

Exclusive rare semileptonic decays of B and B_c mesons in the relativistic independent quark model

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We investigate the exclusive rare semileptonic decays $B(B_c) \rightarrow K(D_{(s)})\bar{l}l/\nu_l\bar{\nu}_l$ ($l = \mu, \tau$) in the framework of relativistic independent quark model based on an average flavor independent confining potential in equally mixed scalar-vector harmonic form. The invariant weak form factors, parametrizing the matrix elements between participating meson states, are calculated in the parent meson rest frame. The momentum transfer dependence of the form factors is reliably determined in the whole accessible kinematical range $q_{\min}^2 \leq 0 \leq q_{\max}^2$. Our predicted branching fractions for $B \rightarrow K\mu^+\mu^-/\tau^+\tau^-$ ($\nu_l\bar{\nu}_l$), $B_c \rightarrow D(D_s)\mu^+\mu^-$, $B_c \rightarrow D(D_s)\tau^+\tau^-$, and $B_c \rightarrow D(D_s)\nu_l\bar{\nu}_l$, obtained in order of 10^{-7} (10^{-6}), 10^{-9} (10^{-7}), 10^{-9} (10^{-8}), and 10^{-8} (10^{-7}), respectively, are in reasonable agreement with other Standard Model predictions and Lattice QCD results. The averaged values of the lepton polarization asymmetries for decay modes are obtained as $\langle P_L(B \rightarrow K\mu^+\mu^-) \rangle = -0.97$, $\langle P_L(B_c \rightarrow D(D_s)\mu^+\mu^-) \rangle = -0.972$, $\langle P_L(B \rightarrow K\tau^+\tau^-) \rangle = -0.224$, $\langle P_L(B_c \rightarrow D\tau^+\tau^-) \rangle = -0.275$, and $\langle P_L(B_c \rightarrow D_s\tau^+\tau^-) \rangle = -0.194$.

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I. INTRODUCTION

The exclusive rare semileptonic $B_{(c)}$ decays are of special interest as they are governed by flavor-changing neutral current, which are forbidden at the tree level in the Standard Model (SM) and proceed at the lowest order through a one-loop diagram [1–8] involving at least one off diagonal element of the Cabbibo-Kobayashi-Maskawa (CKM) matrix element V_{iq} ($q = d, s, b$). Being sensitive to the contribution of new intermediate particles that might appear in the loop, these decays are especially important as they potentially hint at New Physics (NP) beyond SM [9,10]. They also provide a framework for precise determination of the CKM matrix element and leptonic decay constants of $D_{(s)}$ and $B_{(c)}$ mesons.

The rarity of these decays, particularly involving lepton and neutrinos, presents significant experimental challenges due to their extremely low branching fractions (BFs). However, the first observed data on the BF $\mathcal{B}(B \rightarrow Kl^+l^-) = (0.75_{-0.21}^{+0.25} \pm 0.09) \times 10^{-6}$, $l = e, \mu$ by Belle Collaboration [11] was made available in 2001. Since then a series of

experimental probes made by Belle [12–14], BABAR [15–17], CDF [18,19], and the LHCb [20–25] Collaboration have yielded the observed data on number of rare decay properties such as the BFs; forward-backward asymmetry; direct CP , lepton flavor, and isospin asymmetries; angular analysis and test of lepton-flavor universality; etc. A comparison of existing observed data and theoretical predictions provides one of the most rigid constraints on the NP scenario [26]. The first-ever measurement of the BF by Belle II Collaboration, $\mathcal{B}(B^+ \rightarrow K^+\nu_l\bar{\nu}_l) = (2.3 \pm 0.7) \times 10^{-5}$ [27], which is nearly 3.5σ higher than the SM prediction, raises intriguing possibilities for NP beyond SM. The data in B_c sector is scant. However, with the advances in the Large Hadron Collider beauty experiments at LHC, there is a fair chance of around 5×10^{10} B_c -events per year [28,29]. This provides a good platform to explore the B_c rare decays to pseudoscalar (D, D_s) and $\bar{l}l/\nu_l\bar{\nu}_l$.

The theoretical analysis of the rare weak decays is based on the effective electroweak Hamiltonian, derived from the leading and next-to-leading loop diagrams in the SM using the operator product expansion and renormalization group techniques [2,5]. The QCD corrections to these decays due to hard gluon exchange require one to resum large logarithms, which is done with the help of renormalization group methods. The operator product expansion allows one to separate the perturbative short-distance effect encoded in the Wilson coefficients of the effective Hamiltonian from the nonperturbative long-distance part encoded in the hadronic form factors. This part of the calculation is model

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dependent since it involves the nonperturbative QCD. Therefore, it is important to have a reliable estimate of the hadronic form factors for the exclusive rare semileptonic $B_{(c)}$ decays and predict the decay properties within and beyond the SM.

The exclusive rare semileptonic $B_{(c)}$ meson decays have been studied by several theoretical approaches. Although we may not be able to list them all, some noteworthy among them include the relativistic constituent quark model (RCQM) [30,31], light front (LF) and constituent quark model (CQM) [32], light front quark model [33,34], light cone quark model (LCQM) [35,36], light cone sum rule [37–39], QCD sum rule (QCD SR) [40,41], perturbative QCD [42], and lattice QCD (LQCD) [43–53] approaches. In view of the potential prospects of recent experimental techniques and, in particular, the facilities available for the Large Hadron beauty experiments, dedicated to heavy flavor physics at LHC, the unmeasured exclusive rare semileptonic decays can hopefully be measured in the near future. The observed data in this sector [11–14,16,17,20–25] can distinguish the predictions from several theoretical approaches on rare semileptonic b -flavored mesons. The prospects in the ongoing and future experiments in the heavy flavor sector and theoretical predictions available in the literature motivate us to analyze the exclusive rare semileptonic decays of b -flavored mesons in the framework of our relativistic independent quark (RIQ) model.

The RIQ model, developed by our group, has been applied in describing wide-ranging hadronic properties including static properties of hadrons [54–58] such as the elastic form factor, magnetic moment, charge radii of hadrons, and the mass spectra, etc. and several decay properties of hadrons including the semileptonic decays [59–64] of heavy-flavored mesons. In this paper, we would like to extend the applicability of the RIQ model and study the decays $B(B_c) \rightarrow K(D_{(s)})\bar{l}l$ and $B(B_c) \rightarrow K(D_{(s)})\nu_l\bar{\nu}_l$, where l stands for the leptons e, μ , or τ . These decays are, in fact, governed by three independent invariant hadronic form factors ($f_+(q^2)$, $f_-(q^2)$, and $f_T(q^2)$). The form factors $f_+(q^2)$, $f_-(q^2)$ represent the hadronic matrix element of the vector-axial vector current and $f_T(q^2)$ represents the matrix element of the tensor current. The invariant weak decay form factors expressed as overlap integrals of the meson wave functions have been obtained in the corresponding calculation of mass spectra and other static [54–58] and several decay properties including those of Refs. [59–64] in the RIQ model framework in good agreement with the observed data and other SM predictions. In this model, the weak form factors are reliably extracted, where their dependence on the momentum transfer squared (q^2) is automatically encoded kinematical range. This is unlike in some other theoretical approaches available in the literature, where the form factors are determined first, with an endpoint normalization at

minimum q^2 (maximum recoil) or maximum q^2 (minimum recoil) and then extrapolated to the entire kinematical region using some monopole/dipole/Gaussian ansatz, which makes the form factor estimation less reliable. In the present context, we would calculate the form factors [$f_+(q^2)$, $f_-(q^2)$, and $f_T(q^2)$] as well as BFs and longitudinal lepton polarization asymmetry (LPA) within the SM framework, using our RIQ model conventions. The LPA is an important asymmetry and a parity-violating observable [30,34,65,66], which could be accessible in the LHCb experiment. This could be more accessible experimentally in τ -channels, since it is proportional to the lepton mass in the SM. Various theoretical approaches [30,33,34,67–69] have predicted this observable, using the general form of the effective Hamiltonian including all possible forms of interaction. In this paper, we would like to address these issues including the lepton mass effect in the rare semileptonic decay within our RIQ model framework.

The paper is organized as follows. In Sec. II we discuss the general formalism and kinematics for rare decays of $B(B_c) \rightarrow K(D_{(s)})\bar{l}l$ and $B(B_c) \rightarrow K(D_{(s)})\nu_l\bar{\nu}_l$, which include a brief description of the effective Hamiltonian, invariant form factors and differential decay distribution, helicity amplitude and differential decay rate, and longitudinal lepton polarization asymmetry. In Sec. III, we extract the invariant transition form factors in the relativistic independent quark (RIQ) model. Our numerical analysis and discussion on the form factors, branching fractions, and longitudinal polarization asymmetries are described in Sec. IV. Section V encompasses our Summary and Conclusion. A brief description of the RIQ model conventions, quark orbitals, and momentum probability amplitude of the constituent quark and antiquark in the RIQ model are given in the Appendix.

II. GENERAL FORMALISM AND KINEMATICS

A. Effective Hamiltonian

The SM description of rare b -flavored meson decays is based on the low energy effective Hamiltonian, obtained by integrating out the degrees of freedom (the top quark and W bosons). The effective Hamiltonians for $b \rightarrow q_i\bar{l}l$ ($q_1 = d, q_2 = s$) and $b \rightarrow q_i\nu_l\bar{\nu}_l$, renormalized at a scale $\mu = m_b$, are taken from Ref. [70] as

$$\begin{aligned}\mathcal{H}_{\text{eff}}^{\bar{l}l} &= -\frac{4G_F}{\sqrt{2}}V_{tb}V_{tq_i}^*\sum_{i=1}^{10}C_i\mathcal{O}_i, \\ \mathcal{H}_{\text{eff}}^{\nu_l\bar{\nu}_l} &= -\frac{4G_F}{\sqrt{2}}V_{tb}V_{tq_i}^*C_L^{\nu_l}\mathcal{O}_L^{\nu_l},\end{aligned}\quad (1)$$

where G_F is the Fermi coupling constant, V_{ij} is the Cabibbo-Kobayashi-Maskawa matrix elements, and C_i are the Wilson coefficients. $\mathcal{O}_i$ are the SM operator basis

that enter into the description of the $b \rightarrow q_i \bar{l} l$ transitions here and are taken from the original work of Buras *et al.* and Buchalla *et al.* from Refs. [2] and [5], respectively as

$$\begin{aligned}\mathcal{O}_7 &= \frac{e^2}{32\pi^2} m_b (\bar{q}_i \sigma^{\mu\nu} (1 + \gamma_5) b) F_{\mu\nu}, \\ \mathcal{O}_9 &= \frac{e^2}{32\pi^2} (\bar{q}_i \gamma^\mu (1 - \gamma_5) b) (\bar{l} \gamma_\mu l), \\ \mathcal{O}_{10} &= \frac{e^2}{32\pi^2} (\bar{q}_i \gamma^\mu (1 - \gamma_5) b) (\bar{l} \gamma_\mu \gamma_5 l).\end{aligned}\quad (2)$$

