bar{D}_1(2420)D \rightarrow \pi D\bar{D}^*$, which is further enhanced by the triangle diagram that includes the FSI. In contrast, the $\pi^- J/\psi$ ($\rho^- \eta_c$) distribution features two cusps. One from the T matrix and the other from the pure triangle loop integral. Both of them occur exactly at the $D\bar{D}^*$ threshold and play a primary role on the peak structure. Therefore, the Z_c observed in this distribution is identified as a threshold cusp. This interpretation provides a possible explanation to why $Z_c(3900)$ is observed only in e^+e^- collisions and not in B decays, where the production of the Y state is more challenging.

Furthermore, we search for poles of the T matrix on the four Riemann sheets adjacent to the physical region and find only one pole approximately 80 MeV below the $D\bar{D}^*$ threshold on RS^{--} . However, due to its distant location from the threshold, this pole has little impact on the Z_c structure.

Additionally, we calculate the finite volume energy spectra for the T_1^+ irreducible representation, the only one that couples to $J^P = 1^+$. All the energy levels in the relevant region we calculate are very close to the free energies due to the weak interactions between channels. This result is qualitatively consistent with the current findings from several different LQCD collaborations [33,34,36], further supporting our interpretation of the Z_c structure. To be more specific, no signal would be observed in the two-body correlation from lattice simulations if Z_c is indeed from a threshold cusp effect.

Based on the results of this work, we conclude that the Z_c structure near 3.9 GeV in the three distributions is due to the threshold cusp arising from coupled channels. However, this conclusion still faces some uncertainties. For instance, we acknowledge that the $\pi\pi$ FSI is not rigorously treated here, which prevents us from accurately reproducing the experimental $\pi\pi$ invariant mass distribution in the $e^+e^- \rightarrow \pi\pi J/\psi$ process. In Ref. [26], the $\pi\pi$ FSI is carefully incorporated using a dispersion relation. Compared to their previous work [25], the pole is found on the same Riemann sheet but its position shifts by approximately 20 MeV. Therefore, to achieve a more reliable and comprehensive analysis, it will be necessary to incorporate the $\pi\pi$ FSI in a self-contained manner in future research. We also suggest that experimental researchers improve the quality of data, especially near the $D\bar{D}^*$ threshold to further elucidate the nature of the Z_c .

ACKNOWLEDGMENTS

The authors want to thank the useful discussions with Changzheng Yuan, Feng-Kun Guo, Chuan Liu, and Bing-Song Zou. This work is supported by the National Natural Science Foundation of China under Grants No. 12175239, No. 12221005, and No. 12275046, and by the National Key Research and Development Program of China under Contract No. 2020YFA0406400, and by the Chinese Academy of Sciences under Grant No. YSBR-101, and by the KAKENHI under Grants No. 23K03427 and No. 24K17055, and by the Xiaomi Foundation/Xiaomi Young Talents Program.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: NUMERICAL SOLUTION OF LSE

The Lippmann-Schwinger equation in the partial-wave representation can be solved numerically. For simplicity, all

indices labeling coupled channels and angular momentum are omitted here. Our goal is to solve the following one-dimensional integral equation

$$T(k', k; E) = V(k', k) + \int_0^\infty q^2 dq V(k', q) G(q; E + i\epsilon) T(q, k; E), \quad (\text{A1})$$

where the infinitesimal $i\epsilon$ can be treated explicitly,

$$T(k', k; E) = V(k', k) + \mathcal{P} \int q^2 dq V(k', q) G(q; E) T(q, k; E) - i\pi \bar{q} \mu(E) V(k', \bar{q}) T(\bar{q}, k; E) \theta(E - m_1 - m_2),$$

where $\bar{q}$ is defined by $\omega_1(\bar{q}) + \omega_2(\bar{q}) = E$, $\mu(E) = \frac{1}{(2\pi)^3 4E}$ and θ function vanishes unless E is above the threshold. The symbol $\mathcal{P}$ denotes the Cauchy principal value of the integral. $\bar{q}$ would be a simple pole if V is nonsingular. Notice that

$$\mathcal{P} \int \frac{1}{q^2 - \bar{q}^2} dq = 0. \quad (\text{A2})$$

Thus, it is a free lunch to do a subtraction to regularize the integrand,

$$\begin{aligned} & \mathcal{P} \int q^2 dq V(k', q) G(q; E) T(q, k; E) \\ &= \int q^2 dq V(k', q) G(q; E) T(q, k; E) \\ & \quad - V(k', \bar{q}) \bar{q}^2 \frac{-2\mu(E)}{q^2 - \bar{q}^2} T(\bar{q}, k; E). \end{aligned} \quad (\text{A3})$$

To apply the Gauss-Legendre quadrature method, the integration domain must first be transformed into a bounded interval. This is done by splitting the domain $(0, \infty)$ into $(0, \hat{q})$ and $(\hat{q}, \infty)$, where $\hat{q}$ is an arbitrary positive constant. The integrals over these intervals for any function f can be rewritten as follows:

$$\int_0^{\hat{q}} f(q) dq = \hat{q} \int_0^1 f(x\hat{q}) dx, \quad (\text{A4})$$

$$\int_{\hat{q}}^\infty f(q) dq = \hat{q} \int_0^1 \frac{1-b}{(1-x)^2} f\left(\frac{1-bx}{1-x} \hat{q}\right) dx, \quad \forall b < 1. \quad (\text{A5})$$

