

Geometric phases distinguish entangled states in wormhole quantum mechanics

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We establish a relation between entanglement in simple quantum mechanical qubit systems and in wormhole physics as considered in the context of the AdS/CFT correspondence. We show that in both cases, states with the same entanglement structure, indistinguishable by any local measurement, nevertheless are characterized by a different Berry phase. This feature is experimentally accessible in coupled qubit systems where states with different Berry phase are related by unitary transformations. In the wormhole case, these transformations are identified with a time evolution of one of the two throats.

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In recent years, significant progress has been achieved in establishing new relations between geometry and gravity on the one hand, and quantum entanglement on the other. An important example is the Ryu-Takayanagi formula [1] in the context of the AdS/CFT correspondence [2] relating the entanglement entropy of a conformal field theory (CFT) to the area of a minimal surface in anti-de Sitter (AdS) space. Moreover, in the ER = EPR conjecture [3] it is argued that the entanglement in a thermofield double (TFD) state is holographically realized by a geodesic in a nontraversable wormhole in AdS space. The length of the geodesic, stretching between the two boundaries of AdS space, quantifies the amount of entanglement [4].

Within a simpler setting, an early example is provided by the semiclassical Wheeler wormhole [5,6]. An important feature of this solution is that the magnetic field involved cannot be globally written in terms of a vector potential. This amounts to a nonexact symplectic form, yielding a quantized flux, similarly to a magnetic monopole [7].

Recently, H. Verlinde [8] studied quantum-mechanical examples of wormholes by analyzing their partition functions. For systems with a nonexact symplectic form, the thermal partition function becomes a functional integral over

a two-dimensional surface, and corresponds to the Renyi entropy of a thermal mixed state.

Here we show further novel similarities between a simple coupled two-qubit system in quantum mechanics and spacetime wormholes in gravity, in form of a nonfactorized structure of the respective Hilbert spaces. Although in quantum mechanics there is no notion of a horizon causally separating two regions, the nonfactorized structure demonstrates how entanglement between the two subsystems plays the pivotal role in building the full Hilbert space. For quantum mechanics as well as for gravity, the nonfactorization manifests as a class of states with the same entanglement but different Berry phases.

For the gravity analysis, we focus on the entanglement structure of an eternal black hole in AdS space. This black hole is dual to a maximally entangled state of two decoupled CFTs living on the left (L) and right (R) boundaries in the TFD state [9] (Fig. 1). This state can be derived from the Hartle-Hawking wave functional for the bulk geometry [10–12]. As shown in [13,14], there exists a class of time-shifted wormholes dual to states related to the original TFD state by phases α_n . These arise from unequal time evolution at the boundaries. From the perspective of the bulk AdS geometry, these phase-shifted states correspond to the same wormhole geometry, but with different asymptotic identifications of boundary times. Such phase-shifted states have the same entanglement properties as the TFD state. However, their Berry phases are different. This is due to the fact that a gravitational system in the presence of black holes does not have a globally defined time.

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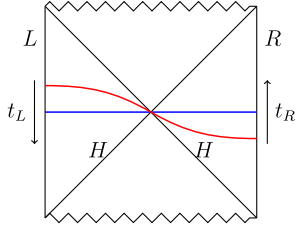


FIG. 1. Kruskal diagram of the eternal Schwarzschild black hole in AdS with time in the vertical and spatial coordinates in the horizontal direction. Vertical lines denote the left (L) and right (R) asymptotic boundaries where the CFTs are defined. The jagged lines are the singularities and diagonal lines represent the black hole horizon (H). The blue line corresponds to the wormhole dual to the original TFD state while the red one corresponds to a time-shifted wormhole. Arrows indicate the directions of time in the boundary theories.