Here $F_{\mu\nu}$ denotes the electromagnetic field strength tensor. The operator that enters into the description of the $b \rightarrow q_i \nu_l \bar{\nu}_l$ transitions is

$$\mathcal{O}_L^{\nu_l} = \frac{e^2}{32\pi^2} (\bar{q}_i \gamma^\mu (1 - \gamma_5) b) (\bar{\nu}_l \gamma_\mu (1 - \gamma_5) \nu_l). \quad (3)$$

In SM, $C_R^{\nu_l}$ is negligible while $C_L^{\nu_l} = \frac{-X(x_t)}{\sin^2 \theta_w}$ with $x_t = m_t^2/m_W^2$ and $X(x_t)$ is an Inami-Lim function [4,5], which is given by

$$X(x_t) = \frac{x}{8} \left(\frac{2+x}{x-1} + \frac{3x-6}{(x-1)^2} \ln x \right). \quad (4)$$

With the effective Hamiltonians in the form (1–4), the structure of the free quark decay amplitudes are

$$\begin{aligned}\mathcal{M}(b \rightarrow q_i \bar{l} l) &= \frac{G_F}{\sqrt{2}} \frac{\alpha_{em}}{2\pi} V_{ib} V_{iq_i}^* \left[C_9^{\text{eff}} (\bar{q}_i \gamma^\mu (1 - \gamma_5) b) (\bar{l} \gamma_\mu l) \right. \\ &\quad + C_{10} (\bar{q}_i \gamma^\mu (1 - \gamma_5) b) (\bar{l} \gamma_\mu \gamma_5 l) \\ &\quad \left. - C_7^{\text{eff}} \frac{2m_b}{q^2} (\bar{q}_i i\sigma^{\mu\nu} q_\nu (1 + \gamma_5) b) (\bar{l} \gamma_\mu l) \right],\end{aligned}\quad (5)$$

$$\begin{aligned}\mathcal{M}(b \rightarrow q_i \nu_l \bar{\nu}_l) &= \frac{G_F}{\sqrt{2}} \frac{\alpha_{em}}{2\pi} V_{ib} V_{iq_i}^* C_L^{\nu_l} (\bar{q}_i \gamma^\mu (1 - \gamma_5) b) \\ &\quad \times (\bar{\nu}_l \gamma_\mu (1 - \gamma_5) \nu_l)\end{aligned}\quad (6)$$

B. Form factors and differential decay distribution

For application of the quark level expressions in the description of rare semileptonic decays of $B_{(c)}$ mesons, it is necessary to calculate the matrix elements of the appropriate current operators (5–6) between the initial $|H_{in}(\vec{p}, S_{H_{in}})\rangle$ and final $|H_f(\vec{k}, S_{H_f})\rangle$ meson states (H_{in} denotes either B or B_c and H_f denotes the pseudoscalar mesons K, D, D_s). Such calculation requires an application of nonperturbative approaches, in which one needs to calculate the matrix elements $\langle H_f(\vec{k}, S_{H_f}) | \bar{q}_i \gamma^\mu (1 - \gamma_5) b | H_{in}(\vec{p}, S_{H_{in}}) \rangle$ and $\langle H_f(\vec{k}, S_{H_f}) | \bar{q}_i i\sigma^{\mu\nu} q_\nu (1 + \gamma_5) b | H_{in}(\vec{p}, S_{H_{in}}) \rangle$. Here, the parts of the transition current containing

γ_5 do not contribute. So we consider only the $\bar{q}_i \gamma^\mu b$ and also $\bar{q}_i i\sigma^{\mu\nu} q_\nu b$ parts in the calculation of hadronic matrix elements.

Considering Lorentz and parity invariance, the matrix elements of the weak currents for rare decays of meson H_{in} to pseudoscalar meson H_f , can be parametrized by the invariant form factors ($f_+(q^2), f_-(q^2), f_T(q^2)$) as

$$J_V^\mu \equiv \langle H_f(k) | \bar{q}_i \gamma^\mu b | H_{in}(p) \rangle = f_+(q^2) \mathcal{P}^\mu + f_-(q^2) q^\mu, \quad (7)$$

$$\begin{aligned}J_T^\mu &\equiv \langle H_f(k) | \bar{q}_i i\sigma^{\mu\nu} q_\nu b | H_{in}(p) \rangle \\ &= \frac{1}{M+m} [\mathcal{P}^\mu q^2 - q^\mu \mathcal{P} q] f_T(q^2).\end{aligned}\quad (8)$$

We specify our choice of momenta of participating particles as $p = k + k_1 + k_2$, with $p^2 = M^2$, $k^2 = m^2$, and $k_1^2 = k_2^2 = \mu^2$, where p, k represent the momenta of the initial and final meson; k_1 and k_2 are momenta of lepton l^+ and l^- . M, m , and μ are masses of initial meson H_{in} , final meson H_f , and lepton, respectively. We also specify our notation: $q = p - k$, $\mathcal{P} = p + k$ and the four-momentum transfer to the lepton pair $q = p - k$, which lies within the limit of $4\mu^2 < q^2 < (M - m)^2$.

For $B(B_c) \rightarrow K(D_{(s)}) \bar{l} l$ decays, the matrix element is written here as

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} \frac{\alpha_{em}}{2\pi} V_{ib} V_{iq_i}^* [H_1^\mu (\bar{l} \gamma_\mu l) + H_2^\mu (\bar{l} \gamma_\mu \gamma_5 l)], \quad (9)$$

where the hadronic parts H_i^μ are obtained in terms of the invariant form factors and Wilson coefficients as

$$H_i^\mu = F_+^{(i)} \mathcal{P}^\mu + F_-^{(i)} q^\mu \quad (i = 1, 2), \quad (10)$$

with

$$\begin{aligned}F_+^{(1)} &= C_9^{\text{eff}} f_+(q^2) - C_7^{\text{eff}} f_T(q^2) \frac{2m_b}{M+m}, \\ F_-^{(1)} &= C_9^{\text{eff}} f_-(q^2) + C_7^{\text{eff}} f_T(q^2) \frac{2m_b}{M+m} \frac{\mathcal{P} q}{q^2}, \\ F_\pm^{(2)} &= C_{10} f_\pm(q^2).\end{aligned}\quad (11)$$

From the invariant matrix element (9), the differential angular decay distribution for rare weak decays can be expressed in terms of hadronic $H_{ij}^{\mu\nu}$, $H^{\mu\nu}$, and leptonic tensors $L_{\mu\nu}^{(k)}$ in the form

$$\begin{aligned}H_{ij}^{\mu\nu} &= H_i^\mu H_j^{\nu\dagger}, \quad H^{\mu\nu} = H^\mu H^{\nu\dagger}, \\ L_{\mu\nu}^{(1)} &= k_{1\mu} k_{2\nu} + k_{2\mu} k_{1\nu}, \quad L_{\mu\nu}^{(2)} = g_{\mu\nu}, \quad L_{\mu\nu}^{(3)} = i\epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta.\end{aligned}\quad (12)$$

Then it is straightforward to find the differential angular decay distribution for $B(B_c) \rightarrow K(D_{(s)})\bar{l}l$ in the form derived for the first by Faessler *et al.*, which is published in [30] and subsequently used by many authors in their SM description of these decays including those as shown here in Ref. [31,34] as

$$\frac{d^2\Gamma}{dq^2 d\cos\theta} = \frac{G_F^2}{(2\pi)^3} \left(\frac{\alpha_{em} |V_{tb} V_{tq_i}^*|}{2\pi} \right)^2 \frac{|\vec{k}|}{8M^2} \sqrt{\left(1 - \frac{4\mu^2}{q^2}\right)} \frac{1}{2} \left\{ L_{\mu\nu}^{(1)} (H_{11}^{\mu\nu} + H_{22}^{\mu\nu}) - \frac{1}{2} L_{\mu\nu}^{(2)} (q^2 H_{11}^{\mu\nu} + (q^2 - 4\mu^2) H_{22}^{\mu\nu}) + L_{\mu\nu}^{(3)} (H_{12}^{\mu\nu} + H_{21}^{\mu\nu}) \right\}. \quad (13)$$

For $B(B_c) \rightarrow K(D_{(s)})\nu_l\bar{\nu}_l$ decays, the invariant matrix element is

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} \frac{\alpha_{em}}{2\pi} V_{tb} V_{tq_i}^* C_L^{\nu_l} [H^\mu \bar{\nu}_l \gamma_\mu (1 - \gamma_5) \nu_l],$$

Note that for such a decay mode, the form factor $f_T(q^2)$ does not contribute, since it is related to the photon vertex ($\sigma^{\mu\nu} q_\nu$). Therefore, the hadronic part H^μ is obtained here in the simple form

$$H^\mu = f_+(q^2) \mathcal{P}^\mu + f_-(q^2) q^\mu. \quad (14)$$

Then the differential angular decay distribution for $B(B_c) \rightarrow K(D_{(s)})\nu_l\bar{\nu}_l$ is obtained in the form [31,34]

$$\frac{d^2\Gamma}{dq^2 d\cos\theta} = \frac{G_F^2}{(2\pi)^3} \left(\frac{\alpha_{em} |V_{tb} V_{tq_i}^*|}{2\pi} \right)^2 \frac{3|\vec{k}|}{8M^2} |C_L^{\nu_l}|^2 \left\{ L_{\mu\nu}^{(1)} - \frac{q^2}{2} L_{\mu\nu}^{(2)} + L_{\mu\nu}^{(3)} \right\} H^{\mu\nu}, \quad (15)$$

where factor 3 comes due to the contribution from all three neutrino modes.

C. Helicity amplitude and differential decay rate

It is well known that a physical observable such as the hadron tensor is conveniently expressed on the helicity basis [30], in which the helicity form factors are obtained in terms of the Lorentz invariant form factors. In the same approach, the exclusive semileptonic B_c decays into radially excited charmonium and charm meson states have also been described [64] in our RIQ model framework. While doing the Lorentz contraction in Eqs. (13) and (15) using the helicity amplitudes, one considers four covariant helicity projections $\epsilon^\mu(m)$ with $m = +, -, 0$ and $m = t$. In our attempt to study the lepton mass effects in the rare semileptonic decay modes, we need to consider the time component of the polarization $\epsilon^\mu(m = t)$ in addition to its other three components with $m = +, -, 0$.

