

Magnetic charge and black hole supersymmetric quantum statistical relation

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We study the thermodynamics in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit of anti-de Sitter black holes realizing the topological twist. We use a limiting procedure that allows us to reach the extremal point along a trajectory in the space of supersymmetric Euclidean solutions. We show that on this space we can write a quantum statistical relation, which is well-defined in the BPS limit and relies on imposing a suitable constraint among the chemical potentials, due to supersymmetry and regularity. We stress the importance of this in relating the thermal partition function of the dual field theory to the topologically twisted index.

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I. INTRODUCTION

The partition function is an important theoretical observable in the study of supersymmetric field theories, and its exact computation via localization on supersymmetry-preserving backgrounds has been a fertile field of research. It is a function of the background fields (the chemical potentials) and the AdS/CFT correspondence relates it, in certain limits, to the renormalized on-shell action of an asymptotically locally anti-de Sitter (AdS) gravity solution with conformal boundary structure equal to the field theory background. The Legendre transform of the partition function counts the number of states with prescribed charges; correspondingly, in the dual bulk picture in presence of a black hole, the Legendre transform of the on-shell action gives the entropy of the black hole as a function of the charges.

Developing a thermodynamic picture for Bogomol'nyi-Prasad-Sommerfield (BPS) black holes is challenging, since in this case the length of the Euclidean time coordinate is infinite, so the on-shell action naively diverges [1]. One way to circumvent the issue is performing a specific limiting procedure [2], starting with a family of nonsupersymmetric thermal black holes in Lorentzian signature and Wick-rotating to Euclidean signature, hence obtaining solutions that are locally the product of a disc and

a sphere. One then imposes supersymmetry, and obtains solutions that are formally not extremal, since the periodicity of the Euclidean time coordinate is finite, but cannot be Wick-rotated back to a Lorentzian well-defined black hole, except for the extremal ones, for which the period of the Euclidean time diverges. The thermodynamics introduced on the family of supersymmetric but not extremal solutions can thus be extended to the BPS extremal ones by taking a well-defined limit, and matches the expectations obtained from the large- N limit of the dual field theory.

This picture, combining the AdS/CFT dictionary [3] and Euclidean quantum gravity [4], has been warranted in recent years by an extensive analysis of BPS rotating electrically charged black holes, starting from five dimensions [2,5], whose entropy can be computed by taking the Legendre transform of the large- N limit of the superconformal index in a dual field theory.

In four dimensions, it is possible to have magnetically charged BPS black holes, in addition to the electric ones found in [6–8] and investigated in [9–17]. In contrast to the latter, these magnetic solutions, first found in [18], and later on generalized in [19–25], have a well defined static limit, and preserve supersymmetry in a different way, realizing the so-called (partial) “topological twist.” The presence of the magnetic charge indeed changes the conditions to preserve supersymmetry in the bulk [26], and thus the boundary ones as well [27]. In fact, the partition function of the dual field theory in the supersymmetric background computes the topologically twisted index [33], instead of the superconformal index. A lot of progress has been made in relating the black hole entropy to the topologically twisted index (see [34] for a review), yet an investigation of

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the thermodynamics in the BPS limit via the limiting procedure described above is still missing [35]. The aim of this letter is to derive a supersymmetric quantum statistical relation for topologically twisted dyonic solutions and, at the same time, elucidating the field theory interpretation of the regularity conditions of the gravity solutions.

II. SUPERSYMMETRIC QUANTUM STATISTICAL RELATION

As mentioned, in order to achieve a well defined BPS limit for black hole thermodynamics, we start from non-supersymmetric nonextremal black holes, and perform a Wick rotation to compute the holographically renormalized Euclidean on-shell action, for which the quantum statistical relation,

$$I = -S + \beta \left(E - \Omega J - \sum_a \Phi_a Q_a \right), \quad (1)$$

holds [4]. In (1), I is the Euclidean on-shell action, S the entropy, E , J , Q_a the energy, angular momentum and electric charges, β is the inverse temperature, and Ω , Φ_a the angular velocity and electric potentials. The Wick-rotated solution is topologically the product of a Riemann surface and a disc, and the Killing generator of the horizon ξ vanishes at the origin. We impose supersymmetry, and to ensure global regularity of the supersymmetry spinor ϵ , it must be antiperiodic around the circle generated by ξ

$$e^{i\beta \mathcal{L}_\xi} \epsilon = -\epsilon. \quad (2)$$

On the family of supersymmetric solutions, we introduce reduced chemical potentials