Here we show that by distinguishing between states with the same entanglement, the Berry phases with respect to this phase-shifted parameter provide a precise analogue to the entanglement structure of coupled quantum spins in a magnetic field. By determining the Berry phase for entangled states of two coupled spins with respect to the magnetic field, we indeed directly find that different states related by unitary transformations, and thus sharing the same entanglement structure, have nevertheless different Berry phases. Furthermore, as we will show following [8], in both our quantum spin system and in gravity, this behavior traces back to the presence of a nonexact symplectic form. The quantum spin model we consider is realized for instance by a hydrogen atom with hyperfine coupling between proton and electron spin. In the quantum mechanics context, finding different Berry phases for states with the same entanglement can be probed experimentally by quantum-state tomography [15] on this specific qubit pair.

I. UNITARY TRANSFORMATIONS AND BERRY PHASE: COUPLED QUANTUM SPINS

We first consider the entanglement structure of a hydrogen atom with hyperfine coupling J between proton and electron spins in magnetic field. In first approximation, the Zeeman coupling to the proton spin can be neglected, leading to the Hamiltonian,

$$H = JS_1 \cdot S_2 - 2\mu_B B S_{1z}, \quad (1)$$

where the second term is the electronic Zeeman interaction. The ground state with energy $E_0 = -J/4 - \sqrt{J^2/4 + (\mu_B B)^2}$ is given by $|\psi_0\rangle = -\sin(\alpha/2)|1, 0\rangle + \cos(\alpha/2)|0, 0\rangle$, in terms of the singlet and triplet states $|s, m\rangle$ and $\tan \alpha = 2\mu_B B/J$. For our analysis we generalize the Hamiltonian (1) to an arbitrary coupling term $\propto \mathbf{B} \cdot \mathbf{S}_1$. Then we apply unitary transformations on the spins designed such that the interaction term reduces back

to the form in (1). While this restricts the transformation on the first spin to $U_1(\varphi, \theta) = e^{-i\varphi S_{1z}} e^{-i\theta S_{1y}} e^{i\varphi S_{1z}}$, we are free to choose what transformation to use on the second spin. Thus, we set $U = U_1 \otimes U_2$, where $U_2(\varphi, \theta) = U_1(\lambda\varphi, \lambda\theta)$, with a parameter $\lambda \in [0, 1]$ interpolating between $U_2 = \mathbb{1}$ for $\lambda = 0$ and $U_2 = U_1$ for $\lambda = 1$.

The entanglement entropy between the spins is not affected by the application of a local unitary transformation $U = U_1 \otimes U_2$, in which U_1 and U_2 only act on their respective subsystem [16], i.e., $S_{EE}^i(\text{tr}_j(|\psi_0\rangle\langle\psi_0|)) = S_{EE}^i(\text{tr}_j(U|\psi_0\rangle\langle\psi_0|U^\dagger))$, where $i, j \in \{1, 2\}$ and $i \neq j$. Thus, we obtain a class of states, parametrized by λ , with the same entanglement entropy. As we now discuss, these states are nevertheless distinguished by the Berry phase, which is sensitive to λ .

II. BERRY PHASE PROBING THE HYPERFINE STRUCTURE OF ENTANGLEMENT

To define the Berry connection for the setup discussed above, we make use of the Maurer-Cartan form [17]. This is the natural connection on a group manifold $\mathcal{M}$ and is defined for any group element $\sigma \in \mathcal{M}$ as $A_{MC} = \sigma^{-1} d\sigma$. The group element we choose in our example is the unitary transformation U and d is the exterior derivative acting on the phase space variables φ and θ . The Berry connection is then given by the expectation value of the Maurer-Cartan form for the ground state,

$$A_B(\lambda) = i\langle\psi_0|A_{MC}|\psi_0\rangle = i\langle\psi_0|(U^\dagger dU)|\psi_0\rangle = \frac{\sin \alpha}{2} \{ (1 - \cos \theta) - \lambda [1 - \cos(\lambda\theta)] \} d\varphi. \quad (2)$$