Using orthogonality and the completeness relations satisfied by helicity projections, the Lorentz contraction in Eqs. (13) and (15) can be written as in Ref. [30]

$$\begin{aligned} L^{\mu\nu(k)} H_{\mu\nu}^{ij} &= L^{(k)}(m, n) g_{mm'} g_{nn'} H^{ij}(m', n'), \\ L^{\mu\nu(k)} H_{\mu\nu} &= L^{(k)}(m, n) g_{mm'} g_{nn'} H(m', n') \end{aligned} \quad (16)$$

with $g_{mn} = \text{dia}(+, -, -, -)$. Here the lepton and hadron tensors are introduced in the space of helicity components as

$$\begin{aligned} L^{(k)}(m, n) &= \epsilon^\mu(m) \epsilon^{\dagger\nu}(n) L_{\mu\nu}^{(k)}, \\ H^{ij}(m', n') &= \epsilon^\mu(m') \epsilon^{\dagger\nu}(n') H_{\mu\nu}^{ij}, \\ H(m', n') &= \epsilon^\mu(m') \epsilon^{\dagger\nu}(n') H_{\mu\nu}. \end{aligned} \quad (17)$$

For the sake of convenience, we consider here two frames of reference: (i) the $\bar{l}l$ and $\nu_l\bar{\nu}_l$ center-of-mass frame and (ii) the parent $B_{(c)}$ -meson rest frame. We evaluate lepton tensor $L^{(k)}(m, n)$ in the $\bar{l}l$ and $\nu_l\bar{\nu}_l$ center-of-mass frame and hadron tensor $H^{ij}(m', n')$ and $H(m', n')$ in the $B_{(c)}$ -rest frame. In the $B_{(c)}$ -rest frame, the four momenta (p^μ , k^μ , q^μ) and polarization vectors ($\epsilon^\mu(\pm)$, $\epsilon^\mu(0)$, and $\epsilon^\mu(t)$) are specified as $p^\mu = (M, 0, 0, 0)$, $k^\mu = (E_k, 0, 0, -|\vec{k}|)$, $q^\mu = (q^0, 0, 0, |\vec{k}|)$, and $\epsilon^\mu(t) = \frac{1}{\sqrt{q^2}}(q^0, 0, 0, |\vec{k}|)$, $\epsilon^\mu(\pm) = \frac{1}{\sqrt{2}}(0, \mp 1, -i, 0)$, $\epsilon^\mu(0) = \frac{1}{\sqrt{q^2}}(|\vec{k}|, 0, 0, q^0)$, where $E_k = \frac{M^2 + m^2 - q^2}{2M}$, $q^0 = \frac{M^2 - m^2 + q^2}{2M}$. The hadronic amplitude H_i^μ is then cast in the helicity frame.

For $B_{(c)} \rightarrow K(D_{(s)})\bar{l}l$ decays, the helicity form factors $H^i(m)$ are obtained in terms of invariant form factor as

$$\begin{aligned}
H^i(t) &= \frac{1}{\sqrt{q^2}} (\mathcal{P}qF_+^{(i)} + q^2 F_-^{(i)}), \\
H^i(\pm) &= 0, \\
H^i(0) &= \frac{2M|\vec{k}|}{\sqrt{q^2}} F_+^{(i)}.
\end{aligned} \tag{18}$$

Similarly, for $B(B_c) \rightarrow K(D_{(s)})\nu_l\bar{\nu}_l$ decays, the helicity form factors $H(m)$ are obtained in the form

$$\begin{aligned}
H(t) &= \frac{1}{\sqrt{q^2}} (\mathcal{P}qf_+(q^2) + q^2 f_-(q^2)), \\
H(\pm) &= 0, \\
H(0) &= \frac{2M|\vec{k}|}{\sqrt{q^2}} f_+(q^2).
\end{aligned} \tag{19}$$

Using the space and time components of the four momenta (q^μ , k^μ , q^μ) and helicity projections in the $\vec{l}\vec{l}$ and $\nu_l\bar{\nu}_l$ -center-of-mass frame,

$$\begin{aligned}
q^\mu &= (\sqrt{q^2}, 0, 0, 0), \\
k_1^\mu &= (E_l, |\vec{k}_1| \sin \theta \cos \chi, -|\vec{k}_1| \sin \theta \sin \chi, |\vec{k}_1| \cos \theta), \\
k_2^\mu &= (E_l, -|\vec{k}_1| \sin \theta \cos \chi, -|\vec{k}_1| \sin \theta \sin \chi, -|\vec{k}_1| \cos \theta),
\end{aligned} \tag{20}$$

with $E_l = \sqrt{q^2}/2$ and $|\vec{k}_1| = \sqrt{q^2 - 4\mu^2}/2$ for two lepton in final states and $E_l = |\vec{k}_1| = \sqrt{q^2}/2$ for two neutrinos in final states, the longitudinal and time components of the polarization vector are reduced to $\epsilon^\mu(0) = (0, 0, 0, 1)$, $\epsilon^\mu(t) = (1, 0, 0, 0)$, whereas its transverse part $\epsilon^\mu(\pm)$ remains unchanged. Using these factors, it is straightforward to calculate the helicity representation $L^{(k)}(m, n)$ of the lepton tensor (17).

In the present analysis, we do not consider the azimuthal χ distribution of the lepton pair and therefore integrate over the azimuthal angle dependence of the lepton tensor leading to the differential $(q^2, \cos \theta)$ distribution of the rare semileptonic decays of $B(B_c)$ mesons.

(a) For $B(B_c) \rightarrow K(D_{(s)})\vec{l}\vec{l}$, the differential $(q^2, \cos \theta)$ distribution is obtained in the form

$$\begin{aligned}
\frac{d^2\Gamma}{dq^2 d(\cos \theta)} &= \frac{3}{8} (1 + \cos^2 \theta) \cdot \frac{1}{2} \left(\frac{d\Gamma_U^{11}}{dq^2} + \frac{d\Gamma_U^{22}}{dq^2} \right) + \frac{3}{4} \sin^2 \theta \cdot \frac{1}{2} \left(\frac{d\Gamma_L^{11}}{dq^2} + \frac{d\Gamma_L^{22}}{dq^2} \right) \\
&+ \sqrt{1 - \frac{4\mu^2}{q^2}} \cdot \frac{3}{4} \cos \theta \cdot \frac{d\Gamma_P^{12}}{dq^2} + \frac{3}{4} \sin^2 \theta \cdot \frac{1}{2} \frac{d\tilde{\Gamma}_U^{11}}{dq^2} - \frac{3}{8} (1 + \cos^2 \theta) \cdot \frac{d\tilde{\Gamma}_U^{22}}{dq^2} \\
&+ \frac{3}{2} \cos^2 \theta \cdot \frac{1}{2} \frac{d\tilde{\Gamma}_L^{11}}{dq^2} - \frac{3}{4} \sin^2 \theta \cdot \frac{d\tilde{\Gamma}_L^{22}}{dq^2} + \frac{1}{4} \frac{d\tilde{\Gamma}_S^{22}}{dq^2}.
\end{aligned} \tag{21}$$

Then integrating over $\cos \theta$, the differential decay width is obtained as

$$\begin{aligned}
\frac{d\Gamma}{dq^2} &= \frac{1}{2} \left(\frac{d\Gamma_U^{11}}{dq^2} + \frac{d\Gamma_U^{22}}{dq^2} + \frac{d\Gamma_L^{11}}{dq^2} + \frac{d\Gamma_L^{22}}{dq^2} \right) \\
&+ \frac{1}{2} \frac{d\tilde{\Gamma}_U^{11}}{dq^2} - \frac{d\tilde{\Gamma}_U^{22}}{dq^2} + \frac{1}{2} \frac{d\tilde{\Gamma}_L^{11}}{dq^2} - \frac{d\tilde{\Gamma}_L^{22}}{dq^2} + \frac{1}{2} \frac{d\tilde{\Gamma}_S^{22}}{dq^2}.
\end{aligned} \tag{22}$$

Out of nine terms in the rhs of Eq. (22), five terms identified by tilde rates $\tilde{\Gamma}_i$ are linked with the lepton mass and other terms identified by Γ_i are lepton mass independent. Both are related via a flip factor $\frac{2\mu^2}{q^2}$ as

$$\frac{d\tilde{\Gamma}_X^{ij}}{dq^2} = \frac{2\mu^2}{q^2} \frac{d\Gamma_X^{ij}}{dq^2}. \tag{23}$$

The tilde rates do not contribute to the decay rate in the vanishing lepton mass limit. They have minimal contributions for rare weak decays to e and μ modes whereas their contribution to decays to τ mode is significant. Therefore, the tilde modes are crucial in the evaluation of the lepton mass effect in the rare semileptonic decay modes. The differential partial helicity rates $d\Gamma_X^{ij}/dq^2$ are obtained in the form

$$\frac{d\Gamma_X^{ij}}{dq^2} = \frac{G_F^2}{(2\pi)^3} \left(\frac{\alpha_{em} |V_{tb} V_{tq}^*|}{2\pi} \right)^2 \frac{|\vec{k}|^2 q^2}{12M^2} \sqrt{1 - \frac{4\mu^2}{q^2}} H_X^{ij}. \tag{24}$$

Here H_X^{ij} ($X = U, L, P, S; i, j = 1, 2$) represents a standard set of helicity structure functions given by linear combinations of helicity components of hadron tensor $H^{ij}(m', n') = H^i(m') H^{\dagger j}(n')$ and $H(m, n) = H(m) H^\dagger(n)$:

$$\begin{aligned}
H_U^{ij} &= \text{Re}(H_+^i H_+^{\dagger j}) + \text{Re}(H_-^i H_-^{\dagger j}) : \text{Unpolarized} - \text{transverse}, \\
H_P^{ij} &= \text{Re}(H_+^i H_+^{\dagger j}) - \text{Re}(H_-^i H_-^{\dagger j}) : \text{Parity} - \text{odd}, \\
H_L^{ij} &= \text{Re}(H_0^i H_0^{\dagger j}) : \text{Longitudinal}, \\
H_S^{ij} &= 3\text{Re}(H_i^i H_i^{\dagger j}) : \text{Scalar}.
\end{aligned} \tag{25}$$

(b) For $B(B_c) \rightarrow K(D_{(s)})\nu_l\bar{\nu}_l$, the differential ($q^2, \cos\theta$) distribution is obtained as

$$\begin{aligned}
\frac{d^2\Gamma}{dq^2 d(\cos\theta)} &= \frac{G_F^2}{(2\pi)^3} \left(\frac{\alpha_{em} |V_{tb} V_{tq_i}^*|}{2\pi} \right)^2 \frac{3|\vec{k}|q^2}{16M^2} \\
&\times C_L^{\nu_l} \sin^2\theta |H(0)|^2.
\end{aligned} \tag{26}$$

Integrating over $\cos\theta$, the differential decay width is reduced to the form

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2}{(2\pi)^3} \left(\frac{\alpha_{em} |V_{tb} V_{tq_i}^*|}{2\pi} \right)^2 \frac{|\vec{k}|q^2}{4M^2} |C_L^{\nu_l}|^2 |H(0)|^2. \tag{27}$$

