

Elastic scattering at $\sqrt{s} = 6$ GeV up to $\sqrt{s} = 13$ TeV: Proton-proton; proton-antiproton, and proton-neutron

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In the framework of the Regge-eikonal model of hadron interaction based on the analyticity of the scattering amplitude, taking into account the hadron structure, a simultaneous analysis is carried out with 90 sets of data. These sets include data obtained at low energies ($\sqrt{s} > 3.6$ GeV) and high energies at FNAL, ISR, *SP̄S*, TEVATRON, and LHC with 4326 experimental points, including the polarization data of analyzing power. The energy and momentum transfer dependence of separate sets of data is analyzed on the basis of the eikonalized Born amplitude, taking into account two additional anomalous terms. Different origins of the nonlinear behavior of the slope of the scattering amplitude are compared. No contribution from the hard Pomeron in elastic hadron scattering is found. The importance of odderon's contribution is presented. The form and energy dependence of different terms of the cross-even and cross-odd parts of the elastic nucleon-nucleon scattering amplitude is determined in the framework of the high-energy general structure model. In this framework, using the electromagnetic and gravitomagnetic form factors, the differential cross sections in the Coulomb nuclear interference region and at large momentum transfer are described well simultaneously in a wide energy region. It is shown that the cross-even part, including the soft Pomeron and an additional term with a large slope, has energy dependence $\ln^2(s)$. The cross-odd part includes the maximal odderon term with $\ln^2(s)$ and an additional oscillation term with $\ln(s)$. It is shown that the additional terms with large slope are proportional to charge distributions, but the maximal odderon term and oscillation term are proportional to matter distributions. Additionally, a good description of proton-neutron differential scattering with 526 experimental points is obtained on the basis of the amplitudes taken from the analysis of pp and $p\bar{p}$ scattering. An adequate description of the polarization data was also obtained.

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I. INTRODUCTION

One of the most important tasks of modern physics is researching the basic properties of hadron interactions. The dynamics of strong interactions finds its most complete representation in elastic scattering. It is this process that allows the verification of the results obtained from the main principles of quantum field theory: The concept of the scattering amplitude as a unified analytic function of its kinematic variables connecting different reaction channels was introduced in the dispersion theory by Bogolyubov [1]. Now, many of the questions about hadron interactions are connected with modern problems of astrophysics such as unitarity and the optical theorem [2], as well as problems of baryon-antibaryon symmetry

and CP -invariance violation [3]. Many models predict that soft hadron interactions will enter a new regime at the LHC: Given the huge energy, the S -matrix reaches the unitarity limit. Small angles are the main domain of elastic scattering. Only in this region of interactions can we measure the basic properties that define the hadron structure. On the one hand, their values are connected with the large-scale structure of hadrons, and, on the other hand, they are connected with the first principles that lead to theorems on the behavior of scattering amplitudes at asymptotic energies [4,5].

The research of the structure of the elastic hadron scattering amplitude at superhigh energies and small momentum transfer t can give a connection between the experimental knowledge and the basic asymptotic theorems based on first principles [5–7]. This gives information about the hadron interaction at large distances where the perturbative QCD does not work [8–10] and a new theory, as instantons or string theories must be developed, for example.

The structure of hadrons reflected in generalized parton distributions (GPDs) is now one of the most interesting questions about the physics of strong interactions (see, for

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example, Refs. [11,12]). This is tightly connected with the spin physics of the hadron [13]. Some modern accelerator experiments have developed extensive programs for deep studies of different issues related to this problem. For example, the Jefferson Laboratory Electron-Ion Collider (EIC) team intends to extract generalized parton distributions (GPDs) [14]. The same problem is posed for future experiments at the SPD of NICA-JINR [15].

Recent studies of elastic scattering of high-energy protons lead to several unexpected results reviewed, e.g., in Refs. [9,16]. Spin amplitudes of the elastic NN scattering constitute a spin picture of the nucleon. Without knowing the spin NN amplitudes, it is impossible to understand the spin observable of nucleon scattering off nuclei. In the modern picture, the structure of hadrons is determined by GPDs, which include the corresponding parton distribution functions (PDFs). The sum rules [17–20] allow one to obtain the elastic form factor (electromagnetic and gravitomagnetic) through the first and second integration moments of GPDs [21]. This leads to remarkable properties of GPDs, some corresponding to inelastic and elastic scattering of hadrons. Now, different models examining the nonperturbative instanton contribution lead to sufficiently large spin effects at superhigh energies [22–25]. The research of such spin effects will be a crucial stepping stone for different models and will help us to understand the interaction and structure of particles, especially at large distances. There are many programs researching spin effects at different accelerators. In particular, we note the programs at the NICA, where the polarization of both the collider beams will be constructed. Thus, it is very important to obtain reliable predictions for spin asymmetries at these energies. In this paper, we extend the model predictions to spin asymmetries in the NICA energy domain.

The unique experiment carried out by the TOTEM Collaboration at the LHC at 13 TeV gave excellent experimental data on the elastic proton-proton scattering in a wide region of transfer momenta [26,27]. It is especially necessary to note the experimental data obtained at a small momentum transfer in the Coulomb-hadron interference region. The experiment reaches very small $t = 8 \times 10^{-4} \text{ GeV}^2$ with small Δt , giving a large number of experimental points in a sufficiently small region of momentum transfer. This allows one to carry out a careful analysis of the experimental data to explore some properties of hadron elastic scattering.

There are two sets of data—at a small momentum transfer [26] and at a middle and large momentum transfer [27]. They overlap in some region of the momentum transfer, which supplies practically the same normalization of both sets of differential cross sections of elastic proton-proton scattering. Recently, the first set of data has created much discussion on the determination of the total cross section and the value of $\rho(t=0)$ [28].

One very important characteristic of the elastic scattering amplitude is the ratio of the real part to the imaginary part of the scattering amplitude, $\rho(s, t)$. It is tightly connected with the integral and differential dispersion relations. Of course, especially after different results were obtained by the UA4 and UA4/2 Collaborations, physicists understand that $\rho(s, t=0)$ is not a simple experimental value but that it heavily depends on theoretical assumptions about the momentum dependence of the elastic scattering amplitude. Our analysis of experimental data obtained by both the UA4 and UA4/2 Collaborations shows a small difference value of $\rho(s, t=0)$ obtained in both experiments if the nonlinear behavior of the elastic scattering amplitude is taken into account [29]. Hence, this is not an experimental problem but a theoretical one [30].

For extraction of the sizes of σ_{tot} and $\rho(t=0)$, the Coulomb hadron region of momentum transfer is used (for example, see Ref. [26]). However, the form of the scattering amplitude assumed for small t and satisfying the existing experimental data at small momentum transfer can essentially be different from experimental data at large t . One should take into account the analysis of the differential cross section at 13 TeV where the diffraction minimum impacts the form of $d\sigma/dt$ already at $t = -0.45 \text{ GeV}^2$.

The analysis of new effects discovered on the basis of the experimental data at 13 TeV [30–32] and associated with the specific properties of the hadron potential at large distances was carried out by taking into account all sets of experimental data on elastic pp scattering obtained by the TOTEM and ATLAS Collaborations in a wide momentum transfer region; this analysis gave a quantitative description of all examined experimental data with minimum fitting parameters.

The nonlinear behavior of the slope of the differential cross sections at a small momentum transfer, which was shown by the TOTEM Collaboration in the proton-proton elastic scattering at 8 TeV, shows that the complex (complicated) form of strong interactions searched at low energies remains at superhigh energies, too. This means that the strong interaction is not simplified at superhigh energies but includes many different parts of the hadron potential.

Using the existing model of nucleon elastic scattering at high energies $\sqrt{s} = [3.6 - 14] \text{ TeV}$ [33,34], which involves a minimum of free parameters, we develop its extended version aimed to describe all available data on cross sections and spin-correlation parameters at lower energies down to the SPD NICA region. The model is based on the usage of known information on GPDs in the nucleon, and electromagnetic and gravitomagnetic form factors of the nucleon, taking into account analyticity and unitarity requirements and providing compatibility with the high-energy limit, where the Pomeron exchange dominates.

The structure of the paper is as follows: In Sec. II the basic picture of diffraction scattering is discussed, including the nonlinear behavior of the slope of the differential

cross section at small momentum transfer. In Sec. III the experimental data base used in the analysis is reviewed. In Sec. IV the asymptotic structure of the scattering amplitude is discussed briefly, and the form factors used are considered, in particular. Section V presents the different model approximations used in the model and the structure of the scattering amplitude. In Sec. VI the results of the fitting procedure and the behavior of the imaginary and real parts of the elastic scattering amplitude are considered. In Sec. VII the impact of the value of $\rho(s, t)$ and odderon contributions on the differential cross sections is analyzed. Section VIII discusses new effects discovered during the model analysis of experimental data. In Sec. IX the results obtained for proton-neutron elastic scattering are presented. In Sec. X the results of the model calculations of the spin correlation parameter $A_N(s, t)$ are presented. Finally, the results obtained and some predictions of our model are discussed in Sec. XI.

II. SMALL MOMENTUM TRANSFER REGION

A. Electromagnetic scattering

The t dependence of the elastic peak is an important issue that can also be related to optics. At high energy elastic scattering at small angles has some similarity with light scattering in the Fresnel region. The amplitude for the electric field propagating in the z direction is given by the Rayleigh-Sommerfeld equation

$$E(x, y, z) = -\frac{i}{\lambda} \int \int_{-\infty}^{+\infty} E(x', y', 0) \frac{e^{ikr}}{r} \cos \theta dx' dy', \quad (1)$$

where $r = \sqrt{(x - x')^2 + (y - y')^2 + z^2}$ and $\cos \theta = \frac{z}{r}$. The integral can be performed analytically only for the simplest geometrical cases. At $x = kR \gg 1$, we can obtain [35]

$$\mathcal{A}(\theta) \propto \frac{iJ_1(x \sin \theta)}{\sin \theta}.$$

Hence, even in the simplest case, the elastic peak cannot be described by a simple exponential. Note that this form of the scattering amplitude is used in diffractive hadron scattering.

B. Hadron scattering

In Ref. [36] the TOTEM Collaboration announced the observation of the nonexponential behavior of the differential elastic cross sections at 8 TeV and small momentum transfer $|t|$. Of course, the form of the elastic peak also depends on the structure of particles and the dynamics of the interaction. These two features can be parametrized by the profile function $\rho(\vec{b})$ (in the space of the impact parameter, $\vec{b}$) or by the elastic form factor (in momentum space). For example, in Ref. [37] different forms of the

profile function determined by the density distributions taking part in the interaction were considered:

$$\begin{aligned} &\text{circle of radius } b_0 = 4 \text{ GeV}^{-1} \\ &\Rightarrow \mathcal{A}(t) \propto J_0(4\sqrt{|t|}); \end{aligned} \quad (2)$$

$$\begin{aligned} &\text{hollow disk near } b_0 = 2.8 \text{ GeV}^{-1} \\ &\Rightarrow \mathcal{A}(t) \propto e^{2.8t} J_0(2.8\sqrt{|t|}); \end{aligned} \quad (3)$$

$$\begin{aligned} &\text{black disk of radius } 6 \text{ GeV}^{-1} \\ &\Rightarrow \mathcal{A}(t) \propto J_1(6\sqrt{|t|})/(6\sqrt{|t|}); \end{aligned} \quad (4)$$

$$\rho(b) \sim e^{-cb} \Rightarrow \mathcal{A}(t) \propto e^{5t}; \quad (5)$$

$$\rho(b) \sim e^{\mu\sqrt{b_0^2+b}} \Rightarrow \mathcal{A}(t) \propto e^{5(\sqrt{4\mu^2-t}-2\mu)}. \quad (6)$$

The t dependence of these amplitudes is shown in Fig. 1. Clearly, the simple exponential form (5) is a special, separate case. Except for the last case [Eq. (6)], we can see that if we take a small interval of t (for example, $0.01 < |t| < 0.15 \text{ GeV}^2$) the curves give practically the same result, especially the lines of Eqs. (3), (5), and (6). However, they give essentially different results on some larger values of t or as $t \rightarrow 0$. Hence, to obtain the true form of the scattering amplitude, it is necessary to take experimental data in a wider interval of t , tending to the limit $t \rightarrow 0$.

This simplest approximation of the differential cross section by one exponent with a constant slope can lead to new artificial effects. For example, in the experiment at the Protvino accelerator on elastic proton-proton scattering, “oscillations” were found at a small momentum transfer [38]. However, in Ref. [39] it was shown that such oscillations can appear when the differential cross section

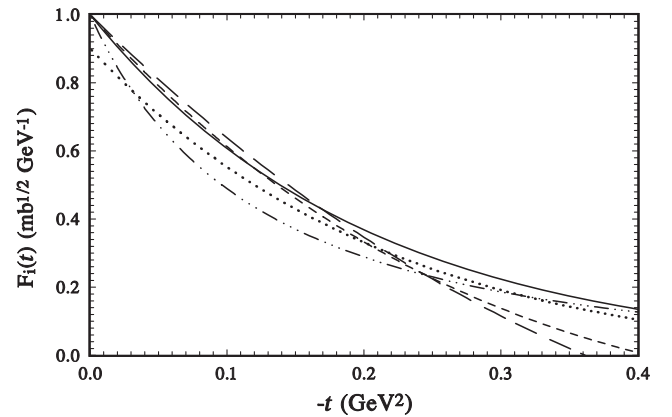


FIG. 1. Possible t dependence of the scattering amplitude $\mathcal{A}(t)$, rescaled to unity at $t = 0$. The solid line is for Eq. (5); the long-dashed line is for Eq. (2); the short-dashed line is for Eq. (4); the dotted line is for Eq. (3); and the dashed-dotted line is for Eq. (6).

is described by two exponentials with different slopes and one exponential with a constant slope.

The investigation of the energy and t dependence of the slope of the elastic peak can lead to new information about the structure of interacting particles. Early measurements at ISR revealed four slopes of differential cross sections in different regions of momentum transfer, which result from the complicated structure of nucleons.

C. Nonlinear slope

The complicated t dependence of the slope can have many origins. First, it comes from the unitarization procedure of the Born term of the elastic scattering amplitude. In many purely phenomenological analyses, it is represented by the Ct^2 term in the slope that mimics the unitarization procedure. The hadron structure is reflected not only at a large momentum transfer but also at small t . This especially concerns the meson clouds whose interaction in many models is added to the central part of hadrons (for example, in the Pumplin model [40] and the Dubna dynamical model [41], leading to the $\sqrt{t_0 + t}$ dependence of the scattering amplitude). Such a complicated hadron structure can be reflected in the presence of two form factors—the electromagnetic and matter form factors of hadrons [for example, the high-energy general structure (HEGS) model [33,34]].