$$\tau \equiv \beta \frac{\Omega - \Omega_*}{2\pi i}, \quad \varphi_a \equiv \beta \frac{\Phi_a - \Phi_{a*}}{2\pi i}, \quad (3)$$

where the subscript $*$ indicates the value of the chemical potential of the BPS black hole. These chemical potentials remain finite everywhere on the family of supersymmetric solutions, so expressing the on-shell action in terms of these allows us to define the on-shell action of BPS black holes via a well-defined limit. Moreover, it allows us to write the quantum statistical relation (1) as

$$I = -S - 2\pi i \tau J - 2\pi i \sum_a \varphi_a Q_a + \beta \left(E - \Omega_* J - \sum_a \Phi_{a*} Q_a \right). \quad (4)$$

As shown in [2,10] for a number of Wick-rotated electrically charged black holes in various dimensions, the bracketed term vanishes for supersymmetric configurations

because it coincides with the supersymmetry condition (the so-called “BPS bound”) $E = J + \sum_a Q_a$, since $\Omega_* = \Phi_{a*} = 1$. Therefore, for these supersymmetric solutions, the quantum statistical relation (1) can be reformulated just in terms of reduced chemical potentials

$$I|_{\text{SUSY}} = -S - 2\pi i \tau J - 2\pi i \sum_a \varphi_a Q_a. \quad (5)$$

When used in conjunction with the constraint imposed on the chemical potentials by the existence of an antiperiodic Killing spinor, this formula shows that the reduced chemical potentials are the most suitable conjugate variables to the charges in order to express the entropy of the extremal black holes as the constrained Legendre transform of the on-shell action. Moreover, on the field theory side, the reduced chemical potentials (3) are also the appropriate variables in order to show that the field theory partition function is the index of a supercharge [39].

In the following, we show in two families of examples that the modified BPS bound in presence of magnetic charges found in [26] still allows us to write the supersymmetric quantum statistical relation (5), and to relate the bulk supersymmetry to the boundary one. Contrary to their electrically charged cousins, the magnetic solutions have a well defined static limit; especially for this reason, we focus our attention on static configurations, with and without scalar profiles, while commenting on the rotating case.

III. MINIMAL GAUGED SUPERGRAVITY

We begin with solutions of minimal gauged supergravity with the following metric and gauge field [18,41] (our conventions are summarized in the Supplemental Material [42])

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sinh^2 \theta d\phi^2) \\ \mathcal{A} = \frac{q}{r}dt + p \cosh \theta d\phi + \alpha dt, \quad (6)$$

where α is a constant and

$$V(r) = r^2 - 1 - \frac{2\eta}{r} + \frac{q^2 + p^2}{r^2}. \quad (7)$$

We use the coordinates θ , ϕ to write a local form of the constant curvature metric on the Riemann surface Σ_g , normalized so that $\text{Vol}(\Sigma_g) = 4\pi(g-1)$ for $g > 1$.

This solution describes a family of asymptotically locally AdS static dyonic black holes with horizon diffeomorphic to Σ_g located at r_h , the largest positive root of $V(r)$. The null generator of the horizon is the Killing vector $\xi = \partial_t$. The entropy of the horizon is given by the Bekenstein-Hawking formula

$$S = \frac{r_h^2}{4G_4} \text{Vol}(\Sigma_g). \quad (8)$$

The electrostatic potential of the solution is

$$\Phi \equiv \iota_{\xi} \mathcal{A}|_{r=r_h} - \iota_{\xi} \mathcal{A}|_{r \rightarrow \infty} = \frac{q}{r_h}. \quad (9)$$

We perform a Wick rotation by defining $t = -it_E$. Regularity of the metric allows us to compute the temperature of the horizon as

$$\beta = \frac{4\pi}{V'(r_h)}, \quad (10)$$

and regularity of the gauge field at the horizon requires $\iota_{\xi} \mathcal{A}|_{r=r_h} = 0$, so that $\alpha = -q/r_h = -\Phi$. The resulting solution is a Euclidean metric on $\mathbb{R}^2 \times \Sigma_g$, and an imaginary gauge field which can be made real defining $q = iq_E$.

