

Higher-derivative corrections to black hole entropy at zero temperature

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There is a well-established connection between the higher-derivative corrections to the black hole entropy and the black hole extremality bound. The particular combination of effective field theory (EFT) coefficients c_i that controls the mass shift at fixed charge and temperature also controls the entropy shift at fixed charge and mass, in the limit where the mass approaches the *uncorrected* value for the extremal mass. In this paper, we use the classical entropy function formalism to examine the entropy corrections at exactly zero temperature, or at the *corrected* value for the extremal mass. We find that the zero-temperature entropy shift (a) is unrelated to the mass shift, (b) is $\mathcal{O}(c_i)$ in the EFT coefficients, rather than $\mathcal{O}(\sqrt{c_i})$, as is the constant-mass entropy shift, and (c) is negative in the example of the EFT arising at low energies from QED plus gravity.

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I. INTRODUCTION

Gravity is well described by general relativity at low energies, but it is expected that the full quantum theory will introduce higher-derivative corrections to the Einstein-Hilbert action. There are certain situations where the absolute sign of these corrections can have a large effect. In field theory, for instance, corrections with a certain sign can lead to superluminal propagation on some backgrounds, thereby disallowing such corrections on causality grounds [1]. In gravity, the corrections can affect the extremality bound of black holes. As first pointed out in [2], this is relevant for the weak gravity conjecture (WGC) [3], which roughly requires the existence of a superextremal state. The WGC can be satisfied by the black hole spectrum alone provided a certain combination of higher-derivative coefficients is positive. This means that the extremal mass decreases for black holes with a fixed charge.

Recently there has been a great deal of progress on understanding if the required inequality does indeed hold using diverse arguments including consistency of the S matrix [4–7], RG running [8–10], modular invariance [11], the c theorem [12], and investigating various specific examples [13–15] and consequences [16–18]. A particularly intriguing argument [19] is that the combination of EFT coefficients which appears in the WGC calculation of

[2] is (-1) times the same combination that appears in the shift to the black hole entropy at extremality. As a result, the mass shift will be negative, and the black hole WGC satisfied, in any theory where the entropy corrections at constant charge and mass are positive. This requirement was shown to hold for any UV completion which arises from integrating out states at tree level, and conjectured to hold more generally [19]. We call this result the “entropy-extremality relation,” which schematically takes the form

$$\Delta m_{\text{ext}} \sim -T \Delta S|_{m=q}. \quad (1)$$

A more general proof unrelated to black holes was given in [20], and further details and examples were discussed in [5,13,14,21].

Generically, the higher-derivative entropy gets first-order corrections from both the higher-derivative operators appearing in the Wald formula, and from the corrections to the horizon radius. From a geometric point of view, the entropy-extremality relation hinges on the fact that at $m = q$, the horizon shift corrections dominate the Wald-formula corrections. The corrections to the horizon radius also control the corrections to the mass shift, establishing the entropy-extremality relation in this limit. Later it was clarified [5] that the horizon shift at $m = q$ is of order $\sqrt{c_i}$, where c_i are the coefficients of the higher-derivative operators appear in the corrected action. The entropy-extremality relation is still valid because $\Delta m \sim c_i$, $\Delta S \sim \sqrt{c_i}$, and $T \sim \sqrt{c_i}$. These corrections to T are inevitable if we hold the mass fixed because the relationship between T and m gets shifted and $m = q$ is no longer extremal. Thus in the corrected theory, T is slightly positive at $m = q$ —specifically; it becomes order $\sqrt{c_i}$.

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A common feature of the WGC and the positive entropy shift conjecture is that both depend on the mass. Relatedly, computing the mass shift and the entropy shift at fixed mass both require understanding of the boundary structure of spacetime. This makes them well suited for studying spacetimes which are asymptotically flat or anti-de Sitter (AdS), but for asymptotically dS spaces or other more cosmological solutions, it is not clear how they might be used as Swampland criteria (though see [22] for a discussion of the WGC in dS). Given this, it may be interesting to try to understand how higher-derivative corrections affect quantities which are defined without reference to the boundary. Examples include any quantities which can be computed using only the near-horizon geometry.