Given N -point nodes x_i and weights w_i of Gauss-Legendre quadrature on interval $(0,1)$, we define the following quantities:

$$q_i = \begin{cases} \bar{q}(E) & i = 0 \\ x_i \hat{q} & 1 \leq i \leq N \\ \frac{1-bx_{i-N}}{1-x_{i-N}} \hat{q} & N < i \leq 2N \end{cases}, \quad (\text{A6})$$

$$\tilde{G}_0(E) = \frac{q_0^2 \hat{q}}{2E} \left(\sum_{i=1}^N \frac{w_i}{q_i^2 - q_0^2} + \sum_{i=N+1}^{2N} \frac{1-b}{(1-x_{i-N})^2} \frac{w_i}{q_i^2 - q_0^2} \right) - \frac{i\pi q_0}{4E} \theta(E - m_1 - m_2), \quad (\text{A7})$$

$$\tilde{G}_i(E) = \begin{cases} w_i q_i^2 \hat{q} G(q_i; E) & 1 \leq i \leq N \\ w_i q_i^2 \frac{1-b}{(1-x_{i-N})^2} \hat{q} G(E; q_i) & N < i \leq 2N \end{cases}, \quad (\text{A8})$$

and $T_{ij} = T(q_i, q_j)$, $V_{ij} = V(q_i, q_j)$. Consequently, Eq. (20) can be converted into the algebraic equation

$$T_{ij}(E) = V_{ij} + \sum_{k=0}^{2N} V_{ik} \tilde{G}_k(E) T_{kj}(E) \quad (\text{A9})$$

or in a matrix notation,

$$[T] = [V] + [V][\tilde{G}][T]. \quad (\text{A10})$$

The solution is given by

$$[T] = (1 - [V][\tilde{G}])^{-1} [V]. \quad (\text{A11})$$

The on-shell T -matrix T_{00} can be obtained directly. For half-on-shell or off-shell T matrix, given any off-shell momentum p , it only needs to extend the definitions in Eqs. (A6) and (A8) by introducing $q_{2N+1} = p$ and set $G_{2N+1} = 0$.

APPENDIX B: NUMERICAL EVALUATION OF THE TRIANGLE LOOP

In this section we introduce how to numerically evaluate the Eq. (37). For convenience, we introduce a simplified notation,

$$\mathcal{Q} = \int_0^\infty dp \int_{-1}^1 dx \frac{g(p, x; \lambda)}{A(p; \lambda) - \sqrt{B(p; \lambda) + C(p; \lambda)x} + i0^+}, \quad (\text{B1})$$

where λ denotes the external kinematic variables $\sqrt{s}$ and $\sqrt{s_a}$, which are fixed during the integration. The functions in the denominator are given by

$$A = \sqrt{s} - \sqrt{p_\pi^2 + m_\pi^2} - \sqrt{p^2 + m_{D^*}^2}, \quad (\text{B2})$$

$$B = p^2 + p_\pi^2 + m_D^2, \quad (\text{B3})$$

$$C = 2pp_\pi. \quad (\text{B4})$$

Given p and λ , the integrand may has a pole if $A > 0$ and $|\frac{A^2-B}{C}| < 1$ with $x \in [-1, 1]$. Therefore, the integral with respect to x requires careful treatment. A straightforward calculation shows that the integrand is regular when $s_a + m_D^2 - m_{D^*}^2 < 2m_{D^*}\sqrt{s_a}$, or equivalently, $\sqrt{s_a} < m_D + m_{D^*}$. In this case, the integral can be numerically evaluated using a two-dimensional Gauss quadrature method. However, when $\sqrt{s_a} > m_D + m_{D^*}$, the domain of p , $(0, \infty)$, should be split into two parts R and $R^\perp$. Here R is the domain where $A > 0$ and $|\frac{A^2-B}{C}| < 1$, while $R^\perp$ is its complement. Without showing details we find

$$R = \left[|p_-|, \min(p_+, \sqrt{(\sqrt{s} - \omega_\pi)^2 - m_{D^*}^2}) \right], \quad (\text{B5})$$

where

$$p_\pm = \frac{\bar{s}_a}{s_a} \left[(\sqrt{s} - \omega_\pi) \sqrt{\frac{1}{4} - \frac{s_a}{\bar{s}_a} m_{D^*}^2} \pm \frac{1}{2} p_\pi \right]. \quad (\text{B6})$$

In the range of $\sqrt{s_a}$ considered, $\min(\dots) \equiv p_+$. When $p \in R^\perp$, the two-dimensional Gauss quadrature formula can be applied directly. For $p \in R$, the $i0^+$ prescription should be worked out manually as

$$\begin{aligned} \mathcal{Q} &\supset \int_R dp \left(\mathcal{P} \int_{-1}^1 \frac{g(p, x) dx}{A - \sqrt{B + Cx}} - i\pi \frac{2A}{C} g(p, x_0) \right) \\ &= \int_R dp \left[\mathcal{P} \int_{-1}^1 \left(\frac{g(p, x) dx}{A - \sqrt{B + Cx}} + \frac{2A}{C} \frac{g(p, x_0)}{x - x_0} \right) dx \right. \\ &\quad \left. - \frac{2A}{C} g(p, x_0) \log \left(\frac{1 - x_0}{1 + x_0} \right) - i\pi \frac{2A}{C} g(p, x_0) \right] \\ &= \int_R dp \int_{-1}^1 \left[\frac{g(p, x)}{A - \sqrt{B + Cx}} + \frac{2A}{C} g(p, x_0) \left(\frac{1}{x - x_0} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \log \left(\frac{1 - x_0}{1 + x_0} \right) - \frac{1}{2} i\pi \right) \right] dx, \end{aligned} \quad (\text{B7})$$

where $x_0 = \frac{A^2-B}{C}$. In the last equation, the symbol $\mathcal{P}$ is dropped since the integrand becomes regular after subtracting a function with the same pole structure as the original integrand. This subtraction enables numerical evaluation using the two-dimensional Gauss quadrature formula. Note that we here adopt $\frac{1}{x-x_0}$ for subtraction instead of $\frac{1}{x^2-x_0^2}$, which would introduce an superfluous pole at $x = -x_0$ within the interval $(-1, 1)$.