The holographic charges of the solutions are

$$\begin{aligned} Q &= \frac{q}{4\pi G_4} \text{Vol}(\Sigma_g), & P &= -\frac{p}{4\pi G_4} \text{Vol}(\Sigma_g), \\ E &= \frac{\eta}{4\pi G_4} \text{Vol}(\Sigma_g), & J &= 0, \end{aligned} \quad (11)$$

and the holographically renormalized Euclidean action

$$I = -\frac{\beta}{8\pi G_4} \left(r_h^3 - \eta + \frac{q^2 - p^2}{r_h} \right) \text{Vol}(\Sigma_g). \quad (12)$$

These quantities satisfy the quantum statistical relation (1) in the form

$$I = -S + \beta(E - \Phi Q). \quad (13)$$

The black holes (6) are supersymmetric provided [41]

$$\eta = 0, \quad p^2 = \frac{1}{4}, \quad (14)$$

with electric charge unconstrained. This corresponds to the BPS bound in presence of magnetic charges derived in [26],

$$E = 0, \quad P^2 = \left(\frac{\text{Vol}(\Sigma_g)}{8\pi G_4} \right)^2. \quad (15)$$

We stress once again that this is *different* from the BPS bound in absence of magnetic charge, namely $E = J + Q$.

In addition to regularity of the metric and gauge field, for the Wick-rotated supersymmetric solutions we require a globally well-defined Killing spinor ϵ . This imposes that it must be antiperiodic around the circle generated by ξ , which in the chosen gauge for $\mathcal{A}$ imposes

$$\beta \frac{\Phi}{\pi i} = n_0 \quad (16)$$

with *odd* n_0 . As advertised, in presence of nonvanishing electric charge the Lorentzian supersymmetric solutions are naked singularities [41]. Indeed, inserting (14) in $V(r)$ (7) leads to two branches of solutions with complex r_h^2 and generically complex chemical potentials (unless we consider a pure imaginary q).

On the other hand, the Lorentzian supersymmetric solution with $q = 0$ describes a well-defined extremal black hole. The potentials of the BPS black holes are $\beta_* \rightarrow \infty$ and $\Phi_* = 0$, using which we can introduce the reduced electrostatic potential φ as defined in (3). Substituting (16), this is further constrained to be half-integer, $\varphi = n_0/2$, in order to have an antiperiodic Killing spinor. Indeed, substituting the value of r_h in (9), for the actual bulk solutions we find $n_0 = \pm 1$ depending on the branch.

Inserting the reduced chemical potentials in (13) singles out the BPS bound (15), so that on the supersymmetric locus we can write the supersymmetric quantum statistical relation (5) in the form

$$I|_{\text{SUSY}} = -S + \beta(\Phi - \Phi_*)Q + \beta E = -S - 2\pi\varphi Q. \quad (17)$$

This is clearly satisfied by the on-shell action of the supersymmetric solutions obtained via direct computation

$$I|_{\text{SUSY}} = -\frac{1}{8G_4} \text{Vol}(\Sigma_g) = \mp 2\pi\varphi P. \quad (18)$$

Here we have rewritten the expression using the reduced electrostatic potential and the magnetic charge $P = \text{Vol}(\Sigma_g)/8\pi G_4$ in order to highlight the fact that we are studying gravity in the grand-canonical ensemble for the electric charge and the canonical ensemble for the magnetic charge.

The on-shell action $I|_{\text{SUSY}}$ does not depend on β , so the judicious use of variables on the supersymmetric locus allows us to define the on-shell action of the BPS black hole as a limit. To compute its entropy, we perform the Legendre transform of $I|_{\text{SUSY}}$ with respect to φ , subject to the constraint $\varphi = \pm 1/2$, which follows directly (17)

$$S = \pi(P \mp iQ). \quad (19)$$

Notice that the two branches of complex solutions have formally complex conjugate entropy. Imposing reality of the resulting function corresponds to the extremality condition $Q = 0$, which leads to $S_* = \pi P$. Thus, restricting the right-hand side of (17) to the extremal solution, justifies the relation

$$I|_{\text{SUSY}} = -S_*, \quad (20)$$

which had been observed via other routes in [36–38,43]. Notice that the constraint on the charges imposing reality of the entropy is equivalent to the extremality condition and it is linear, reminiscent of the extremality condition for asymptotically flat black holes (see e.g., [44,45]), and in contrast to the nonlinear relation of asymptotically AdS black holes without a twist [10].