D. Longitudinal lepton polarization asymmetry

An interesting observable, the lepton polarization asymmetry (LPA), is defined as

$$P_L(q^2) = \frac{d\Gamma_{h=-1}/dq^2 - d\Gamma_{h=+1}/dq^2}{d\Gamma_{h=-1}/dq^2 + d\Gamma_{h=+1}/dq^2} \tag{28}$$

where $h = +1(-1)$ denotes right (left) handed l^- in the final state. From the differential decay width expressions for $h = +1(-1)$, it is straightforward to obtain $P_L(q^2)$ in terms of the invariant form factors and Wilson coefficients as

$$P_L(q^2) = \frac{2\sqrt{1 - \frac{4\mu^2}{q^2}} |\vec{k}|^2 C_{10} f_+(q^2) \left[f_+(q^2) \text{Re} C_9^{\text{eff}} - \frac{2m_b}{M+m} C_7^{\text{eff}} f_T(q^2) \right]}{|\vec{k}|^2 \left(1 + \frac{2\mu^2}{q^2} \right) \mathcal{F}_1 + 3\mu^2 \mathcal{F}_2}, \tag{29}$$

where

$$\begin{aligned}
\mathcal{F}_1 &= \left| f_+(q^2) C_9^{\text{eff}} - \frac{2m_b}{M+m} C_7^{\text{eff}} f_T(q^2) \right|^2 + |C_{10} f_+(q^2)|^2 \\
\mathcal{F}_2 &= |C_{10}|^2 \left[\left(1 + \frac{m^2}{M^2} - \frac{q^2}{2M^2} \right) |f_+(q^2)|^2 + \left(1 - \frac{m^2}{M^2} \right) f_+(q^2) f_-(q^2) + \frac{q^2}{2M^2} |f_-(q^2)|^2 \right]
\end{aligned}$$

Due to the experimental difficulties in the study of the polarizations of each lepton, which depends on q^2 and the Wilson coefficients, it is better to eliminate the q^2 dependence of the LPA by considering its averaged form over the entire kinematical range. The averaged LPA is defined by

$$\langle P_L \rangle = \frac{\int_{4\mu^2}^{(M-m)^2} P_L(q^2) \frac{dBF}{dq^2} dq^2}{\int_{4\mu^2}^{(M-m)^2} \frac{dBF}{dq^2} dq^2}. \tag{30}$$

III. INVARIANT TRANSITION FORM FACTORS IN THE RELATIVISTIC INDEPENDENT QUARK MODEL

The decay process physically occurs in the momentum eigenstate of the participating mesons. Therefore, in the field-theoretic description of the decay process, the participating meson-bound states are represented here by the appropriate momentum wave packets with suitable momentum and spin distribution between the constituent quark-antiquark pair in the corresponding meson core. In the relativistic independent particle picture of the RIQ model, the constituent quark and antiquark are assumed to be in the state of independent and relativistic motion inside

the meson-bound state $|M(\vec{p}, S_M)\rangle$, for example, with constituent quark and antiquark momentum q_1 and $\bar{q}_2$, respectively. The momentum probability amplitude of the constituent quark and antiquark in the participating meson ground state is extracted here from the momentum projection of the static quark orbitals, obtainable in the present model framework. The corresponding expressions for the momentum probability amplitude $\mathcal{G}_{q_1}(\vec{p}_{q_1})$ for the constituent quark q_1 and $\tilde{\mathcal{G}}_{q_2}(\vec{p}_{q_2})$ for the antiquark $\bar{q}_2$ are given in (A6) of the Appendix. In this model, we construct an effective momentum distribution function $\mathcal{G}_M(\vec{p}_{q_1}, \vec{p}_{q_2})$ in the form $\mathcal{G}_M(\vec{p}_{q_1}, \vec{p}_{q_2}) = \sqrt{G_{q_1}(\vec{p}_{q_1}) \tilde{G}_{q_2}(\vec{p}_{q_2})}$ in a straightforward extension of ansatz of Margolis and Mendel [71] in

their bag model description of the meson bound state. With the effective momentum distribution function $\mathcal{G}_M(\vec{p}_{q_1}, \vec{p}_{q_2})$, representing the bound-state character, the meson bound state in the RIQ model is taken in the form

$$|M(\vec{p}, S_M)\rangle = \hat{\Lambda}(\vec{p}, S_M) |(\vec{p}_{q_1}, \lambda_{q_1}); (\vec{p}_{q_2}, \lambda_{q_2})\rangle \\ = \hat{\Lambda}(\vec{p}, S_M) \hat{b}_{q_1}^\dagger(\vec{p}_{q_1}, \lambda_{q_1}) \hat{b}_{q_2}^\dagger(\vec{p}_{q_2}, \lambda_{q_2}) |0\rangle, \quad (31)$$

where $|(\vec{p}_{q_1}, \lambda_{q_1}); (\vec{p}_{q_2}, \lambda_{q_2})\rangle$ is the Fock-space representation of the unbound quark and antiquark in a color-singlet configuration with their respective momentum and spin, $(\vec{p}_{q_1}, \lambda_{q_1})$ and $(\vec{p}_{q_2}, \lambda_{q_2})$. Here $\hat{b}_{q_1}^\dagger(\vec{p}_{q_1}, \lambda_{q_1})$ and $\hat{b}_{q_2}^\dagger(\vec{p}_{q_2}, \lambda_{q_2})$ are the quark and antiquark creation operators and $\hat{\Lambda}(\vec{p}, S_M)$ is a bag-like operator taken here in the integral form

$$\hat{\Lambda}(\vec{p}, S_M) = \frac{\sqrt{3}}{\sqrt{N_M(\vec{p})}} \sum_{\lambda_{q_1}, \lambda_{q_2}} \zeta_{\lambda_{q_1}, \lambda_{q_2}}^M \\ \times \int d\vec{p}_{q_1} d\vec{p}_{q_2} \delta^{(3)}(\vec{p}_{q_1} + \vec{p}_{q_2} - \vec{p}) \mathcal{G}_M(\vec{p}_{q_1}, \vec{p}_{q_2}), \quad (32)$$

where $\sqrt{3}$ is the effective color factor, and $\zeta_{\lambda_{q_1}, \lambda_{q_2}}^M$ is the $SU(6)$ spin-flavor coefficients for the meson state $|M(\vec{p}, S_M)\rangle$. Finally, $N_M(\vec{p})$ represents the meson state normalization, which is obtained in the integral form

$$N_M(\vec{p}) = \frac{1}{(2\pi)^3 2E_p} \int d\vec{p}_{q_1} |\mathcal{G}_M(\vec{p}_{q_1}, \vec{p}_{q_2})|^2 = \frac{\bar{N}_M(\vec{p})}{(2\pi)^3 2E_p} \quad (33)$$

by imposing a normalization condition $\langle M(\vec{p}') | M(\vec{p}) \rangle = (2\pi)^3 2E_p \delta^{(3)}(\vec{p} - \vec{p}')$. A brief description of the quark orbitals obtained in the Dirac framework using the RIQ model conventions and corresponding momentum probability amplitudes ($G_{q_1}(\vec{p}_{q_1})$, $\tilde{G}_{q_2}(\vec{p}_{q_2})$) for the constituent quark and antiquark are given in the Appendix.

Now using the wave packet representation of the initial $|H_{in}\rangle$ and final $|H_f\rangle$ meson states, the invariant hadronic matrix elements J_V^μ and J_T^μ (7, 8) are calculated in the parent meson rest frame, yielding

$$J_V^\mu \equiv \langle H_f(k) | \bar{q}_i \gamma^\mu b | H_{in}(0) \rangle = \sqrt{\frac{4E_k M}{\bar{N}_{H_{in}}(0) \bar{N}_{H_f}(\vec{k})}} \int \frac{d\vec{p}_b}{\sqrt{2E_{p_b} 2E_{p_b+k}}} \\ \times \mathcal{G}_{H_{in}}(\vec{p}_b, -\vec{p}_b) \mathcal{G}_{H_f}(\vec{p}_b + \vec{k}, -\vec{p}_b) \langle S_{H_f} | J_V^\mu(0) | S_{H_{in}} \rangle, \quad (34)$$

$$J_T^\mu \equiv \langle H_f(k) | \bar{q}_i i\sigma^{\mu\nu} q_\nu b | H_{in}(0) \rangle = \sqrt{\frac{4E_k M}{\bar{N}_{H_{in}}(0) \bar{N}_{H_f}(\vec{k})}} \int \frac{d\vec{p}_b}{\sqrt{2E_{p_b} 2E_{p_b+k}}} \\ \times \mathcal{G}_{H_{in}}(\vec{p}_b, -\vec{p}_b) \mathcal{G}_{H_f}(\vec{p}_b + \vec{k}, -\vec{p}_b) \langle S_{H_f} | J_T^\mu(0) | S_{H_{in}} \rangle, \quad (35)$$

where E_{p_b} and E_{p_b+k} stand for the energy of the nonspectator quark of the parent and daughter meson, respectively. $\langle S_{H_f} | J_V^\mu(0) | S_{H_{in}} \rangle$ and $\langle S_{H_f} | J_T^\mu(0) | S_{H_{in}} \rangle$ represent symbolically the spin matrix elements of vector and tensor current, respectively.

For $0^- \rightarrow 0^-$ transitions, the axial vector current does not contribute. The spin matrix elements corresponding to the nonvanishing vector and tensor currents are obtained in the form:

$$\langle S_{H_f} | V^0 | S_{H_{in}} \rangle = \frac{(E_{p_b} + m_b)(E_{p_b+k} + m_{d/s}) + |\vec{p}_b|^2}{\sqrt{(E_{p_b} + m_b)(E_{p_b+k} + m_{d/s})}}, \quad (36)$$

$$\langle S_{H_f} | V^i | S_{H_{in}} \rangle = \frac{(E_{p_b} + m_b)k^i}{\sqrt{(E_{p_b} + m_b)(E_{p_b+k} + m_{d/s})}}, \quad (37)$$

$$\langle S_{H_f} | T^0 | S_{H_{in}} \rangle = -\frac{(E_{p_b} + m_b)|\vec{k}|^2}{\sqrt{(E_{p_b} + m_b)(E_{p_b+k} + m_{d/s})}}. \quad (38)$$

With the above spin matrix elements, the expressions for hadronic amplitudes (34) and (35) are compared with corresponding expressions from Eqs. (7) and (8) yielding the model expression of form factors $f_\pm(q^2)$ and $f_T(q^2)$ in the form

$$f_{\pm}(q^2) = \frac{1}{2M} \sqrt{\frac{ME_k}{\bar{N}_{H_{in}}(0)\bar{N}_{H_f}(\vec{k})}} \int d\vec{p}_b \mathcal{G}_{H_{in}}(\vec{p}_b, -\vec{p}_b) \mathcal{G}_{H_f}(\vec{p}_b + \vec{k}, -\vec{p}_b) \times \left[\frac{(E_{p_b} + m_b)(E_{p_b+k} + m_{d/s}) \pm (M \mp E_k)(E_{p_b} + m_b) + |\vec{p}_b|^2}{\sqrt{E_{p_b}E_{p_b+k}}(E_{p_b} + m_b)(E_{p_b+k} + m_{d/s})} \right], \quad (39)$$

$$f_T(q^2) = -\frac{(M+m)}{2M} \sqrt{\frac{ME_k}{\bar{N}_{H_{in}}(0)\bar{N}_{H_f}(\vec{k})}} \int d\vec{p}_b \mathcal{G}_{H_{in}}(\vec{p}_b, -\vec{p}_b) \mathcal{G}_{H_f}(\vec{p}_b + \vec{k}, -\vec{p}_b) \sqrt{\frac{(E_{p_b} + m_b)}{E_{p_b}E_{p_b+k}(E_{p_b+k} + m_{d/s})}}. \quad (40)$$

IV. NUMERICAL RESULTS AND DISCUSSION

In this section, we present our numerical results and discussion on exclusive rare semileptonic decays $B(B_c) \rightarrow K(D_{(s)})\bar{l}l$ and $B(B_c) \rightarrow K(D_{(s)})\nu_l\bar{\nu}_l$. The input parameters used in the numerical calculation, the quark masses ($m_u = m_d$, m_s , m_c , m_b) and potential parameters (a , V_0), have been fixed in this model by fitting our predicted meson masses with the data of heavy and heavy-light flavored mesons in their ground state [72]. The input parameters, so fixed from hadron spectroscopy, are shown in Table I.