APPENDIX C: POTENTIAL

In this section, we present the tree-level amplitude $i\mathcal{M}$ derived from the Lagrangians in Eqs. (13)–(17). First, the propagators of heavy meson and charmonium in terms of residual momentum within the framework of HEFT are given by

$$\frac{i}{2v \cdot k - \frac{k^2}{m_H} + i\epsilon} \approx \frac{i}{(k + m_H v)^2 - m_H^2 + i\epsilon} m_H^2 \quad (C1)$$

for (pseudo)scalar particles and

$$\frac{i(-g^{\mu\nu} + v^\mu v^\nu)}{2v \cdot k - \frac{k^2}{m_H} + i\epsilon} \approx \frac{i(-g^{\mu\nu} + v^\mu v^\nu)}{(k + m_H v)^2 - m_H^2 + i\epsilon} m_H^2 \quad (C2)$$

for (pseudo)vector particles. Second, the polarization vectors of heavy mesons and charmonium are treated to be static at leading order, i.e.,

$$\epsilon_0^\mu = \epsilon_0^\mu(\mathbf{0}) = (0, 0, 0, 1), \quad (C3)$$

$$\epsilon_{\pm 1}^\mu = \epsilon_{\pm 1}^\mu(\mathbf{0}) = \frac{1}{\sqrt{2}}(0, \mp 1, -i, 0), \quad (C4)$$

which satisfy the condition $v \cdot \epsilon = 0$ and the spin sum $\sum_\sigma \epsilon_\sigma^{\mu*} \epsilon_\sigma^\nu$ gives the numerator of Eq. (C2).

With these ingredients, the amplitude in terms of full momentum can be obtained. In Ref. [43], the amplitude $i\mathcal{M}_{[D\bar{D}^*] \rightarrow [D\bar{D}^*]}$ has already been provided so we only need to present $i\mathcal{M}_{\alpha(k) \rightarrow [D\bar{D}^*](p)}$, $\alpha = \pi J/\psi, \rho\eta_c$. Due to the G-parity symmetry:

$$i\mathcal{M}_{\alpha \rightarrow [D\bar{D}^*]} = i\sqrt{2}\mathcal{M}_{\alpha \rightarrow D^0 D^{*-}} \equiv i\mathcal{M}^\alpha. \quad (C5)$$

For $\alpha = \pi J/\psi$,

$$\frac{i\mathcal{M}_1^{\pi\psi}}{\sqrt{m_\psi m_{D^*} m_D^3}} = \frac{i4gg'(\epsilon_\sigma \cdot (2\mathbf{p} + \mathbf{k}))(\epsilon_\sigma^* \cdot \mathbf{k})}{f_\pi \frac{q^2 - m_D^2 + i\epsilon}{q^2 - m_D^2 + i\epsilon}}, \quad (C6)$$

$$\frac{i\mathcal{M}_2^{\pi\psi}}{\sqrt{m_\psi m_{D^*} m_D^3}} = \frac{igg'((\mathbf{k} \times \epsilon_\sigma^*) \cdot (2\mathbf{p} + \mathbf{k}) \times \epsilon_\sigma)}{f_\pi \frac{q^2 - m_{D^*}^2 + i\epsilon}{q^2 - m_{D^*}^2 + i\epsilon}}, \quad (C7)$$

$$\frac{i\mathcal{M}_3^{\pi\psi}}{\sqrt{m_\psi m_{D^*} m_D^3}} = \frac{i4gg'}{f_\pi} \frac{k_i(k-2p)_j}{q^2 - m_{D^*}^2 + i\epsilon} (\delta^{ij} \delta_{\sigma\sigma'} + \epsilon_\sigma^i \epsilon_\sigma^{j*} - \epsilon_\sigma^j \epsilon_\sigma^{i*}), \quad (C8)$$

where $g = 0.6$ is the decay constant of D^* . For $\alpha = \rho\eta_c$,

$$\begin{aligned} \frac{i\mathcal{M}_1^{\rho\eta_c}}{\sqrt{m_{\eta_c} m_{D^*} m_D^3}} &= \frac{g'g_V}{q^2 - m_{D^*}^2 + i\epsilon} [\beta \epsilon_\sigma^0(\mathbf{k})(\mathbf{k} - 2\mathbf{p}) \cdot \epsilon_\sigma^* \\ &\quad + 2\lambda(\mathbf{k} \cdot (\mathbf{k} - 2\mathbf{p}))\delta_{\sigma\sigma'} \\ &\quad - (\mathbf{k} - 2\mathbf{p}) \cdot \epsilon_\sigma(\mathbf{k})\mathbf{k} \cdot \epsilon_\sigma^*], \end{aligned} \quad (C9)$$

$$\frac{i\mathcal{M}_2^{\rho\eta_c}}{\sqrt{m_{\eta_c} m_{D^*} m_D^3}} = \beta g'g_V \epsilon_\sigma^0(\mathbf{k}) \frac{(\mathbf{k} + 2\mathbf{p}) \cdot \epsilon_\sigma^*}{q^2 - m_D^2 + i\epsilon}, \quad (C10)$$