The purpose of this paper is calculate such a quantity: the entropy shift of 4D Reissner-Nordström black holes at $T = 0$ (i.e., at $m = q + \Delta m$) in the corrected theory. We do so using Sen's classical entropy function formalism [23], which only makes reference to the near-horizon geometry. We then check this calculation using the explicit higher-derivative-corrected Reissner-Nordström solution. We find that the $T = 0$ entropy shift is $\mathcal{O}(c_i)$, in contrast to the $m = q$ entropy shift, which is $\mathcal{O}(\sqrt{c_i})$. We also demonstrate this numerically. As a result, we find that the entropy shift at $T = 0$ is unrelated to the extremality bound. After showing that the $T = 0$ entropy shift can be written with a field-redefinition invariant set of four-derivative operators, we check it explicitly for the EFT that arises at low energies from QED with gravity, and we find that it is negative, in contrast to the fixed- m entropy shift of [19].

II. 4D REISSNER-NORDSTRÖM BLACK HOLES

We will consider the following Lagrangian:

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{4} - \frac{1}{4} F^2 + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_4 R F^2 + c_5 R_{\mu\nu} F^{\mu\rho} F^{\nu}_{\rho} + c_6 R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} + c_7 (F^2)^2 + c_8 F_{\mu\nu} F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu} \right). \quad (2)$$

This theory admits solutions that are Reissner-Nordström (RN) plus corrections, which are order c_i . Such solutions take the form

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_{S^2}^2, \quad (3)$$

$$F_{tr} = \frac{q}{r^2},$$

where

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} + \Delta f(r),$$

$$g(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} + \Delta g(r). \quad (4)$$

In this paper, Δ will always denote a quantity which vanishes when $c_i \rightarrow 0$; such quantities are corrected with respect to the two-derivative solution with the same charge q . We can see the *unshifted* black holes ($\Delta f = \Delta g = 0$) will be extremal when $m = q$, with horizon radius $r_h = q$.

III. ENTROPY FUNCTION

The classical entropy function is a method for computing the black hole entropy of extremal black holes [23] (see also [24] for a review). The method relies on the fact that extremal black holes have near-horizon geometries which factorize into $\text{AdS}_2 \times S^2$. The most general electrically charged such solution is

$$ds^2 = v_1 \left(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2} \right) + v_2 d\Omega^2, \quad (5)$$

$$F_{rt} = e.$$

The idea, based on the logic of the attractor mechanism, is that the constants v_1 and v_2 are fixed by extremizing the entropy function, which equals the Wald entropy for the extremal RN black hole. The entropy function is constructed first by considering the function $f(v_1, v_2, e)$ defined by

$$f(v_1, v_2, e) = \int d\theta d\phi \sqrt{-g} \mathcal{L}, \quad (6)$$

where $\mathcal{L}$ is the Lagrangian density defined by (2). The entropy function is defined as the Legendre transform of this function,

$$\mathcal{E} = 2\pi(eq - f(v_1, v_2, e)), \quad (7)$$

where e can be written in terms of q using the condition

$$\frac{\partial f}{\partial e} = q. \quad (8)$$

The result for this procedure is

$$e = q \frac{v_1}{v_2} - \frac{4q}{v_2} \left(\left(2 - \frac{2v_1}{v_2} \right) c_4 + c_5 + 2c_6 + \frac{4q^2 v_1}{v_2^2} c_7 + \frac{2q^2 v_1}{v_2^2} c_8 \right). \quad (9)$$

Now it is a simple matter to extremize the entropy by demanding

$$\frac{\partial \mathcal{E}}{\partial v_1} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_2} = 0. \quad (10)$$