The phenomenological input parameters—the Fermi coupling constant G_F , participating meson masses, lifetime τ of decaying $B_{(c)}$ mesons, and the precise global fit values of CKM parameters—taken from PDG [73] are listed in Table II. Although our predictions [72] of ground state heavy meson masses are overall in good agreement with the corresponding experimental data, we used the observed masses of $(B_{(c)}^+, D_{(s)}^+)$ and K^+ mesons [73] in our calculation to reduce possible theoretical uncertainties.

TABLE I. Quark masses m_q (in GeV) and potential parameters: a (in GeV^3) and V_0 (in GeV).

$m_u = m_d$	m_s	m_c	m_b	a	V_0
0.26	0.49	1.64	4.92	0.023	-0.307

TABLE II. Numerical inputs.

Parameter	Value [73]
G_F	$1.1663788 \times 10^{-5} \text{ GeV}^{-2}$
$M_{B_c^+}$	$6274.47 \pm 0.27 \pm 0.17 \text{ MeV}$
M_{B^+}	$5279.42 \pm 0.08 \text{ MeV}$
m_{D^+}	$1869.5 \pm 0.4 \text{ MeV}$
$m_{D_s^+}$	$1969.0 \pm 1.4 \text{ MeV}$
m_{K^+}	$493.677 \pm 0.005 \text{ MeV}$
$\tau_{B_c^+}$	$0.51 \pm 0.009 \text{ ps}$
τ_{B^+}	$1.638 \pm 0.004 \text{ ps}$
$ V_{td} $	$0.00858^{+0.00019}_{-0.00017}$
$ V_{ts} $	$0.04111^{+0.00077}_{-0.00068}$
$ V_{tb} $	$0.999118^{+0.000029}_{-0.000034}$

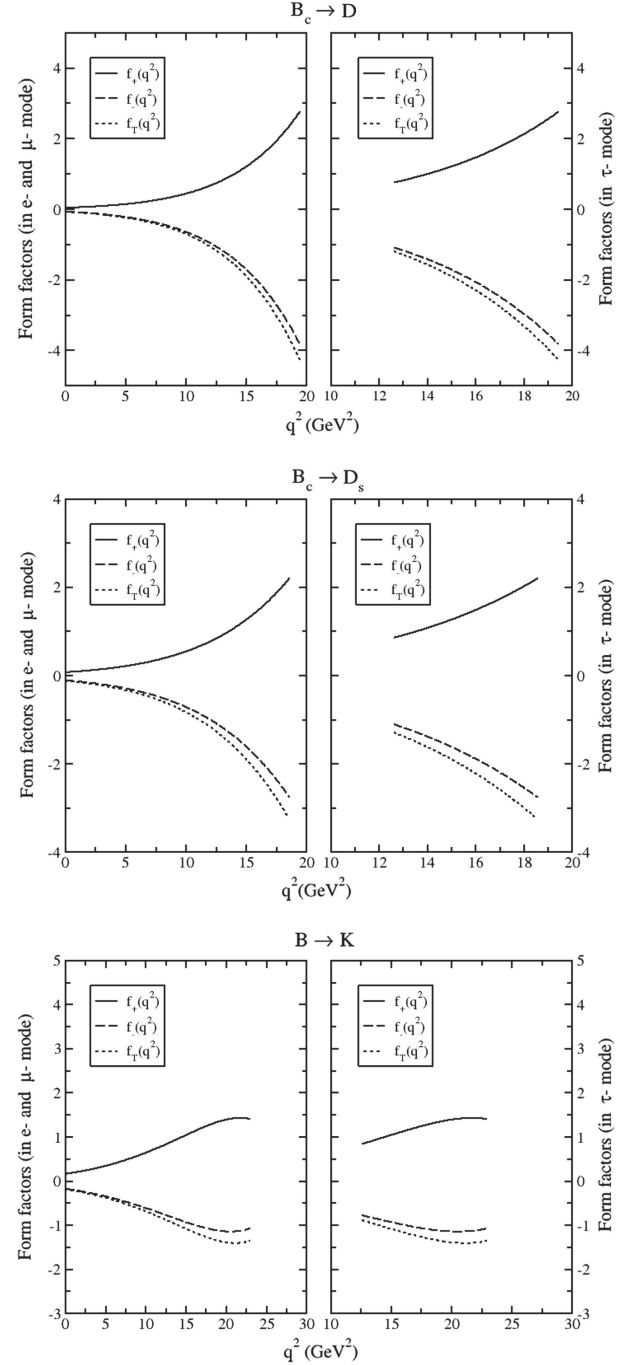


FIG. 1. q^2 dependence of the form factors for $B_c \rightarrow D_{(s)}\bar{l}l$ and $B \rightarrow K\bar{l}l$ decays.

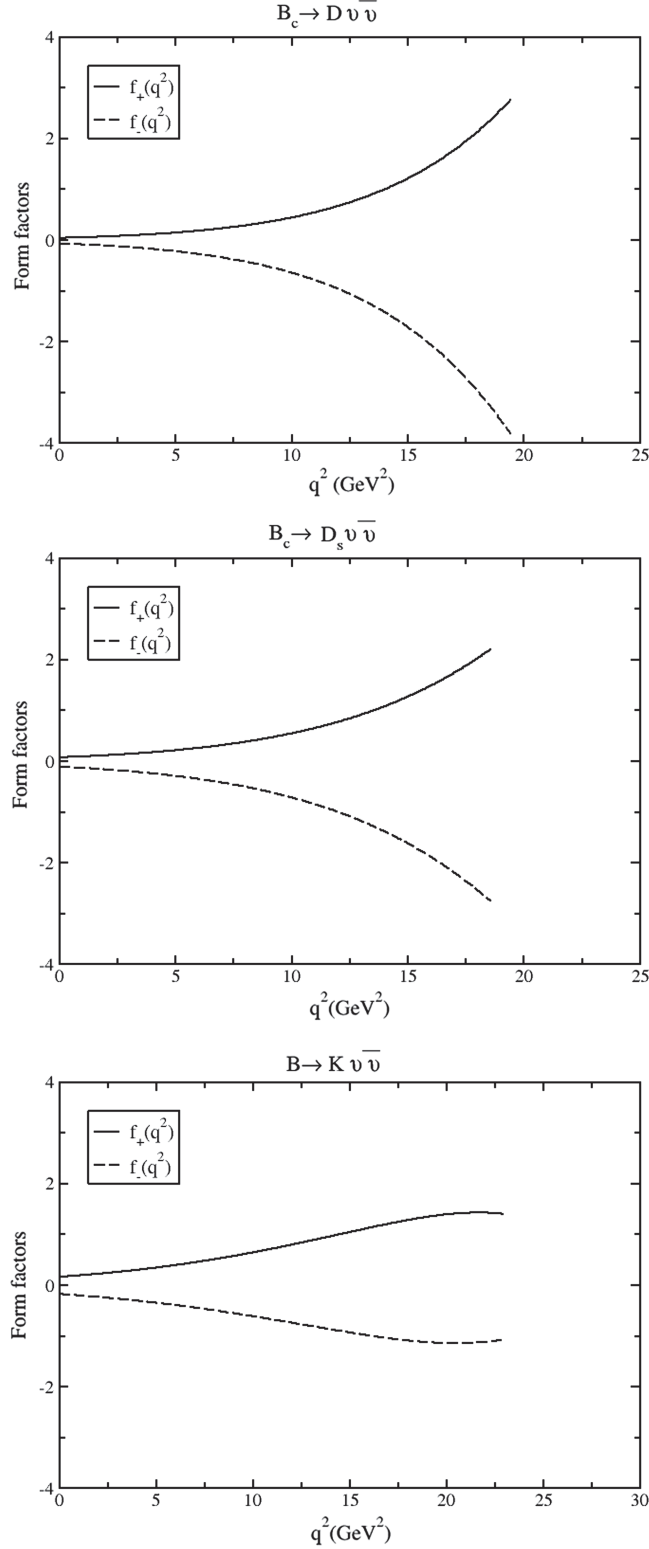


FIG. 2. q^2 dependence of the form factors for $B_c \rightarrow D_{(s)}\nu\bar{\nu}$ and $B \rightarrow K\nu\bar{\nu}$ decays.

We use the values of the Wilson coefficients for decays to two lepton final states, $C_7^{\text{eff}} = -0.313$, $C_9^{\text{eff}} = 4.344$, and $C_{10} = -4.669$ from Refs. [2,41]. For decays to two neutrino final states, the value of the effective Wilson

coefficient $C_L^{\nu_l} = -6.32(7)$ used in the present calculation is taken from Refs. [10,74–79].

With the input parameters (Table I), the Lorentz invariant form factors (f_+ , f_- , f_T) representing decay amplitudes are calculated from the overlapping integral of participating meson wave functions. Before evaluating numerically the physical quantities of interest, we study the q^2 dependence of the invariant form factors in the accessible kinematical range. Since one of our objectives here is to study the Lepton mass effect in the rare semileptonic decays, we plot the q^2 dependence of the form factors in $B_{(c)}$ -meson decays to their e , μ , and τ modes. Our results are shown in Figs. 1

TABLE III. Results for form factors at $q^2 \rightarrow q_{\min}^2$ of $B(B_c) \rightarrow K(D_{(s)})\mu^+\mu^-/\nu_l\bar{\nu}_l$ decays.

Decay mode		$f_+(q^2)$	$f_-(q^2)$	$f_T(q^2)$
$B_c \rightarrow D\mu^+\mu^-/\nu_l\bar{\nu}_l$	This work	0.044	-0.067	-0.072
	[31]	0.081	0.081	0.061
	[34]	0.079	-0.070	-0.108
$B_c \rightarrow D_s\mu^+\mu^-/\nu_l\bar{\nu}_l$	This work	0.078	-0.107	-0.122
	[31]	0.129	0.129	0.098
	[34]	0.126	-0.099	-0.168
$B \rightarrow K\mu^+\mu^-/\nu_l\bar{\nu}_l$	This work	0.166	-0.173	-0.186
	[31]	0.242	0.242	0.258

TABLE IV. Results for form factors at $q^2 \rightarrow q_{\max}^2$ of $B(B_c) \rightarrow K(D_{(s)})\mu^+\mu^-/\nu_l\bar{\nu}_l$ decays.