IV. $F = -iX^0X^1$ MODEL

We now extend the considerations of the previous sections to a nonminimal case by looking at the model characterized by prepotential $F = -iX^0X^1$. The bosonic sector consists of the metric $\mathcal{G}$, two $U(1)$ gauge fields $\mathcal{A}_1$, $\mathcal{A}_2$, a scalar X and a pseudoscalar χ . The solution we consider is

$$\begin{aligned} ds^2 &= -\frac{\mathcal{Q}}{W} dt^2 + \frac{W}{\mathcal{Q}} dr^2 + W(d\theta^2 + \sinh^2\theta d\phi^2), \\ \mathcal{A}_1 &= \frac{q_1}{r-\Delta} dt + p_1 \cosh\theta d\phi + \alpha_1 dt, \\ \mathcal{A}_2 &= \frac{q_2}{r+\Delta} dt + p_2 \cosh\theta d\phi + \alpha_2 dt, \\ X &= \left(\frac{r+\Delta}{r-\Delta}\right)^{\frac{1}{2}}, \quad \chi = 0. \end{aligned} \quad (21)$$

The functions $\mathcal{Q}$ and W appearing in the metric are

$$\mathcal{Q} = r^4 + a_2 r^2 - 2a_1 r + a_0, \quad W = r^2 - \Delta^2, \quad (22)$$

with

$$\begin{aligned} a_0 &= \frac{q_1^2 + q_2^2 + p_1^2 + p_2^2}{2} + \Delta^2(1 + \Delta^2), \\ a_1 &= -\frac{q_1^2 - p_1^2 - q_2^2 + p_2^2}{4\Delta}, \\ a_2 &= -1 - 2\Delta^2. \end{aligned} \quad (23)$$

The charges are further constrained by [46]

$$q_1 p_1 - q_2 p_2 = 0. \quad (24)$$

As in the previous case, it describes a family of AlAdS static dyonic black holes with Σ_g horizon at $r = r_h$, the largest positive root of $\mathcal{Q}(r)$, with Killing generator $\xi = \partial_t$, and entropy

$$S = \frac{W(r_h)}{4G_4} \text{Vol}(\Sigma_g). \quad (25)$$

The electrostatic potentials are

$$\Phi_1 = \frac{q_1}{r_h - \Delta}, \quad \Phi_2 = \frac{q_2}{r_h + \Delta}. \quad (26)$$

Regularity of the Wick-rotated metric and gauge field fixes the temperature and the gauge shifts α_a

$$\beta = \frac{4\pi W(r_h)}{\mathcal{Q}'(r_h)}, \quad \alpha_a = -\Phi_a, \quad a = 1, 2. \quad (27)$$

Using holographic renormalization, we also compute the charges of the solution

$$\begin{aligned} Q_a &= \frac{q_a}{8\pi G_4} \text{Vol}(\Sigma_g), \quad P_a = -\frac{p_a}{8\pi G_4} \text{Vol}(\Sigma_g), \\ E &= \frac{a_1}{4\pi G_4} \text{Vol}(\Sigma_g), \quad J = 0, \end{aligned} \quad (28)$$

and the on-shell Euclidean action

$$\begin{aligned} I &= -\frac{\beta}{8\pi G_4} \left[r_h^3 - \Delta^2 r_h - a_1 + \frac{1}{2} \frac{q_1^2 - p_1^2}{(r_h - \Delta)} \right. \\ &\quad \left. + \frac{1}{2} \frac{q_2^2 - p_2^2}{(r_h + \Delta)} \right] \text{Vol}(\Sigma_g), \end{aligned} \quad (29)$$

where we need to impose the constraint among charges (28). Notice that this expression for the on-shell action reduces to the minimal case if we set $\Delta = 0$ and $p_1 = p_2$ [50]. These quantities are related by the quantum statistical relation

$$I = -S + \beta(E - \Phi_1 Q_1 - \Phi_2 Q_2). \quad (30)$$

As showed in the Supplemental Material [42], the solution is supersymmetric provided

$$\begin{aligned} a_1 &= 0, \quad p_a^2 = \frac{1}{4} \\ \Leftrightarrow E &= 0, \quad Q_1 = Q_2, \quad P_a^2 = \left(\frac{\text{Vol}(\Sigma_g)}{16\pi G_4} \right)^2. \end{aligned} \quad (31)$$

Again, we stress that this BPS bound [51] is different from that derived in absence of magnetic charges $E = J + Q_1 + Q_2$ [7]. The Lorentzian solutions are naked singularities, unless we set $q_1 = q_2 = 0$, in which case $\beta_* \rightarrow \infty$, and we find BPS black holes with $Q_{1*} = Q_{2*} = 0$ and vanishing electrostatic potentials [20].

