

WKB asymptotics for multipoint Virasoro conformal blocks and applications

Aleksandr Artemev^{1,2,*} and Dmitry Khromov^{1,2,3}

¹*Landau Institute for Theoretical Physics, 142432, Chernogolovka, Russia*

²*Skolkovo Institute of Science and Technology, 121205, Moscow, Russia*

³*Moscow Institute of Physics and Technology, 141700, Dolgoprudny, Russia*



(Received 2 April 2026; accepted 14 July 2026; published 4 August 2026)

We study multipoint Virasoro conformal blocks on the sphere in the comb channel. We arrive at the asymptotic expression for these blocks at large intermediate dimensions, applying the WKB method for the “classical Belavin-Polyakov-Zamolodchikov equation,” which is used to study (classical) Virasoro blocks via the monodromy method. Several applications of this asymptotic are discussed, such as the possibility to generalize Zamolodchikov’s elliptic recursion and numerical evaluation of amplitudes in minimal string theory. Our expressions pass nontrivial checks, such as agreement with known exact expressions for five-point blocks in special cases and the usual series expansion of Virasoro blocks computed using Alday-Gaiotto-Tachikawa correspondence.

DOI: [10.1103/9y4x-9b4p](https://doi.org/10.1103/9y4x-9b4p)

I. INTRODUCTION

Virasoro conformal blocks introduced in [1] are important functions in two-dimensional conformal field theory (CFT) and string theory. They are the kinematical “building blocks” for correlators and are determined completely by Virasoro symmetry. In principle, it means that one can write down an expression for any correlation function in CFT with known spectrum and structure constants. However, due to technical difficulties, apart from some special limits (such as global blocks [2,3] or blocks with perturbatively heavy operators [4]—cases relevant, e.g., for $\text{AdS}_3/\text{CFT}_2$ correspondence), questions involving Virasoro blocks on the sphere with more than four external points or higher genus blocks seem to remain relatively unexplored. The present paper intends to establish some results useful for analytic or numerical studies that use multipoint Virasoro blocks in genus zero.

The main motivation for our study concerns theories of “Liouville gravity.” This is a class of string theories defined by a world sheet CFT consisting of three uncoupled sectors: Liouville theory, Becchi-Rouet-Stora-Tyutin (BRST) ghosts, and some matter CFT, with the total central charge of the three sectors equal to zero. Several interesting examples of such theories were conjectured to have a dual

description in terms of “matrix models”/“topological recursion.” This dual approach produces relatively simple and explicit predictions for the values of perturbative string amplitudes. On the level of the world sheet, this correspondence is very nontrivial (even when analyzing simple analytic properties; see [5]) and explicit checks (numerical or analytic) require a good understanding of the structure of correlation functions in CFT (especially in Liouville theory).

At the moment, a lot of numerical studies of the amplitudes involving integration over complex one-dimensional moduli spaces (four-point tachyon amplitude on the sphere and one point on the torus) are available in the literature. They include amplitudes in minimal string (MS) theory (for A series [6,7] and, more recently, D and E [8]), Virasoro minimal string (VMS) [9], and complex Liouville string [10]. The only calculation of higher-dimensional moduli space integrals that we are aware of is the two-point torus amplitude in $c = 1$ string performed in [11].

Two main technical steps that become more difficult for higher-point amplitudes are how to compute efficiently higher-point conformal blocks and how to perform the moduli space integral itself. The most straightforward approach to general conformal blocks would be to use the formulas that follow from Alday-Gaiotto-Tachikawa (AGT) correspondence [12,13], giving an explicit combinatorial expression for their coefficients. For spherical blocks, this gives a series expansion in the cross ratios that has a limited radius of convergence. Moreover, one needs a large number of terms in the expansion to get reliable results after integration, and computing the coefficients to high orders using AGT formulas takes a lot of time. For four-point amplitudes, it is known that the problem is greatly

*Contact author: artemev.aa@phystech.edu

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](https://creativecommons.org/licenses/by/4.0/). Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI. Funded by SCOAP³.