Decay mode		$f_+(q^2)$	$f_-(q^2)$	$f_T(q^2)$
$B_c \rightarrow D\mu^+\mu^-/\nu_l\bar{\nu}_l$	This work	2.755	-3.807	-4.258
	[34]	0.754	-0.721	-1.012
$B_c \rightarrow D_s\mu^+\mu^-/\nu_l\bar{\nu}_l$	This work	3.924	-2.749	-3.250
	[31]	2.662	0.669	2.151
$B \rightarrow K\mu^+\mu^-/\nu_l\bar{\nu}_l$	[34]	0.874	-0.722	-1.155
	This work	1.408	-1.074	-1.357
	[31]	2.991	0.6601	0.794

TABLE V. Results for form factors at $q^2 \rightarrow q_{\min}^2$ and $q^2 \rightarrow q_{\max}^2$ of $B(B_c) \rightarrow K(D_{(s)})\tau^+\tau^-$ decays.

Decay mode	Valued at	$f_+(q^2)$	$f_-(q^2)$	$f_T(q^2)$
$B_c \rightarrow D\tau^+\tau^-$	$q^2 = q_{\min}^2$	0.763	-1.089	-1.202
	$q^2 = q_{\max}^2$	2.755	-3.807	-4.258
	$q^2 = q_{\min}^2$	0.863	-1.110	-1.296
$B_c \rightarrow D_s\tau^+\tau^-$	$q^2 = q_{\max}^2$	3.924	-2.749	-3.250
	$q^2 = q_{\min}^2$	0.849	-0.779	-0.891
$B \rightarrow K\tau^+\tau^-$	$q^2 = q_{\max}^2$	1.408	-1.074	-1.357

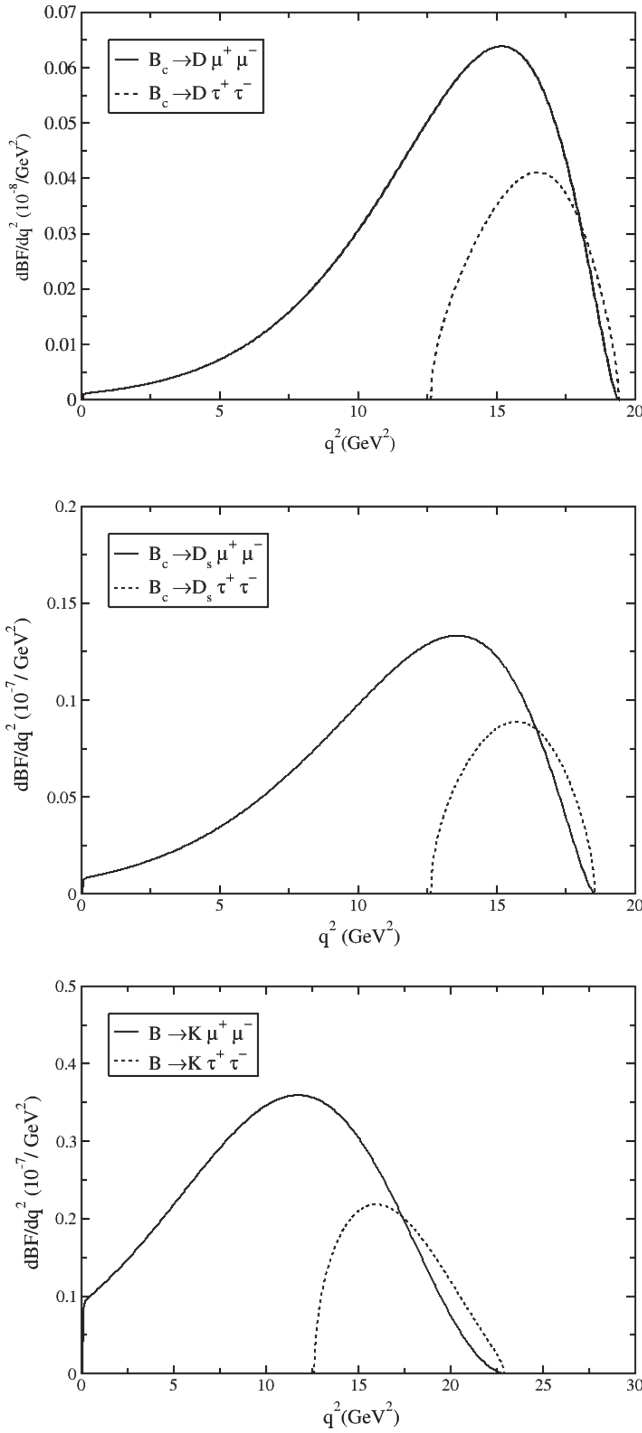


FIG. 3. Predicted differential branching fractions for $B_{(c)} \rightarrow X\mu^+\mu^-$ (solid line) and $B_{(c)} \rightarrow X\tau^+\tau^-$ (dotted line), $X = D, D_s, K$.

and 2, where we find that the behavior of form factors in e and μ mode overlap in the entire kinematic range ($q_{\min}^2 \simeq 0$) $\leq q^2 \leq q_{\max}^2$. This is because of an insignificant change in the phase space boundary going from e to μ mode. As one can see here, the maximal lepton energy shift $\frac{m_\mu^2 - m_e^2}{2M}$ is invisible at the usual scale of the plot. On the other hand, for the decays in the τ mode, the relevant form factors

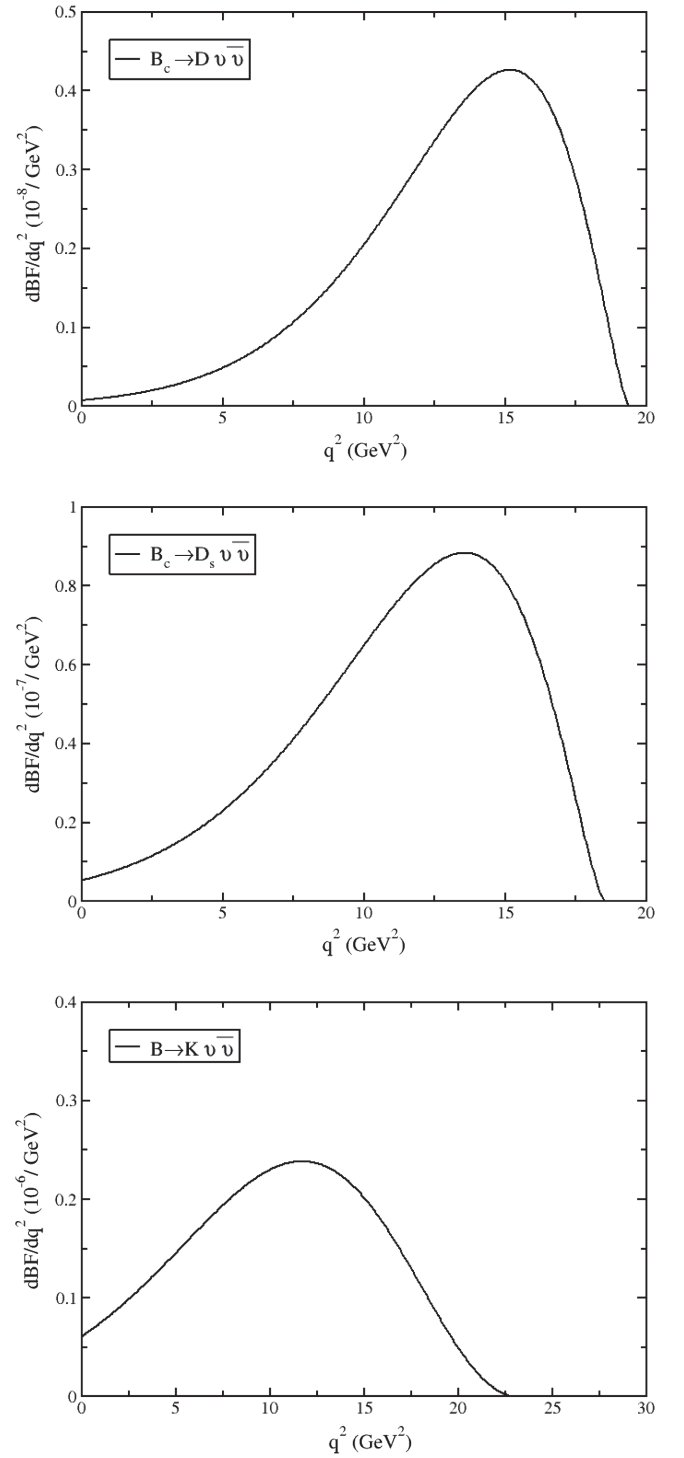


FIG. 4. Predicted differential branching fractions for $B_{(c)} \rightarrow X\nu\bar{\nu}$, $X = D, D_s, K$.

behave differently throughout the accessible kinematical range of $q_{\min}^2 \leq q^2 \leq q_{\max}^2$, where $q_{\min}^2$ is +ve and away from $q^2 \rightarrow 0$. The τ - phase space, as compared to those in decays to e and μ modes, is considerably reduced and shifted to a large q^2 region. In the present study we, therefore, consider decays to their $\mu^+\mu^-$ and $\tau^+\tau^-$ modes

TABLE VI. Comparison of our predictions of BF's of $B \rightarrow K\bar{l}l/\nu_l\bar{\nu}_l$ decays with experimental data (in order 10^{-7}).

Modes	This work	Belle [13]	BABAR [15,16]	CDF [18,19]	LHCb [23]	Belle II [27]
$B \rightarrow K\mu^+\mu^-$	$4.968^{+0.761}_{-0.723}$	$4.8^{+0.5}_{-0.4} \pm 0.3$	$3.4 \pm 0.7 \pm 0.2$	$5.9 \pm 1.5 \pm 0.4$	$1.56^{+0.19+0.06}_{-0.15-0.04}$	
$B \rightarrow K\tau^+\tau^-$	$1.494^{+0.095}_{-0.093}$		$< 2.25 \times 10^4$			
$B \rightarrow K \sum \nu_l\bar{\nu}_l$	$33.012^{+4.622}_{-4.491}$					$(2.3 \pm 0.5^{+0.5}_{-0.4}) \times 10^2$

only for evaluating the lepton mass effect on the rare semileptonic $B_{(c)}$ meson decays.

