Letter

Physical meaning of the Belinfante-Rosenfeld ambiguity

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The current literature lacks consensus on how to theoretically describe spin polarization and its transport in matter. The underlying reason is the presence of the Belinfante-Rosenfeld (BR) ambiguity in the form of the energy-momentum and spin tensors. I re-examine the BR ambiguity in Einstein-Cartan spacetimes using the biform formalism in a manner applicable to any conserved current and its corresponding gauge field. This allows us to physically interpret the BR ambiguity as the choice of splitting a conserved current into its matter and gauge field contributions. I argue this follows from Noether's second theorem and show that certain examples of BR ambiguities reduce to large gauge transformations.

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Introduction. Experiments at the Relativistic Heavy Ion Collider (RHIC) have shown that the quark gluon plasma generated in heavy-ion collisions behaves as a nearly perfect fluid with a large angular momentum [1–4]. The experimental results, however, cannot be fully described by standard hydrodynamics, which takes into account only the orbital part of the fluid angular momentum (vorticity). This discrepancy brought about a surge in papers analyzing the contribution of spin angular momentum to both the thermodynamics and hydrodynamics of matter [5–25]. Despite this, no consensus exists in the literature on what spin hydrodynamics is or whether it is even useful to talk about spin transport.

The underlying reason is an ambiguity in the local definition of the energy, momentum, and angular momentum of matter. Namely, we are free to shift the energy momentum, $T_{\mu\nu}$, and spin, $S_{\mu\nu\rho}$, tensors without affecting the conservation equations they satisfy, as[1]

$$\delta T_{\mu\nu} = \partial^\lambda \Phi_{\lambda\mu\nu}, \quad \delta S_{\mu\nu\rho} = 2\Phi_{\mu[\nu\rho]} + \partial^\lambda \phi_{\lambda\mu\nu\rho}. \tag{1}$$

Equation (1) is the Belinfante-Rosenfeld transformation (BRt), and the choice of particular superpotentials, Φ, ϕ, is the Belinfante-Rosenfeld (BR) ambiguity [26,27] (see also [28,29]).

Recent examinations of the BR ambiguity have focused on the covariance properties of thermodynamics with spin under (1) [30,31] (see also [32]) or on providing an origin for certain choices of superpotentials [33,34]. In this paper, I re-examine the BRt (1) from a cohomological point of view and show it is exact with respect to a suitably defined differential operator. This allows us to derive and extend the BRt in flat Minkowski as well as curved and torsionfull Einstein-Cartan (EC) spacetimes. I argue these extensions lead to a physical interpretation of the superpotentials and the BRt: The BRt shifts the local energy momentum and angular momentum of matter to the background gravitational field. Thus, to use an E&M analogy, different Φ, ϕ correspond to different matter "polarizations." This discussion generalizes to any conserved current and the gauge field it couples to.

Biforms. In the following I briefly explain the biform formalism [35–43] employed in the sequel.

Biforms are tensors that carry two sets of indices, each set taking its values in a different vector space—the L and R space, respectively. Biforms are completely antisymmetric in each set of indices, and a generic (p, q) biform W reads in components

$$W = \frac{1}{p!q!} W_{\mu_1\mu_2\ldots\mu_p|\nu_1\nu_2\ldots\nu_q} dx^{\{\mu_p\}} \otimes dy^{\{\nu_q\}}, \tag{2}$$

where $\mu_1, \ldots, \mu_p$ and $\nu_1, \ldots, \nu_q$ take their values in L and R, respectively, and range from 0 to $n-1$, n being the spacetime dimension. The Einstein summation convention is used for repeated upper and lower indices, $\otimes$ stands for the tensor product between vector spaces, and

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[1]We have $\Phi_{\lambda\mu\nu} = -\Phi_{\mu\lambda\nu}$, $\phi_{\lambda\mu\nu\rho} = -\phi_{\mu\lambda\nu\rho} = -\phi_{\lambda\mu\rho\nu}$, and $\Phi_{\mu[\nu\rho]} = (\Phi_{\mu\nu\rho} - \Phi_{\mu\rho\nu})/2$.

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$$dx^{\{\mu_p\}} \equiv dx^{\mu_1} \wedge dx_2^{\mu} \wedge \ldots \wedge dx^{\mu_p} \tag{3}$$

is a local basis of p forms in the L space built out of the wedge product $\wedge$ between forms. Similarly for $dy^{\{\nu_q\}}$.[2] The vertical bar separates the L and R indices.