simplified if the conformal blocks are expanded in elliptic variable q . It is related to cross ratio z by a well-known “pillow map” [14,15] $q = \exp(-\pi \frac{K(1-z)}{K(z)})$, which realizes four-punctured sphere moduli space $\mathcal{M}_{0,4}$ as a sixfold covering of one-punctured torus moduli space $\mathcal{M}_{1,1}$ (the inverse map is also known as the “modular lambda function”). Such an expansion can be efficiently computed using the celebrated Zamolodchikov recursion [14] (see also [16] for the $\mathcal{N} = 1$ supersymmetry case and [17] for the torus one-point block). This recursion is based on an asymptotic form of four-point conformal blocks when internal dimension $\Delta \equiv \frac{c-1}{24} + P^2 \rightarrow \infty$:

$$\begin{aligned} & \mathfrak{F}^{(c)}(P_1, P_2, P_3, P_4; P|z) \\ & \sim (16q)^{P^2} z^{-\frac{c-1}{24}-P_1^2-P_2^2} (1-z)^{-\frac{c-1}{24}-P_2^2-P_3^2} \theta_3(q)^{-\frac{c-1}{6}-4\sum_i P_i^2} \end{aligned} \quad (1.1)$$

and known analytic structure in Δ . It turns out that in the four-point case, only a few terms in this expansion apart from the leading asymptotic are actually needed to compute string amplitudes with reasonable precision. The main goal of the present paper is to derive expressions similar to (1.1) and recursive representation for higher-point blocks in genus zero and test them in practice. We note that our recursion representation is different from the “ h recursion” previously derived in [18], where a different limit was taken to compute the asymptotics, and the block was computed as a series in usual cross ratios instead of elliptic variables.

The structure of the paper is as follows: In Sec. II, we recall the standard method used to compute classical ($c \rightarrow \infty$) Virasoro conformal blocks for the case of a five-point block at genus zero. It involves solving an auxiliary ordinary differential equation (ODE) and finding the values of parameters for which the solutions of the ODE have specified monodromy properties. The asymptotics at large internal dimension in this approach follow from the WKB-type expansion for the solution of this equation; with some additional approximations, the WKB series can be expressed in terms of elliptic functions. We then check the obtained expressions against the available exact results and asymptotics of the AGT formula. In Sec. III, we give a short discussion on how it can be generalized for higher-point conformal blocks on the sphere. In Sec. IV, we provide a geometric interpretation of our results. Section V is devoted to applications of these asymptotics to obtain a Zamolodchikov-like elliptic recursion for multipoint blocks and checks of this recursion. Then, we demonstrate how these formulas can be applied in practice for correlators in Liouville gravity. Before the calculation, we shortly introduce the problem and the example we study—a five-point amplitude involving one “ground ring” operator. Finally, in Sec. VI we conclude and discuss directions for future work.

II. WKB COMPUTATION FOR FIVE-POINT BLOCKS

Let us first introduce our notations. We use “Liouville parametrization” of the central charge of the CFT in terms of parameter b :

$$c \equiv 1 + 6Q^2, \quad Q \equiv b + b^{-1}.$$

The semiclassical limit $c \rightarrow \infty$ then corresponds to $b \rightarrow 0$. The holomorphic conformal dimensions Δ of primary operators are parametrized with “Liouville momenta” P : operator V_P has dimension

$$\Delta(P) \equiv \frac{Q^2}{4} + P^2.$$

In this notation, degenerate fields with dimensions from the Kac table have

$$P = P_{m,n} = \frac{i(mb + nb^{-1})}{2};$$

they will also be denoted as $V_{m,n}$. External momenta for n -point conformal blocks will be denoted $P_1, \dots, P_n$ and internal $P_{i1}, \dots, P_{in-3}$. To shorten the notation, we will also sometimes denote by $\{P_k\}$ the collection of external momenta and by $\{P_{ik}\}$ the collection of internal momenta.