The form factors at the maximum recoil point (i.e., $q^2 = q_{\min}^2$) correspond to the final state meson recoiling with maximum three momentum $|\vec{k}| = M^2 - m^2/2M$ in the rest frame of the decaying $B_{(c)}$ meson, whereas at the zero recoil point (i.e., $q^2 = q_{\max}^2$) they correspond to the overlap integral of the initial and final state meson wave functions. Our prediction on the values of the form factors ($f_+(q^2)$, $f_-(q^2)$ and $f_T(q^2)$) at $q^2 \rightarrow q_{\min}^2$ and $q^2 \rightarrow q_{\max}^2$ for the decay modes $B(B_c) \rightarrow K(D_{(s)})\mu^+\mu^-/\nu_l\bar{\nu}_l$ are shown in Tables III and IV, respectively. Our results on the form factors in $B(B_c) \rightarrow K(D_{(s)})\mu^+\mu^-/\nu_l\bar{\nu}_l$ at $q^2 \rightarrow q_{\min}^2$ are comparable to those of Ref. [31] in their calculation in the framework of the QCD-motivated relativistic quark model based on the quasipotential approach and Ref. [34] in their calculation based on the light front quark model using coulomb plus harmonic oscillator (HO) and coulomb plus linear confining potential. However, our predicted form factors at $q^2 = q_{\max}^2$ in the above decay process along with those in the decays $B \rightarrow K\mu^+\mu^-/\nu_l\bar{\nu}_l$ at $q_{\min}^2$ and $q_{\max}^2$ are found somewhat underestimated compared to the results of Refs. [31,34] and those of other SM predictions based on the RCQM [30], LF, and CQM [32] and QCD SR [41] approach. The upcoming experimental probes planned at Tevatron and LHCb may distinguish these different model predictions. Our predicted form factors of the decays $B(B_c) \rightarrow K(D_{(s)})\tau^+\tau^-$ at $q_{\min}^2$ and $q_{\max}^2$ are shown in Table V, which can be tested in future experiments.

Now using the expressions (39) and (40) of the form factors $f_+(q^2)$, $f_-(q^2)$, and $f_T(q^2)$, the differential decay

rates (22) and (27) for $B(B_c) \rightarrow K(D_{(s)})l^+l^-$ and $B(B_c) \rightarrow K(D_{(s)})\nu_l\bar{\nu}_l$ are obtained in terms of the model quantities. Before calculating numerically the physical quantities of interest, BF's and LPA, we study the q^2 dependence of the differential branching fractions of $B_{(c)} \rightarrow X\mu^+\mu^-/\tau^+\tau^-$ and $B_{(c)} \rightarrow X\nu_l\bar{\nu}_l$ over the accessible kinematical range, where $X = D, D_s, K$. This is depicted in Figs. 3 and 4. Then using the input parameters (Table I) and relevant phenomenological input parameters (Table II) taken from PDG [73], we perform the numerical calculation of the BF's for different rare semileptonic decays. The uncertainties in our predicted values of BF's are obtained by varying all input parameters simultaneously within $\pm 10\%$ of the central values and taking the largest variation of the BF's. Our results are shown in Tables VI and VII in comparison with the available experimental data [13,15,16,18,19,23,27] and other SM predictions [30,31,34,41] and LQCD results [80,81], cited in the literature.

We first compare our predicted BF's of $B \rightarrow K\mu^+\mu^-$ and $B \rightarrow K\nu_l\bar{\nu}_l$ with the available experimental data, as cited in Table VI. Our result for $B \rightarrow K\mu^+\mu^-$ compares well with the available experimental data from Belle [13] and CDF [18,19] Collaborations. A stringent test of our prediction on the differential decay distribution with the available observed data is demonstrated in Fig. 5, where we confront our predictions with the experimental data. Our result for $B \rightarrow K\tau^+\tau^-$ is found to lie within the experimental upper limit of BABAR [16] and that for $B \rightarrow K\nu_l\bar{\nu}_l$ is obtained one order of magnitude lower than the recently observed data of Belle II [27]. However, compared with the SM predictions, our results for $B \rightarrow K\mu^+\mu^-/\tau^+\tau^-/\nu_l\bar{\nu}_l$ are in reasonable

TABLE VII. Branching fractions in order 10^{-7} in comparison with other SM predictions.

Modes	This work	[30]	[31]	HO(LO) [34]	[41]	[80]	[81]
$B_c \rightarrow D\mu^+\mu^-$	$0.053^{+0.011}_{-0.009}$	0.038	0.037	0.011(0.018)	0.031 ± 0.006	...	...
$B_c \rightarrow D_s\mu^+\mu^-$	$1.259^{+0.242}_{-0.218}$	0.86	1.16	0.51(0.54)	0.61 ± 0.15	1.00 ± 0.11	...
$B \rightarrow K\mu^+\mu^-$	$4.968^{+0.761}_{-0.723}$	5.1	4.19	...	...	...	7.03 ± 0.55
$B_c \rightarrow D\tau^+\tau^-$	$0.019^{+0.002}_{-0.002}$	0.009	0.015	0.004(0.008)	0.013 ± 0.003	...	...
$B_c \rightarrow D_s\tau^+\tau^-$	$0.389^{+0.033}_{-0.032}$	0.18	0.33	0.13(0.14)	0.23 ± 0.05	0.245 ± 0.018	...
$B \rightarrow K\tau^+\tau^-$	$1.494^{+0.095}_{-0.093}$	0.87	1.17	...	...	...	1.83 ± 0.13
$B_c \rightarrow D \sum \nu_l\bar{\nu}_l$	$0.356^{+0.067}_{-0.061}$	0.328	0.216	0.081(0.131)	0.384 ± 0.071	...	...
$B_c \rightarrow D_s \sum \nu_l\bar{\nu}_l$	$8.363^{+1.488}_{-1.364}$	7.3	6.5	3.7(3.9)	4.9 ± 1.2	...	...
$B \rightarrow K \sum \nu_l\bar{\nu}_l$	$33.012^{+4.622}_{-4.491}$	41.9	26.1	...	...	...	55.8 ± 3.7

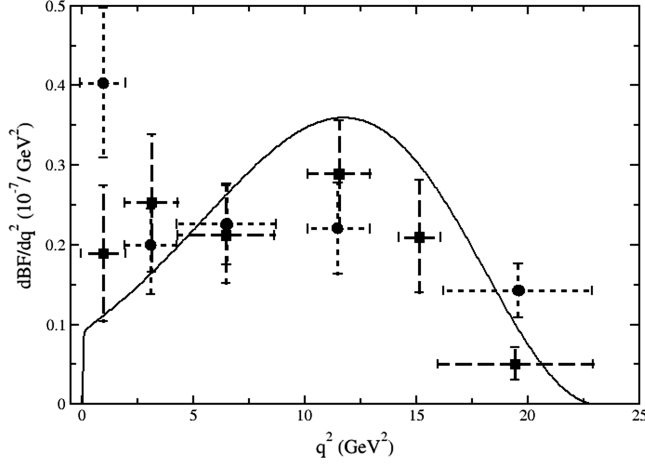


FIG. 5. Comparison of our prediction for the differential branching fractions $B \rightarrow K\mu^+\mu^-$ decay with available experimental data. Belle data are given by filled circles with dotted error bars and CDF data are presented by filled squares with dashed error bars.

agreement with those of Refs. [30,31], Refs. [31,81], and Refs. [31,81], respectively.

In the rare semileptonic B_c meson decays, where observed data are yet to be made available, we compare our results with other SM predictions and LQCD results as shown in Table VII. For decays $B_c \rightarrow D_s\mu^+\mu^-$, $B_c \rightarrow D_{(s)}\tau^+\tau^-$, and $B_c \rightarrow D_{(s)}\nu_l\bar{\nu}_l$, our predicted BF's are found in good agreement with those of Refs. [30,31], Ref. [31], and Refs. [31,41], respectively. Our results on BF's of $B_c \rightarrow D\mu^+\mu^-$, although consistent in the same order of magnitude, are found to be somewhat overestimated compared to those of Refs. [30,31,41] by a factor of about ~ 1.5 . Finally, for $B \rightarrow K\nu_l\bar{\nu}_l$, our result on BF is obtained comparable to those of Refs. [30,31].

Finally, before calculating the LPAs in different decay modes analyzed in the present study, we show their q^2 dependence in Fig. 6 for $B(B_c) \rightarrow K(D_{(s)})\mu^+\mu^-$ in solid lines and for $B(B_c) \rightarrow K(D_{(s)})\tau^+\tau^-$ in dashed lines in their allowed kinematical range. Here, the upper panels in the figure refer to the decays to $\tau^+\tau^-$ and lower panels to the decays to $\mu^+\mu^-$ mode. We note that for all the decay to $\mu^+\mu^-$ mode considered here, the LPAs (P_L) are zero near the endpoints ($q^2 \rightarrow q_{\min}^2$ and $q^2 \rightarrow q_{\max}^2$) and approximate to -1 away from the endpoints. In fact, the P_L for the muon decay is insensitive to the form factors, e.g., our P_L away from the endpoint regions is well approximated by [34,82,83]

$$P_L \simeq 2 \frac{C_{10}\text{Re}C_9^{\text{eff}}}{|C_9^{\text{eff}}|^2 + |C_{10}|^2} \simeq -1 \quad (41)$$

in the limit of $C_7^{\text{eff}} \rightarrow 0$. It also shows that the P_L for the μ dilepton channel is insensitive to the little variation of C_7^{eff} , as expected. On the other hand, the P_L for the τ dilepton channel is somewhat sensitive to the form factors.

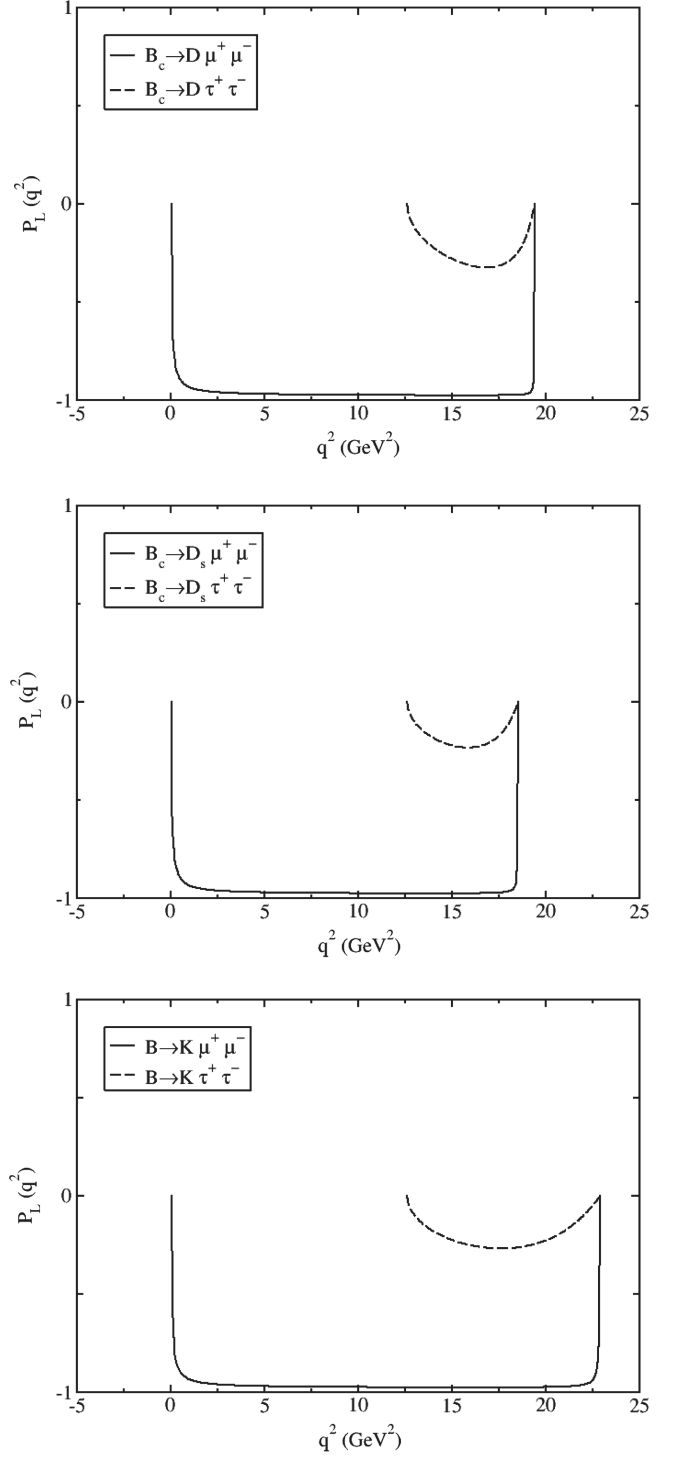


FIG. 6. Predicted longitudinal lepton polarization asymmetries for $B(B_c) \rightarrow K(D_{(s)})\mu^+\mu^-$ (solid line) and $B(B_c) \rightarrow K(D_{(s)})\tau^+\tau^-$ (dashed line).