Another term of the slope, which is commonly represented as

$$\sqrt{4\mu^2 - t} - 2\mu, \quad (7)$$

is used in many phenomenological descriptions of the elastic differential cross sections to explain the “break” in the differential cross sections. Note that an additional term in the slope, like $\sqrt{t_0 - t}$, was obtained earlier; its first approximation can be related to absorption corrections that can produce a set of canceling Regge cuts [42,43] and can lead to the Schwarz-type trajectories [44], $\alpha(t) = 1 + \gamma t^{1/2}$. A more complicated form was obtained in [45], $\alpha_{\pm}(t) = 1 \pm \gamma t^{1/2} + 2\rho(1/2\gamma^2 t)^{3/2}(-\ln(t))^{1/2}$. The appearance of a complex trajectory greatly complicates the picture and requires additional research. Hence, in [46], based on the works [47,48], it was proposed to use the simplest form $\alpha(t) = 1.041 - 0.15\sqrt{t_0 - t}$. However, as we show, such behavior has a really pure phenomenological basis and can be replaced by either a simpler form [for example, Eq. (7)] or a more complicated form as used in the HEGS-model [34].

Many models based on the famous works [49,50] researched the nonlinear behavior of the scattering amplitude. Based on the works in [51], it was found that

$$\alpha_P(q^2) = 1 - C_P q^2 - (\sigma_{\pi\pi}/32\pi^2)h_1(q^2), \quad (8)$$

where

$$h_1(q^2) = \frac{q^2}{\pi}, \quad \left[\frac{8\mu^2}{q^2} - \left(\frac{4\mu^2 + q^2}{q^2} \right)^{3/2} \ln \frac{\sqrt{4\mu^2 + q^2} + q}{\sqrt{4\mu^2 + q^2} - q} + \ln \frac{m^2}{\mu^2} \right], \quad (9)$$

with $q^2 = -t$. Note that we have removed the misprint in this equation, as made in [52,53]. They obtained the limits of the representation in the brackets at $q^2 \gg 4\mu^2$; hence, the slope grows at order q^2 with small logarithmic suppression. At small t ($q^2 \ll 4\mu^2$) those works predicted that the representation in the brackets goes to $\ln(m^2/\mu^2) - 8/3$. Note that the authors aimed to explain the deviation of the slope from the constant at a nonsmall momentum transfer (in the domain $-t = 0.4$ GeV²). However, in this domain the impact of the diffraction minimum is already felt. Hence, this domain of t is usually described by an additional term like ct^2 , which was proposed earlier (for example, Ref. [54]).

Practically at the same time, a similar equation was obtained in [55],

$$D_R^{NN}(t) = n \left[A - \left(\frac{(4\mu^2 + q^2)^{3/2}}{q} \right) \ln \frac{\sqrt{4\mu^2 + q^2} + q}{\sqrt{4\mu^2 + q^2} - q} \right]^{-1}. \quad (10)$$

That work proposed approximating this representation by

$$D_R^{NN}(t) \sim \left[\frac{1}{5\mu^2 - t} + C_{\text{const}} \right]. \quad (11)$$

For numerical calculations and comparison with experimental data, the authors found $C_{\text{const}} = 24.3$ GeV⁻².

In [52], using the calculations in [51] and removing the misprint, the authors proposed for h_1 , taking into account the meson form factor (with $\Lambda_\pi^2 = m_\rho^2$ GeV²), the following equation:

$$h_1(q^2) = \frac{4}{\tau} f_\pi^2(t) \left[2\tau - (1 + \tau)^{3/2}, \right. \\ \left. \ln \frac{\sqrt{1 + \tau} + 1}{\sqrt{1 + \tau} - 1} + \ln \frac{m^2}{\mu^2} \right], \quad (12)$$

where $\tau = 4\mu^2/q^2$ and

$$f_\pi(t) = \frac{\Lambda_\pi^2}{\Lambda_\pi^2 - t}. \quad (13)$$

For the slope, which is multiplied by q^2 , there are two divergence terms with different signs plus a small constant term. The divergence terms cancel each other, and the rest have a slow t dependence (see Fig. 2). The cancelation of the two terms as $t \rightarrow 0$ is not complete, and the rest also

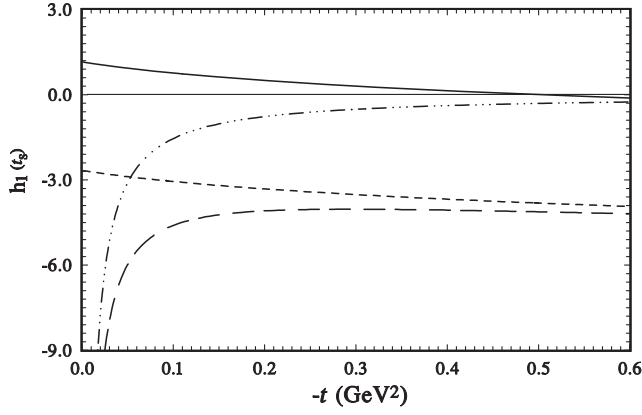


FIG. 2. The t dependence of the different parts of the terms of Eq. (9). The solid line is the sum of all three terms in the brackets; the dot-dashed line is the first term multiplied by (-1) ; the long-dashed line is the second term; and the short-dashed line is the sum of the two first terms.

have some indefiniteness and strongly depend on t . In Fig. 2 the t dependence of the different parts of the term of Eq. (9) in the brackets is shown. The cancelation of the two diverse terms leads to the negative contribution to the standard constant slope. However, the sum of all three terms gives a positive additional contribution with a small t dependence.

In Fig. 3 we show the t dependence of a different form of the slopes with the full kinematic coefficient. One can see a very different t dependence, especially in the form $\sqrt{4\mu^2 + q^2}$, which was proposed in [46] without the extraction of 2μ . Some models of hadron interactions at high energy suppose that the slope has a slow t dependence. For example, the Dubna dynamical model [41], which takes into account the contribution to hadron interactions from the meson cloud of the nucleon and uses the standard eikonal form of the unitarization, leads to the scattering amplitude in the form

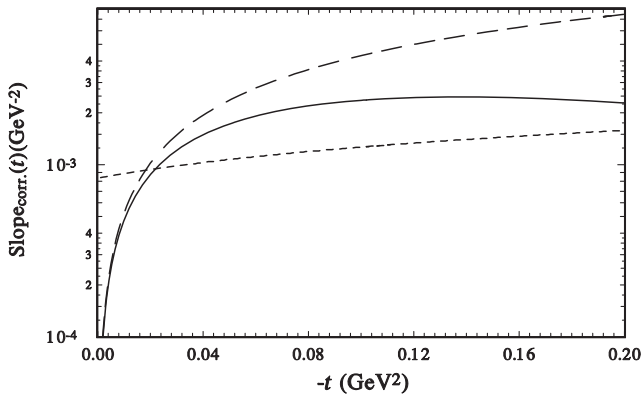


FIG. 3. The t dependence of the additional terms on $\alpha't$. The solid line is Eq. (9); the long-dashed line is Eq. (7); and the short-dashed line is Eq. (7) without the extraction of 2μ and multiplied by 0.1.

$$T(s, t) = -is \sum_{n=1}^{\infty} \frac{\mu}{(n^2 \mu^2 - t)^{3/2}} \times \left(1 - b\sqrt{n^2 \mu^2 - t}\right) e^{-b\sqrt{n^2 \mu^2 - t}}. \quad (14)$$

The analysis of the high-energy data on the proton-antiproton scattering in the framework of this model shows the obvious nonexponential behavior of the differential cross sections. For $\sqrt{s} = 540$ GeV it shows the change in the slope from 16.8 GeV^{-2} at $t = -0.001 \text{ GeV}^2$ up to 14.9 GeV^{-2} at $t = -0.12 \text{ GeV}^2$. For the same t the size of $\rho(s, t)$ changes from 0.141 up to 0.089. For Tevatron energy $\sqrt{s} = 1800$ GeV it shows the change of the slope from 18.1 GeV^{-2} at $t = -0.001 \text{ GeV}^2$ up to 15.9 GeV^{-2} at $t = -0.12 \text{ GeV}^2$, and again the size of $\rho(s, t)$ changes from 0.182 up to 0.143. Hence, the model shows the continuously decreasing slope and ρ at small t . Practically the same results were obtained in [40] in the framework of the model that also uses the eikonal unitarization.

The analysis [29] of the high precision data obtained at $SP\bar{P}S$ at $\sqrt{s} = 541$ GeV in the UA4/2 experiment shows the existence of the term proportional to $\sqrt{|t|}$ in the slope. This term can be related to the nearest π meson threshold, or as was shown in [29], such behavior of the differential cross sections can reflect the presence of the contribution of the spin-flip amplitude. As was noted in [56], the analytic S-matrix theory, perturbative quantum chromodynamics, and the data require Regge trajectories to be nonlinear complex functions [57,58]. The Pomeron trajectory has threshold singularities, the lowest one being due to the two-pion exchange required by the t -channel unitarity [51]. This threshold singularity appears in different forms in various models (see [56]). In the recent HEGS model [34], a small additional term is introduced into the slope, which reflects some possible small nonlinear properties of the intercept. As a result, the slope is taken in the form

$$B(s, t) = -\alpha' \ln(\hat{s}) [1 - d_1 t / \ln(\hat{s}) e^{d_2 \alpha_1 t \ln(\hat{s})}]. \quad (15)$$

This form leads to the standard form of the slope as $t \rightarrow 0$ and $t \rightarrow \infty$. Note that our additional term at large energies has a similar form as the additional term to the slope coming from the π loop examined in [51] and recently in [52].

III. EXPERIMENTAL DATABASE

In our research, we use the widest region of experimental data. The energy region begins at $\sqrt{s} = 3.5\text{--}3.8$ GeV for proton-antiproton scattering. At these energies new data were obtained in the high precision experiment on elastic $p\bar{p}$ scattering at small angles. There are four sets at different energies in the momentum transfer region $|t| = [0.000986 - 0.02] \text{ GeV}^2$ [59]. This experiment is very important as it obtained the value of $\rho(s, t = 0)$ with a remarkably small error and with a size near zero. It is essentially different from

the value of $\rho(s, t = 0)$ obtained in the framework of the dispersion relation analysis by Kroll, which was carried out on the basis of old experimental data with large errors.

Low-energy proton-antiproton data from $\sqrt{s} = 11.54$ GeV up to the final ISR energy $\sqrt{s} = 62$ GeV are represented in 11 sets [60]. Then, we include seven sets of experimental data obtained at the $SP\bar{P}S$ collider at energies around $\sqrt{s} = 540\text{--}630$ GeV. The Tevatron data at $\sqrt{s} = 1800\text{--}1960$ GeV are represented in four sets. The latter data present the maximal energy obtained for the proton-antiproton scattering at accelerators. On the whole, for the $p\bar{p}$ elastic scattering we have 33 sets of different experiments.

For the proton-proton elastic scattering we take into account 65 sets of different experiments from the low energy $\sqrt{s} = 6.1$ GeV up to the maximal LHC energy $\sqrt{s} = 13$ TeV [61,62]. We especially note the high precision experimental data obtained at a small momentum transfer by the FNAL at $\sqrt{s} = 9.8, 9.9, 10.6, 12.3$ GeV and at $\sqrt{s} = 19.4, 22.2, 23.9, 27.4$ GeV, starting from a very small momentum transfer $t = -0.00049$ GeV². These results can be compared with the data obtained by the UA4/2 Collaboration at the $SP\bar{P}S$ collider, which start from $t = -0.000875$ GeV², and with the data obtained at the LHC by the TOTEM Collaboration at $\sqrt{s} = 13$ TeV, which reached $t = -0.00029$ GeV². Of course, the last case achieved the smallest possible angles of scattering. We take into account the experimental data at a high value of momentum transfer up to $-t = 10\text{--}14$ GeV². The corresponding experimental data were obtained at energies $\sqrt{s} = 19.4, 27.4, 52.8$ GeV.

Summarizing all the sets of experimental data on elastic scattering at angles that are not large, we take into account 115 sets of different experiments, which include 4326 experimental points on proton-proton and proton-antiproton elastic scattering.

In our simultaneous analysis we include the data for the spin correlation parameter $A_N(s, t)$ of the polarized proton-proton elastic scattering. This set of data includes 235 experimental data for relatively small energies $\sqrt{s} = 3.63$ GeV and up to $\sqrt{s} = 23.4$ GeV. During our fitting procedure, sufficiently good descriptions of the experimental data were obtained.

For the first time, we also include in our research the elastic proton-neutron experimental data. The corresponding data are taken into account beginning from $\sqrt{s} = 4.5$ GeV up to maximal energy proton-neutron collisions $\sqrt{s} = 27.19$ GeV obtained at accelerators. It should be noted that such energy represents an average between minimum and maximum energies. For example, the maximum energy obtained at the FNAL averaged between $P_L = 340$ GeV/c and $P_L = 400$ GeV/c. It is very important that in such experiments a very small momentum transfer was reached, for example, $-t_{\min} = 0.2310 \cdot 10^{-4}$ GeV² at $\sqrt{s} = 23.193$ GeV. On the whole, we take into account 24 sets of experimental

data of different experiments that supply 526 experimental data. Hence, on the whole we take 5027 experimental data on elastic nucleon-nucleon scattering in our analysis.

IV. MAIN AMPLITUDES OF THE HEGS MODEL

A. Asymptotic part of the scattering amplitude

The HEGS model is based on the idea that at high energies a hadron interaction in the nonperturbative regime is determined by the nonperturbative Regge gluons exchange. The cross-even part of this amplitude can have two nonperturbative parts—a possible standard nonperturbative QCD Pomeron (P_{2np}) and a cross-even part of three nonperturbative gluons (P_{3np}) (see Fig. 4). The interaction of these two objects is proportional to two different form factors of the hadron, and both form factors are discussed in the next subsection. This is the main assumption of the model. Both terms have the same intercept. This corresponds to the maximal odderon, which was introduced by Lukaszuk and Nicolescu [63].

Note that the odderon arises naturally in perturbative QCD, for example [64], with the intercept $\alpha_{odd}(0) = 1$. This shows that shadowing corrections decrease odderon contributions. However, the experimental data show the odderon contribution at ISR energy $\sqrt{s} = 52.8$ GeV and at Tevatron energy [65], and probably at LHC energy $\sqrt{s} = 13$ TeV. For example, in [66] it was noted that it is possible to phenomenologically introduce the odderon as an object that does not violate first principles or axiomatic theorems. Reference [28] stated that the new TOTEM result is a definitive confirmation of the experimental discovery of the maximal form of the odderon.

In our fitting procedure we checked for such a possibility and made the fit with two intercepts. As a result we obtained the odderon intercept $\alpha_{0-odd} - 1 = 0.1101 \pm 0.0004$ (with fixed Pomeron intercept $\alpha_{0-Pom} - 1 = 0.11$). Note that χ^2 did not change. Hence, this confirms our assumption about the equality of both intercepts.