A. Classical conformal blocks via the monodromy method

Now we recall how classical conformal blocks can be computed. Consider a correlation function in Liouville theory with the insertion of the degenerate field $V_{2,1}$. The corresponding Verma module has a singular vector on level 2 of the form $(L_{-1}^2 + b^2 L_{-2})V_{2,1}$. Factorization over the corresponding submodule (i.e., putting this singular vector to zero) after using conformal Ward identities to express the action of Virasoro modes as differential operators yields the celebrated Belavin-Polyakov-Zamolodchikov (BPZ) equation [1]

$$\begin{aligned} & \left[\frac{\partial^2}{\partial z^2} + b^2 \sum_{k=1}^n \left(\frac{\Delta_k}{(z - z_k)^2} + \frac{\partial_k}{z - z_k} \right) \right] \\ & \times \left\langle V_{2,1}(z) \prod_{k=1}^n V_{P_k}(z_k) \right\rangle = 0. \end{aligned} \quad (2.1)$$

Being holomorphic, this equation is also valid for all conformal blocks that enter the decomposition of this correlator.

Now consider the semiclassical limit when $b \rightarrow 0$. The path integral representation for Liouville correlators suggests that in this limit the correlation functions admit asymptotic expansion in $1/c$ or b^2 after taking the logarithm: $\langle \prod V_{P_i}(z_i) \rangle \sim \exp(b^{-2} S_{\text{cl}}(bP_i, z_i) + \dots)$. The

leading term S_{cl} (“classical Liouville action”) is determined by the saddle point in the path integral. It is only affected by operator insertions if their conformal dimensions also scale in the classical limit: $\Delta = b^{-2}\delta$, $b \rightarrow 0$ with δ finite (in this limit, momenta P also tend to infinity with bP finite). The same is valid for conformal blocks (this statement was proven in [19] for four-point blocks and more recently in [20] in more generality):

$$\mathfrak{F}^{(b)}(\{P_k\}; \{P_{ik}\} | \{z_k\}) \sim \exp(b^{-2}\mathfrak{F}^{\text{cl}}(\{bP_k\}; \{bP_{ik}\} | \{z_k\}) + \dots).$$

For $V_{2,1}$, the conformal dimension stays finite in the semiclassical limit. It means that from the path integral perspective the corresponding correlation function is computed by finding the saddle configuration for other n insertions and simply evaluating $V_{2,1}(z)$ on this saddle configuration. All dependence of the correlator on z in this limit is in a preexponential factor which is of order 1 when $c \rightarrow \infty$, while the exponent still depends only on z_k . For conformal blocks, we can write a similar factorized expression

$$\mathfrak{F}^{(b)}(\{P_k\}, P_{2,1}; \{P_{ik}\} | \{z_k\}, z) \sim \Psi(z) \exp(b^{-2}\mathfrak{F}^{\text{cl}}(\{bP_k\}; \{bP_{ik}\} | \{z_k\})).$$

Substituting this into the BPZ equation, we can ignore the derivatives ∂_k acting on Ψ in comparison with their action on the exponential, which brings down a large factor b^{-2} . Thus, such derivatives can be replaced by multiplying by “accessory parameters” $c_k = \frac{\partial \mathfrak{F}^{\text{cl}}}{\partial z_k}$. The BPZ equation becomes a second-order ODE in the variable z . Only $n - 3$ unknown accessory parameters enter this equation; 3 can be expressed in terms of the other using global Ward identities.

For example, let us specialize to the five-point conformal block on the sphere with insertions $\{P_1, \dots, P_5\}$ at points $\{0, x, y, 1, \infty\}$. Then, the classical BPZ equation takes the form

$$\Psi''(z) - \left[\frac{x(1-x)c_x}{z(1-z)(x-z)} + \frac{y(1-y)c_y}{z(1-z)(y-z)} - Q(z, x, y) \right] \Psi(z) = 0, \quad (2.2)$$

where

$$Q(z, x, y) = \frac{\delta_1}{z^2} + \frac{\delta_2}{(z-x)^2} + \frac{\delta_3}{(z-y)^2} + \frac{\delta_4}{(1-z)^2} + \frac{\delta_1 + \delta_2 + \delta_3 + \delta_4 - \delta_5}{z(1-z)} \quad (2.3)$$