Since biforms are forms in each space, we can employ all standard operators acting on forms. The most relevant for us are the L-exterior derivative, the L-Hodge star, and the L-interior product. The L-exterior derivative d increases the L degree of a form by 1 via[3]

$$dW = \frac{1}{p!q!}\partial^x_{[\mu} W_{\{\mu_p\}]|\{\nu_q\}} dx^{\mu\{\mu_p\}} dy^{\{\nu_q\}}, \tag{4}$$

where the brackets denote antisymmetrization of indices as in (1). Most importantly, d is nilpotent, i.e., $d^2 = 0$, because we antisymmetrize over commuting derivatives.

We further define the L-Hodge star, $\star$, as usual via

$$\star W = \frac{1}{p!q!(n-p)!}\epsilon_{\{\mu_p\}\{\mu_{n-p}\}} W^{\{\mu_p\}}{}_{|\{\nu_q\}} dx^{\{\mu_{n-p}\}} \otimes dy^{\{\nu_q\}}, \tag{5}$$

where $\epsilon_{\mu_1\mu_2\ldots\mu_n}$ is the completely antisymmetric Levi-Civita symbol, satisfying $\epsilon_{012\ldots n} = -1$. It is contracted to the L indices of W via the Minkowski metric. A useful property of $\star$ is $\star^2 W = (-1)^{p(n-p)+1} W$, i.e., it squares to the identity operator up to a sign.

Finally, the L-interior product operator ι_v is defined with respect to an L vector v and acts on a biform with a contraction

$$\iota_v W = \frac{1}{(p-1)!q!} v^{\mu} W_{\mu\{\mu^{p-1}\}|\{\nu_q\}} dx^{\{\mu_{p-1}\}} dy^{\{\nu_q\}}. \tag{6}$$

We may also define operators acting on both L and R indices. The most relevant for us is the R antisymmetrizer ε_R. It acts on a (p, q) biform W as

$$\varepsilon_R(W) = \frac{(-1)^{q+1}}{(p-1)!q!} W_{\{\mu_{p-1}\}|[\mu_p\{\nu_q\}]} dx^{\{\mu_{p-1}\}} dy^{\mu_p\{\nu_q\}}. \tag{7}$$

The R antisymmetrizer essentially turns an L index of W into an R one. The transmutation of indices is performed via the (1, 1) biform $\eta = \eta_{\mu|\nu} dx^{\mu} dy^{\nu}$ by contracting the L index of η with an L index of W, using the Minkowski metric on L. We may interpret η also as a metric when $L = R$, but in general, it is more akin to a vielbein field [44], as explained in a later section.

[2]I should emphasize that biforms are tensors at a *single* point, say r, in spacetime. However, I use L and R coordinate systems to express r's coordinates, $x^{\mu}(r)$ and $y^{\nu}(r)$.

[3]Henceforth, I suppress the wedge and tensor products.

Belinfante-Rosenfeld transform in Minkowski space. The BRt (1) is a symmetry transformation of the relativistic energy momentum and angular momentum conservation equations. In the present section, I rederive and extend the BRt through the use of biforms.

The energy momentum and angular momentum conservation equations in Minkowski spacetime are

$$\partial^{\mu} T_{\mu\nu} = 0, \qquad -\frac{1}{2}\partial^{\mu} S_{\mu\nu\rho} = T_{[\nu\rho]}. \tag{8}$$

Note that the energy-momentum tensor (EMT) has no symmetry under interchanges of its indices, while the spin tensor is antisymmetric only under the interchange of its last two indices; $S_{\mu\nu\rho} = S_{\mu[\nu\rho]}$.

To express the conservation equations (8) in terms of biforms, first define the EMT and spin biforms:

$$T = T_{\mu|\nu} dx^{\mu} dy^{\nu}, \qquad S = \frac{1}{2} S_{\mu|\nu_1\nu_2} dx^{\mu} dy^{\nu_1\nu_2}. \tag{9}$$

The definition (9) is mostly motivated by the symmetry properties of $T_{\mu\nu}$ and $S_{\mu\nu\rho}$. Further motivation stems from the first order and metric-affine gravity formalisms [44–46], where T and S naturally take their values in two vector spaces, precisely as in (9).