The second important assumption is that we choose the slope of the second term to be 4 times smaller than the slope of the first term, by analogy with the two Pomeron cuts. An estimation of the diffraction slope of the generalized BFKL Pomeron was made in [67] in terms of the correlation radius for the perturbative gluons. There it was shown that the Pomeron trajectory indeed has the finite slope $\alpha'_{ip} \sim R_c^2$, where R_c is the correlation radius for the perturbative gluons. It was noted that in the gBFL dynamics, a dimensionful α'_{ip} is a nonperturbative quantity related to the nonperturbative infrared parameter of the model—the gluon propagation radius R_c . For the preferred $R_c = 0.27$ fm, quite a small $\alpha'_{ip} = 0.072$ GeV² was found. In our work we use the phenomenological properties of the Pomeron (odderon), taking into account that the odderon only feels the center of the proton and not the pion cloud. Therefore, it is reasonable to assume that the odderon slope B_{odd} is lower

than that for the even-signature (Pomeron) amplitude [66]. This takes into account the data of the ISR experiments, which show that the slope of the differential cross sections after the second bump is approximately 4 times smaller than the slope at small t , as well as the conclusion of Landshoff that the odderon contributions are essentially important at large momentum transfer. Our fitting procedure also confirms such an assumption when we take the second slope as a free parameter.

B. Form factors

Since nucleons are not point particles, their structure must be taken into account. Particularly in the 1960s and 1970s, researchers proposed that the hadron form factor was proportional to the charge distribution into the hadron, which can be obtained from electron-nucleon scattering.

This is primarily due to the electromagnetic structure of the nucleon, which can be obtained from the electron-hadron elastic scattering. In the Born approximation, the Feynman amplitude for the elastic electron-proton scattering is

$$M_{ep \rightarrow ep} = \frac{1}{q^2} [e\bar{u}(k_2)\gamma^\mu u(k_1)][e\bar{U}(p_2)\Gamma_\mu(p_1, p_2)U(p_1)], \quad (16)$$

where u and U are the electron and nucleon Dirac spinors,

$$\Gamma^\mu = F_1(t)\gamma^\mu + F_2(t)\frac{i\sigma^{\mu\nu}q_\nu}{2m}. \quad (17)$$

Here, m is the nucleon mass, κ is the anomalous part of the magnetic moment, and $t = -q^2 = -(p - p')^2$ is the square

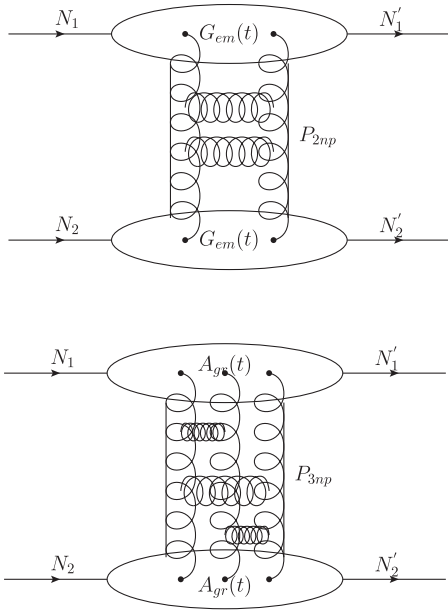


FIG. 4. Born amplitudes of nucleon-nucleon elastic scattering. Top diagram: two nonperturbative gluons, P_{2np} . Bottom diagram: three nonperturbative gluons, P_{3np} .

of the momentum transfer of the nucleon. The functions $F_1(t)$ and $F_2(t)$ are named the Dirac and Pauli form factors, which depend upon the nucleon structure.

However, it is not obvious that strong interactions have to be proportional to the electromagnetic properties of hadrons. Taking this fact into account in one of the famous models, Bourrely-Soffer-Wu [68] used some modification of the form factor with free parameters, which was obtained from the description of the differential cross section of hadron scattering. For example, it takes the form

$$G(t) = \frac{1}{1 - t/m_1^2} \frac{1}{1 - t/m_2^2} \frac{a^2 + t}{a^2 - t}. \quad (18)$$

However, this allows for some freedom in the t dependence of the scattering amplitude. It is necessary to take into account the parton distribution functions (PDFs) in the hadrons. However, PDFs depend on the Bjorken variable x . In the 1980s some relations between PDFs and hadron form factors were proposed.

Note that a function like GPDs ($x, t, \xi = 0$) was already used in the old “Valon” model proposed by Sanielevici and Valin in 1986 [69]. In the model, the hadron elastic form factor was obtained by the integration function $L(x)G(x, t)$, where $L(x)$ corresponds to the parton function and $G(x, t)$ corresponds to the additional function, which depends on momentum transfer and x . In modern language, $L(x)G(x, t)$ exactly corresponds to the GPDs. The scattering amplitude is

$$M_{AB}(s, t) = K_A(q^2)K_B(q^2)V(s, q^2), \quad (19)$$

where $V(s, q^2)$ is the potential of the strong interaction and $K_{AB}(q^2)$ are the corresponding form factors,

$$K_p(q^2) = \frac{1}{3} \int_0^1 dx \left[2L_p^{u,d}(x)T_p^{u,d}(\vec{k}) \right]. \quad (20)$$

Here, $\vec{k} = (1 - x)\vec{q}$, $k^2 = (1 - x)^2 q^2$ and

$$T_p^u(\vec{k}) = e^{6.1k^2}; \quad T_p^d(\vec{k}) = e^{3k^2}.$$

This form factor can be obtained by taking into account the PDF of interacting particles, which is multiplied by some function depending on the momentum transfer t and Bjorken variable x .

Many different forms of the t dependence of GPDs were proposed [70]. In the quark-diquark model [71] the form of GPDs consists of three parts—PDFs, function distributions, and Regge-like functions. In other works (see, e.g., Ref. [72]), the description of the t dependence of GPDs was developed in a more complicated picture using the polynomial forms with respect to x .

Commonly, the form of GPDs(x, ξ, t) is determined by the exclusive deep inelastic processes of type $\gamma^* p \rightarrow Vp$,

where V stands for a photon or a vector meson. However, such processes have a narrow region of momentum transfer, and in most models the t dependence of GPDs is taken in the factorization form with the Gaussian form of the t dependence. This form of GPDs(x, ξ, t) cannot be used to build a space structure of hadrons, as for that one needs to integrate over t in a maximally wide region.

The conjunction between the momentum transfer and the impact parameter allows one to obtain a space parton distribution that has probability conditions [10]. The connections between the deep-inelastic scattering, from which we can obtain the x dependence of parton distributions, and the elastic electron-nucleon scattering, where the form factors of the nucleons are obtained, can be derived by using the sum rules [17–20]. The form factors, which are obtained in different reactions, can be calculated as the Mellin moments of GPDs. Using the electromagnetic (calculated as the zero Mellin moment of GPDs) and gravitomagnetic form factors (calculated as the first moment of GPDs) in the hadron scattering amplitude, one can obtain a quantitative description of hadron elastic scattering in a wide region of energy and transfer momenta.

The proton and neutron Dirac form factors are defined as

$$\begin{aligned} F_1^p(t) &= e_u F_1^u(t) + e_d F_1^d(t), \\ F_1^n(t) &= e_u F_1^u(t) + e_d F_1^d(t), \end{aligned} \quad (21)$$

where $e_u = 2/3$ and $e_d = -1/3$ are the relevant quark electric charges. As a result, the t dependence of the GPDs($x, \xi = 0, t$) can be determined from the analysis of the nucleon form factors for which experimental data exist in a wide region of momentum transfer. This is a unique situation as it unifies elastic and inelastic processes.

In the limit $t \rightarrow 0$, the functions $H^q(x, t)$ are reduced to the usual quark densities in the proton:

$$\mathcal{H}^u(x, t = 0) = u_v(x), \quad \mathcal{H}^d(x, t = 0) = d_v(x),$$

with the integrals

$$\int_0^1 u_v(x) dx = 2, \quad \int_0^1 d_v(x) dx = 1$$

normalized to the number of u and d valence quarks in the proton. The energy-momentum tensor $T_{\mu\nu}$ [18,19,21,73] contains three gravitation form factors (GFF), $A^{Q,G}(t)$, $B^{Q,G}(t)$, and $C^{Q,G}(t)$. We scrutinize the first one, which corresponds to the matter distribution in a nucleon. This form factor contains the quark and gluon contributions

$$A^{Q,G}(t) = A_q^{Q,G}(t) + A_g^{Q,G}(t).$$

To obtain the true form of the proton and neutron form factors, it is important to have the true form of the momentum transfer dependence of GPDs. Let us choose

the t dependence of GPDs in a simple form, $\mathcal{H}^q(x, t) = q(x) \exp[a_+ f(x)t]$, with $f(x) = (1-x)^2/x^\beta$ [74]. The isotopic invariance can be used to relate the proton and neutron GPDs; hence, we have the same parameters for the proton and neutron GPDs.

The complex analysis of the corresponding description of the electromagnetic form factors of the proton and neutron by different PDF sets (24 cases) was carried out in [75]. These PDFs included the leading-order (LO), next-leading-order (NLO), and next-next-leading-order (NNLO) determinations of the parton distribution functions. They used different forms of the x dependence of PDFs. We slightly complicate the form of GPDs in comparison with the equation used in [74], but it is the simplest form compared to other works,

$$\begin{aligned} \mathcal{H}^u(x, t) &= q(x)^u e^{2a_H f(x)_u t}, \\ \mathcal{H}_\perp^d(x, t) &= q(x)^d e^{2a_H f_d(x)_d t}, \end{aligned} \quad (22)$$

$$\begin{aligned} \mathcal{E}^u(x, t) &= q(x)^u (1-x)^{\gamma_u} e^{2a_E f(x)_u t}, \\ \mathcal{E}_\perp^d(x, t) &= q(x)^d (1-x)^{\gamma_d} e^{2a_E f_d(x)_d t}, \end{aligned} \quad (23)$$

where $f_u(x) = \frac{(1-x)^{2+\epsilon_u}}{(x_0+x)^m}$, $f_d(x) = (1+\epsilon_0) \frac{(1-x)^{1+\epsilon_d}}{(x_0+x)^m}$.

The hadron form factors are obtained by integration over x in the whole range of x (0–1). Hence, the obtained form factors depend on the x dependence of the forms of the PDF at the ends of the integration region. The PDF sets from the inelastic processes were determined only in some region of x , which is only approximated to $x = 0$ and $x = 1$. Some PDFs have a polynomial form of x with different power. Others have the exponential dependence of x . As a result, the behavior of the PDFs, when $x \rightarrow 0$ or $x \rightarrow 1$, can impact the form of the calculated form factors.

In [75], 24 different PDFs were analyzed. On the basis of our GPDs with, for example, the PDFs ABM12 [76], we calculated the hadron form factors by numerical integration and then by fitting these integral results with the standard dipole form with some additional parameters,

$$F_1(t) = (4m_p - \mu t)/(4m_p - t) \tilde{G}_d(t),$$

with

$$\tilde{G}_d(t) = 1/(1 + a_1 q + q^2/a_2^2 + a_3^3 q^3)^2,$$

which is slightly different from the standard dipole form by two additional terms with small coefficients ($a_1 = 0.06$, GeV^{-1} , $a_2^2 = 0.78 \text{ GeV}^2$, and $a_3^3 = 0.08 \text{ GeV}^{-3}$). The matter form factor

$$A(t) = \int_0^1 dx [q_u(x) e^{2a_H f(x)_u t} + q_d(x) e^{2a_H f_d(x)_d t}] \quad (24)$$

is fitted by the simple dipole form $A(t) = \Lambda^4/(\Lambda^2 - t)^2$, with $\Lambda^2 = 1.6 \text{ GeV}^2$. These form factors will be used in

our model of proton-proton and proton-antiproton elastic scattering as well as in one of the vertices of pion-nucleon elastic scattering.

To check the momentum dependence of the spin-dependent part of GPDs, $E_{u,d}(x, \xi = 0, t)$, we can calculate the magnetic transition form factor, which is determined by the difference of $E_u(x, \xi = 0, t)$ and $E_d(x, \xi = 0, t)$. For the magnetic $N \rightarrow \Delta$ transition form factor $G_M^*(t)$, in the large N_c limit, the relevant $GPD_{N\Delta}$ can be expressed in terms of the isovector GPD yielding the sum rules [21].

The experimental data exist up to $-t = 8 \text{ GeV}^2$, and our results show a sufficiently good coincidence with experimental data. It is confirmed that the form of the momentum transfer dependence of $E(x, \xi, t)$ determined in our model is correct.

Now let us calculate the moments of the GPDs with the inverse power of x , which gives us the Compton form factors. The results of our calculations of the Compton form factors coincide well with the existing experimental data. Note that $R_V(t)$ and $R_T(t)$ have a similar momentum transfer dependence but differ essentially in size. On the contrary, the axial form factor R_A has an essentially different t dependence.

A good description of the variable form factors and elastic scattering of hadrons gives large support to our determination of the momentum transfer dependence of GPDs. Based on this determination of GPDs, one can calculate the gravitomagnetic radius of the nucleon using the integral representation of the form factor, and one can make the numerical differentiation over t as $t \rightarrow 0$. This method allows us to obtain a concrete form of the form factor by fitting the result of the integration of the GPDs over x . As a result, the gravitomagnetic radius is determined as

$$\langle r_A^2 \rangle = -\frac{6}{A(0)} \frac{dA(t)}{dt} \Big|_{t=0}. \quad (25)$$

We use the same procedure as for our calculations of the matter radius. As a result, the Dirac radius is determined from the zero Mellin moment of GPDs,

$$\langle r_D^2 \rangle = -\frac{6}{F(0)} \frac{dF(t)}{dt} \Big|_{t=0}, \quad (26)$$

where $F(t) = \int_0^1 (e_u q_u(x) + e_d q_d(x)) e^{-\alpha t f(x)} dx$.

One can obtain gravitational form factors of quarks, which are related to the second moments of GPDs. For $\xi = 0$, one finds

$$\int_0^1 dx x \mathcal{H}_q(x, t) = A_q(t); \quad \int_0^1 dx x \mathcal{E}_q(x, t) = B_q(t). \quad (27)$$

The parameters of the phenomenological form of GPDs can be obtained from the analysis of the experimental data for the proton and neutron electromagnetic form factors

simultaneously. Our determination of the momentum transfer dependence of GPDs of hadrons allows us to obtain good quantitative descriptions of different form factors, including the Compton, electromagnetic, transition, and gravitomagnetic form factors simultaneously.

V. MODEL APPROXIMATION AND THE STRUCTURE OF THE ELASTIC SCATTERING AMPLITUDE

The differential cross sections of nucleon-nucleon elastic scattering can be written as a sum of different helicity amplitudes:

$$\frac{d\sigma}{dt} = \frac{2\pi}{s^2} \left(|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 + |\Phi_4|^2 + 4|\Phi_5|^2 \right), \quad (28)$$

and the spin correlation parameter $A_N(s, t)$ is

$$A_N \frac{s^2}{4\pi} \frac{d\sigma}{dt} = -[\text{Im}(\Phi_1(s, t) + \Phi_2(s, t) + \Phi_3(s, t) - \Phi_4(s, t)\Phi_5^*(s, t))]. \quad (29)$$

The HEGS model [33,34] takes into account all five spiral electromagnetic amplitudes. The electromagnetic amplitude can be calculated in the framework of QED. For the spin-flip amplitudes, with the electromagnetic and hadronic interactions included, every amplitude $\Phi_i(s, t)$ can be described as

$$\Phi_i(s, t) = \Phi_i^{em} \exp(i\alpha\varphi(s, t)) + \Phi_i^h(s, t), \quad (30)$$

where $\varphi(s, t) = \varphi_C(t) - \varphi_{Ch}(s, t)$, and $\varphi_C(t)$ are calculated in the second Born approximation in order to allow for the evaluation of the Coulomb-hadron interference term $\varphi_{Ch}(s, t)$. The quantity $\varphi(s, t)$ has been calculated at a large momentum transfer, including the region of the diffraction minimum [77,78].