From it we define the associated symplectic (Kirillov-Kostant) form, $\omega_{KK} = dA_{MC}$ [17]. Hence,

$$F_B(\lambda) = i\langle\psi_0|\omega_{KK}|\psi_0\rangle = \frac{\sin \alpha}{2} (\sin \theta - \lambda^2 \sin(\lambda\theta)) d\theta \wedge d\varphi. \quad (3)$$

The Berry phase follows by integrating (3) over the phase space

$$\Phi_B = \int_0^{2\pi} \int_0^\pi F_B(\lambda) = \pi \sin \alpha \{ 2 - \lambda [1 - \cos(\lambda\pi)] \}, \quad (4)$$

which is nontrivial as long as $\lambda \neq 1$, i.e., $U_2 \neq U_1$ yields a nontrivial Berry phase. *The key observation is that according to (4), two states $|\psi_0\rangle$ and $U|\psi_0\rangle$ with the same entanglement entropy nevertheless have a different Berry phase, characterized by a different value of λ . This feature can be converted into an experimental tool to differentiate between two such states with indistinguishable entanglement properties.*

The nonzero Berry phase (4) is related to the nonexactness of the Berry curvature (3). The space in our case being just compact S^2 , the symplectic form, and equivalently the Berry curvature cannot be globally exact. A nontrivial Berry phase can be understood as the holonomy [18] associated to this nonexact symplectic form.

III. BERRY PHASE AND NONEXACT SYMPLECTIC STRUCTURE

Let us compare the above results to gravitational wormholes, focussing on the nonfactorization of spacetime in the presence of a wormhole. To do so, we reformulate the symplectic structure of the Hamiltonian (1) and the unitary transformation U by a group theoretic analysis.

A single spin is represented in $\mathbf{CP}^1$ variables z_i as, $\mathbf{S} = (1/2)z_i^* \boldsymbol{\sigma}^{ij} z_j$. For the two-spin model we may introduce $\mathbf{CP}^1$ variables for each spin, which in absence of an external field leads to a tensor product structure $\mathbf{CP}^1 \times \mathbf{CP}^1$ that can be realized as a diagonal embedding into $\mathbf{CP}^3$. The factorized $\mathbf{CP}^1 \times \mathbf{CP}^1$ structure is sufficient to explain local properties. However to understand nonlocal correlations, leading to wormholelike structures, it is necessary to consider the full $\mathbf{CP}^3$ symmetry. In particular, including the Zeeman term as in (1) breaks the tensor product structure. Using the $\mathbf{CP}^3$ representation we can understand the Berry phase arising for different unitaries.

As a coset space $\mathbf{CP}^3 = \text{SU}(4)/\text{U}(3)$, so the $\text{U}(1)$ factor in $\text{U}(3)$ can be used to define a $\text{U}(1)$ bundle. The bundle connection follows from the Maurer-Cartan form, with symplectic form given by the exterior derivative of the connection. Since in the presence of the magnetic field, the ground state does not lie inside the natural $\mathbf{CP}^1 \times \mathbf{CP}^1 \subset \mathbf{CP}^3$ submanifold, we have to consider the Berry connection inside the full $\text{U}(1)$ bundle given by $\mathbf{CP}^3$. In particular, the state evolution given by $U(\lambda) = U_1 \otimes U_2(\lambda)$ implies a Berry connection (3) on the full $\mathbf{CP}^3$ bundle.

For a general $g \in \text{SU}(4)$ in the coset construction of $\mathbf{CP}^3$, the connection in the $\text{U}(1)$ fiber is defined as

$$A = \text{itr}(t_{\text{U}(1)} A_{\text{MC}}(g)). \quad (5)$$

Here, $t_{\text{U}(1)}$ corresponds to the generator of $\text{SU}(4)$ yielding the $\text{U}(1)$ factor of $\text{U}(3)$ given by