In terms of T, S, the conservation equations (8) become

$$\star d \star T = 0, \qquad \star d \star S = \varepsilon_R(T), \tag{10}$$

as can be verified by using standard manipulations of (4), (5) (see, e.g., [47]), and (7).

Equation (10) suggests the natural variables to work with are $\star T$ and $\star S$. So, define

$$B^1 = \star T, \qquad B^2 = \star S, \tag{11}$$

and write (10) as

$$dB^1 = 0, \qquad dB^2 = \eta B^1. \tag{12}$$

The product between η and B^1 is the wedge product between forms in both L and R spaces.

In the form (12), the nilpotency of d, $d^2 = 0$, implies two important corollaries. First, we find

$$\tau \equiv d\eta = 0, \tag{13}$$

by acting with d on (12). Thus, η must be irrotational.

Second, it is clear that

$$\delta B^1 = d\Xi^1, \qquad \delta B^2 = d\Xi^2 - \eta \Xi^1 \tag{14}$$

is a symmetry of (12). Equation (14) is a rewrite of the BRt (1) with Ξ^1 and Ξ^2 the $\star$ of Φ and ϕ, respectively.

Thus, expressing the conservation equations in terms of biforms makes manifest the BRt due to $d^2 = 0$. However, we gain even more from biforms: The constraint (13) is invariant under $\delta\eta = d\xi$, with ξ a (0, 1) biform. To interpret this invariance, note that (13) locally implies $\eta = dx_R$ with x_R the position covector in R.[4] Hence, the shift of η is induced by a translation of x_R; $\delta x_R = \xi$. Consequently, we must interpret η as a gauge field of translations,[5] which implies τ is geometrically the torsion of spacetime [45].

More importantly, R translations by ξ can be extended to a symmetry of the conservation equations (12). In particular, (12) is invariant under R translations if

$$\delta_\xi \eta = d\xi, \qquad \delta_\xi B^2 = \xi B^1, \qquad \delta_\xi B^1 = 0. \tag{15}$$

Translations by ξ mix B^1 and B^2, implying we should think of them as the doublet $B^I = (B^1, B^2)$. In terms of B^I, the conservation equations (12) become a single vectorial equation:

$$dB^I - \eta C^I_J B^J \equiv DB^I \equiv \mathcal{F}^I = 0. \tag{16}$$

In (16), I first introduced the matrix $C^I_J = \delta^I_2 \delta^1_J$, then the covariant derivative operator $D = d\delta^I_J - \eta C^I_J$ acting on B^I. Finally, I defined $\mathcal{F}^I = DB^I$ for later convenience.

Taking the square of D, we find

$$D^2 B^I = -\tau C^I_J B^J, \tag{17}$$

implying D is the exterior covariant derivative of translations and B^I transforms under the R translation representation generated by C^I_J.

The introduction of ξ then extends the BRt (14) to

$$\delta\eta = d\xi, \qquad \delta B^I = D\Xi^I + \xi C^I_J B^J. \tag{18}$$

See [48] on how to use (18) to fix the form of B^I.

Equation (18) is, of course, also a symmetry of $\mathcal{F}^I$, since

$$\delta\mathcal{F}^I = \xi C^I_J \mathcal{F}^J - \tau C^I_J \Xi^J = 0. \tag{19}$$

As a comment, note that the conservation equations are invariant under the BRt (18) only when the microscopic equations of motion hold, i.e., when the theory is *on-shell*. This implies that (18) is a symmetry of the solution space of a microscopic theory and does not necessarily follow from a symmetry of the microscopic Lagrangian. This is also suggested in [49].

The physical meaning of (18) can be gleaned from its action on the globally conserved charges. These are the total energy momentum and angular momentum

$$P = \int_{\Sigma_L} B^1, \qquad J = \int_{\Sigma_L} x_R B^1 - B^2, \tag{20}$$

where the integral is taken over a codimension one L surface. The BRt (18) acts on P and J via

$$\delta P = \int_{S_L} \Xi^1, \qquad \delta J = \int_{S_L} x_R \Xi^1 - \Xi^2, \tag{21}$$

which tells us the BRt (14) shifts the total energy momentum and angular momentum by the fluxes of the superpotentials through a codimension two surface S_L. In contrast, R translations leave the charges invariant. The latter fact clarifies the physical origin of the transformation of B^2 under R translations: B^2 must transform under R translations in order to compensate for the orbital angular momentum added to the system due to the shift in the spacetime origin.