In the standard perturbative picture the spin-flip amplitudes decrease as energy grows. However, there are different nonperturbative approaches which show nondying spin-flip amplitudes, for example, Refs. [22–25]. One approach is by Kuraev [79,80], which we used for our nondying part of the spin-flip amplitude. We also use the second part of the spin-flip amplitude, which has a Regge form [Eq. (50)] and energy dependence of order $1/s$.

A. Electromagnetic amplitudes and phase factor

The electromagnetic amplitude can be calculated in the framework of QED in the one-photon approximation,

$$\begin{aligned}
\Phi_1^{em}(t) &= \alpha f_1^2 \frac{s - 2m^2}{t}, \quad \Phi_3^{em}(t) = \Phi_1^{em}(t), \\
\Phi_2^{em}(t) &= \alpha \frac{f_2^2(t)}{4m^2}, \quad \Phi_4^{em}(t) = -\phi_2^{em}(t), \\
\Phi_5^{em}(t) &= \alpha \frac{s}{2m\sqrt{|t|}} f_1^2.
\end{aligned} \tag{31}$$

The Coulomb-hadron interference phase is calculated with the dipole electromagnetic form factor [78]. For calculation of the Coulomb-hadron phase, we take the energy dependence of the slope in the form $b_{sl} = 6 + 0.75 \ln(s)$.

We take $\Lambda^2 = 0.71$ and $\gamma = 0.577215665$:

$$\begin{aligned}
\phi_a &= q^2(2\Lambda^2 + q^2)/\Lambda^4 \ln(\Lambda^2 + q^2)^2/(\Lambda^2 q^2), \quad \phi_b = \\
&= (\Lambda^2 + q^2)^2/(\Lambda^4(4\Lambda^2 + q^2)^2 q \sqrt{4\Lambda^2 + q^2}), \quad (4\Lambda^4(\Lambda^2 + 7q^2) + \\
&+ q^4(10\Lambda^2 + q^2)) \ln(4\Lambda^2/(\sqrt{(4\Lambda^2 + q^2) + q^2})), \quad \phi_c = (2\Lambda^4 - \\
&- 17\Lambda^2 q^2 - q^4)/(4\Lambda^2 + q^2)^2, \\
\phi_{1-3} &= \phi_a + \phi_b + \phi_c, \quad \phi_{CN} = -\ln b_{sl} q^2/2 + \gamma + \ln(1 + 8/ \\
&(b_{sl} \Lambda^2)) - \phi_{1-3}.
\end{aligned}$$

B. Hadron scattering amplitude

Let us define the hadron spin-nonflip amplitude as

$$F_{nf}^h(s, t) = [\Phi_{1h}(s, t) + \Phi_{3h}(s, t)]/2. \tag{32}$$

At small momentum transfer in the Coulomb nuclear interference (CNI) region, there are two contributions coming from the electromagnetic and the strong interactions.

On the one hand, the interference of such contributions gives the possibility to determine the size of the real part of the scattering amplitude. On the other hand, to determine the form of the imaginary part of the hadronic amplitude, it is necessary to extract the Coulomb contribution and the interference term. As the electromagnetic amplitude and its main contribution as $t \rightarrow 0$ are well known, the measure of experimental data at very small t gives the possibility to improve the normalization of the differential cross sections.

The existence of the different sets of experimental data also allows one to improve separate normalizations of experimental data. To compare different sets, an independent basis is needed.

Let us take the calculations of the differential cross sections carried out in the framework of the new HEGS model [33,34] as such a basis. The model has only a few free parameters, and it quantitatively describes experimental data in a wide domain of the momentum transfer, including the data in the CNI region, in a very wide energy region [from $\sqrt{s} = 6$ GeV up to LHC energies], simultaneously with the same number of the free parameters. The HEGS model assumes Born terms for the scattering amplitude, which have unitarization via the standard eikonal representation to obtain the full elastic scattering amplitude.

The scattering amplitude has exact $s \leftrightarrow u$ crossing symmetry as it is written in terms of the complexified

Mandelstam variable $\hat{s} = se^{-i\pi/2}$ that determines its real part. The scattering amplitude also satisfies the integral dispersion relation at large s . It can be thought of as the simplest unified analytic function of its kinematic variables connecting different reaction channels without additional terms for separate regions of momentum transfer or energy. Note that the model reproduces the diffraction minimum of the differential cross section in a wide energy region [81]. The HEGS model describes the experimental data at low momentum transfer, including the Coulomb-hadron interference region, and hence includes all five electromagnetic spin amplitudes and the Coulomb-hadron interference phase.

Let us determine the Born terms of the elastic nucleon-nucleon scattering amplitude using both (electromagnetic and gravitomagnetic) form factors,

$$\begin{aligned}
F_{\text{Pom}2}^{\text{Born}}(\hat{s}, t) &= h_{\text{Pom}2} G_{em}^2(t) F_a(\hat{s}, t), \\
F_{\text{Pom}3}^{\text{Born}}(\hat{s}, t) &= h_{\text{Pom}3} A_{gr}^2(t) F_b(\hat{s}, t), \\
F_{\text{odd}3}^{\text{Born}}(\hat{s}, t) &= h_{\text{odd}3} A_{gr}^2(t) F_b(\hat{s}, t),
\end{aligned} \tag{33}$$

where $F_a(\hat{s}, t)$ and $F_b(\hat{s}, t)$ have the standard Regge form:

$$F_a(s, t) = \hat{s}^{\alpha_0 + \alpha_1 t}, \quad F_{b,c}(s, t) = \hat{s}^{\alpha_0 + \alpha_2 t}, \tag{34}$$

with $\hat{s} = se^{-i\pi/2}/s_0$ and $s_0 = 4m_p^2 \text{ GeV}^2$. The intercept of all main parts of the scattering amplitudes, $1 + \alpha_0 = 1.11$, was chosen as an arithmetic means from different works, including those on inelastic scattering. Hence, at the asymptotic energy we have the universality of the energy behavior of the elastic hadron scattering amplitudes. The main part of the slope of the scattering amplitude has the standard logarithmic dependence on the energy $B(s) = \alpha_{1,2} \ln(\hat{s})$, with $\alpha_1 = 0.24 \text{ GeV}^{-2}$ and $\alpha_2 = \alpha_1/4 \text{ GeV}^{-2}$, taken with some correction [Eq. (15)].

Both the hadron electromagnetic and gravitomagnetic form factors are used in the framework of the HEGS model of elastic nucleon-nucleon scattering. This allows us to build the model with a minimum number of fitting parameters [33,34,82]. The Born term of the elastic hadron amplitude can now be written as

$$\begin{aligned}
F_h^{\text{Born}}(s, t) &= h_1 G^2(t) F_a(s, t) (1 + r_1/\hat{s}^{0.5}), \\
&+ h_2 A^2(t) F_b(s, t), \\
&\pm h_{\text{odd}} A^2(t) F_b(s, t) (h_{as} + r_2/\hat{s}^{0.75}) \\
&\times (-t)/(1 - r_d t),
\end{aligned} \tag{35}$$

where both (electromagnetic and gravitomagnetic) form factors are used. The corresponding parameters are determined by the fitting procedure.

The final elastic hadron scattering amplitude is obtained after unitarization of the Born term [83]. There are different approaches for summing the ladder diagrams—K-matrix, U-matrix, standard eikonal, and eikonal (with some

inelastic corrections [84]). We analyzed these forms and compared the results given by these approaches in some of our other works (for example, see [83,85,86]). In most cases, other approaches give worse results at high energy (especially at LHC energy). Moreover, we introduce some parameters reflecting the Kaidalov corrections and find that their sizes are near zero or unity. Thus, this conserves the standard eikonal form. In a recent work [87] the authors compared the U-matrix with the eikonal form and concluded that the eikonal approach better agrees with the ATLAS 13 TeV data, as seen in Fig. 3 of that work.

Thus, we use the eikonal representation, and at first, we have to calculate the eikonal phase

$$\chi(s, b) = -\frac{1}{2\pi} \int d^2q e^{i\vec{b}\cdot\vec{q}} F_h^{\text{Bom}}(s, q^2).$$

Then, we obtain the final hadron scattering amplitude,

$$F_h(s, t) = is \int b J_0(bq) (1 - \exp[\chi(s, b)]) db.$$

The essential property of the model is that the real part of the scattering amplitude is obtained automatically through the complex $\hat{s}$ only. The scattering amplitude has exact $s \leftrightarrow u$ crossing symmetry.

Thus, to extend the model to low energies, it is necessary to take into account the contributions of the second Reggions. To avoid a substantially increasing number of fitting parameters, we introduce the effective terms which represent the contributions of the sums of different Reggions. For terms with the energy dependence order $1/\sqrt{\hat{s}}$, we take, for the pp scattering, the cross-even part in the form

$$F_{R1} = ih_{R1}/\sqrt{\hat{s}} e^{b_{R1}t \ln \hat{s}}, \quad (36)$$

and for the $p\bar{p}$ the cross-odd term, in the form

$$F_{R1} = h_{R1}/\sqrt{\hat{s}} e^{b_{R1}t \ln \hat{s}}, \quad (37)$$

where the value b_{R1} is fixed by unity. For the cross-even part that is quickly decreasing with growing energy, we take a term in the form

$$F_{R2} = h_{R2}/\hat{s} e^{b_{R2}t \ln \hat{s}}. \quad (38)$$

As a result, the low-energy terms require only three additional fitting parameters.

The model is very simple from the viewpoint of the number of terms of the scattering amplitude and fitting parameters. There are no artificial functions or cuts which bound the separate parts of the amplitude by some region of momentum transfer. In the framework of the model, the description of experimental data was obtained

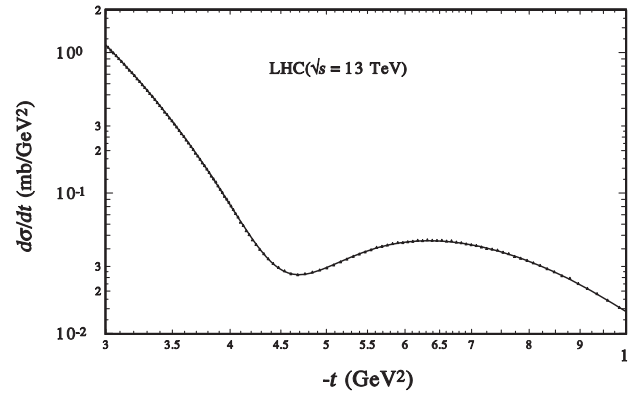


FIG. 5. HECS model calculation of $d\sigma/dt$ of pp scattering at $\sqrt{s} = 13$ TeV (points are experimental data of the TOTEM Collaboration [27]).

simultaneously at large momentum transfer and in the Coulomb-hadron region in the energy region from $\sqrt{s} = 6$ GeV up to LHC energies. The model gives a very good quantitative description of the recent experimental data at $\sqrt{s} = 13$ TeV in the region of the diffraction minimum [31] (see Fig. 5).

As a result, good descriptions of the experimental data of pp and $p\bar{p}$ elastic scattering (90 experimental sets including 4326 points and 256 polarization data) were obtained. Moreover, a good description of pn data was also obtained. Some examples of such descriptions are presented in Figs. 5–7. Of particular interest is Fig. 6, which gives the comparison of the HECS model calculation of $d\sigma/dt$ of pp scattering at $\sqrt{s} = 13$ TeV, $p\bar{p}$ scattering at $\sqrt{s} = 53$ GeV, and pn scattering at $\sqrt{s} = 23$ GeV at small momentum transfer. For the first time, a simultaneous study of proton-proton, proton-antiproton, and proton-neutron elastic scattering has been carried out in a wide energy (from

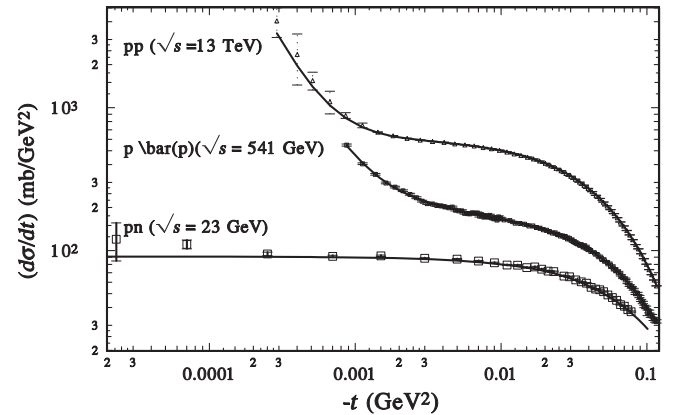


FIG. 6. Comparison of the HECS model calculation of $d\sigma/dt$ of pp scattering at $\sqrt{s} = 13$ TeV (experimental data [88]), $p\bar{p}$ scattering at $\sqrt{s} = 53$ GeV (experimental data [60]), and pn scattering at $\sqrt{s} = 23$ GeV (experimental data [60]).

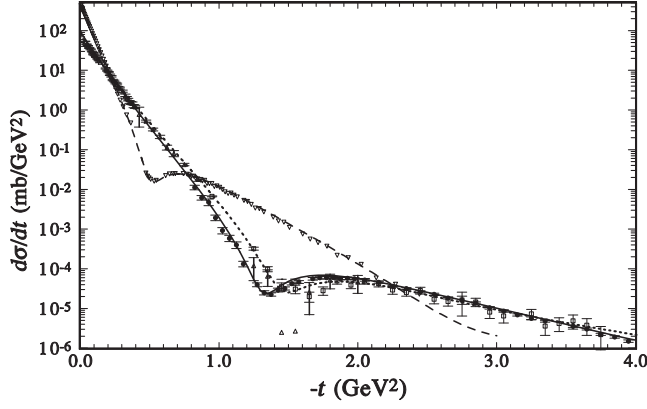


FIG. 7. Comparison of the HEGS model calculation of $d\sigma/dt$ of pp scattering in the dip-bump regions at $\sqrt{s} = 7$ TeV (dashed line and open down triangles), $\sqrt{s} = 53.8$ GeV (solid line and circles), and at $\sqrt{s} = 19.4$ GeV (tiny dashed line and boxes).

3.6 GeV up to 13 TeV) and momentum transfer region (from $|t| = 2 \times 10^{-4}$ GeV² up to $|t| = 14$ GeV²).

In the fitting procedure based on the modern version of FUMILY [89,90], we use only statistical errors. Systematic errors, which are mostly determined by indefiniteness of luminosity, are taken into account as an additional normalization coefficient, which is the same for all the data of a given set. The different normalization coefficients have practically random distributions. As a result, a wide range of possible forms of the scattering amplitudes decrease.