Performing the Wick rotation we find two branches of supersymmetric solutions labeled by $Q_1 = Q_2$ and Δ , with complex chemical potentials, and, following the definitions spelled out in (3), we define the following reduced electrostatic potentials:

$$\varphi_1 = \pm \frac{1}{2} \left(1 + \frac{\Delta}{r_h} \right), \quad \varphi_2 = \pm \frac{1}{2} \left(1 - \frac{\Delta}{r_h} \right) \quad (32)$$

constrained to satisfy

$$\varphi_1 + \varphi_2 = \pm 1, \quad (33)$$

where the different signs correspond to the two branches. Again, this constraint guarantees that the supersymmetry Killing spinor is antiperiodic around the circle generated by ξ , as it should be in order to have disc topology. The introduction of the reduced electrostatic potentials allows us to modify (30) into the supersymmetric quantum statistical relation (5)

$$I|_{\text{SUSY}} = -S - 2\pi i \sum_a \varphi_a Q_a, \quad (34)$$

which indeed holds with

$$I|_{\text{SUSY}} = -\frac{1}{8G_4} \text{Vol}(\Sigma_g) = \mp 2\pi(\varphi_1 P_2 + \varphi_2 P_1), \quad (35)$$

choosing $P_a = \text{Vol}(\Sigma_g)/16\pi G_4$ [52]. Notice that the on-shell action of these solutions is the same as in minimal supergravity, even in presence of the scalar. The constrained Legendre transform leads to $P_1 = P_2$, $Q_1 = Q_2$ and

$$S = 2\pi(P_1 \mp iQ_1), \quad (36)$$

and the reality condition is equivalent to the extremality condition. Moreover, substituting the extremality condition directly in (34) gives again the relation (20) between the supersymmetric on-shell action and the BPS entropy.

The methods described here involve finding supersymmetric saddles of the Euclidean gravitational path integral starting from nonsupersymmetric nonextremal Lorentzian black holes [54]. This explains why we restricted ourselves to considering static black holes, as in the theories we considered there are no known rotating dyonic BPS solutions with compact horizons. Rotating dyonic-twisted BPS solutions with spherical horizon do exist for the *STU* model [55], which is an extension of the models investigated here, but their nonsupersymmetric generalizations are still unknown.

V. TOPOLOGICALLY TWISTED INDEX

Finally, we look at the conformal boundary of the supersymmetric solutions described above, and highlight the importance of a nontrivial holonomy of the R -symmetry gauge field—and thus an antiperiodic supercharge—for the thermal partition function of a three-dimensional $\mathcal{N} = 2$ theory on the supersymmetric background to reproduce the topologically twisted index.

The conformal boundary of the supersymmetric dyonic solutions (21) with (31) is a direct product spacetime $S^1 \times \Sigma_g$ with two background gauge fields A_1, A_2 , whereas the leading scalar sources χ_1, χ vanish

$$\begin{aligned} ds^2 &= dt_E^2 + d\theta^2 + \sinh^2 \theta d\phi^2, \\ A_a &= p_a \cosh \theta d\phi + i\Phi_a dt_E. \end{aligned} \quad (37)$$

Additionally, the bulk Killing spinors determine boundary spinors that define three-dimensional $\mathcal{N} = 2$ supersymmetric backgrounds for conformal supergravity [56]. In fact, expanding the bulk Killing spinor equations near the boundary shows that the boundary spinor is charged under the sum of the two gauge fields, so from the boundary viewpoint, the background gauge field coupling to the $U(1)_R$ R -symmetry is

$$\begin{aligned} A^{(R)} &= \frac{A_1 + A_2}{2} = \frac{1}{2} \cosh \theta d\phi + \frac{i}{2} (\Phi_1 + \Phi_2) dt_E \\ &\equiv \frac{1}{2} \cosh \theta d\phi + i\Phi^{(R)} dt_E. \end{aligned} \quad (38)$$

The holonomy of $A^{(R)}$ around S^1_β is constrained by (33) to be $\Phi^{(R)} = \pm \pi i / \beta$, and this guarantees that the boundary spinor is antiperiodic around S^1_β [as can be seen by expanding near the boundary the bulk spinor or by solving directly the conformal Killing spinor equation in the background (37)]. The difference of the two gauge fields, instead, has the role of a flat background gauge field coupling to a flavor symmetry

$$\begin{aligned} A^{(f)} &= \frac{A_1 - A_2}{2} = \frac{i}{2} (\Phi_1 - \Phi_2) dt_E \\ &= i \frac{q_1 \Delta}{r_h^2 - \Delta^2} dt_E \equiv i\Phi^{(f)} dt_E. \end{aligned} \quad (39)$$