Finally, let us explore the group structure of (18) by calculating the algebra of successive transformations. Consider the transformations with parameters (ξ_1, Ξ^I_1) and (ξ_2, Ξ^I_2), respectively. Calculating the commutator of these two transformations on B^I, we find it closes to a third BRt, i.e.,

$$[\delta_1, \delta_2] B^I = \delta_3 B^I, \tag{22}$$

with δ_3 parametrized by (see [48])

$$(\xi_3, \Xi^I_3) = (0, \xi_2 C^I_J \Xi^J_1 - \xi_1 C^I_J \Xi^J_2). \tag{23}$$

We see that separately, the ξ's and the Ξ's form Abelian subgroups, since no ξ^2 or $(\Xi^I)^2$ terms appear in δ_3. However, the R translations have a nontrivial action on the superpotentials. Therefore, the full BRt group preserving the conservation equations in Minkowski spacetime is the semidirect product

$$G_{\text{M-BR}} \equiv R^n \ltimes (R^n \times R^{n(n-1)/2}) = R^n \ltimes R^{n(n+1)/2}, \tag{24}$$

where the left factor denotes the group of R translations and the right factor the shift of the energy momentum and angular momentum of the system. The $\ltimes$ product indicates that the left factor acts nontrivially on the right one via (23).

Equation (24) constitutes an extension of the usually discussed BRt group and, along with its action (18) on η and B^I, constitutes one of the main results of this paper.

Extension to Einstein-Cartan spacetime. The $G_{\text{M-BR}}$ group in Eq. (24) showcases the importance of local R symmetries in the BRt discussion. It also suggests that extending the present analysis to more general spacetimes with nonzero torsion τ might modify our discussion. For these reasons, in

[4] In index notation, $\eta_{\mu\nu} = \partial_\mu x_\nu$.

[5] We cannot identify η as a vielbein in Minkowski spacetime because of this gauge field interpretation.

this section I derive the BR group in an EC spacetime, i.e., a spacetime with nontrivial torsion and curvature [50].

From this point forward, I interpret L as the local tangent space over a point in our spacetime and R as a local Minkowski space above the same point. This allows us to map the biform formalism to the standard formulation of gravity in an EC spacetime [45,46,50]. Namely, we can identify η with the vielbein field in said formulations. The only issue preventing this matching is η's nontrivial transformation under R translations, (18). This can be remedied by examining the conservation equations.

The conservation equations can be read off of [45,46] by restricting to the EC limit. We have[6]

$$\nabla B^1 = (\iota_\eta \tau) \cdot_R B^1 - (\iota_\eta \Omega) \cdot_R B^2, \qquad \nabla B^2 = 0. \tag{25}$$

Several new notations are introduced in (25). First, the exterior covariant derivative ∇ is defined as

$$\nabla = D + [\omega,] = \overset{\omega}{\nabla} - \eta. \tag{26}$$

ω is the spin connection with components $(\omega_\mu)^a{}_b$, and $[\omega, \cdot]$ is an L-graded commutator defined as [51]

$$[\omega, W] = \omega \cdot W + (-1)^{p+1} W \cdot \omega. \tag{27}$$

I denote by $\cdot$ the matrix product in R. For example,

$$(W \cdot \omega)^a_{bc} = W_{bd} \wedge \omega^d{}_c + W_{dc} \wedge \omega^d{}_b, \qquad \omega \cdot W = 0 \tag{28}$$

for a $(p, 2)$ biform W and analogously for general R tensors.

The (2, 1) and (2, 2) biforms τ and Ω are the torsion and curvature of the EC spacetime defined via

$$\nabla^2 B^I = -\tau C^I_J B^J + [\Omega, B^J], \tag{29}$$

$$\tau = \nabla \eta, \quad \Omega = d\omega + \frac{1}{2}[\omega, \omega]. \tag{30}$$

Further, in (25) I treat η as an L vector by raising its L index with a general L metric g. Thus, the ι_η contraction changes an L index to an R index (without antisymmetrization). For example, $(\iota_\eta \tau)_{\mu a|b} = g^{\nu\rho} \eta_{\nu|a} \tau_{\rho\mu|b}$. The inner product $\cdot_R$ contracts the R indices of two biforms using the Minkowski metric,[7] for example, $\tau \cdot_R B^1 = \tau^a \wedge B^1_a$ with L indices suppressed. Finally, recall that only B^2 is charged under translations, such that $\nabla B^1 = \overset{\omega}{\nabla} B^1$ and $\nabla B^2 = \overset{\omega}{\nabla} B^2 - \eta B^1$.