$$t_{\text{U}(1)} = \frac{1}{\sqrt{24}} \text{diag}(1, 1, 1, -3). \quad (6)$$

To obtain the connection for the Hamiltonian (1), we pick the transformation U as group element of $\text{SU}(4)$. This group element only belongs to the factorised subgroup $\text{SU}(2) \times \text{SU}(2)$. However, $\text{SU}(4) \supset \text{SU}(2) \times \text{SU}(2) \times \text{U}(1)$ where the $\text{U}(1)$ factor rotates in the first $\text{SU}(2)$ subspace with a positive phase, and in the second $\text{SU}(2)$ subspace with the opposite phase [19]. To account

for this relative sign in our transformation $U(\lambda)$, we have to replace $\sigma_i \rightarrow -\sigma_i$ in the transformation acting only on the second spin, $U_2 \rightarrow \tilde{U}_2 = U_2^T$. Inserting (6) into (5) and considering $g = U_1 \otimes \tilde{U}_2$, we find the same connection as in (2) [20].

Following the argument of [8], we identify the nonexact Berry curvature (3) as signaling a wormhole structure. As discussed above, this structure arises from nondiagonally embedded submanifolds of $\mathbf{CP}^3$. It is not manifest for the diagonal embedding $\mathbf{CP}^1 \times \mathbf{CP}^1$.

We now show that the Berry phase nonexact symplectic structure is also present in actual spacetime wormholes due to the lack of a global time-like Killing vector. Therefore, correlations present in the emergent wormhole show up in the Berry connection whenever different unitary transformations act upon the two spins.

IV. UNITARY TRANSFORMATIONS AND BERRY PHASE: GRAVITATIONAL WORMHOLES

The group-theory and entanglement structures obtained for the two-spin quantum mechanical model arise also for gravitational wormholes in an exactly analogous way. As an example, we consider an eternal black hole in AdS spacetime depicted in Fig. 1.

We find, in exact analogy to the two-spin example, a Berry phase by considering different unitary transformations given by Hamiltonian evolution of the boundary states. The class of wormholes shown in Fig. 1 is dual to the class of states

$$|\text{TFD}_\alpha\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{i\alpha_n} e^{-\beta \frac{E_n}{2}} |n\rangle_L |n\rangle_R^*, \quad (7)$$

where Z is the partition sum, β is the inverse temperature and $E_n = (E_n^{(L)} + E_n^{(R)})/2$ are the sums of the energy eigenvalues corresponding to the left and right energy eigenstates $|n\rangle_{L/R}^{(*)}$. When all phases α_n vanish, (7) reduces to the standard TFD state. These additional phase factors can be incorporated in the Hartle-Hawking derivation when relating left and right boundary states $|n\rangle_L$ and $|n\rangle_R$ by an antiunitary operator Θ [10]. The map $|n\rangle_L = \Theta |n\rangle_R$ is not unique. The fact that it is defined up to a phase gives rise to the phases in (7) [20], which do not change the entanglement properties of the resulting state. When calculating the reduced density matrix of either CFT using (7), the phases α_n drop out, causing the entanglement entropy to remain insensitive [20]. The same conclusion is also reached by computing the mixed two-point correlators $\langle \text{TFD}_\alpha | \mathcal{O}_L \mathcal{O}_R | \text{TFD}_\alpha \rangle$. They are insensitive to the phases α_n [21], implying unaltered entanglement properties.

Each field theory boundary shown in Fig. 1 has a time coordinate t_L and t_R , respectively. As shown in Fig. 2, at the interface I we have to specify an identification between the

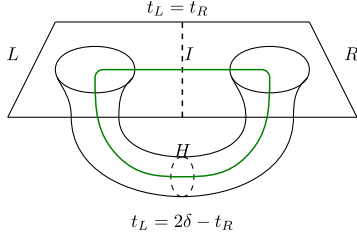


FIG. 2. Schematic representation of a wormhole corresponding to the colored lines in Fig. 1. The times t_L and t_R of the left and right regions are identified at the interface I . In the bulk a different relation is used that accounts for the sign flip of the timelike Killing vector at the horizon H . The Berry phase is accumulated along the closed loop printed in green.