$$\frac{i\mathcal{M}_3^{\rho\eta_c}}{\sqrt{m_{\eta_c} m_{D^*} m_D^3}} = 2\lambda g'g_V \frac{(\mathbf{k} \times \epsilon_\sigma(\mathbf{k})) \cdot ((\mathbf{k} + 2\mathbf{p}) \times \epsilon_\sigma^*)}{q^2 - m_{D^*}^2 + i\epsilon}, \quad (C11)$$

where $g'_V = 2\sqrt{2}g_V = 2\sqrt{2} \times 5.8 = 16.4$, $\lambda = 0.683/\text{GeV}$, and $\beta = 0.687$ (at $\Lambda_{[D\bar{D}^*]} = 1 \text{ GeV}$) are taken from Ref. [43]. $\epsilon_\sigma^\mu(\mathbf{k})$ is the polarization vector of the ρ light meson,

$$\epsilon_\sigma^0(\mathbf{k}) = \frac{1}{m_\rho} \mathbf{k} \cdot \epsilon_\sigma(\mathbf{0}), \quad (C12)$$

$$\epsilon_\sigma^i(\mathbf{k}) = \epsilon_\sigma^i(\mathbf{0}) + \left(\frac{\sqrt{k^2 + m_\rho^2}}{m_\rho} - 1 \right) \frac{k^i(\mathbf{k} \cdot \epsilon_\sigma(\mathbf{0}))}{|\mathbf{k}|^2}. \quad (C13)$$

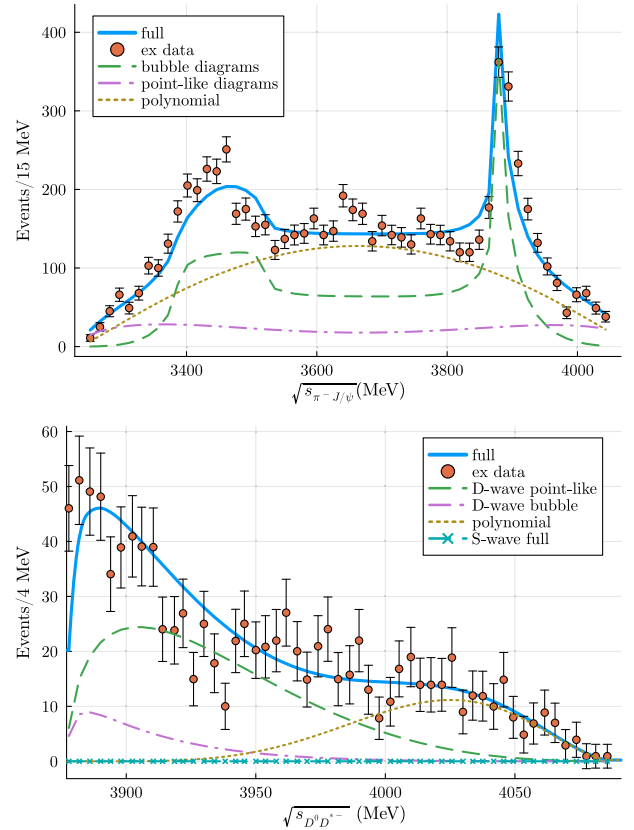


FIG. 10. Line shapes determined by fitting the experimental data at $\sqrt{s} = 4.23 \text{ GeV}$ without $Y \rightarrow D_1 \bar{D}$ vertex. The cutoff $\Lambda_{\pi J/\psi(\rho\eta_c)}$ is fixed at 2.0 GeV. Orange solid circles are experimental data. Blue solid, green dashed pink dot dashed and yellow dotted lines in the top(bottom) row denote the contribution from full amplitude, bubble diagrams [D -wave $Y \rightarrow \pi_D(D\bar{D}^*)$ pointlike diagram], pointlike diagrams [D -wave $Y \rightarrow \pi_D(D\bar{D}^*)$ bubble diagram] and the polynomial function, respectively. Line with markers in the bottom row denote the highly suppressed contribution from S -wave $Y \rightarrow \pi_S(D\bar{D}^*)$ pointlike as well as bubble diagrams.

APPENDIX D: FITTING WITHOUT $Y \rightarrow D_1 \bar{D} + \text{C.C. VERTEX}$

In this appendix we exclude the $Y \rightarrow D_1 \bar{D}$ vertex and attempt to fit the experimental data using only pointlike and bubble diagrams. Specifically, the cascade decay $Y \rightarrow D_1 \bar{D} \rightarrow \pi \alpha$ and the triangle diagram are omitted, while the FSIs are still considered.

As discussed in the main text, the peak in $D\bar{D}^*$ distribution cannot be reproduced using only S -wave $Y \rightarrow \pi_S(D\bar{D}^*)$ vertex. To address this, we introduce an extra D -wave $Y \rightarrow \pi_D(D\bar{D}^*)$ vertex, which ultimately lead to a term proportional to p_π^4 in $|\mathcal{M}|^2$. We present the fitting for $\pi J/\psi$ and $D\bar{D}^*$ distribution at $\sqrt{s} = 4.23$ GeV in Fig. 10.