The BRt group in EC spacetimes is, thus, the group of transformations preserving (25). I suggest the infinitesimal form of the BRt is

$$\delta\eta = \eta \cdot \theta + \nabla\xi, \quad \delta\omega = \nabla\theta, \quad \delta B^2 = B^2 \cdot \theta + \nabla\Xi^2 + \xi B^1,$$
$$\delta B^1 = B^1 \cdot \theta + \nabla\Xi^1 - (\iota_\eta \tau) \cdot_R \Xi^1 + (\iota_\eta \Omega) \cdot_R \Xi^2. \tag{31}$$

The newly introduced matrix $\theta^a{}_b$ parametrizes local R rotations, and the $\cdot_R$ product follows the same rules, as explained below (27). The resulting transformation for $\mathcal{F}^I = \nabla B^I$ is also modified to

$$\delta \mathcal{F}^I = \mathcal{F}^I \cdot \theta + \xi C^I_J \mathcal{F}^J. \tag{32}$$

Importantly, notice that when $\xi \neq 0$, the angular momentum conservation equation is shifted by a term proportional to the left-hand side of the energy-momentum conservation equation. This means that upon enforcing (25), the angular momentum conservation equation is not satisfied in general. In short, we have found that R translations are no longer a symmetry of our conservation equations, and they must be frozen. This is natural from the point of view of differential geometry, since there is no canonical map that would allow us to associate two vector spaces at two different points in spacetime.

Therefore, the true BRt of (25) is

$$\delta\eta = \eta \cdot \theta, \qquad \delta\omega = \nabla\theta, \qquad \delta B^2 = B^2 \cdot \theta + \nabla\Xi^2,$$
$$\delta B^1 = B^1 \cdot \theta + \nabla\Xi^1 - (\iota_\eta \tau) \cdot_R \Xi^1 + (\iota_\eta \Omega) \cdot_R \Xi^2. \tag{33}$$

Essentially, (33) tells us that η is, in fact, the vielbein field between the local tangent space and Minkowski spacetime [45].

In index notation, the BRt (33) reads at $\theta = 0$

$$\delta T_{\mu|a} = \frac{1}{\sqrt{-g}} \overset{\omega}{\nabla}_\nu (\sqrt{-g} \Phi^\nu{}_{\mu|a}) + \tau_{\nu a}{}^{|b} \Phi^\nu{}_{\mu|b} - \Omega_{\nu a}{}^{|c_1 c_2} \phi^\nu{}_{\mu|c_1 c_2}, \tag{34}$$

$$\delta S_{\mu|ab} = \frac{1}{\sqrt{-g}} \overset{\omega}{\nabla}_\nu (\sqrt{-g} \phi^\nu{}_{\mu|ab}) + 2\Phi_{\mu|[ab]}, \tag{35}$$

with g the L-metric determinant, $\Phi = (-1)^n \star \Xi^1$, $\phi = (-1)^n \star \Xi^2$, $\tau_{\mu a|b} = (\iota_\eta \tau)_{\mu a|b}$ and similarly for $\Omega_{\mu a|bc}$.

The EC-BR group corresponding to (33) is (see [48])

$$G_{\text{EC-BR}} = SO(n-1, 1) \ltimes R^{n(n+1)/2}. \tag{36}$$

Finally, I should note that most of (33) appears already in [45]. However, the curvature term in (34) is new as far as I am aware. It is precisely this new term that allows us to give a concrete physical meaning to the BRt parameters.

The physical meaning of the superpotentials. We have now all the necessary tools to understand the physical meaning

[6] See [46] for coordinate expressions.