includes the terms dependent on external dimensions and, as stated before,

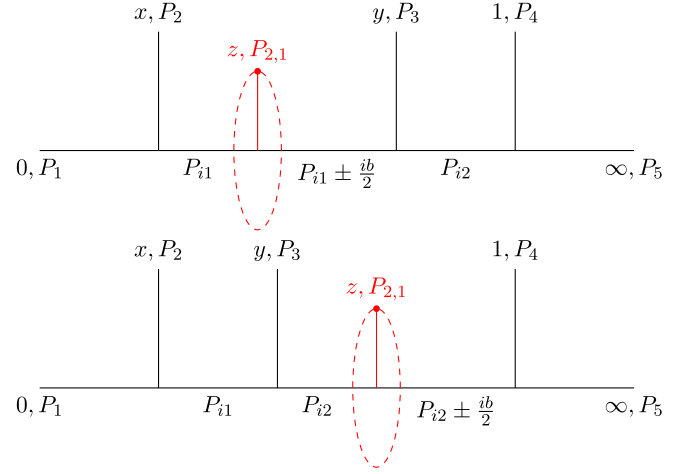


FIG. 1. Monodromy in the comb channel of the five-point block.

$$c_x(x, y) = \frac{\partial \mathfrak{F}^{\text{cl}}(x, y)}{\partial x}, \quad c_y(x, y) = \frac{\partial \mathfrak{F}^{\text{cl}}(x, y)}{\partial y}. \quad (2.4)$$

The condition on c_x and c_y such that the solutions of this equation actually correspond to some conformal blocks can be formulated as follows: Solutions $\Psi(z)e^{b^{-2}\mathfrak{F}^{\text{cl}}}$ should give six-point conformal blocks with one degenerate insertion or some linear combination of them. The “comb channel” six-point blocks with degenerate field illustrated in Fig. 1 in the classical limit tend to five-point comb channel blocks and have diagonal monodromy when z goes around $(0, x)$ or $(1, \infty)$ determined by the internal momenta. Instead of finding this diagonal basis for both cases, we can just impose two conditions on the trace of the monodromy matrix for the system of solutions, which is basis independent:

$$\text{tr}M_{(0,x)} = -2 \cosh 2\pi b P_{i1}, \quad \text{tr}M_{(1,\infty)} = -2 \cosh 2\pi b P_{i2}. \quad (2.5)$$

These are two equations that allow us to find c_x and c_y as functions of x and y .

B. WKB at large internal momenta: Leading orders

Now we proceed to actually solving these equations. Our goal is to obtain the asymptotic expression for the block when internal dimensions $b^2\Delta(P_{i1,2})$ are large compared to external dimensions, which are of order 1. As we will see later, in this limit, accessory parameters c_x, c_y are also of order $(bP_{i1,2})^2$, i.e., large (this is true in the example of the four-point conformal block as well). This means that Eq. (2.2) is amenable to the WKB method: An equation of the form

$$\psi''(z) - (\hbar^{-2}t_0(z) + t_2(z))\psi(z) = 0 \quad (2.6)$$

with small parameter $\hbar$ admits a solution

$$\begin{aligned}\psi(z) &= \exp\left(\int^z p(t, \hbar) dt\right), \\ p(t, \hbar) &= \sum_{k=-1}^{\infty} p_k(t) \hbar^k.\end{aligned}\quad (2.7)$$

This is not only a formal solution, but in good cases, $p(t, \hbar)$ is an asymptotic (Borel summable) series [21]. Coefficients p_k can be found recursively by substituting the solution into the equation and collecting terms of given order in $\hbar$; in the end, they are expressed algebraically in terms of t_0 , t_2 , and their derivatives. The first few are

$$\begin{aligned}p_{-1}^{\pm}(z) &= \pm t_0(z)^{1/2}, \quad p_0^{\pm}(z) = -\frac{1}{4}(\ln t_0(z))', \\ p_1^{\pm}(z) &= \pm \frac{-\frac{5t_0'(z)^2}{16t_0(z)^2} + \frac{t_0''(z)}{4t_0(z)} + t_2(z)}{2\sqrt{t_0(z)}} \dots\end{aligned}\quad (2.8)$$