The corrected geometry corresponds to the values of v_1 and v_2 which extremize the entropy. We find

$$\begin{aligned} v_1 &= q^2 + 8c_4 + 4c_5 + 8c_6 + 8c_7 + 4c_8, \\ v_2 &= q^2 + 8c_4 - 8c_7 - 4c_8. \end{aligned} \quad (11)$$

The extremal near-horizon geometry is easily recovered from the entropy functional formalism. When we plug our values of v_1 and v_2 back into the entropy function we obtain

$$\boxed{\mathcal{E} = \pi q^2 - 4\pi(2c_2 + 4c_3 + c_5 + 2c_6 + 2c_7 + c_8)}. \quad (12)$$

This is the entropy of the extremal black hole—we see that it is given by $\mathcal{E} = A/4 + \Delta S$.

IV. EXTREMALITY AND ENTROPY FROM THE GEOMETRY SHIFT

To confirm and to better understand this result, let us consider the geometry shift. In [2,19], it was shown that this takes the form

$$\begin{aligned} \Delta g(r) &= -\frac{4q^2}{5r^6}((c_2 + 4c_3)(12q^2 - 30mr + 20r^2) \\ &\quad + c_4(30q^2 - 70mr + 40r^2) + c_5(11q^2 - 25mr + 15r^2) \\ &\quad + c_6(16q^2 - 35mr + 20r^2) + c_7(2q^2) + c_8(q^2)). \end{aligned} \quad (13)$$

A. Extremality shift

First we compute the shift to extremality, which is defined as the mass shift at zero temperature, again holding q fixed throughout the calculation. The extremal mass is $m_{\text{ext}} = q + \Delta m$. At this value of the mass, there is also a shift to the horizon radius, $r_h = q + \Delta r$.

If we work exactly at extremality, $m = m_{\text{ext}}$, then $g(r_h) = g'(r_h) = 0$. This gives us two conditions, which we can solve for Δm and Δr (see the Supplemental Material [25] for the expression in terms of g). The result is

$$\begin{aligned} \Delta m &= -\frac{2}{5q}(2c_2 + 8c_3 + c_5 + c_6 + 2c_7 + c_8), \\ \Delta r &= \frac{2}{q}(2c_4 - 2c_7 - c_8). \end{aligned} \quad (14)$$

The mass shift here is the result of [2]. The radius shift is known to be related to the mass shift away from extremality

and at unshifted extremality $m = q$ [5,19]. However, we see that the radius shift is not proportional to Δm at $T = 0$.

B. Entropy shift

The Wald entropy formula [26] allows us to compute the entropy of higher-derivative-corrected black holes. For spherically symmetric spacetimes, this takes the simplified form

$$S = -2\pi A \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} \Big|_{\text{hor}}. \quad (15)$$

Higher-derivative corrections contribute in two ways: first, they shift the radius, and therefore A . Second, the interactions themselves introduce Riemann tensors into the Lagrangian which must be differentiated. We will call these two contributions ΔS_h and ΔS_i respectively. How they are computed is nicely explained in [19], so we will not review the details here. The results are

$$\begin{aligned} \Delta S_h &= \frac{\Delta A}{4} = 2\pi r \Delta r = 4\pi(2c_4 - 2c_7 - c_8), \\ \Delta S_i &= -4\pi(2c_2 + 4c_3 + 2c_4 + c_5 + 2c_6). \end{aligned} \quad (16)$$

The total entropy is the sum of both of these,

$$\Delta S = -4\pi(2c_2 + 4c_3 + c_5 + 2c_6 + 2c_7 + c_8), \quad (17)$$

in agreement with the result of the entropy function (12). This expression is not proportional to the mass shift (14).

C. Numerical test

The geometry shift given above actually allows us to probe what happens for any value of the mass, not just at zero temperature. In particular, we should be able to see that the horizon shift is $\mathcal{O}(c_i)$ at all points except for at $m = q$. To do this, let us set $q = 1$, and choose $c_7 = 10^{-4}$ with all other $c_i = 0$. Then we can study the behavior of $g(r)$ numerically.