[7] R indices due to ι_η are not contracted.

of the superpotentials Ξ^I. To do so, consider the pure "gauge" modes of $G_{\mathrm{EC-BR}}$. These are the EMT and spin tensors BR equivalent to zero, i.e.,

$$B^1_{\mathrm{pg}} = \nabla\Xi^1 - (\iota_\eta\tau)\cdot_R \Xi^1 + (\iota_\eta\Omega)\cdot_R \Xi^2, \tag{37}$$

$$B^2_{\mathrm{pg}} = \nabla\Xi^2 = \overset{\omega}{\nabla}\Xi^2 - \eta\Xi^1. \tag{38}$$

The most important feature of Eqs. (37) and (38) is that they are formally the same as the equations satisfied by a dynamical EC theory of gravity [45,46].[8] To be precise, in EC gravity, we have

$$\nabla H^1 = B^1 + (\iota_\eta\tau)\cdot_R H^1 - (\iota_\eta\Omega)\cdot_R H^2, \tag{39}$$

$$\overset{\omega}{\nabla} H^2 = B^2 + \eta H^1, \tag{40}$$

where

$$H^1 = -\frac{\partial L_{\mathrm{EC}}}{\partial\tau}, \qquad H^2 = \frac{\partial L_{\mathrm{EC}}}{\partial\Omega}, \tag{41}$$

$$B^1 = B_{\mathrm{mat}} + \iota_\eta L_{\mathrm{EC}}, \qquad B^2 = B^2_{\mathrm{mat}}, \tag{42}$$

with L_{EC} the gravitational Lagrangian $(n,0)$ form and B_{mat} the matter EMT, and similarly for B^2_{mat}. The fields H^1 and H^2 are called the gravitational momenta of L_{EC}.

We see then that the BRt (33), when applied to Eqs. (39) and (40), shifts parts of the energy momentum and spin of matter to the gravitational momenta. This implies that by choosing specific superpotentials, we define which part of the energy momentum and angular momentum of matter we consider to be part of the gravitational field (see Fig. 1). This is the second major result of this work.

To showcase the naturality of this interpretation, I note that the formalism I employ in this paper works for any background field, not just curvature and torsion. Thus, for example, consider E&M coupled to matter. In this case, we know that the BRt amounts to shifting the polarization and magnetization tensor from the bound current and charge in matter to the E&M field [52–54].

The conservation equations for a U(1) current $(n-1,0)$ form j coupled to an electromagnetic field described by the Maxwell $(2,0)$ form F are, in biform notation,

$$dB^1 = (\iota_\eta F)j, \quad dj = 0. \tag{43}$$

It is straightforward to guess the BRt preserving (43) by noting the similarity between (43) and (25). It reads

$$\delta B^1 = d\Xi^1 - \iota_\eta F\Xi^2, \quad \delta j = d\Xi^2. \tag{44}$$

[8]Note our spin tensor is minus the spin tensor in [45,46].

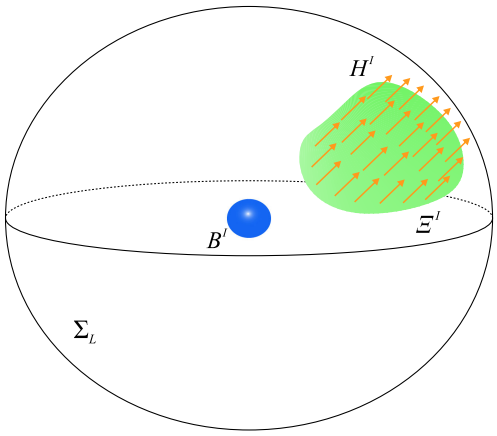


FIG. 1. A conserved charge (blue sphere) B^I in a volume Σ_L can be (partially or fully) delocalized on the boundary of Σ_L using a BRt described by Ξ^I (green region). The resulting surface charge can be interpreted as the flux (orange arrows) of the gauge momenta H^I through the boundary of Σ_L.

The pure "gauge" modes of (44) correspond to the dynamical equations

$$dH^1 = B^1 + \iota_\eta F H^2, \qquad dH^2 = j, \tag{45}$$

where

$$B^1 = B_{\mathrm{mat}} + \iota_\eta L_{\mathrm{EM}}, \qquad H^2 = -\frac{\partial L_{\mathrm{EM}}}{\partial F}, \tag{46}$$

and L_{EM} is the electromagnetic Lagrangian $(n,0)$ form.

The first equation of (45) tells us that it is the EMT of both matter and the electromagnetic field, $B^1 + \iota_\eta F H^2$, that is conserved, while the second gives the Maxwell equations in matter. Thus, upon using (44), we see Ξ^2 shifts bound currents from j to H^2, as expected. We can similarly extend our discussion to any conserved current.