Choosing a sign for p_{-1} gives two linearly independent solutions $\psi^{\pm}$. In our case, for Eq. (2.2) we can put $x(1-x)c_x$ to be the large parameter $\hbar^{-2}$, so

$$\begin{aligned}t_0(z) &= \frac{1}{z(1-z)(x-z)} + \frac{\frac{y(1-y)c_y}{x(1-x)c_x}}{z(1-z)(y-z)}, \\ t_2(z) &= -Q(z).\end{aligned}$$

We will then obtain WKB approximations to $\Psi_{\pm}(z)$ as a series in $1/\sqrt{c_x}$, which, as we expect, later can be transformed into an expansion in $1/P_{i1,2}$. E.g., the first two orders in $\hbar$ give (up to overall normalization that can be accounted for in the lower integration limit a —“normalization point”)

$$\begin{aligned}\Psi_{\pm}(z) &= \left(\frac{x(1-x)c_x}{z(1-z)(x-z)} + \frac{y(1-y)c_y}{z(1-z)(y-z)}\right)^{-\frac{1}{4}} \\ &\times \exp\left[\pm \int_a^z dt \sqrt{\frac{x(1-x)c_x}{t(1-t)(x-t)} + \frac{y(1-y)c_y}{z(1-z)(y-z)}}\right].\end{aligned}\quad (2.9)$$

The solutions normalized at point $a = 0$ have diagonal monodromy around $(0, x)$, and for $a = 1$ around $(1, \infty)$. The conditions (2.5) then can be written as

$$\begin{aligned}\left(\frac{1}{2} \oint_{(0,x)} = \int_0^x\right) dt \sqrt{\frac{x(1-x)c_x}{t(1-t)(x-t)} + \frac{y(1-y)c_y}{t(1-t)(y-t)}} \\ = \pi b P_{i1} \equiv \pi b(P + \delta P), \\ \int_1^{\infty} dt \sqrt{\frac{x(1-x)c_x}{t(1-t)(x-t)} + \frac{y(1-y)c_y}{t(1-t)(y-t)}} \\ = \pi b P_{i2} \equiv \pi b P.\end{aligned}\quad (2.10)$$

Note that the minus sign in (2.5) is supplied by the preexponential factor.¹

The left-hand side of these equations is a generic hyperelliptic integral associated with the curve of genus two defined by the equation

$$w^2 = z(1-z)(x-z)(y-z)(v-z), \quad (2.11)$$

where

$$v = \frac{xy((1-x)c_x + (1-y)c_y)}{x(1-x)c_x + y(1-y)c_y} \quad (2.12)$$

is the zero of the integrand. As one can see, v is determined by the ratio of internal momenta P_{i1}/P_{i2} . Obviously, this cannot be solved for c_x, c_y for generic values of large $P_{i1,2}$. However, consider the case where $\delta P/P$ is also a small parameter. Two integrals become equal to each other when $c_y \rightarrow 0$, so in the limit $\delta P \ll P$ we expect that $c_y \ll c_x$ as well. We can thus expand the lhs in c_y/c_x and then translate this expansion into the one in $\delta P/P$. Note that in the limit $c_y \rightarrow 0$ and $v \rightarrow y$, the curve degenerates into an elliptic one. All terms in the expansion of the lhs can then be evaluated in terms of elliptic integrals with parameter x . The same will be valid in subleading orders in $\hbar$.

One more technical point is worth mentioning: When computing the contributions to monodromy from $p_1^{\pm}(z)$, we will encounter powerlike divergences at the boundaries of integration, since t_2 has a double pole at $z = 0, x, 1, \infty$. They appear because WKB approximation breaks down near these double poles, and the WKB solution should be “glued” with the one accounting for singular terms exactly (for a thorough analysis of such cases in the WKB approach, see, e.g., [22]). In practice, this procedure simply means that when computing the monodromy, we shift the boundaries of integration (e.g., from 0 to x) by a small ϵ and discard the singular in ϵ terms.

Now we present the final results of the described procedure—the expansions for accessory parameters in

¹In this equation, we ignore the possible Stokes phenomena that can happen for general complex values of parameters and affect monodromy matrices. For real $x < y < 1$, P_{ik} , they do not occur; we expect that they should not affect the asymptotic in the general case as well.