A simultaneous description of the cross sections and spin correlation parameter of different nucleon-nucleon reactions, including 90 sets of experimental data, with the total number of data $N = 4326$ gives a very reasonable result, $\sum_{i,j} \chi_{i,j}^2 = 4826$. The pn case with 526 experimental data, where the basic parameters are fixed from pp and $p\bar{p}$ scattering, gives $\sum_{i,j} \chi_{i,j}^2 = 585$.

The data of the TOTEM Collaboration at $\sqrt{s} = 7$ TeV are consistent, and their mean value is equal to 98.5 mb. The ATLAS Collaboration, using their differential cross section data in a region of t where the Coulomb-hadron

interference is negligible, obtained the value $\sigma_{\text{tot}} = 95.35 \pm 2.0$ mb. The difference between the two results, $\sigma_{\text{tot}}(s)(T.) - \sigma_{\text{tot}}(s)(A.) = 3.15$ mb, is about 1σ . At $\sqrt{s} = 8$ TeV the measured value of σ_{tot} grows, especially in the case of the TOTEM Collaboration, and the difference between the results of the two collaborations grows to $\Delta(\sigma_{\text{tot}}(s)(T.) - \sigma_{\text{tot}}(s)(A.)) = 5.6$ mb, i.e., 1.9σ . This is reminiscent of the previous situation with the measurement of the total cross sections at the Tevatron at $\sqrt{s} = 1.8$ TeV via the luminosity-independent method by different collaborations.

Of course, we cannot say that the normalization of the ATLAS data is better than that of the TOTEM data simply because it coincides with the HEGS predictions. However, this exercise may point to the main reason for the different values of the total cross sections obtained by the two collaborations. This does not exclude further problems with the analysis of experimental data, e.g., those related to the analysis of the TOTEM data at $\sqrt{s} = 7$ TeV [91].

One of the most important properties of the diffraction scattering is the form and energy dependence of the dip-bump structure. In Fig. 7 our model calculations are compared with the experimental data of pp scattering at ISR energies ($\sqrt{s} = 19.4$ GeV and $\sqrt{s} = 52.8$ GeV) and at LHC energies $\sqrt{s} = 7$ TeV. At $\sqrt{s} = 19.4$ GeV the real part of the pp scattering changes its sign at $t = 0$, and we can see that it gives a small contribution to the diffraction minimum, too. It is interesting that at $\sqrt{s} = 13$ TeV, the diffraction minimum also has a sharp form. The ratio of the maximum to minimum of the differential cross section at $\sqrt{s} = 19.4$ GeV is 2.4, and it equals 1.8 at the LHC energy. The position of the diffraction minimum $t_{\text{min}}(s, t)$ moves to low momentum transfer continuously [81] and is described by the model very well. It is interesting that the velocity of changing the position of the diffraction minimum is very slow. For example, from ISR energy $\sqrt{s} = 53$ GeV up to SPS energy $\sqrt{s} = 540$ GeV, this position changes with a speed of 0.11 GeV² per 100 GeV. Between $\sqrt{s} = 540$ GeV and $\sqrt{s} = 7$ TeV, such a speed is 2 times less and equals

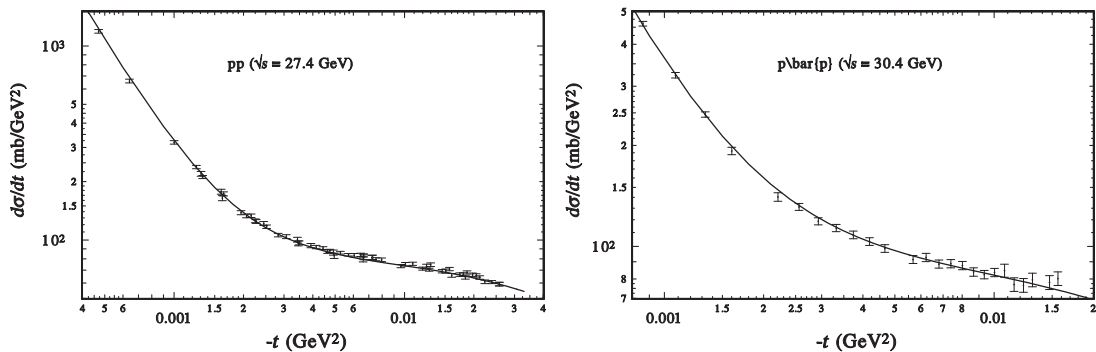


FIG. 8. Comparison of the HEGS model calculation of $d\sigma/dt$ of pp scattering at $\sqrt{s} = 27.4$ GeV (left) and $p\bar{p}$ scattering at $\sqrt{s} = 30$ GeV (right).

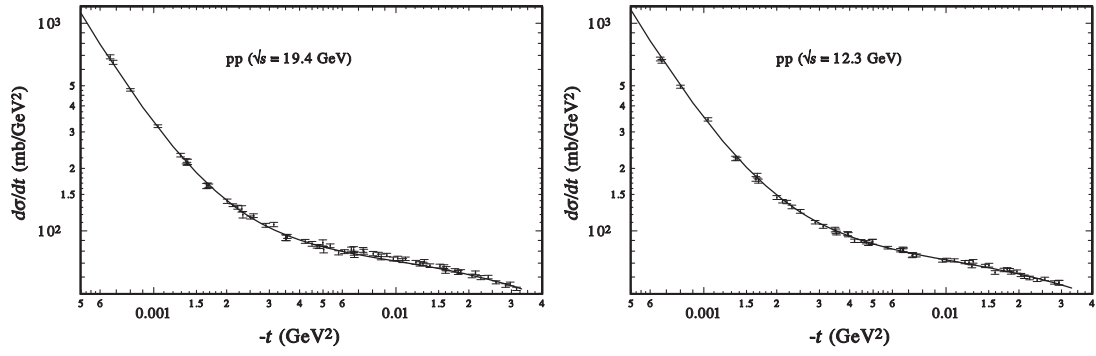


FIG. 9. Comparison of the HEGS model calculation of $d\sigma/dt$ of pp scattering at $\sqrt{s} = 19.45$ GeV (left) and at $\sqrt{s} = 12.3$ GeV (right).

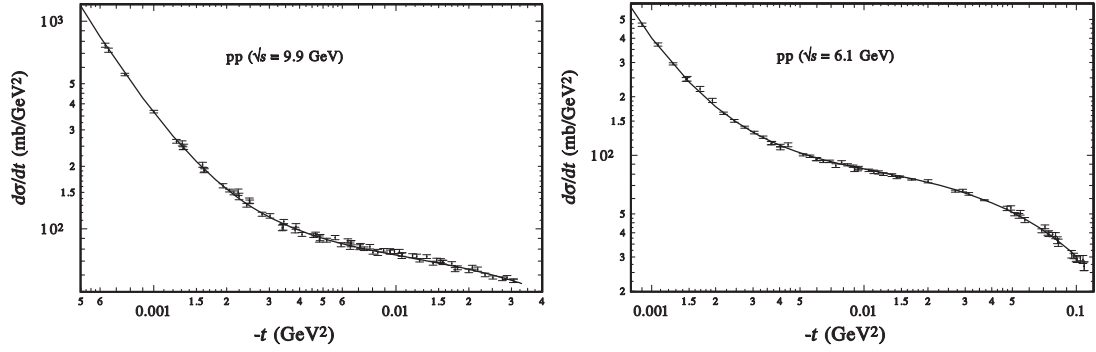


FIG. 10. Comparison of the HEGS model calculation of $d\sigma/dt$ of pp scattering at $\sqrt{s} = 9.9$ GeV (left) and at $\sqrt{s} = 6.1$ GeV (right).

0.006. Finally, the position of the minimum between 7 and 13 TeV changes with a speed of 0.002 per 100 GeV. The scaling of this process can approximately be represented as $t_{\min} \ln s/s_0 = \text{const}$. After the second bump the slope of the differential cross sections increases with energy. It corresponds to the increase of the slope of the diffraction peak. In Figs. 8–10 the model calculations are compared with experimental data at low energies at small momentum transfer for pp and $p\bar{p}$ scattering. It can be seen that the model reproduces well the existing experimental data of both reactions in that region.

The point of t , where the imaginary part changes its sign, determines the position of the diffraction minimum. However, it moves slightly at some large t from the contribution of the real part of the elastic hadron scattering amplitude. The behavior of the imaginary part of the scattering amplitude over momentum transfer calculated in the framework of the HEGS model is presented in Fig. 11 for energies $\sqrt{s} = 0.51, 7, 13$ TeV. Again, we can see the point of crossover in the region of $|t| = 0.2$ GeV^2 . Despite the essential growth of the imaginary part of the scattering amplitude at a very small momentum transfer, its slope slightly changes with t in the region of the Coulomb nuclear interference. The size of the slope is practically proportional to the size of the total cross section in that region. However, at

larger t , for example, at $|t| = 0.1$ GeV^2 , it essentially grows faster.

It should be noted that the size of the slope of the differential cross sections is determined in that region of momentum transfer by the CNI interference term, which is proportional to α_{em}/t (where α_{em} is a fine electromagnetic structure constant). This allows us to analyze [92] the first

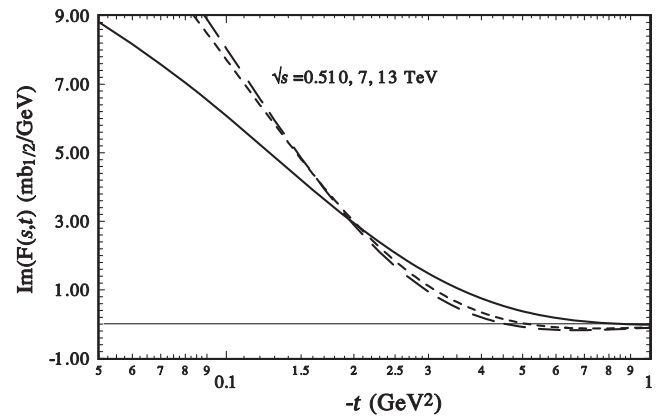


FIG. 11. Dependence of the imaginary part of the hadron scattering amplitude on s and t calculated in the model at energy $\sqrt{s} = 0.51, 7, 13$ TeV.

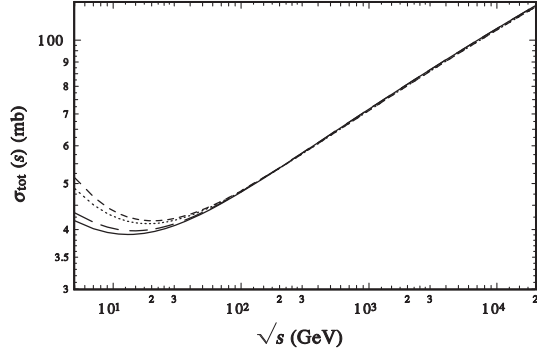


FIG. 12. The calculations σ_{tot} in the HEGS model. The dashed line is for $p\bar{p}$ scattering, the solid line is pp scattering. The same is shown with other lines but without the second Regge contributions: Tiny-dashed line is $p\bar{p}$ scattering; long-dashed line is pp scattering.

points of the unique experiment carried out by the ATLAS Collaboration [93].

The s dependence of the total cross sections of pp (solid line) and $p\bar{p}$ (dashed line) scattering is shown in Fig. 12. The difference of $\sigma_{\text{tot}}^{p\bar{p}}$ and σ_{tot}^{pp} is less than 1 mb at $\sqrt{s} = 20$ TeV.

C. Real part of the elastic scattering amplitude

It is interesting that the form of the real part of the hadron elastic scattering amplitude is similar to its imaginary part. Of course, they are not proportional to each other as their connection has to satisfy the dispersion relations [30], which require, for example, the changing size of the real part at sufficiently small momentum transfer. In fact, in Fig. 13, we see that the real part changes its sign essentially earlier than the position of the diffraction minimum. At LHC energies this happens in the area of momentum transfer approximately equal to 0.2 GeV^2 . However, before that, we can see the crossing point at $|t| \sim 0.06 \text{ GeV}^2$. It is likely that the behavior of the imaginary part at the slope of

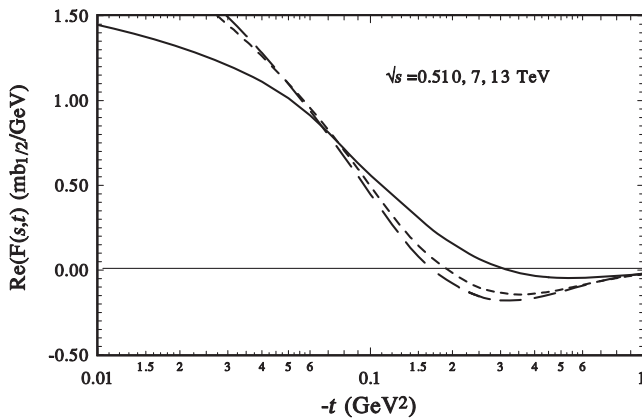


FIG. 13. Dependence of the real part of the hadron scattering amplitude on s and t calculated in the model at energies $\sqrt{s} = 0.51, 7, 13$ TeV.

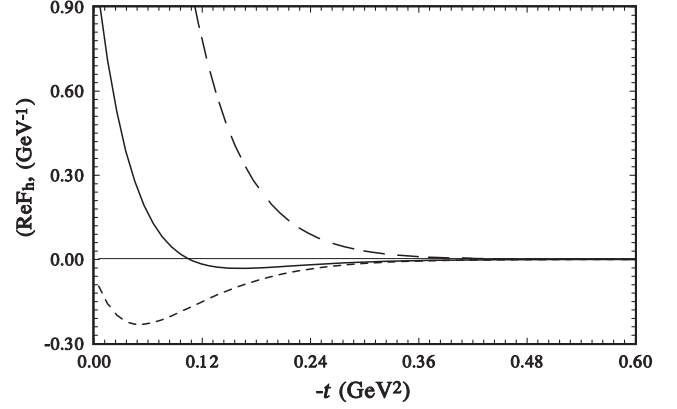


FIG. 14. Real part of the scattering amplitude coming from complex $\hat{s}$ in the Regge representation. The short-dashed line shows complex $\hat{s}$ only in the exponential, Eq. (41); the solid line shows the full representation, Eq. (42).

the real part at very small momentum transfer is also practically proportional to the size of the total cross section at different energies and that it grows at larger momentum transfers. It can be seen that the real part at LHC energies has the negative maximum at approximately $|t| = 0.3 \text{ GeV}^2$ situated near the diffraction minimum. Hence, it essentially impacts the form and size of the diffraction minimum in the differential cross sections.

The size and energy dependence of the real part of the hadronic amplitude especially impact the differential cross sections in the CNI region. The model descriptions of the differential cross section at a small momentum transfer and at low energies are presented in Figs. 6 and 8–10. These figures show the experimental data with high precision and only statistical errors, which were taken into account in our fitting procedure. Obviously, the model reproduces the experimental data very well in a wide energy region.

One of the origins of the nonlinear behavior of the differential cross sections may be the different t dependencies of the imaginary and real parts of the scattering amplitude. For the most part, in different approaches it is supposed that this t dependence is the same for both parts of the scattering amplitude. It should be noted that the importance of determining the size of the real part of the scattering amplitude was emphasized in many works of Martin [94]. If at the LHC the value of $\rho(s, t)$ is measured at high precision, it is possible to check the validity of the dispersion relations [56].