I believe the universality of this conclusion stems from Noether's second theorem [55,56], which states that gauging a symmetry implies the *off-shell* conservation of its corresponding matter current. As a result, the BR ambiguity in the current must also result in an ambiguity in the gauge theory it couples to (see also [48]). The above abstract analysis is a proof of that statement for Poincaré and U(1) gauge theories.

One may then wonder whether a BRt can be realized as a gauge field transformation. This is true when

$$\delta H^I = \Xi^I \tag{47}$$

can be solved in terms of the corresponding gauge field. Whether this is possible relies on the precise form of the gauge field Lagrangian. In the simplest case of E&M in vacuum, (47) simplifies to (see [48])

$$d\delta\eta = -\star\Xi^1, \qquad d\delta A = -\star\Xi^2, \tag{48}$$

with A a U(1) gauge field. We can solve for $\delta\eta$ and δA only if $d\star\Xi^{1,2} = 0$, to find

$$\delta\eta = \lambda^1 + d\xi, \qquad \delta A = \lambda^2 + d\chi, \tag{49}$$

with $d\lambda^I = -\star\Xi^I$ and ξ, χ arbitrary. Equation (49) is an example of large R translation and U(1) gauge transformations [57], consistent with the fact that a BRt may change the total energy momentum and charge of the system. Thus, gauge transformations are a subclass of BRts, at least for the particular case of vacuum E&M. Whether this conclusion is true in general should be confirmed with an in-depth analysis of Eq. (47).

Conclusions and outlook. In this paper, I employed the biform formalism in order to understand the Belinfante-Rosenfeld transformation group of any conserved current in the presence of any background field. In particular, I applied this formalism to the case of conserved energy-momentum and angular momentum tensors in both Minkowski and Einstein-Cartan spacetimes. This led to extensions of the Belinfante-Rosenfeld transformation, (18) and (33), respectively. These extensions were crucial in deriving the true group structure of the Belinfante-Rosenfeld transformation, (24) and (36), as well as the physical meaning of the Belinfante-Rosenfeld transformation parameters. Namely, I have found that the superpotentials shift the conserved current from matter to the gauge field coupling to said current [see discussion around (39)]. That is, the Belinfante-Rosenfeld transformation parameters are analogous to the polarization and magnetization tensors in E&M.

An important outlook of this work is in spin hydrodynamics. In particular, my results imply that comparing spin-hydrodynamics analyses performed for different choices of the superpotentials amounts to comparing systems with different "free" energy momentum and angular momentum. Thus, to compare systems for different choices of superpotentials, we need to construct a Belinfante-Rosenfeld-covariant spin hydrodynamics. The authors of [58] provide one such construction by considering fluctuations of the thermal partition function around equilibrium and the Ward identities it satisfies. We may also construct a Belinfante-Rosenfeld-covariant spin hydrodynamics by following the example of [54,59,60] for polarized matter. Namely, we need to incorporate torsion and curvature into the thermodynamics of matter.[9] This is also suggested by the discussion around Eq. (47), where H^I is used to compensate a BRt. I must emphasize that this is necessary only to compare spin hydrodynamics at different superpotentials. To construct a canonical theory of spin hydrodynamics, we should simply set the superpotentials to zero. This amounts to using the canonical energy-momentum and spin tensors, as suggested in [49] more generally and explicitly carried out in [19].

I have formulated the Belinfante-Rosenfeld transformation in terms of exact forms with respect to a suitably defined operator. I have also shown it reduces to a large gauge transformation if restricted appropriately. It would then be interesting to understand if we can go further and reformulate the Belinfante-Rosenfeld symmetry as a Becchi–Rouet–Stora–Tyutin gauge symmetry [62].[10]

Finally, we have seen that the superpotentials imply the presence of ambiguities in the definition of the gravitational momenta. Therefore, it would be interesting to understand the Belinfante-Rosenfeld ambiguity from the point of view of gravity, where similar ambiguities arise in the covariant phase space approach [64–66]. Can the conventions used in this setting to fix the ambiguity in the conserved currents also be used to fix the energy-momentum and spin tensors?

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Data availability. The data that support the findings of this article are not publicly available. The data are available from the author upon reasonable request.

[9]Part of this analysis is performed in [61], where curvature effects are taken into account for spinless matter.

[10]Work in this direction appeared recently in [63], independently of this work, for the case of magnetohydrodynamics.

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