$1/P$ and $\delta P/P$. They can be expressed in terms of standard elliptic integrals $K(x), E(x), \Pi(y, x)$:

$$c_x = c_{x,-1} b^2 P^2 + c_{x,1} + O(P^{-2}), \quad c_y = c_{y,-1} b^2 P^2 + c_{y,1} + O(P^{-2})$$

with

$$\begin{aligned} c_{x,-1} = & \frac{\pi^2}{4x(1-x)K(x)^2} + \frac{\pi(1-y)(xK(x) + (y-x)\Pi(y, x))}{x(1-x)K(x)^2 \sqrt{y(1-y)(y-x)}} \frac{\delta P}{P} \\ & + \frac{1}{2x(1-x)y(y-x)K(x)^2} \left[y(y-x)E(x)K(x) + (2x^2(1-y) - y^2(1-x))K(x)^2 \right. \\ & \left. + 4x(y-x)(1-y)K(x)\Pi(y, x) + 2(y-x)^2(1-y)\Pi(y, x)^2 \right] \left(\frac{\delta P}{P} \right)^2 + O\left(\frac{\delta P}{P} \right)^3, \end{aligned} \quad (2.13)$$

$$c_{y,-1} = -\frac{\pi}{\sqrt{y(1-y)(y-x)}K(x)} \frac{\delta P}{P} - \left(\frac{3x-2y-2xy+y^2}{2y(y-x)(1-y)} + \frac{2\Pi(y, x)}{yK(x)} \right) \left(\frac{\delta P}{P} \right)^2 + O\left(\frac{\delta P}{P} \right)^3, \quad (2.14)$$

and

$$\begin{aligned} c_{x,1} = & \frac{1}{4x(1-x)K(x)} \left[3E(x) + (x-2)K(x) - 4E(x)\delta_1 - 4(E(x) - xK(x))\delta_2 - 4(E(x) - K(x))\delta_4 \right. \\ & \left. - 4(E(x) - (1-x)K(x))\delta_5 + \frac{2(y(1-x)K(x) - (y-x)E(x))\delta_3}{y-x} \right] + O\left(\frac{\delta P}{P} \right), \end{aligned} \quad (2.15)$$

$$c_{y,1} = \frac{(x-2yx+y(3y-2))}{2y(1-y)(y-x)} \delta_3 + O\left(\frac{\delta P}{P} \right). \quad (2.16)$$

What is left is to integrate these formulas with respect to x or y to find the classical conformal block. The result is conveniently expressed in terms of variables

$$q = e^{i\pi\tau(x)}, \quad \tau(x) = i \frac{K(1-x)}{K(x)}, \quad u(y, x) = \frac{\pi}{4K(x)} \int_0^{\frac{y-x}{x(y-1)}} \frac{dt}{\sqrt{t(1-t)(1-xt)}}, \quad (2.17)$$

and standard Jacobi theta functions. $u(y, x)$ can be thought of as a coordinate on the torus with modular parameter τ , or, alternatively, one of the periods of the curve (2.11) in the degeneration limit when $v \rightarrow y$ (see Appendix B for details).

The terms in the expansion computed above are enough to find the leading terms for the five-point conformal block at fixed δP and external dimensions Δ_i when $P \rightarrow \infty$:

$$\mathfrak{F}^{(b)}(\{P_k\}; P + \delta P, P|x, y) = \mathfrak{f}^{(b)}(\{P_k\}; P + \delta P, P|x, y) \left(1 + O\left(\frac{1}{P} \right) \right) \quad (2.18)$$

with

$$\begin{aligned} \mathfrak{f}^{(b)}(\{P_k\}; P + \delta P, P|x, y) = & (16q)^{P^2} e^{4iP\delta P(u-i\log 2)} x^{\frac{Q^2}{4} + \delta P^2 - \Delta_1 - \Delta_2} (1-x)^{\frac{Q^2}{4} + \delta P^2 - \Delta_2 - \Delta_4} \\ & \times (y(1-y)(y-x))^{-\frac{1}{2}(\delta P^2 + \Delta_3)} \Theta_3(u|q)^{-4\delta P^2} \Theta_3(q)^{3Q^2 + 6\delta P^2 - 2(2\Delta_1 + 2\Delta_2 + 2\Delta_4 + 2\Delta_5 + \Delta_3)}. \end{aligned} \quad (2.19)$$