On this background we can formulate a three-dimensional $\mathcal{N} = 2$ field theory while preserving two supercharges with opposite $\mathfrak{u}(1)_R$ charge [57]. As we quantize the theory on Σ_g , we find a Hilbert space $\mathcal{H}_{\Sigma_g}$ with states labeled by the eigenvalues of the Hamiltonian H and the generators of the flavor symmetry J_f . The supercharges preserved by the background have anticommutation relation

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = H - 2\pi\sigma_f J_f, \quad (40)$$

where σ_f is the real mass present in the background vector multiplet coupling to $U(1)_f$. The partition function with thermal boundary conditions on the background (37) has the form

$$\begin{aligned} Z &= \text{Tr}_{\mathcal{H}_{\Sigma_g}} e^{-\beta H + \beta \Phi^{(R)} R + \beta \Phi^{(f)} J_f} \\ &= \text{Tr}_{\mathcal{H}_{\Sigma_g}} (-1)^R e^{-\beta \{\mathcal{Q}, \mathcal{Q}^\dagger\} + \beta (\Phi^{(f)} - 2\pi\sigma_f) J_f} \\ &= \text{Tr}_{\mathcal{H}_{\Sigma_g}} (-1)^R e^{-\beta \{\mathcal{Q}, \mathcal{Q}^\dagger\} + 2\pi i u_f J_f}. \end{aligned} \quad (41)$$

Here, we massaged the first expression inserting (40), the constraint on the holonomy of $A^{(R)}$, and the complex combination of the gauge field holonomy and scalar of the flavor background vector multiplet

$$u_f \equiv -\frac{1}{2\pi} \int_{S^1_\beta} A^{(f)} + i\beta\sigma_f. \quad (42)$$

The manipulations above connect the thermal partition function to a supersymmetric index for the supercharge $\mathcal{Q}$, graded by the R -symmetry, and refined by the flavor symmetry. Crucially, it is the presence of an R -symmetry background with nontrivial holonomy that allows a grading while the fermions have antiperiodic boundary conditions. The resulting expression is the topologically twisted index of the field theory [33,58,59], which in presence of flavor symmetries generated by f^α is defined as [60]

$$\mathcal{I} = \text{Tr}_{\mathcal{H}_{\Sigma_g}} (-1)^F e^{-\beta\{\mathcal{Q}, \mathcal{Q}^\dagger\} + 2\pi i \varphi^{(R)}(R+F) + 2\pi i \sum_\alpha u_{f^\alpha} J_{f^\alpha}}. \quad (43)$$

Notice the role of the holonomy of the R -symmetry background around S^1_β , $\varphi^{(R)} = \beta\Phi^{(R)}/2\pi i \in \frac{1}{2}\mathbb{Z}$, which determines the spin structure on S^1_β and in turn on the entire $S^1_\beta \times \Sigma_g$ [61,62]. In the cases just considered, $\varphi^{(R)}$ is half-integer, which is needed in order to be able to “fill” the background and have a bulk gravity dual. On the other hand, if $\varphi^{(R)}$ is integer, the Killing spinor is periodic around S^1_β , the topologically twisted index is graded by the fermion number F , but this circle cannot bound a disk.

As a final check, we can look at the large- N limit of the topologically twisted index of the specific mass-deformed

ABJM dual to the solutions considered, since the $F = -iX^0X^1$ model is a truncation of M -theory on S^7 [63,64]. In the large- N limit, one finds that in presence of the background gauge fields (38) and (39) [53,65]

$$\log Z_{\text{ABJM}} \sim \pm \frac{2\pi\sqrt{2}}{3} N^{\frac{3}{2}} \mathfrak{p}_R \varphi^{(R)}, \quad (44)$$

where $\mathfrak{p}_R = g - 1$ is the first Chern number of the R -symmetry bundle, provided $\varphi^{(R)} = \pm 1/2$. This matches the above expectation that the leading large- N behavior requires the antiperiodic spin structure around S^1_β , and matches (35), including the constraint, using the standard dictionary $1/G_4 = 2\sqrt{2}N^{\frac{3}{2}}/3$.