We now offer some observations. First, the peak in $\pi^- J/\psi$ distribution can indeed be produced if $Y \rightarrow D\bar{D}^* \pi$ is significantly much stronger than $Y \rightarrow \pi \pi J/\psi$. Second, the peak in $D\bar{D}^*$ distribution can also be obtained with the help of a D -wave $Y \rightarrow \pi_D(D\bar{D}^*)$ vertex. This is easy to understand because p_π increases as $\sqrt{s_{D^0 D^{*-}}}$ approaches threshold. However, the contribution from S -wave $Y \rightarrow \pi_S(D\bar{D}^*)$ is highly suppressed. This is quite unnatural since the phase space of $Y \rightarrow D\bar{D}^* \pi$ is not large and the typical p_π is small, the high partial wave is expected to be suppressed for naturalness. If the D -wave contribution is significantly larger than the S -wave contribution, then it would be difficult to justify ignoring a G or even higher partial wave. Lastly, the reduced χ^2 is considerably worse than the fit in the main text.

In conclusion, though the experiment data can more or less be fitted without $Y \rightarrow D_1 \bar{D}$ vertex, the result is physically and theoretically unsound. Therefore, this vertex should be included and the triangle diagram must be incorporated into the formalism for self-consistency.

APPENDIX E: FITTED PARAMETERS

In this appendix, we present the results for the most relevant fitting parameters except g' for scheme 1 at $\Lambda_{\pi J/\psi(\rho\eta_c)} = 1.5$ GeV, as shown in Table III.

The magnitude of parameters such as $\mathcal{N}_{D\bar{D}^*} |g_{YD_1 D}|^2$ is large but reasonable since it related to the normalization to the number of events by a overall factor $\mathcal{N}_{D\bar{D}^*}$ and only the line shapes are fitted here. Some parameters such as $\mathcal{N}_{\pi J/\psi} |c_{\pi J/\psi}^D|^2$ suffer large uncertainties since they have little influence on the quality of fit. Other parameters are referred to the incoherent background, which generate the similar line shape shown in Refs. [21,25].

TABLE III. Fitting results for the parameters except g' . (Dimensional parameters are on the order of MeV.) $\tilde{b}_0^{\pi J/\psi, D\bar{D}^*}$ is the dimensionless quantity defined as $\tilde{b}_0^\alpha = b_0^\alpha(\sqrt{s} - M_\alpha)^{b_1^\alpha + b_2^\alpha}$ with $M_\alpha = m_{\alpha_1} + m_{\alpha_2} + m_\pi$.

	$\sqrt{s} = 4.23$ GeV	$\sqrt{s} = 4.26$ GeV
$\mathcal{N}_{\pi J/\psi} g_{YD_1 D} ^2 / 10^{11}$	42 ± 9	7.7 ± 2.6
$\mathcal{N}_{\rho\eta_c} g_{YD_1 D} ^2 / 10^{11}$	27 ± 7	...
$\mathcal{N}_{D\bar{D}^*} g_{YD_1 D} ^2 / 10^{11}$	0.93 ± 0.21	0.14 ± 0.05
$\mathcal{N}_{\pi J/\psi} c_{\pi J/\psi}^S ^2 / 10^5$	4.4 ± 1.8	6.1 ± 2.9
$\mathcal{N}_{\pi J/\psi} c_{\pi J/\psi}^D ^2 / 10^{-6}$	8.0 ± 7.0	2.1 ± 2.1
$\mathcal{N}_{\pi J/\psi} c_{D\bar{D}^*}^S ^2$	1.41 ± 0.32	0.015 ± 0.05
$\tilde{b}_0^{\pi J/\psi} / 10^3$	1.67 ± 0.29	0.91 ± 0.16
$b_{1.2}^{\pi J/\psi}$	1.85 ± 0.13	1.72 ± 0.13
$\tilde{b}_0^{D\bar{D}^*} / 10^3$	0.46 ± 0.10	1.54 ± 0.27
$b_1^{D\bar{D}^*}$	3.54 ± 0.60	6.55 ± 0.40
$b_2^{D\bar{D}^*}$	2.03 ± 1.70	2.92 ± 0.90

- [1] S. K. Choi *et al.* (Belle Collaboration), Observation of a narrow charmonium-like state in exclusive $B^\pm \rightarrow K^\pm \pi^+ \pi^- J/\psi$ decays, *Phys. Rev. Lett.* **91**, 262001 (2003).
- [2] M. Ablikim *et al.* (BESIII Collaboration), Observation of a charged charmoniumlike structure in $e^+ e^- \rightarrow \pi^+ \pi^- J/\psi$ at $\sqrt{s} = 4.26$ GeV, *Phys. Rev. Lett.* **110**, 252001 (2013).
- [3] Z. Q. Liu *et al.* (Belle Collaboration), Study of $e^+ e^- \rightarrow \pi^+ \pi^- J/\psi$ and observation of a charged charmoniumlike state at Belle, *Phys. Rev. Lett.* **110**, 252002 (2013); **111**, 019901(E) (2013).
- [4] T. Xiao, S. Dobbs, A. Tomaradze, and K. K. Seth, Observation of the charged hadron $Z_c^\pm(3900)$ and evidence for the neutral $Z_c^0(3900)$ in $e^+ e^- \rightarrow \pi \pi J/\psi$ at $\sqrt{s} = 4170$ MeV, *Phys. Lett. B* **727**, 366 (2013).