Our calculated values of the averaged LPAs are shown in Table VIII. Averaged LPA ($\langle P_L \rangle$) for decays to $\mu^+\mu^-$ modes are obtained close to -1 , i.e., $\langle P_L(B_c \rightarrow D_{(s)}\mu^+\mu^-) \rangle = -0.972$ and $\langle P_L(B \rightarrow K\mu^+\mu^-) \rangle = -0.97$. However, for decays to corresponding $\tau^+\tau^-$ mode, we

TABLE VIII. Longitudinal lepton polarization asymmetry.

Modes	$\langle P_L \rangle$
$B_c \rightarrow D\mu^+\mu^-$	-0.972
$B_c \rightarrow D_s\mu^+\mu^-$	-0.972
$B \rightarrow K\mu^+\mu^-$	-0.970
$B_c \rightarrow D\tau^+\tau^-$	-0.275
$B_c \rightarrow D_s\tau^+\tau^-$	-0.194
$B \rightarrow K\tau^+\tau^-$	-0.224

obtain $\langle P_L(B_c \rightarrow D\tau^+\tau^-) \rangle = -0.275$, $\langle P_L(B_c \rightarrow D_s\tau^+\tau^-) \rangle = -0.194$, and $\langle P_L(B \rightarrow K\tau^+\tau^-) \rangle = -0.224$.

V. SUMMARY AND CONCLUSION

In this work, we investigate the rare semileptonic $B(B_c) \rightarrow K(D_{(s)})\bar{l}l/\nu_l\bar{\nu}_l$ ($l = \mu, \tau$) decays in the framework of the RIQ model based on the average flavor independent confining potential in equally mixed scalar-vector harmonic form. The weak form factors ($f_\pm(q^2), f_T(q^2)$), expressed through the overlap integrals of the initial and final meson wave functions, are obtained in the whole accessible kinematical range $q_{\min}^2 \leq 0 \leq q_{\max}^2$, without using any *ad hoc* assumptions and extrapolations. The meson wave functions, obtained previously in the investigations of the meson mass spectra and several decay properties of hadrons in this model, are used for the present numerical calculations. On the basis of the form factors, the differential decay distributions and branching fractions for the above rare semileptonic $B_{(c)}$ meson decays are predicted.

First we tested our model by confronting its predictions for $B \rightarrow K\mu^+\mu^-$ decays with the available experimental data from Belle and CDF Collaboration, where our results are found in reasonable agreement with their data. For $B \rightarrow K\nu_l\bar{\nu}_l$, our prediction on BF is found to be one order lower than the observed data of Belle II Collaboration. Our prediction on BFs for $B \rightarrow K\mu^+\mu^-$ and $B \rightarrow K\nu_l\bar{\nu}_l$ are found in agreement with other SM predictions and LQCD results.

Next, we provide our numerical results on the BFs of different decay modes. Our predicted BFs for $B \rightarrow K\mu^+\mu^-/\tau^+\tau^-$ ($\nu_l\bar{\nu}_l$), $B_c \rightarrow D(D_s)\mu^+\mu^-$, $B_c \rightarrow D(D_s)\tau^+\tau^-$, and $B_c \rightarrow D(D_s)\nu_l\bar{\nu}_l$, obtained in order of 10^{-7} (10^{-6}), 10^{-9} (10^{-7}), 10^{-9} (10^{-8}), and 10^{-8} (10^{-7}), respectively, are in reasonable agreement with other SM predictions and LQCD results. For longitudinal lepton polarization asymmetries (LPAs), which provide valuable information on the flavor changing loop effects in the SM, our predicted averaged values are: $\langle P_L(B \rightarrow K\mu^+\mu^-) \rangle = -0.97$, $\langle P_L(B_c \rightarrow D_{(s)}\mu^+\mu^-) \rangle = -0.972$, $\langle P_L(B \rightarrow K\tau^+\tau^-) \rangle = -0.224$, $\langle P_L(B_c \rightarrow D\tau^+\tau^-) \rangle = -0.275$, and $\langle P_L(B_c \rightarrow D_s\tau^+\tau^-) \rangle = -0.194$.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: QUARK ORBITALS AND MOMENTUM PROBABILITY AMPLITUDE OF THE CONSTITUENT QUARK AND ANTIQUARK IN RIQ MODEL FRAMEWORK

In the RIQ model, a meson is picturized as a color-singlet assembly of a quark and an antiquark independently confined by an effective and average flavor-independent potential in the form

$$U(r) = \frac{1}{2}(1 + \gamma^0)V(r),$$

where $V(r) = (ar^2 + V_0)$ with $a > 0$ is believed to provide zeroth order constituent quark dynamics inside the meson-bound state. Here (a, V_0) are the potential parameters and r is the relative distance between constituent quark and antiquark inside the meson core. It is believed that the zeroth-order quark dynamics generated by the phenomenological confining potential $U(r)$, taken in equally mixed scalar-vector harmonic form, can adequately describe the decay process. With the interaction potential $U(r)$ put into the zeroth-order quark Lagrangian density, the ensuing Dirac equation admits a static solution of positive and negative energy as

$$\begin{aligned} \psi_\xi^{(+)}(\vec{r}) &= \left(\frac{ig_\xi(r)}{r} \right) \chi_{ljm_j}(\hat{r}), \\ \psi_\xi^{(-)}(\vec{r}) &= \left(\frac{i(\vec{\sigma} \cdot \vec{r})f_\xi(r)}{r} \right) \tilde{\chi}_{ljm_j}(\hat{r}), \end{aligned} \quad (A1)$$

where $\xi = (nlj)$ represents a set of Dirac quantum numbers specifying the eigenmodes. $\chi_{ljm_j}(\hat{r})$ and $\tilde{\chi}_{ljm_j}(\hat{r})$ are the spin angular parts given by

$$\begin{aligned} \chi_{ljm_j}(\hat{r}) &= \sum_{m_l, m_s} \left\langle lm_l \frac{1}{2} m_s \middle| jm_j \right\rangle Y_l^{m_l}(\hat{r}) \chi_{\frac{1}{2}}^{m_s}, \\ \tilde{\chi}_{ljm_j}(\hat{r}) &= (-1)^{j+m_j-l} \chi_{l j-m_j}(\hat{r}). \end{aligned} \quad (A2)$$

With the quark binding energy parameter E_q and quark mass parameter m_q , written in the form $E'_q = (E_q - V_0/2)$, $m'_q = (m_q + V_0/2)$, and $\omega_q = E'_q + m'_q$, one can obtain solutions to the radial equation for $g_\xi(r)$ and $f_\xi(r)$ in the form

$$\begin{aligned}
g_{nl} &= \mathcal{N}_{nl} \left(\frac{r}{r_{nl}} \right)^{l+1} \exp(-r^2/2r_{nl}^2) L_{n-1}^{l+1/2}(r^2/r_{nl}^2), \\
f_{nl} &= \frac{\mathcal{N}_{nl}}{r_{nl}\omega_q} \left(\frac{r}{r_{nl}} \right)^l \exp(-r^2/2r_{nl}^2), \\
&\times \left[\left(n + l - \frac{1}{2} \right) L_{n-1}^{l-1/2}(r^2/r_{nl}^2) + n L_n^{l-1/2}(r^2/r_{nl}^2) \right],
\end{aligned} \tag{A3}$$

where $r_{nl} = (a\omega_q)^{-1/4}$ is a state independent length parameter, $\mathcal{N}_{nl}$ is an overall normalization constant given by

$$\mathcal{N}_{nl}^2 = \frac{4\Gamma(n)}{\Gamma(n+l+1/2)} \frac{(\omega_q/r_{nl})}{(3E'_q + m'_q)}, \tag{A4}$$

and $L_{n-1}^{l+1/2}(r^2/r_{nl}^2)$ are associated Laguerre polynomials. With the radial solutions (A3), an independent quark bound-state condition for constituent quark and antiquark is obtained in the RIQ model in the form of a cubic equation

$$\sqrt{(\omega_q/a)(E'_q - m'_q)} = (4n + 2l - 1). \tag{A5}$$

From the solution of the cubic equation (A5), the zeroth-order binding energies of the confined quark and antiquark are obtained for all possible eigenmodes.

In the relativistic independent particle picture of this model, the relativistic constituent quark and antiquark are thought to move independently inside the meson bound-state $|M(\vec{p}, S_M)\rangle$ with their momentum $\vec{p}_{q_1}$ and $\vec{p}_{q_2}$, respectively. To study the decay process that takes place in the momentum eigenstates of participating mesons, we Fourier transform the quark orbitals (A1) to momentum space and obtain the momentum probability amplitude $G_{q_1}(\vec{p}_{q_1})$ of the quark and $\tilde{G}_{q_2}(\vec{p}_{q_2})$ of the antiquark. For ground state mesons ($n = 1, l = 0$), the momentum probability amplitudes for the constituent quark and antiquark are obtained in the forms

$$\begin{aligned}
G_{q_1}(\vec{p}_{q_1}) &= \frac{i\pi\mathcal{N}_{q_1}}{2\alpha_{q_1}\omega_{q_1}} \sqrt{\frac{(E_{p_{q_1}} + m_{q_1})}{E_{p_{q_1}}}} (E_{p_{q_1}} + E_{q_1}) \\
&\times \exp\left(-\frac{\vec{p}_{q_1}^2}{4\alpha_{q_1}}\right), \\
\tilde{G}_{q_2}(\vec{p}_{q_2}) &= -\frac{i\pi\mathcal{N}_{q_2}}{2\alpha_{q_2}\omega_{q_2}} \sqrt{\frac{(E_{p_{q_2}} + m_{q_2})}{E_{p_{q_2}}}} (E_{p_{q_2}} + E_{q_2}) \\
&\times \exp\left(-\frac{\vec{p}_{q_2}^2}{4\alpha_{q_2}}\right).
\end{aligned} \tag{A6}$$

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