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- [1] Generically, the supersymmetry and extremality conditions do not coincide. Adopting the nomenclature of [2], we denote by “BPS” the extremal supersymmetric solutions.
 - [2] A. Cabo-Bizet, D. Cassani, D. Martelli, and S. Murthy, *J. High Energy Phys.* **10** (2019) 062.
 - [3] E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998).
 - [4] G. W. Gibbons and S. W. Hawking, *Phys. Rev. D* **15**, 2752 (1977).
 - [5] S. M. Hosseini, K. Hristov, and A. Zaffaroni, *J. High Energy Phys.* **07** (2017) 106.
 - [6] V. A. Kostelecky and M. J. Perry, *Phys. Lett. B* **371**, 191 (1996).
 - [7] M. Cvetič, G. W. Gibbons, H. Lu, and C. N. Pope, *arXiv: hep-th/0504080*.
 - [8] K. Hristov, S. Katmadas, and C. Toldo, *Phys. Rev. D* **100**, 066016 (2019).
 - [9] S. Choi, C. Hwang, S. Kim, and J. Nahmgoong, *J. Korean Phys. Soc.* **76**, 101 (2020).
 - [10] D. Cassani and L. Papini, *J. High Energy Phys.* **09** (2019) 079.
 - [11] N. Bobev and P. M. Cricigno, *J. High Energy Phys.* **12** (2019) 054.
 - [12] J. Nian and L. A. Pando Zayas, *J. High Energy Phys.* **03** (2020) 081.
 - [13] F. Benini, D. Gang, and L. A. Pando Zayas, *J. High Energy Phys.* **03** (2020) 057.
 - [14] S. M. Hosseini, C. Toldo, and I. Yaakov, *J. High Energy Phys.* **07** (2020) 131.
 - [15] A. González Lezcano, M. Jerdee, and L. A. Pando Zayas, *J. High Energy Phys.* **01** (2023) 044.
 - [16] N. Bobev, S. Choi, J. Hong, and V. Reys, *J. High Energy Phys.* **02** (2023) 027.
 - [17] P. Benetti Genolini, A. Cabo-Bizet, and S. Murthy, *arXiv: 2301.00763*.
 - [18] L. J. Romans, *Nucl. Phys.* **B383**, 395 (1992).
 - [19] J. P. Gauntlett, N. Kim, S. Pakis, and D. Waldram, *Phys. Rev. D* **65**, 026003 (2002).
 - [20] S. L. Cacciatori and D. Klemm, *J. High Energy Phys.* **01** (2010) 085.
 - [21] G. Dall’Agata and A. Gnecci, *J. High Energy Phys.* **03** (2011) 037.
 - [22] K. Hristov and S. Vandoren, *J. High Energy Phys.* **04** (2011) 047.
 - [23] N. Halmagyi and T. Vanel, *J. High Energy Phys.* **04** (2014) 130.
 - [24] S. Katmadas, *J. High Energy Phys.* **09** (2014) 027.
 - [25] N. Halmagyi, *J. High Energy Phys.* **03** (2015) 032.