At $m = q$ we find $\Delta r = \pm 0.0125 \sim \sqrt{c_7}$ but at $m = 1 + \Delta m$, $\Delta r = -0.0004 \approx -4c_7$. Furthermore we can plot the horizon radius as a function of mass, as we have done in Fig. 1(b). This shows us that the horizon radius varies smoothly down to its extremal value, and dips slightly into the negative region very near the true value for the extremal mass.

V. POSITIVITY

To draw any physical conclusions from this calculation, we must translate into a basis which is invariant under field redefinitions. We use the basis $\{d_0, d_3, d_6, d_9\}$ of [19]. The mass and entropy shifts become

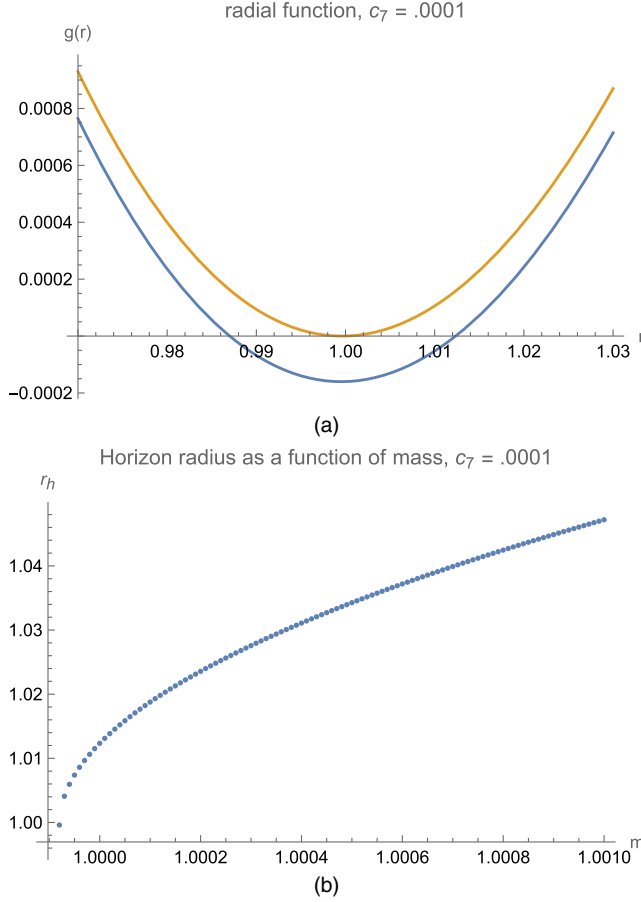


FIG. 1. (a) Radial functions with only c_7 . The blue curve is where $m = q$, the unshifted extremal mass. The orange curve is $m = q + \Delta m$, the shifted extremal mass. (b) Horizon shift with only c_7 , for different values of the mass.

$$\Delta m = -\frac{2}{5q}d_0, \quad \Delta S = -4\pi(d_0 - 2d_3 + d_6). \quad (18)$$

The horizon shift we obtained cannot be written in this form. This is a little strange but ultimately irrelevant for our purposes. It is interesting that d_0 , the combination which is related to the WGC, enters the $T = 0$ entropy shift with the opposite sign as the constant m entropy shift of [19]. Furthermore, consider that for magnetic black holes, $\Delta m \sim -(d_0 + 2d_6)$ [9] (see Supplemental Material [25] for full results on magnetic black holes). Therefore enforcing the WGC (or alternatively, the positivity of the fixed-mass entropy shift) for all charged black holes in 4D tells us

$$d_0 + d_6 > 0. \quad (19)$$

The coefficient d_3 multiplies $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ in the action. We see that if $d_3 > 0$ then the extremal entropy shift can have either sign. But if $d_3 < 0$, then the extremal entropy shift must be negative to be consistent with the black hole WGC. (It is also interesting to note that in AdS, the

positivity of the fixed-mass entropy implies that the coefficient of Riemann squared is positive [21]—the argument does not extend to asymptotically flat space because there are no neutral black holes which are thermodynamically stable.)