For future reference, we note that the term of order δP^2 can alternatively be written as follows (see useful formulas in Appendix A):

$$\left(\frac{x(1-x)\theta_3(q)^6}{\sqrt{y(1-y)(y-x)}\Theta_3(u|q)^4} \right)^{\delta P^2} = \left(\frac{i\Theta'_1(0|q)\theta_2(q)\theta_3(q)\theta_4(q)}{\Theta_1(u|q)\Theta_2(u|q)\Theta_3(u|q)\Theta_4(u|q)} \right)^{\delta P^2} = \left(\frac{2i\Theta'_1(0|q)}{\Theta_1(2u|q)} \right)^{\delta P^2}. \quad (2.20)$$

A few comments are in order:

- (i) The terms independent of x, y cannot be found from integrating the accessory parameters. They define the overall normalization of conformal blocks, which will be fixed shortly.
- (ii) In some of the terms in the exponents we replaced b^{-2} , which would follow from exponentiating the classical conformal block by Q^2 . Using our procedure, we cannot argue rigorously why the “quantum” corrections of order 1 and b^2 should take this form. However, we believe that such a replacement gives an exact asymptotic for the quantum block for a few reasons: It makes the expression symmetric under the exchange of b and b^{-1} , fixes the coordinate asymptotics at small x and y , as well as makes this formula agree with known exact answers in special cases. This will be discussed in the next subsection.
- (iii) In this approximation, the factor depending on internal dimensions (up to a constant independent of x, y) can be written as an exponent of a quadratic form in Liouville momenta

$$\mathfrak{f}^{(b)} \sim \exp \left(i\pi \begin{pmatrix} \delta P & P \\ P & \end{pmatrix} \begin{pmatrix} \frac{1}{\pi i} \log \frac{\Theta'_1(0|q)}{\Theta_1(2u|q)} & \frac{2}{\pi} u \\ \frac{2}{\pi} u & \tau \end{pmatrix} \times \begin{pmatrix} \delta P \\ P \end{pmatrix} \right). \quad (2.21)$$

This quadratic form can be compared with the period matrix of the WKB curve (2.11) in the degeneration limit, which is computed in Appendix B [Eq. (B16)]. In such a limit, one of the diagonal entries of the period matrix diverges, but the ones that are finite agree with the matrix above. Moreover, in our case, the WKB curve degenerates in such a way that one can unambiguously extract the finite coordinate-dependent part of the divergent matrix element. It is natural to take $v - y$ as the cutoff ϵ in (B16). Using (2.12), the divergent term in (B16) gives

$$\begin{aligned} \frac{1}{\pi i} \log(\epsilon u'(y, x)) &\approx \frac{1}{\pi i} \log \frac{-c_y y(1-y)(y-x)u'(y, x)}{c_x x(1-x)} \\ &\approx \frac{1}{\pi i} \log \left(\frac{\delta P}{P} \frac{u'(y, x)}{\frac{\pi}{4\sqrt{y(1-y)(y-x)K(x)}}} \right). \end{aligned} \quad (2.22)$$

Using (A2), we see that coordinate dependence cancels in the divergent term. The second term in (B16) that does depend on coordinates coincides precisely with the corresponding matrix element in (2.21).

Thus, up to an infinite constant the quadratic form in (2.21) coincides with the period matrix of the WKB curve (2.11) in the degeneration limit. We elaborate on this relation in Sec. IV.

C. Checks

1. Coordinate asymptotics

The first thing we check is the correct behavior in the limit $x \rightarrow 0, y \rightarrow 0, y > x$. Using

$$q(x) = \frac{x}{16} + O(x), \quad u(y, x) = \frac{i}{2} \log \frac{4y}{x} + O\left(x, \frac{x}{y}\right), \quad x, y \rightarrow 0, \quad y > x, \quad (2.23)$$

we see that the powers of x and y and the normalization are correct:

$$\begin{aligned} \mathfrak{f}^{(b)}(\{P_k\}; P + \delta P, P|x, y) \\ = 1 \cdot x^{\Delta(P+\delta P)-\Delta_1-\Delta_2} y^{\Delta(P)-\Delta_3-\Delta(P+\delta P)} \left(1 + O\left(y, \frac{x}{y}\right) \right). \end{aligned} \quad (2.24)$$