- [5] M. Ablikim *et al.* (BESIII Collaboration), Observation of a charged $(D\bar{D}^*)^\pm$ mass peak in $e^+ e^- \rightarrow \pi D\bar{D}^*$ at $\sqrt{s} = 4.26$ GeV, *Phys. Rev. Lett.* **112**, 022001 (2014).
- [6] M. Ablikim *et al.* (BESIII Collaboration), Confirmation of a charged charmoniumlike state $Z_c(3885)^\mp$ in $e^+ e^- \rightarrow \pi^\pm (D\bar{D}^*)^\mp$ with double D tag, *Phys. Rev. D* **92**, 092006 (2015).
- [7] M. Ablikim *et al.* (BESIII Collaboration), Determination of the spin and parity of the $Z_c(3900)$, *Phys. Rev. Lett.* **119**, 072001 (2017).
- [8] C.-Z. Yuan, The XYZ states revisited, *Int. J. Mod. Phys. A* **33**, 1830018 (2018).
- [9] K. Chilikin *et al.* (Belle Collaboration), Observation of a new charged charmoniumlike state in $\bar{B}^0 \rightarrow J/\psi K^- \pi^+$ decays, *Phys. Rev. D* **90**, 112009 (2014).

- [10] R. Aaij *et al.* (LHCb Collaboration), Measurement of the resonant and CP components in $\bar{B}^0 \rightarrow J/\psi \pi^+ \pi^-$ decays, *Phys. Rev. D* **90**, 012003 (2014).
- [11] D. Johnson, I. Polyakov, T. Skwarnicki, and M. Wang, Exotic hadrons at LHCb, *Annu. Rev. Nucl. Part. Sci.* **74**, 583 (2024).
- [12] E. Braaten, How the $Z_c(3900)$ reveals the spectra of quarkonium hybrid and tetraquark mesons, *Phys. Rev. Lett.* **111**, 162003 (2013).
- [13] J. M. Dias, F. S. Navarra, M. Nielsen, and C. M. Zanetti, $Z_c^+(3900)$ decay width in QCD sum rules, *Phys. Rev. D* **88**, 016004 (2013).
- [14] C.-F. Qiao and L. Tang, Estimating the mass of the hidden charm $1^+(1^+)$ tetraquark state via QCD sum rules, *Eur. Phys. J. C* **74**, 3122 (2014).
- [15] Z.-G. Wang, Analysis of $Z_{cs}(3985)$ as the axialvector tetraquark state, *Chin. Phys. C* **45**, 073107 (2021).
- [16] L. Maiani, A. D. Polosa, and V. Riquer, The new resonances $Z_{cs}(3985)$ and $Z_{cs}(4003)$ (almost) fill two tetraquark nonets of broken $SU(3)_f$, *Sci. Bull.* **66**, 1616 (2021).
- [17] Q. Wang, C. Hanhart, and Q. Zhao, Decoding the riddle of $Y(4260)$ and $Z_c(3900)$, *Phys. Rev. Lett.* **111**, 132003 (2013).
- [18] F. Aceti, M. Bayar, E. Oset, A. Martinez Torres, K. P. Khemchandani, J. M. Dias, F. S. Navarra, and M. Nielsen, Prediction of an $I = 1$ $D\bar{D}^*$ state and relationship to the claimed $Z_c(3900)$, $Z_c(3885)$, *Phys. Rev. D* **90**, 016003 (2014).
- [19] M. Albaladejo, F.-K. Guo, C. Hidalgo-Duque, and J. Nieves, $Z_c(3900)$: What has been really seen?, *Phys. Lett. B* **755**, 337 (2016).
- [20] Q.-R. Gong, Z.-H. Guo, C. Meng, G.-Y. Tang, Y.-F. Wang, and H.-Q. Zheng, $Z_c(3900)$ as a $D\bar{D}^*$ molecule from the pole counting rule, *Phys. Rev. D* **94**, 114019 (2016).
- [21] J. He and D.-Y. Chen, $Z_c(3900)/Z_c(3885)$ as a virtual state from $\pi J/\psi - \bar{D}^* D$ interaction, *Eur. Phys. J. C* **78**, 94 (2018).
- [22] P. G. Ortega, J. Segovia, D. R. Entem, and F. Fernández, The Z_c structures in a coupled-channels model, *Eur. Phys. J. C* **79**, 78 (2019).
- [23] Z. Yang, X. Cao, F.-K. Guo, J. Nieves, and M. P. Valderrama, Strange molecular partners of the $Z_c(3900)$ and $Z_c(4020)$, *Phys. Rev. D* **103**, 074029 (2021).
- [24] M.-J. Yan, F.-Z. Peng, M. Sánchez Sánchez, and M. Pavaon Valderrama, Axial meson exchange and the $Z_c(3900)$ and $Z_{cs}(3985)$ resonances as heavy hadron molecules, *Phys. Rev. D* **104**, 114025 (2021).
- [25] M.-L. Du, M. Albaladejo, F.-K. Guo, and J. Nieves, Combined analysis of the $Z_c(3900)$ and the $Z_{cs}(3985)$ exotic states, *Phys. Rev. D* **105**, 074018 (2022).