boundary times, which for convenience we choose as $t_L = t_R$. However, since there is no globally defined time for the dual geometry due to the presence of the black hole horizon H , this asymptotic identification cannot hold throughout the bulk. In particular, since the timelike Killing vector flips sign across the horizon, a time shift δ at the horizon needs to be taken into account, via the relation $t_L = 2\delta - t_R$ [14]. Together with the boundary identification $t_L = t_R$, this relation implies $\delta = t_L = t_R$ at the boundaries. Thus, time evolution on both boundaries can be expressed using the emerging parameter δ . As shown in [14], the additional phases appearing in the time-shifted TFD states (7) can be understood as resulting from time evolution through an identification $\alpha_n = -2E_n\delta$. In (7), the α_n clearly have periodicities of 2π ,

$$|\text{TFD}_\alpha\rangle = |\text{TFD}_{\alpha+2\pi}\rangle. \quad (8)$$

Thus, in analogy to the quantum mechanics example, we define a Berry connection for (7), with $\alpha_n = -2E_n\delta$,

$$A_\delta = i\langle \text{TFD}_\alpha | \partial_\delta | \text{TFD}_\alpha \rangle = \frac{2}{Z} \sum_n E_n e^{-\beta E_n}. \quad (9)$$

Note that δ is a phase space variable. We stress here that the existence of δ is a direct consequence of the lack of a global time in the gravitational spacetime.

Equation (9) follows from considering unitary transformations in the asymptotic symmetry group, which is the subgroup of bulk diffeomorphisms leaving the boundary conditions invariant. The gauge parameter of this group is δ [20].

In order to show the analogy between the two-spin system and the gravitational example, we consider below two contrasting scenarios.

- (i) $\lambda = 0$: In this case the quantum mechanical example leading to (2) implies a nontrivial Berry connection. An analogous situation arises for the TFD state for the unitary operation

$$u_0(\delta) = e^{-i(H_L + H_R)\delta} \quad (10)$$

belonging to the asymptotic symmetry group. The transformation (10) can be turned into a one-sided transformation via

$$\tilde{u}_0(\delta) = u_1(\delta) \cdot u_0(\delta) = e^{-2iH_L\delta}, \quad (11)$$

$$u_1(\delta) = e^{-i(H_L - H_R)\delta}, \quad (12)$$

where $u_1(\delta)$ is a trivial transformation for $H_L = H_R$. In order to compute the Berry connection, we note that the time-shifted TFD states (7) are obtained from the TFD states without additional phases by applying (11),

$$|\text{TFD}_\alpha\rangle = \tilde{u}_0(\delta) |\text{TFD}_{\alpha=0}\rangle. \quad (13)$$

With respect to δ , the Berry connection is then given by (9). This is the exact analogue of the quantum mechanical case with $\lambda = 0$. Both for the two-spin system and for the wormhole geometry, nontrivial Berry connections are obtained for one-sided transformation, which can also be understood as a nonzero holonomy [22].

- (ii) $\lambda = 1$: According to (2), the $\lambda = 1$ case in the quantum mechanics example implies a vanishing Berry connection. An analogous situation arises for the TFD state when we consider the unitary operation (12). Since the times t_L and t_R run in opposite direction [26] as shown in Fig. 1, the transformation (12) corresponds to applying the same unitary transformation to the two subsystems as in the two-spin system. The Berry connection for this transformation reads, $A_\delta = i\langle \text{TFD}_{\alpha=0} | u_1^\dagger \partial_\delta u_1 | \text{TFD}_{\alpha=0} \rangle = 0$, in analogy to the $\lambda = 1$ case in (2).