It is tempting to try to relate this to a statement about the Gauss-Bonnet term, which can be related to d_3 by field redefinitions. This term gives a topological contribution to the Wald entropy that appears (with either sign) to allow for violations of the second law of thermodynamics during black hole mergers or formation [27–29]. However, the present case (i.e., involving the full set of coefficients) is more complicated. Even if the Gauss-Bonnet term is zero, we can still have a d_3 contribution, which arises after breaking d_3 into a part which depends on Gauss-Bonnet and a part which does not. See the Supplemental Material [25] for details on field redefinition invariance and Gauss-Bonnet invariance.

A. QED with gravity

The purpose of this work has been to consider the entropy shift at $T = 0$, and to explore the extent to which this might be an interesting quantity for bounding higher-derivative corrections. We find that this shift can also be written in a field-redefinition-invariant basis, which is a basic requirement on physically meaningful quantities. However it is not related to the other shifts, and there does not appear to be anything inconsistent with its negativity [30]. For an explicit example, one can consider QED with gravity, and compute the EFT coefficients that arise from integrating out the electron. These corrections were computed in [31,32]:

$$\begin{aligned} c_7 &= -\frac{5\alpha^2}{180m_e^4}, & c_8 &= \frac{14\alpha^2}{180m_e^4} \\ c_4 &= \frac{\alpha}{144\pi m_e^2}, & c_5 &= -\frac{13\alpha}{360\pi m_e^2}, & c_6 &= \frac{\alpha}{360\pi m_e^2}. \end{aligned} \quad (20)$$

Our conventions imply $M_p = 1/\sqrt{2}$, so we can restore M_p in the ratios to find

$$\Delta S|_{T=0} = -4\pi \left(\frac{16\alpha^2}{180} \frac{M_p^4}{m_e^4} - \frac{22\alpha}{360\pi} \frac{M_p^2}{m_e^2} \right). \quad (21)$$

So, the extremal entropy shift must be negative because $\alpha \gg (M_p/m_e)^2$. This example is particularly interesting because it is likely very close to our universe since the electron is the lightest charged particle. However this has not been confirmed through measurement, and it is possible that other sources change the answer.

Part of the motivation of this is to provide some insight on the entropy-extremality relation, which relates the mass shift to the entropy shift at fixed mass. From a geometric

point of view, the shift to entropy at constant mass is special because it is controlled by the geometry shift $\Delta g(r)$. That means, in particular, that it cannot be affected by topological contributions to the entropy, which are possible in principle but which cannot have any local effects, such as contributions to solutions to the equations of motion such as $g(r)$. Such topological corrections can, in principle, be addressed by the $T = 0$ entropy shift.

Another motivation here is that both the WGC and the positive entropy conjecture of [19] are limited by the fact that they depend on the mass. In particular, this makes it difficult to understand what, if any, form should hold in de Sitter space [34], where there is no positive conserved energy, rendering the mass somewhat ambiguous. It would be interesting to try to extend the ideas discussed in this paper to de Sitter space. This also raises the possibility that some form of these positivity conjectures might be relevant for the cosmological horizon in addition to black hole horizons.

Another possible direction would be to compare the constant-mass and constant-temperature entropy shifts in more

concrete examples (for instance by computing them using the higher-derivative coefficients computed for UV complete holographic theories in [35]). It would also be interesting to continue to investigate what can be learned from the near-horizon limit. In this work we have only computed the classical entropy, but in principle higher-derivative corrections to the *quantum* entropy [36] could be addressed as well, perhaps along the lines of recent work such as [37–39].

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