- [26] Y.-H. Chen, M.-L. Du, and F.-K. Guo, Precise determination of the pole position of the exotic $Z_c(3900)$, *Sci. China Phys. Mech. Astron.* **67**, 291011 (2024).
- [27] Z.-Y. Lin, J.-Z. Wang, J.-B. Cheng, L. Meng, and S.-L. Zhu, Identify the new state $Y(3872)$ as the P-wave $D\bar{D}^*/\bar{D}D^*$ resonance, *arXiv:2403.01727*.
- [28] L.-W. Yan, Z.-H. Guo, F.-K. Guo, D.-L. Yao, and Z.-Y. Zhou, Reconciling experimental and lattice data of $Z_c(3900)$ in a $J/\psi \pi - D\bar{D}^*$ coupled-channel analysis, *Phys. Rev. D* **109**, 014026 (2024).
- [29] M. Sadl, S. Collins, Z.-H. Guo, M. Padmanath, S. Prelovsek, and L.-W. Yan, Charmoniumlike channels 1^+ with isospin 1 from lattice and effective field theory, *arXiv:2406.09842*.
- [30] E. S. Swanson, Z_b and Z_c exotic states as coupled channel cusps, *Phys. Rev. D* **91**, 034009 (2015).
- [31] J.-Z. Wang, D.-Y. Chen, X. Liu, and T. Matsuki, Universal non-resonant explanation to charmoniumlike structures $Z_c(3885)$ and $Z_c(4025)$, *Eur. Phys. J. C* **80**, 1040 (2020).
- [32] L. von Detten, V. Baru, C. Hanhart, Q. Wang, D. Winney, and Q. Zhao, How many vector charmonium(-like) states sit in the energy range from 4.2 to 4.35 GeV?, *Phys. Rev. D* **109**, 116002 (2024).
- [33] S. Prelovsek and L. Leskovec, Search for $Z_c^+(3900)$ in the 1^{+-} channel on the lattice, *Phys. Lett. B* **727**, 172 (2013).
- [34] S. Prelovsek, C. B. Lang, L. Leskovec, and D. Mohler, Study of the Z_c^+ channel using lattice QCD, *Phys. Rev. D* **91**, 014504 (2015).
- [35] S.-h. Lee, C. DeTar, D. Mohler, and H. Na (Fermilab Lattice and MILC Collaborations), Searching for the $X(3872)$ and $Z_c^+(3900)$ on HISQ lattices, *arXiv:1411.1389*.
- [36] G. K. C. Cheung, C. O'Hara, G. Moir, M. Peardon, S. M. Ryan, C. E. Thomas, and D. Tims (Hadron Spectrum Collaboration), Excited and exotic charmonium, D_s and D meson spectra for two light quark masses from lattice QCD, *J. High Energy Phys.* **12** (2016) 089.
- [37] G. K. C. Cheung, C. E. Thomas, J. J. Dudek, and R. G. Edwards (Hadron Spectrum Collaboration), Tetraquark operators in lattice QCD and exotic flavour states in the charm sector, *J. High Energy Phys.* **11** (2017) 033.
- [38] Y. Chen *et al.*, Low-energy scattering of the $(D\bar{D}^*)^\pm$ system and the resonance-like structure $Z_c(3900)$, *Phys. Rev. D* **89**, 094506 (2014).
- [39] C. Liu, L. Liu, and K.-L. Zhang, Towards the understanding of $Z_c(3900)$ from lattice QCD, *Phys. Rev. D* **101**, 054502 (2020).
- [40] T. Chen, Y. Chen, M. Gong, C. Liu, L. Liu, Y.-B. Liu, Z. Liu, J.-P. Ma, M. Werner, and J.-B. Zhang (CLQCD Collaboration), A coupled-channel lattice study on the resonance-like structure $Z_c(3900)$, *Chin. Phys. C* **43**, 103103 (2019).
- [41] Y. Ikeda, S. Aoki, T. Doi, S. Gongyo, T. Hatsuda, T. Inoue, T. Iritani, N. Ishii, K. Murano, and K. Sasaki (HAL QCD Collaboration), Fate of the tetraquark candidate $Z_c(3900)$ from lattice QCD, *Phys. Rev. Lett.* **117**, 242001 (2016).
- [42] Y. Ikeda (HAL QCD Collaboration), The tetraquark candidate $Z_c(3900)$ from dynamical lattice QCD simulations, *J. Phys. G* **45**, 024002 (2018).
- [43] G.-J. Wang, Z. Yang, J.-J. Wu, M. Oka, and S.-L. Zhu, New insight into the exotic states strongly coupled with the $D\bar{D}^*$ from the T_{cc}^+ , *Sci. Bull.* **69**, 3036 (2024).
- [44] D.-Y. Chen, X. Liu, and T. Matsuki, Reproducing the $Z_c(3900)$ structure through the initial-single-pion-emission mechanism, *Phys. Rev. D* **88**, 036008 (2013).
- [45] D.-Y. Chen and X. Liu, Predicted charged charmonium-like structures in the hidden-charm dipion decay of higher charmonia, *Phys. Rev. D* **84**, 034032 (2011).