In the analysis of the experimental data [95], two cases were considered. One is the so-called “central” case, in which the ratio of the real to imaginary parts of the scattering amplitude is independent of momentum transfer or it slightly decreases. The other case, the so-called “peripheral” case, takes into account the assumption that $\rho(s, t)$ grows with momentum transfer. In fact, the last case contradicts the dispersion relations; hence, it has nonphysical motivation.

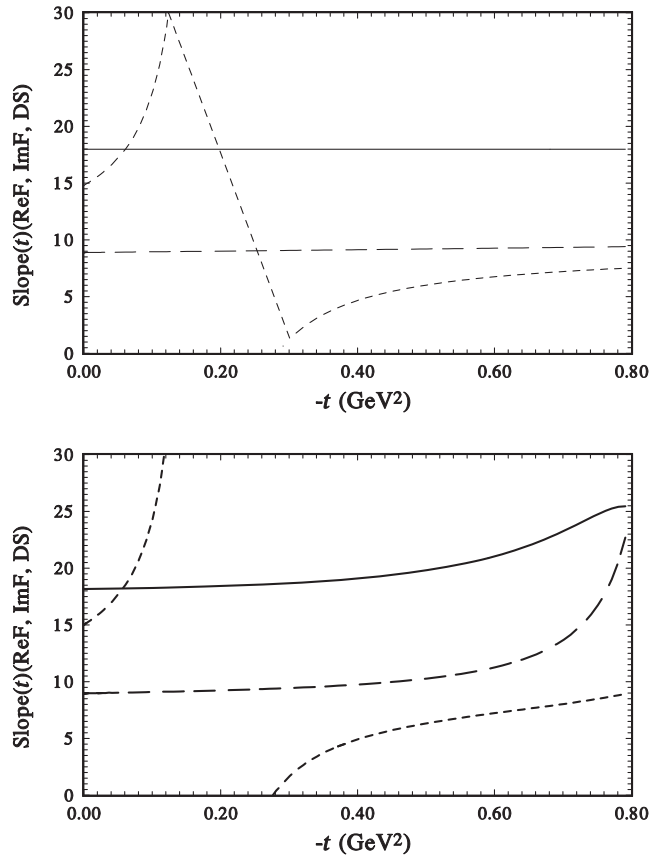


FIG. 15. Slopes of the Born term of the scattering amplitude (top) and the unitarized scattering amplitude (bottom). The long-dashed line is the slope of the imaginary part, the short-dashed line is the slope of the real part, and the solid line is the slope of the differential cross sections.

As the scattering amplitude has to be an analytic function of its kinematic variable, let us take the energy dependence of the scattering amplitude through the complex $\hat{s} = se^{-i\pi/2}$; note that it must satisfy the dispersion relations.

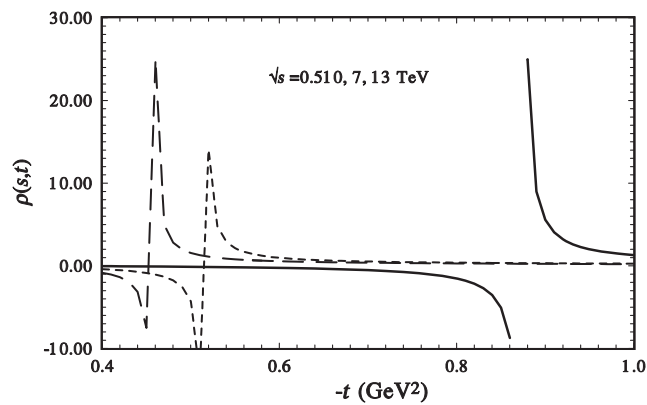


FIG. 16. Size of the $\rho(s, t)$ ratio of the real to the imaginary part of the pp scattering amplitude calculated in the model at energies $\sqrt{s} = 0.51, 7, 13$ TeV (solid, short-dashed, and dashed line, respectively) depending on s and t .

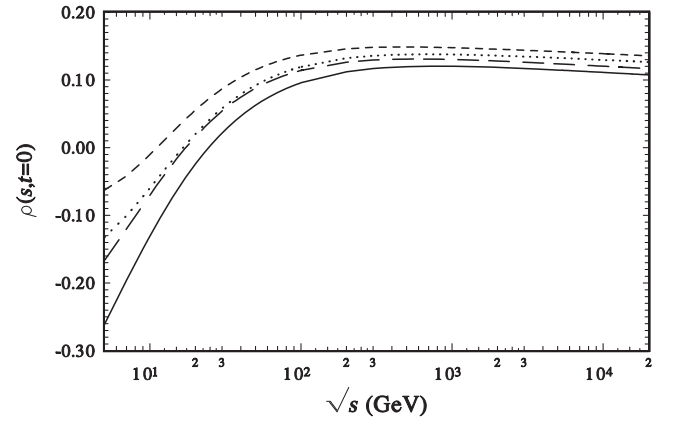


FIG. 17. The calculations of $\rho(s, t=0)$ in the HEGS model. The dashed line is for $p\bar{p}$ scattering, and the solid line shows pp scattering. The same is shown with other lines but without the second Reggion contributions: The tiny-dashed line is $p\bar{p}$ scattering, and the long-dashed line is pp scattering.

For simplicity, very often the so-called local or derivative dispersion relations are used (see, for example, Ref. [45]) to determine the real part of the scattering amplitude. For example, the COMPETE Collaboration used

$$\text{Re}F_+(E, 0) = \left(\frac{E}{m_p}\right)^\alpha \tan\left[\frac{\pi}{2}\left(\alpha - 1 + E\frac{d}{dE}\right)\right] \times \text{Im}F_+(E, 0) \left(\frac{E}{m_p}\right)^\alpha. \quad (39)$$

A different form of the derivative dispersion relation was taken as [96]

$$\text{Re}F_+(E, 0) = \left(\frac{\pi}{\ln(s/s_0)}\right) \frac{d}{d\tau} [\tau \text{Im}F_+(s, t)/\text{Im}_+(s, t=0)] \times \text{Im}F_+(s, t=0), \quad (40)$$

where $\tau = t(\ln(s/s_0))^2$ and as $s \rightarrow \infty$. To satisfy these relations, the scattering amplitude has to be a unified analytic function of its kinematic variables connecting different reaction channels.

In most cases one assumes that the real part is taken proportional to the imaginary part of the scattering amplitude. Thus, the slopes of both parts are equal to each other. There is also an unusual assumption about the growth of the real part of the scattering amplitude at a small momentum transfer relative to its imaginary part (the so-called peripheral case [95]). Obviously, neither assumption satisfies the dispersion relations, especially the last one.

Let us examine the origins of the complicated t dependence of the real part. Take the scattering amplitude in the form

$$F(s, t) = h s^\Delta e^{B i \text{Ln}(\hat{s})}, \quad (41)$$

where the complex $\hat{s}$ is used only in the exponential. In this case, the real part is negative (in Fig. 14, it is shown by the short-dashed line) and essentially differs from the behavior of the imaginary part (long-dashed line). Now let us use $\hat{s}$ in all parts of the scattering amplitude,

$$F(s, t) = h\hat{s}^\Delta e^{Bt \ln(\hat{s})}. \quad (42)$$

The dispersion relations lead to the fact that the slope of the real part of the scattering amplitude must be larger than the slope of the imaginary part. For example, if the imaginary part of the spin-nonflip hadron elastic scattering amplitude takes a simple exponential form $\text{Im}F_+ \sim h e^{Bt}$, then from Eq. (40) we have that the real part of F_+ is $\text{Re}F_+ = (1 + Bt)e^{Bt}$. Hence, it has a zero in the region of momentum transfer around $-0.1 - -0.15 \text{ GeV}^2$.

Last, but not least, it should be noted that the unitarization procedure has a strong influence on the t dependence of the real part of the scattering amplitude. For example, take the standard eikonal form of unitarization. If the Born term is taken in the ordinary exponential form [Eq. (42)], then the imaginary part has a constant slope [see Fig. 15 (top)], and the real part has the first zero at a small momentum transfer. After eikonalization both slopes of the imaginary and real parts of the scattering amplitude have a strong t dependence [Fig. 15 (bottom)].

VI. VALUE OF $\rho(s, t)$ AND ODDERON CONTRIBUTIONS

In Fig. 16 the ratio $\rho(s, t)$ of the real to the imaginary part of the hadron elastic scattering amplitude is presented for different energies. Such a complicated structure of $\rho(s, t)$ is determined by the changes of the sign of the real and imaginary parts of the scattering amplitude. At a small momentum transfer, the size of $\rho(s, t)$ is small as the real part changes sign. On the contrary, when the imaginary part changes its sign, the size of $\rho(s, t)$ grows much faster. The energy dependence of $\rho(s, t)$ is due to the movement of the position of the diffraction minimum and hence to the energy dependence of the imaginary part of the scattering amplitude (see Fig. 17).

At high energies and in the region of small momentum transfer, the difference between the pp and $p\bar{p}$ differential cross sections comes mostly from the CNI term, as the real part of the Coulomb amplitude has a different sign in these reactions. In the standard fitting procedure, one neglects the α_{em}^2 term, and then the equation takes the form

$$d\sigma/dt = \pi[(F_C(t))^2 + (\rho(s, t)^2 + 1)(\text{Im}F_N(s, t))^2 + 2(\rho(s, t) + \alpha\varphi(t))F_C(t)\text{Im}F_N(s, t)], \quad (43)$$

where $F_C(t) = \mp 2\alpha G^2(t)/|t|$ is the Coulomb amplitude (the upper sign is for pp , and the lower sign is for $p\bar{p}$) and $G^2(t)$ is the proton electromagnetic form factor squared;

$\text{Re}F_N(s, t)$ and $\text{Im}F_N(s, t)$ are the real and imaginary parts of the hadron amplitude; and $\rho(s, t) = \text{Re}F_N(s, t)/\text{Im}F_N(s, t)$. Formula (43) is often used for the fit of the experimental data in obtaining hadron amplitudes and the Coulomb-hadron phase in order to obtain the value of $\rho(s, t)$. One can clearly see the impact of the value of $\rho(s, t)$ on the form and energy dependence of the differential cross sections.

The odderon amplitude has a large real part compared to its imaginary part. Hence, the odderon contribution changes the size and t dependence of the real part of the full amplitude. In a recent paper [97] different cases (with and without odderon contributions) were analyzed. It was noted that the effect of incorporating the odderon becomes notably significant when analyzing specific subsets of data. As noted in that work, $\rho^{p\bar{p}}$ will become too large at $\sqrt{s} \sim 541 \text{ GeV}$, in disagreement with the data from the UA4/2 Collaboration. As a result, they restored a previous problem of the value of $\rho(s, t)$ at $\sqrt{s} \sim 540 \text{ GeV}$. However, as was noted in the Introduction, this problem strongly depends on the form of the nonlinear slope and was solved in [29,98]. Now, in the present model, the odderon amplitude essentially decreases as $t \rightarrow 0$. We find $\rho(\sqrt{s} = 541 \text{ GeV}, t = 0) = 0.122$, which coincides with the result of the UA4/2 Collaboration.

In Fig. 18 the ratio of the cross-odd part of the scattering amplitude in the HEGS model of elastic proton-proton scattering to its cross-even part at 13 TeV and at 9.26 GeV is presented. It is seen that this ratio is small at 13 TeV, except for the position of the diffraction minimum. At low energy, this ratio is small at small t but increases essentially in the region of a large momentum transfer.

However, the odderon contribution is very important at the diffraction minimum (see Fig. 19). The real part of the scattering amplitude, which is positive at $t = 0$ and $\sqrt{s} > 30 \text{ GeV}$, changes its sign at larger t ; thus, it corresponds to the dispersion relations, but, in any case, it fulfills the diffraction dip in the differential cross sections.

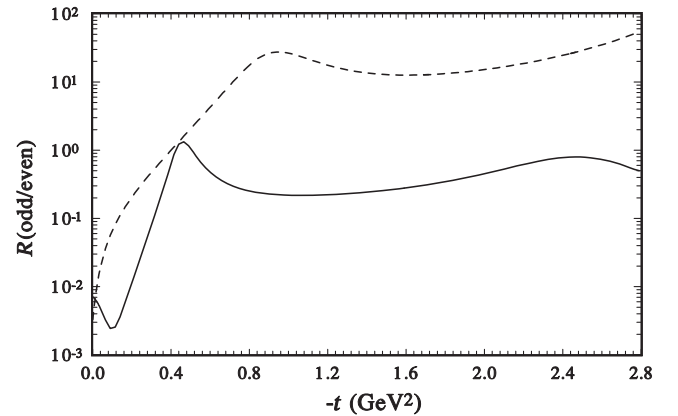


FIG. 18. Ratio $R(s, t)$ of the cross-odd part of the scattering amplitude of elastic proton-proton scattering to its cross-even part at 13 TeV (dashed line) and at 9.26 GeV (solid line).

VII. NEW EFFECTS IN DIFFRACTION ELASTIC SCATTERING AT SMALL ANGLES

In the fitting procedure of the experimental data, only statistical errors were taken into account. As the systematic errors are mostly determined by indefiniteness of luminosity, they are taken into account as an additional normalization coefficient. This method essentially decreases the space of a possible form of the scattering amplitude, which allows us to find the manifestation of small effects at 13 TeV experimental data for the first time [19–21]. Further research, taking into account a wider range of experimental data, confirms the existence of such new effects. We determine the new anomalous term with a large slope as

$$f_{\text{an}}(t) = ih_{0\text{an}} \ln(\hat{s}/s_0)/k(1 + h_{1\text{an}} \ln(\hat{s}/s_0)/k) \times \exp[-\alpha_{\text{an}}(|t| + (2t)^2/t_n) \ln(\hat{s}/s_0)] G_{\text{em}}^2(t), \quad (44)$$

where h_{an} is a constant determining the size of the anomalous term with a large slope, α_{an} ; $G_{\text{em}}^2(t)$ is the electromagnetic form factor, which was determined from the GPDs [75]; $k = \ln(13000^2 \text{ GeV}^2/s_0)$ is introduced for normalization of h_{an} at 13 TeV; and $t_n = 1 \text{ GeV}^2$ is the normalization factor. This form adds only two additional fitting parameters, and it is supposed to grow with energy of order $\ln^2(\hat{s}/s_0)$. The analysis of the data in a wide energy region gives a complicated energy dependence, which is reflected in Eq. (44). The anomalous term has a large imaginary part and a small real part determined by the complex $\hat{s}$. It is remarkable that the anomalous slope is determined with high accuracy, $\alpha_{\text{an}} = 0.513 \pm 0.004 \text{ GeV}^{-2}$, from the analysis of the whole set of experimental data.