- [26] K. Hristov, C. Toldo, and S. Vandoren, *J. High Energy Phys.* **12** (2011) 014.
- [27] There is yet another distinct way to obtain dyonic supersymmetric black holes, by considering C-metric solutions describing accelerating black holes with magnetic charge related to the acceleration parameter [28–32]. In the present paper we restrict our attention to dyonic twisted non-accelerating solutions.
- [28] P. Ferrero, J. P. Gauntlett, J. M. P. Ipiña, D. Martelli, and J. Sparks, *Phys. Rev. D* **104**, 046007 (2021).
- [29] D. Cassani, J. P. Gauntlett, D. Martelli, and J. Sparks, *Phys. Rev. D* **104**, 086005 (2021).
- [30] P. Ferrero, M. Inglese, D. Martelli, and J. Sparks, *Phys. Rev. D* **105**, 126001 (2022).
- [31] F. Faedo and D. Martelli, *J. High Energy Phys.* **02** (2022) 101.
- [32] P. Ferrero, J. P. Gauntlett, and J. Sparks, *J. High Energy Phys.* **01** (2022) 102.
- [33] F. Benini and A. Zaffaroni, *J. High Energy Phys.* **07** (2015) 127.
- [34] A. Zaffaroni, *Living Rev. Relativity* **23**, 2 (2020).
- [35] Though a different limiting procedure to define the on-shell action of the BPS black hole by studying perturbations to the supersymmetric solution has been proposed in the literature [36–38].
- [36] F. Azzurli, N. Bobev, P. M. Crichigno, V. S. Min, and A. Zaffaroni, *J. High Energy Phys.* **02** (2018) 054.
- [37] N. Halmagyi and S. Lal, *J. High Energy Phys.* **03** (2018) 146.
- [38] A. Cabo-Bizet, U. Kol, L. A. Pando Zayas, I. Papadimitriou, and V. Rathee, *J. High Energy Phys.* **05** (2018) 155.
- [39] See [17,40] for a review of the cases of three- and four-dimensional SCFTs.
- [40] O. Aharony, F. Benini, O. Mamroud, and E. Milan, *Phys. Rev. D* **104**, 086026 (2021).
- [41] M. M. Caldarelli and D. Klemm, *Nucl. Phys.* **B545**, 434 (1999).
- [42] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevD.107.L121902> for conventions and details of the Killing spinor equations and holographic renormalization.
- [43] P. Benetti Genolini, J. M. Perez Ipiña, and J. Sparks, *J. High Energy Phys.* **10** (2019) 252.
- [44] M. Heydeman, L. V. Iliesiu, G. J. Turiaci, and W. Zhao, *J. Phys. A* **55**, 014004 (2022).
- [45] K. Hristov, *J. High Energy Phys.* **09** (2022) 204.
- [46] There also exists a family of static solutions with unconstrained charges [47], and rotating extensions thereof [47,48]. Euclidean solutions generalizing (23) have been presented in [49]. However, for the purposes of this paper it is appropriate to work with this solution.
- [47] D. D. K. Chow and G. Compère, *Phys. Rev. D* **89**, 065003 (2014).
- [48] A. Gnechchi, K. Hristov, D. Klemm, C. Toldo, and O. Vaughan, *J. High Energy Phys.* **01** (2014) 127.
- [49] N. Bobev, A. M. Charles, and V. S. Min, *J. High Energy Phys.* **10** (2020) 073.
- [50] For $\Delta = 0$ and $p_1 = p_2$, $q_1 = q_2$ the parameter a_1 is unconstrained by the equations of motion, and can be identified with the parameter η in (7).
- [51] K. Hristov, *J. High Energy Phys.* **03** (2012) 095.
- [52] Rewriting the action in terms of φ_a , P_a is ambiguous. We choose the formula to be consistent with the large- N limit of the partition function of mass-deformed ABJM [53].
- [53] F. Benini, K. Hristov, and A. Zaffaroni, *Phys. Lett. B* **771**, 462 (2017).
- [54] On the other hand, there also exist saddles of the path integral that are intrinsically Euclidean, such as the ones found in [49], that do not have a Lorentzian black hole counterpart and whose on-shell action can be matched to a field theory computation.
- [55] K. Hristov, S. Katmadas, and C. Toldo, *J. High Energy Phys.* **01** (2019) 199.
- [56] C. Klare, A. Tomasiello, and A. Zaffaroni, *J. High Energy Phys.* **08** (2012) 061.
- [57] C. Closset, T. T. Dumitrescu, G. Festuccia, and Z. Komargodski, *J. High Energy Phys.* **01** (2014) 124.
- [58] F. Benini and A. Zaffaroni, *Proc. Symp. Pure Math.* **96**, 13 (2017).
- [59] C. Closset and H. Kim, *J. High Energy Phys.* **08** (2016) 059.
- [60] If $\Sigma_g \cong S^2$, there is an additional isometry, the states in $\mathcal{H}_{S^2}$ are labelled by the angular momentum as well, and it is possible to further refine the index [33]. The bulk dual states would be dyonic rotating solutions with spherical horizons such as [55].
- [61] C. Closset, H. Kim, and B. Willett, *J. High Energy Phys.* **11** (2018) 004.
- [62] C. Closset and H. Kim, *Int. J. Mod. Phys. A* **34**, 1930011 (2019).
- [63] M. Cvetič, M. J. Duff, P. Hoxha, J. T. Liu, H. Lu, J. X. Lu, R. Martinez-Acosta, C. N. Pope, H. Sati, and T. A. Tran, *Nucl. Phys.* **B558**, 96 (1999).
- [64] A. Azizi, H. Godazgar, M. Godazgar, and C. N. Pope, *Phys. Rev. D* **94**, 066003 (2016).
- [65] F. Benini, K. Hristov, and A. Zaffaroni, *J. High Energy Phys.* **05** (2016) 054.