- [46] D.-Y. Chen, X. Liu, and T. Matsuki, Novel charged charmoniumlike structures in the hidden-charm dipion decays of $Y(4360)$, *Phys. Rev. D* **88**, 014034 (2013).
- [47] M. Cleven, Q. Wang, F.-K. Guo, C. Hanhart, U.-G. Meißner, and Q. Zhao, $Y(4260)$ as the first S -wave open charm vector molecular state?, *Phys. Rev. D* **90**, 074039 (2014).
- [48] F.-K. Guo, Triangle singularities and charmonium-like XYZ states, *Nucl. Phys. Rev.* **37**, 406 (2020).
- [49] N. Li, Z.-F. Sun, X. Liu, and S.-L. Zhu, Coupled-channel analysis of the possible $D^{(*)}D^{(*)}\bar{B}^{(*)}\bar{B}^{(*)}$ and $D^{(*)}\bar{B}^{(*)}$ molecular states, *Phys. Rev. D* **88**, 114008 (2013).
- [50] N. Li and S.-L. Zhu, Isospin breaking, coupled-channel effects and diagnosis of $X(3872)$, *Phys. Rev. D* **86**, 074022 (2012).
- [51] K. Yokokawa, S. Sasaki, T. Hatsuda, and A. Hayashigaki, First lattice study of low-energy charmonium-hadron interaction, *Phys. Rev. D* **74**, 034504 (2006).
- [52] X.-H. Liu, F.-K. Guo, and E. Epelbaum, Extracting $\pi\pi S$ -wave scattering lengths from cusp effect in heavy quarkonium dipion transitions, *Eur. Phys. J. C* **73**, 2284 (2013).
- [53] E. E. Jenkins, M. E. Luke, A. V. Manohar, and M. J. Savage, Semileptonic B_c decay and heavy quark spin symmetry, *Nucl. Phys.* **B390**, 463 (1993).
- [54] Y. Yamaguchi, Y. Abe, K. Fukukawa, and A. Hosaka, $\pi J/\psi - D\bar{D}^*$ potential described by the quark exchange diagram, *EPJ Web Conf.* **204**, 01007 (2019).
- [55] R. Blankenbecler and R. Sugar, Linear integral equations for relativistic multichannel scattering, *Phys. Rev.* **142**, 1051 (1966).
- [56] J.-J. Wu, T. S. H. Lee, and B. S. Zou, Nucleon resonances with hidden charm in coupled-channel models, *Phys. Rev. C* **85**, 044002 (2012).
- [57] R. L. Workman *et al.* (Particle Data Group), Review of particle physics, *Prog. Theor. Exp. Phys.* **2022**, 083C01 (2022).
- [58] F.-K. Guo, C. Hanhart, Q. Wang, and Q. Zhao, Could the near-threshold XYZ states be simply kinematic effects?, *Phys. Rev. D* **91**, 051504 (2015).
- [59] F.-K. Guo, X.-H. Liu, and S. Sakai, Threshold cusps and triangle singularities in hadronic reactions, *Prog. Part. Nucl. Phys.* **112**, 103757 (2020).
- [60] X.-H. Liu and G. Li, Exploring the threshold behavior and implications on the nature of $Y(4260)$ and $Z_c(3900)$, *Phys. Rev. D* **88**, 014013 (2013).
- [61] M. Achasov *et al.*, STCF conceptual design report (volume 1): Physics & detector, *Front. Phys. (Beijing)* **19**, 14701 (2024).
- [62] S. X. Nakamura, X. H. Li, H. P. Peng, Z. T. Sun, and X. R. Zhou, Global coupled-channel analysis of $e^+e^- \rightarrow c\bar{c}$ processes in $\sqrt{s} = 3.75\text{--}4.7$ GeV, [arXiv:2312.17658](https://arxiv.org/abs/2312.17658).
- [63] J.-J. Wu, T. S. H. Lee, A. W. Thomas, and R. D. Young, Finite-volume Hamiltonian method for coupled-channels interactions in lattice QCD, *Phys. Rev. C* **90**, 055206 (2014).
- [64] M. Luscher, Two particle states on a torus and their relation to the scattering matrix, *Nucl. Phys.* **B354**, 531 (1991).
- [65] M. T. Hansen and S. R. Sharpe, Multiple-channel generalization of Lellouch-Lüscher formula, *Phys. Rev. D* **86**, 016007 (2012).
- [66] K. Rummukainen and S. A. Gottlieb, Resonance scattering phase shifts on a nonrest frame lattice, *Nucl. Phys.* **B450**, 397 (1995).
- [67] Y. Li, J.-J. Wu, C. D. Abell, D. B. Leinweber, and A. W. Thomas, Partial wave mixing in Hamiltonian effective field theory, *Phys. Rev. D* **101**, 114501 (2020).
- [68] Y. Li, J.-j. Wu, D. B. Leinweber, and A. W. Thomas, Hamiltonian effective field theory in elongated or moving finite volume, *Phys. Rev. D* **103**, 094518 (2021).
- [69] Y. Li, J.-J. Wu, T. S. H. Lee, and R. D. Young, Generalized boost transformations in finite volumes and application to Hamiltonian methods, *J. High Energy Phys.* **08** (2024) 178.
- [70] J. M. M. Hall, A. C. P. Hsu, D. B. Leinweber, A. W. Thomas, and R. D. Young, Finite-volume matrix Hamiltonian model for a $\Delta \rightarrow N\pi$ system, *Phys. Rev. D* **87**, 094510 (2013).
- [71] Z.-W. Liu, W. Kamleh, D. B. Leinweber, F. M. Stokes, A. W. Thomas, and J.-J. Wu, Hamiltonian effective field theory study of the $N^*(1440)$ resonance in lattice QCD, *Phys. Rev. D* **95**, 034034 (2017).
- [72] J.-j. Wu, D. B. Leinweber, Z.-w. Liu, and A. W. Thomas, Structure of the Roper resonance from lattice QCD constraints, *Phys. Rev. D* **97**, 094509 (2018).
- [73] Z.-W. Liu, W. Kamleh, D. B. Leinweber, F. M. Stokes, A. W. Thomas, and J.-J. Wu, Hamiltonian effective field theory study of the $N^*(1535)$ resonance in lattice QCD, *Phys. Rev. Lett.* **116**, 082004 (2016).
- [74] C. D. Abell, D. B. Leinweber, Z.-W. Liu, A. W. Thomas, and J.-J. Wu, Low-lying odd-parity nucleon resonances as quark-model-like states, *Phys. Rev. D* **108**, 094519 (2023).
- [75] Z. Yang, G.-J. Wang, J.-J. Wu, M. Oka, and S.-L. Zhu, Novel coupled channel framework connecting the quark model and lattice QCD for the near-threshold D_s states, *Phys. Rev. Lett.* **128**, 112001 (2022).
- [76] V. Bernard, M. Lage, U.-G. Meißner, and A. Rusetsky, Resonance properties from the finite-volume energy spectrum, *J. High Energy Phys.* **08** (2008) 024.