The anomalous term is proportional to electromagnetic form factors, and the analysis of the experimental data above 6 GeV gives the size of the constant $h_{\text{an}} = 1.13 \pm 0.04$. This result is a little less than that obtained in the fit of only high-energy data, but now it has a more complicated energy dependence and a very small error. It impacts the differential cross sections at small t as well as the size of $\sigma_{\text{tot}}(s)$. However, it barely influences the value of $\rho(s, t)$ as the imaginary and real parts of the scattering amplitude grow proportionally. The contributions of the anomalous term with a large slope to the differential cross sections of pp scattering,

$$\Delta(d\sigma/dt = d\sigma/dt|_{\text{with}} - d\sigma/dt|_{\text{without}}),$$

are shown in Fig. 20 at $\sqrt{s} = 19 \text{ GeV}$ (solid line) and at $\sqrt{s} = 13 \text{ TeV}$ (dashed line). It can be seen that the contribution of the anomalous term disappears after $-t > 0.2 \text{ GeV}^2$. At low energies its relative contribution is less but remains visible and slowly decreases with t .

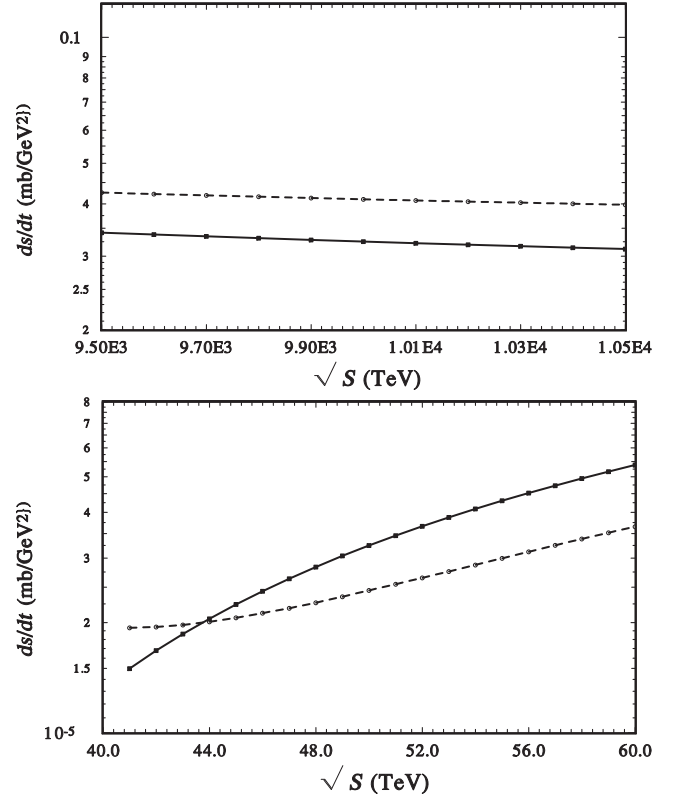


FIG. 19. Contributions of the odderon part to the differential cross sections in the region of the dip. Top: $-t = 0.45 \text{ GeV}^2$. Bottom: $-t = 1.45 \text{ GeV}^2$.

Our method helps us find a small oscillation effect in the differential cross section at small momentum transfer [31]. Such an oscillation can be determined by an additional oscillation term in the scattering amplitude [99]. The theoretical predictions of some oscillations in the differential cross section have a long history, starting from one of the first papers [100] taking into account a multi-Pomeron exchange, as well as papers based on the violation of the

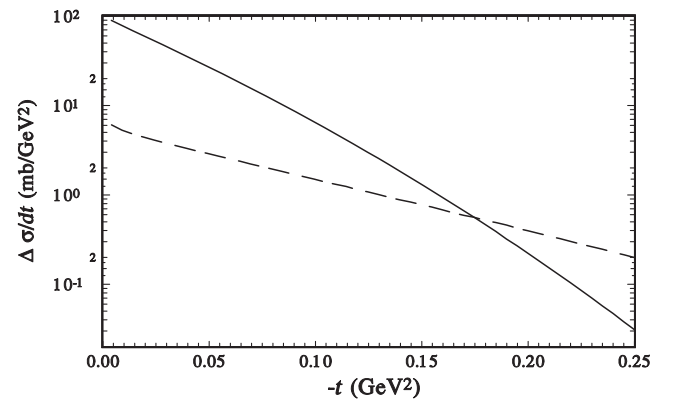


FIG. 20. Contributions $\Delta(d\sigma/dt)$ of the anomalous term with a large slope to the differential cross sections of pp scattering at $\sqrt{s} = 19 \text{ GeV}$ (solid line) and at $\sqrt{s} = 13 \text{ TeV}$ (dashed line).

Pomeranchuk theorem (for example, Ref. [6]), up to more recent papers such as [101]. In our fitting procedure the oscillatory function is

$$f_{\text{osc}}(t) = ih_{\text{osc}}(1 \pm ih_{\text{osc}})(\ln(\hat{s}/s_0)/k + h_s/\hat{s}) \times J_1(\tau)/\tau A^2(t),$$

$$\tau = \pi(\phi_0 - t)/t'_0. \quad (45)$$

Here, $J_1(\tau)$ is the Bessel function of the first order; $t_0 = 1/[a_p/(1+2\ln^2(\hat{s}/s_0)/k)]$, where $a_p = 17.15 \text{ GeV}^{-2}$ is the fitting parameter that leads to the AKM scaling on $\ln(\hat{s}/s_0)$; $A(t)$ is the gravitomagnetic form factor, which was determined from the GPDs [75]; and h_{osc} is the constant that determines the amplitude of the oscillatory term, with the period determined by τ . This form only has a few additional fitting parameters, and it allows one to represent a wide range of possible oscillation functions. The phase ϕ_0 is obtained near zero for pp scattering, and it is slightly small and negative for $p\bar{p}$ scattering; thus, it has a different sign for pp and $p\bar{p}$ scattering. The inclusion of the $p\bar{p}$ elastic scattering data in the fitting procedure shows that part of the oscillation function changes its sign for the crossing reactions. As a result, the plus sign is related to pp and the minus sign to $p\bar{p}$ elastic scattering. Hence, this part is the crossing-odd amplitude, which has the same simple form for pp and $p\bar{p}$ scattering, only with a different sign.

The wider energy region used in this analysis allows one to reveal the logarithmic energy dependence of the oscillation term. The constant (size) of the oscillation function is determined from experimental data at high energies (all data above 500 GeV) as $h_{\text{osc}}^c = 0.270 \pm 0.007 \text{ GeV}^{-2}$, whereas in our analysis of the whole set of data, $\sqrt{s} > 3.6 \text{ GeV}$, we find $h_{\text{osc}}^c = 0.232 \pm 0.009 \text{ GeV}^{-2}$. We obtain a smaller h_{osc} compared to that obtained with only high-energy data; however, the error essentially decreases. Perhaps this reflects a more complicated form of the energy dependence obtained now. Note that, despite the logarithmic growth of the oscillation term, its relative contribution decreases as the main scattering amplitude grows as $\ln^2(s)$.

The oscillation term is represented in Fig. 21 at different energies, $\sqrt{s} = 13 \text{ TeV}$ and $\sqrt{s} = 540 \text{ TeV}$, for the proton-proton and proton-antiproton scattering. It can be seen that the amplitude of the oscillation term increases at high energies, but the period of oscillations decreases with growing energy. In particular, note that the period of oscillation is essentially less for $p\bar{p}$ than for pp scattering. This can explain the difference, which was noted in [102], in the behavior of oscillations of the differential cross sections at $\sqrt{s} = 540 \text{ GeV}$, which were found in [103], and at $\sqrt{s} = 13 \text{ TeV}$ [31]. Of course, the amplitude of oscillations is weak, so it is very difficult to identify it by experiment. However, our analysis of a wide energy region of experimental data gives a small error in the definition of

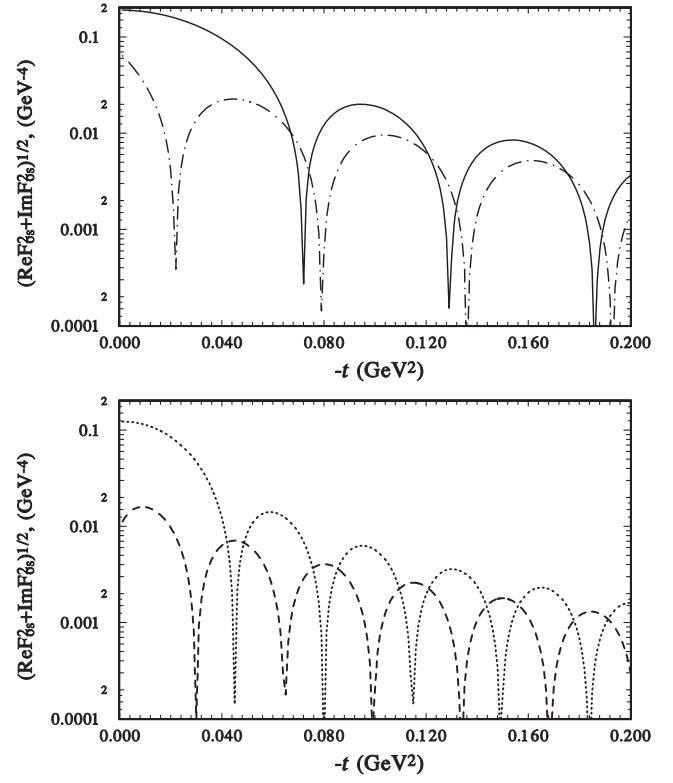
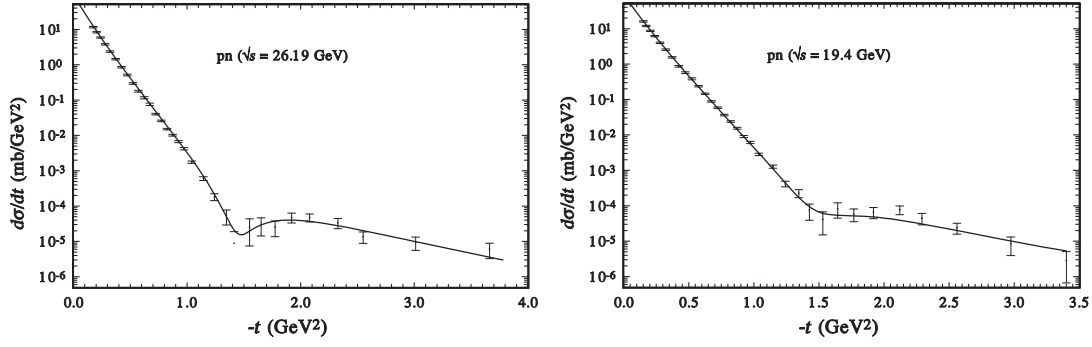
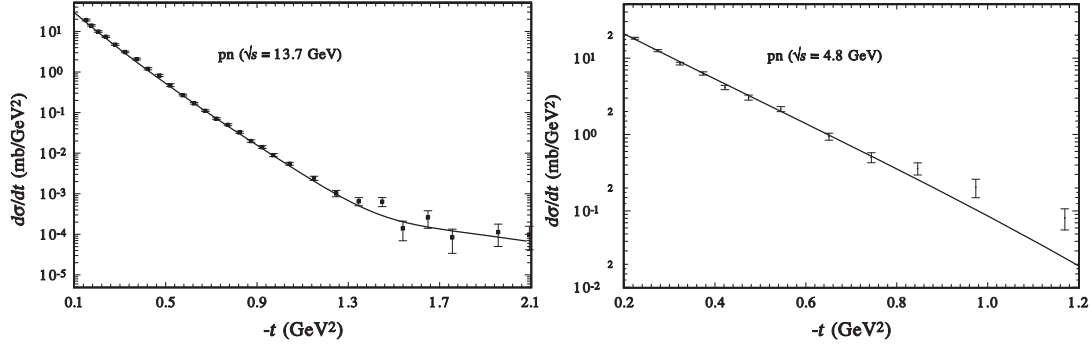


FIG. 21. Absolute value of the oscillation term $(\text{Re}^2 F_{\text{os}} + \text{Im}^2 F_{\text{os}})^{1/2}$ at $\sqrt{s} = 13 \text{ TeV}$ (solid line is pp scattering; dashed line is $p\bar{p}$ scattering) and at $\sqrt{s} = 540 \text{ GeV}$ (dotted line is pp scattering; dashed line is $p\bar{p}$ scattering).

the parameter determining the size of oscillations. This shows that further meticulous work is required, both from the theoretical and experimental points of view.

VIII. PROTON-NEUTRON ELASTIC SCATTERING

In our analysis we take 24 sets of proton-neutron experimental data from $\sqrt{s} > 4.5 \text{ GeV}$ up to a maximum energy $\sqrt{s} > 27.5 \text{ GeV}$, where they correspond to experimental data with a total number of experimental points $N = 526$. As in the case of pp scattering, we include only statistical errors in our fitting procedure, taking into account the systematic errors as additional normalization of a separate set. We take the amplitudes of the model obtained in the case of pp and $p\bar{p}$ scattering and fix the parameters of the main terms and two anomalous terms, except for the terms of the second Reggions. The electromagnetic amplitudes are removed in this case. As a result, in the pn case, only the parameters of the second effective Reggion are obtained from the fitting procedure. In the Pomeron-neutron vertex, $F_{\text{Pom}2n}^{\text{Bom}}$ is taken in the standard dipole form with free $\Lambda_{\text{pom-n}}$. Its value is found to be slightly above the standard electromagnetic $\Lambda_{\text{pom-n}}^2 = 0.82 \text{ GeV}^2$. We find that $\sum_{i,j} \chi_{i,j}^2 = 567$. The corresponding description of the differential cross section is represented in

FIG. 22. HECS model calculation of $d\sigma/dt$ of pn scattering at $\sqrt{s} = 26$ GeV (left) and $\sqrt{s} = 19.4$ GeV (right).FIG. 23. HECS model calculation of $d\sigma/dt$ of pn scattering at $\sqrt{s} = 13.7$ GeV (left) and $\sqrt{s} = 4.8$ GeV (right).

Figs. 6, 22, and 23. It is clear that the HECS model works very well in the case of proton-neutron elastic scattering. In Fig. 22, the differential cross section of pn elastic scattering is represented for a large momentum transfer at energies $\sqrt{s} = 26.19$ GeV and $\sqrt{s} = 19.4$ GeV. At $\sqrt{s} = 26.19$ GeV [Fig. 22 (left)] the diffractive dip-bump structure is clearly represented, whereas at $\sqrt{s} = 19.4$ GeV [Fig. 22 (right)] this structure is leveled out and practically disappears at lower energies (see Fig. 23). As in the case of pn scattering, there are many experimental data at small momentum transfer, where the contributions of our two anomalous terms are important; the successful description of the differential cross section confirms the existence of these terms.

IX. POLARIZATION EFFECTS IN PROTON-PROTON ELASTIC SCATTERING

In the Regge limit t_{fix} and $s \rightarrow \infty$, one can write the Regge-pole contributions to the helicity amplitudes in the s -channel as

$$\Phi_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}^B(s, t) \sim \sum_i g_{\lambda_1, \lambda_2}^i(t) g_{\lambda_3, \lambda_4}^i(t) [\sqrt{|t|}]^{|\lambda_1 - \lambda_2| + |\lambda_3 - \lambda_4|} \times \left(\frac{s}{s_0}\right)^{\alpha_i} (1 \pm e^{-i\pi\alpha_i}). \quad (46)$$

The corresponding spin-correlation values are presented in Eq. (29).