V. NONEXACT SYMPLECTIC FORM WITHIN GRAVITY

These structures may be realized explicitly for a wormhole in Jackiw-Teitelboim (JT) gravity [27–29], consisting of AdS_2 gravity coupled to a dilaton field. The Hamiltonians H_L and H_R are given in terms of the ratio of the dilaton values at the horizon and the AdS boundary, ϕ_h and ϕ_b , as $H_L = H_R = \phi_h^2 / \phi_b > 0$ [30]. Thus, the asymptotic symmetries are given only by time translations with associated gauge parameter δ . Clearly, the difference $H_L - H_R$ is a trivial operator and only $H = H_L + H_R$ generates time evolution. The associated phase space consists of variables δ and ϕ_h , or equivalently δ and H , with symplectic form [30]

$$\omega = \frac{4\phi_h}{\phi_b} d\delta \wedge d\phi_h = d\delta \wedge dH. \quad (14)$$

As in the quantum mechanical example, this symplectic form yields the Berry curvature. The trajectory for the Berry phase is visualized in Fig. 2. It enters the wormhole on one side, exits on the other and then closes the loop to the starting point outside the wormhole. Recall that due to the presence of the horizon, defined by the region where the dilaton ϕ takes the value ϕ_h , δ is π/E -periodic following (8) with $E = \phi_h^2/\phi_b$, and $H > 0$ [31]. From this we find that the connection (9) and its symplectic form (14) are defined only on the complement of the origin of $\{\delta, H\}$ phase space, i.e., the phase space has the topology of $\mathbf{R}^2 \setminus \{0\}$, where δ and $\phi_h \propto H$ correspond to angular and radial coordinate, respectively. In JT gravity, where $E = \phi_h^2/\phi_b$ and (9) evaluates to $A_\delta = 2\phi_h^2/\phi_b$, this yields a Berry phase

$$\Phi_B^{\text{JT}} = \oint A_\delta d\delta = 2 \frac{\phi_h^2}{\phi_b} \int_0^{\frac{\pi}{E}} d\delta = 2\pi, \quad (15)$$

associated with the holonomy of the closed path. Since δ is periodic while ϕ_h is not, this holonomy is understood as a winding number of a circle, thus establishing that the symplectic structure (14) is nonexact. The holonomy (15) is due to the fact that in presence of the horizon, the path through the wormhole cannot be shrunk to a point. Equation (15) is analogous to the quantum mechanical result (4) for the special case $\lambda = 0$.

VI. DISCUSSION AND CONCLUSION

The nonexactness of the symplectic structure and the corresponding Berry phases associated to a gravitational wormhole are equivalently found in analyzing the quantum mechanical two-spin system. This further exemplifies the surprising similarity between quantum mechanical models and gravitational wormholes. In the gravitational setup, wormholes arise naturally from the nonlocal structure of the spacetime due to the presence of the black hole horizon. In quantum mechanical systems, where locality is manifest, this feature is realized by intricate group theoretical structures. Using group theoretic arguments, we have actually identified the entangled degrees of freedom in

our quantum mechanical system which are responsible for creating a wormhole geometry in spin space. This may be viewed as a manifestation of *entanglement creating space-time* which lies at the heart of understanding AdS/CFT duality [3,4,32,33].

In particular, we found that both for wormholes and in simple qubit systems, there are entangled states having different Berry phases sharing the same entanglement entropy. This analysis might be probed on a number of experimental platforms. Generally measurements of both quantum entanglement and of Berry phases involve interference between the original starting state and a rotated one for an ensemble of identical quantum states. Apart from the two-spin systems discussed above, which are toy models for the multiple spin-qubits accessible in liquid-state nuclear magnetic resonance [34], also quantum dots coupled to an optical cavity [35] and superconducting quantum circuits offer experimental platforms for quantum tomography on controlled qubit pairs [36–38]. A further avenue arises from the fact that TFD states can be experimentally prepared to a high accuracy using quantum approximate optimization algorithm for transverse field Ising models [39]. Modifying this algorithm to realize time-shifted TFD states as in (7) will provide an experimental realization of the proposed entanglement structure in this context.

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