Neglecting the $\Phi_2(s, t) - \Phi_4(s, t)$ contribution, the spin correlation parameter $A_N(s, t)$ can be written by taking into account the phases of separate spin-nonflip and spin-flip amplitudes as $\varphi_{nf}(s, t), \varphi_{sf}(s, t)$; the analyzing power is

$$A_N(s, t) = -\frac{4\pi}{s^2} [|F_{nf}(s, t)| |F_{sf}(s, t)|] \times \sin(\varphi_{nf}(s, t) - \varphi_{sf}(s, t)) / \frac{d\sigma}{dt}. \quad (47)$$

It is clearly seen that, despite the large spin-flip amplitude, the analyzing power can be near zero if the difference of the phases is zero in some region of momentum transfer. The experimental data at some point of the momentum transfer show the energy independence of the size of the spin correlation parameter $A_N(s, t)$. Hence, the small value of $A_N(s, t)$ at some t (for example, very small t) does not serve as a proof that it will be small in other regions of momentum transfer.

It is usually assumed that the imaginary and real parts of the spin-nonflip amplitude have exponential behavior with the same slope, and the imaginary and real parts of the spin-flip amplitudes, without the kinematic factor $\sqrt{|t|}$ [104], are proportional to the corresponding parts of the nonflip amplitude. This is not the case as regards the t dependence shown in Ref. [68], where F_h^{fl} is multiplied by a special function dependent on t . Moreover, one mostly takes the

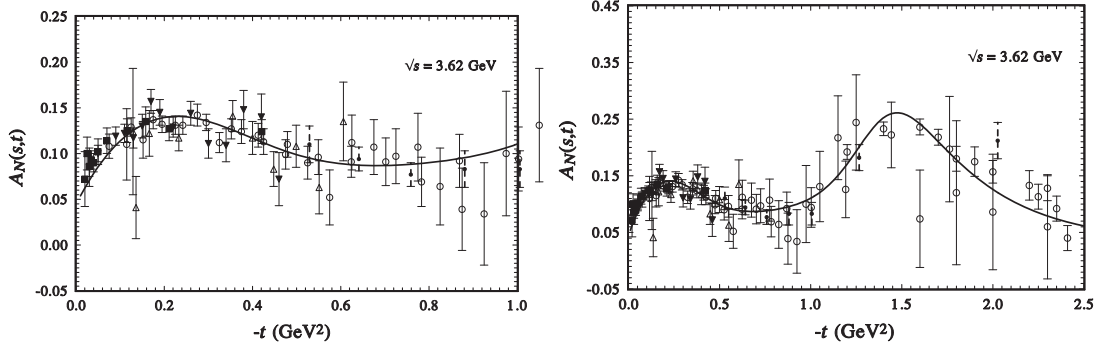


FIG. 24. Analyzing power A_N of pp scattering [left] at $\sqrt{s} = 3.62$ GeV (small t , left; larger t , right).

energy independence of the ratio of the spin-flip parts to the spin-nonflip parts of the scattering amplitude. This is our theoretical uncertainty [4,105].

In [79,80], on the basis of generalization of the constituent-counting rules of the perturbative QCD, the proton current matrix elements $J_p^{\pm\delta\delta}$ for a full set of spin combinations corresponding to the number of spin-flipped quarks were calculated. This leads to part of the spin-flip amplitude,

$$F_h^{sl} \sim \sqrt{-t} / \left(\frac{4}{9} m_p^2 \right) \sqrt{-t} / \left(\frac{4}{9} m_p^2 \right) \sqrt{-t} / \left(\frac{4}{9} m_p^2 \right). \quad (48)$$

Hence, such an amplitude gives large contributions at large momentum transfer.

Of course, at lower energies we need to take into account the energy-dependent parts of the spin-flip amplitudes. Thus, the form of the spin-flip amplitude is determined as

$$F_{sf1}(s, t) = i h_{sf1} q^3 (1 + q^3 / \sqrt{s}) G_{em}^2 e^{2t \ln \hat{s}}. \quad (49)$$

We take the second part of the spin-flip amplitude in the form

$$F_{sf2}(s, t) = i \sqrt{|t|} G_{em}^2 (h_5 + h_6 (1 + i h_4) / s s c^2) e^{2t \ln \hat{s}}, \quad (50)$$

which works mostly at low energies.

The minimal energy at which the analyzing power $A_N(s, t)$ of pp scattering is examined in the HEGS model is $\sqrt{s} = 3.62$ GeV. For our high-energy model it is a very small energy. These results are shown in Fig. 24 at small t (left) and at larger t (right). These model calculations are compared with the results of four experiments. Obviously, we obtain a very good description of the experimental data. At these energies, the diffraction minimum is practically fill up by the real part of the spin-nonflip amplitude and the contribution of the spin-flip amplitude; however, the t dependence of the analyzing power is reproduced very well in this region of the momentum transfer.

Our calculation for $A_N(t)$ is shown in Fig. 25 at $\sqrt{s} = 4.9$ GeV (left) and $\sqrt{s} = 6.8$ GeV (right). The description of the existing data is sufficient. Note that the magnitude and the energy dependence of this parameter depend on the energy behavior of the zeros of the imaginary part of the spin-flip amplitude and the real part of the spin-nonflip amplitude. Figure 26 shows $A_N(t)$ at $\sqrt{s} = 9.2$ GeV and $\sqrt{s} = 13.7$ GeV. At these energies the diffraction minimum deepens, and its form affects the form of $A_N(t)$. Finally, $A_N(t)$ is shown at large energies $\sqrt{s} = 19.4$ GeV and $\sqrt{s} = 23.4$ GeV in Fig. 27. The diffraction dip in the

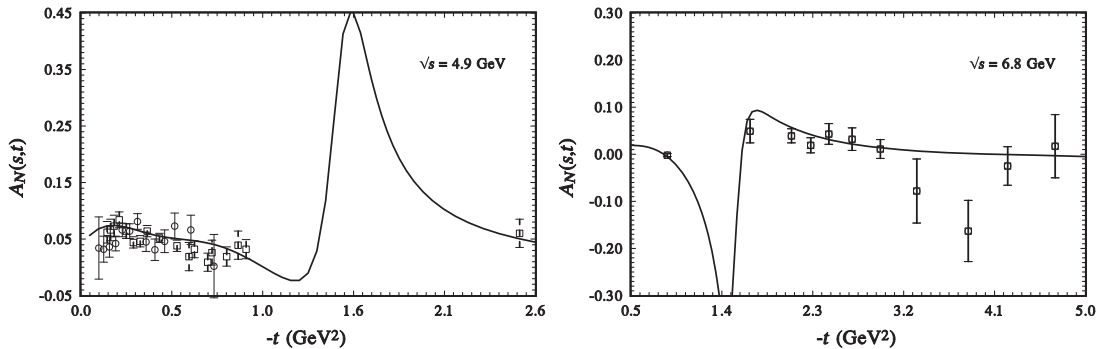


FIG. 25. Analyzing power A_N of pp scattering calculated at $\sqrt{s} = 4.9$ GeV (left, experimental data [60]) and at $\sqrt{s} = 6.8$ GeV (right; points, experimental data [60]).

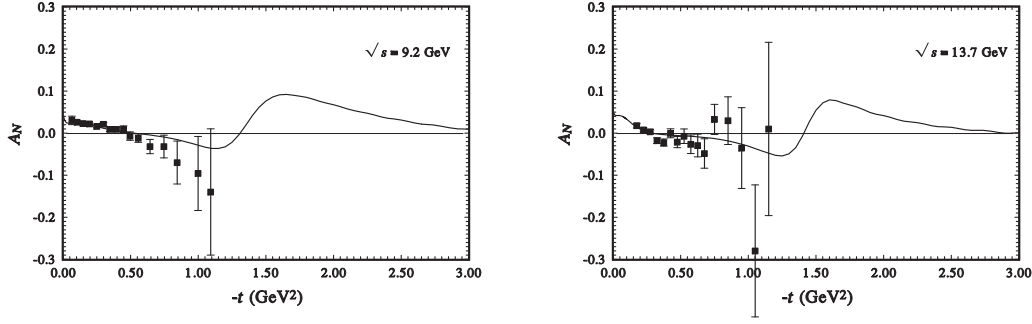


FIG. 26. Analyzing power A_N of pp scattering calculated at $\sqrt{s} = 9.2$ GeV (left, experimental data [60]) and at $\sqrt{s} = 13.7$ GeV (right; points, experimental data [60]).

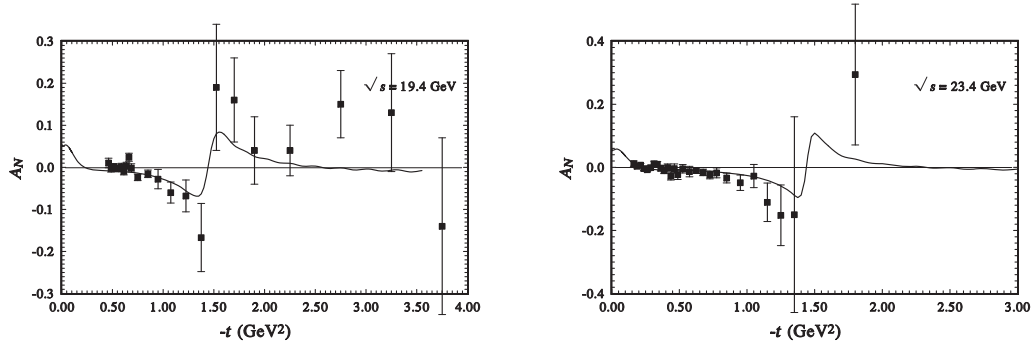


FIG. 27. Analyzing power A_N of pp scattering calculated at $\sqrt{s} = 19.4$ GeV (left, experimental data [60]) and at $\sqrt{s} = 23.4$ GeV (right; points, existing experimental data [60]).

differential cross section has a sharp form, and it affects the sharp form of $A_N(t)$. The maximum negative values of A_N coincide closely with the diffraction minimum.

We have found that the contribution of the spin-flip amplitude to the differential cross sections is much less than the contribution of the spin-nonflip amplitude in the examined region of momentum transfers. Note that A_N is determined in the domain of the diffraction dip by the ratio

$$A_N \sim \text{Im}f_- / \text{Re}f_+. \quad (51)$$

The size of the analyzing power changes from -45% to -50% at $\sqrt{s} = 50$ GeV up to -25% at $\sqrt{s} = 500$ GeV. These numbers give the magnitude of the ratio, Eq. (51), which does not strongly depend on the phase between the spin-flip and spin-nonflip amplitudes. This picture implies that the diffraction minimum is mostly filled by the real part of the spin-nonflip amplitude and that the imaginary part of the spin-flip amplitude increases in this domain as well.

X. CONCLUSIONS

For the first time, simultaneous research of proton-proton, proton-antiproton, and proton-neutron elastic

scattering has been carried out in a wide energy (from 3.6 GeV up to 13 TeV) and momentum transfer region (from $|t| = 2.10^{-4}$ GeV² up to $|t| = 14$ GeV²). In the fitting procedure we used only statistical errors. Systematic errors, which are mostly determined by indefiniteness of luminosity, are taken into account as additional normalization coefficients. As a result, a wide range of possible forms of the scattering amplitudes decreases considerably. In this paper a simultaneous description of the cross sections and spin correlation parameters of different nucleon-nucleon reactions, including 90 sets of experimental data, with the total number of data $N = 4326$, gives a very reasonable $\sum_{i,j} \chi_{i,j}^2 = 4826$. In the pn case with 526 experimental data, where the basic parameters are fixed from pp and $p\bar{p}$ scattering, $\sum_{i,j} \chi_{i,j}^2 = 585$.

Our analysis is carried out by using the successful development of the HEGS model, which can be applied in a wide energy and momentum transfer region. The model of hadron interaction is based on the analyticity of the scattering amplitude, taking into account the hadron structure, which is represented by GPDs. Different origins of the nonlinear behavior of the slope of the scattering amplitude are analyzed. The possible contribution of a meson threshold is compared with different forms of the approximations for a nonlinear slope at a small momentum transfer.

The relative contributions of the possible different parts of the scattering amplitude were analyzed. It is remarkable that in the model, the main Pomeron and odderon amplitudes have the same intercept. After eikonalization this leads to the $\ln^2(s)$ energy dependence. In this sense we have a maximal odderon case. In the model the odderon amplitude has a special kinematic factor and does not give a visible contribution at a zero momentum transfer.

It was found that the new anomalous term with a large slope has complicated logarithmic energy dependence and cross-even properties. Hence, it is part of the Pomeron amplitude, and it is also proportional to charge distributions. Our analysis of the contribution of the so-called hard Pomeron with a large intercept does not show a visible contribution of this term. The second additional term, which represents the additional oscillation properties of the scattering amplitude at a small momentum transfer with the cross-odd properties, has a logarithmic energy dependence and is proportional to the gravitomagnetic form factor. Hence, it belongs to the odderon contribution to the scattering amplitude.

As was noted in many models, the oscillations of the scattering amplitude are connected with the broken Pomanchuk theorem, and the scattering amplitude grows to a maximal possible extent without breaking the Froissart boundary. In our opinion this oscillation of the scattering amplitude is connected with the behavior of the hadron interaction potential at large distances [31,106,107]. Note that in [108] a quark-antiquark potential as a function of the separation distance r between the particles for the perturbative case in the framework of the refined Gribov-Zwanziger model was calculated. It includes a combination of oscillating terms. This can be compared to the perturbative approach at short distances and lattice QCD in the IR regime. Thus, research into the properties of the oscillations term can give very important information about the hadron interactions at large distances. Of course, this requires further theoretical and experimental study.

The new fitting procedure helps reduce some tension between the TOTEM and ATLAS data. No contribution is shown from the hard-Pomeron to elastic hadron

scattering. However, the importance of odderon's contribution is shown. A good description of proton-neutron differential scattering with 526 experimental points, including the experimental data which reach an extremely small momentum transfer $t = 210^{-4} \text{ GeV}^2$, is also obtained on the basis of the amplitudes taken from the analysis of pp and $p\bar{p}$ scattering. An adequate description of the polarization data is also obtained, which reflects the true phases of the spin-nonflip and spin-flip amplitudes; thus, we find $A_N(s, t) \sim \varphi_{\text{spin-nonflip}} - \varphi_{\text{spin-flip}}$.

Our work supports the idea that GPDs reflect the basic properties of the hadron structure and provide a bridge between many different reactions. The new form of the momentum transfer dependence of GPDs allows one to obtain different form factors, including Compton form factors, electromagnetic form factors, transition form factors, and gravitational form factors. The chosen form of the t dependence of GPDs of the pion (the same as the t dependence of the nucleon) allows us to describe the electromagnetic and gravitomagnetic form factors of pion and pion-nucleon scattering.

The impact of the different t dependence of the real and imaginary parts of the elastic scattering amplitude at small t should be noted. The dispersion relation shows that the real part has zero at small t , approximately in the domain $-t = 0.1$; of course, this depends on the energy. Hence, the contrary behavior of the real part (which grows at small t , the so-called peripheral case of the phase of the scattering amplitude) which is examined in [36] has no physical meaning.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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