2. Exact formulas for conformal blocks

In [23], the authors derived integral representations for discrete families of conformal blocks with one degenerate field $V_{1,2}(y)$. Such conformal blocks satisfy the BPZ equation, which can be transformed into a generalized Lamé heat equation for the function $\Psi(u|q)$ defined by

$$\begin{aligned} \mathfrak{F}^{(b)}(P_1, P_2, P_{1,2}, P_4, P_5; P \pm i/2b, P|x, y) \\ = y^{\frac{1}{2b^2}} (1-y)^{\frac{1}{2b^2}} \frac{(y(1-y)(y-x))^{\frac{1}{4}} x^{-P_1^2} (1-x)^{-P_4^2}}{(x(1-x))^{\frac{2\Delta(P_2)}{3} - \frac{1}{3}(P_1^2+P_4^2+P_5^2) + \frac{1}{12}}} \\ \times \frac{(-i\Theta_1(u|q))^{b-2}}{\Theta'_1(0|q)^{\frac{b-2+1}{3}}} \Psi_P^\pm(u|q). \end{aligned} \quad (2.25)$$

The prefactor differs from asymptotics $\mathfrak{f}$ by the following factor:

$$\begin{aligned} y^{\frac{1}{2b^2}} (1-y)^{\frac{1}{2b^2}} \frac{(y(1-y)(y-x))^{\frac{1}{4}} x^{-P_1^2} (1-x)^{-P_4^2}}{(x(1-x))^{\frac{2\Delta(P_2)}{3} + \frac{1}{12} - \frac{1}{3}(P_1^2+P_4^2+P_5^2)}} \\ \times \frac{(-i\Theta_1(u|q))^{b-2}}{\Theta'_1(0|q)^{\frac{b-2+1}{3}}} \\ = \frac{\mathfrak{f}^{(b)}(P_1, P_2, P_{1,2}, P_4, P_5; P \pm i/2b, P)}{(16q)^{P^2} e^{\mp 2Pb^{-1}(u-i \log 2)}} \\ \times \Theta'_1(0|q)^{\frac{b^2+4}{3} + \frac{1}{3}(P_1^2+P_2^2+P_4^2+P_5^2)}. \end{aligned} \quad (2.26)$$

If the momenta P_k take the values

$$P_k = \frac{ib}{4} + \frac{im_k b}{2} + \frac{in_k}{b}, \quad m_k, n_k \in \mathbb{Z}_{\geq 0}, \quad (2.27)$$

then integral representations for $\Psi(u|q)$ have been obtained in [23]. In the simplest case $P_1 = P_2 = P_4 = P_5 = ib/4$, the solution with appropriate normalization is simply

$$\Psi_P^\pm(u|q) = (16q)^{P^2} e^{\mp 2b^{-1}P(u-i\log 2)}. \quad (2.28)$$

This, together with (2.26), means that in this case our asymptotic formula for the conformal block is exact:

$$\mathfrak{F}^{(b)}\left(\frac{ib}{4}, \frac{ib}{4}, P_{1,2}, \frac{ib}{4}, \frac{ib}{4}; P \pm \frac{i}{2b}, P \middle| x, y\right) = \mathfrak{f}^{(b)}\left(\frac{ib}{4}, \frac{ib}{4}, P_{1,2}, \frac{ib}{4}, \frac{ib}{4}; P \pm \frac{i}{2b}, P \middle| x, y\right). \quad (2.29)$$

For $P_1 = P_4 = P_5 = ib/4$, $P_2 = ib/4 + imb/2 + in/b$ the solution is (up to some normalization factor)

$$\begin{aligned} \Psi_P^\pm(u|q) &\sim \Theta'_1(0|q)^{\frac{2n}{3}(1-\frac{2}{b^2})} \int \dots \int \prod_{k=1}^m \left(\frac{\Theta_1(v_k|q)}{\Theta'_1(0|q)^{\frac{1}{3}}} \right)^{mb^2+2n} \frac{E_1(u+v_k)}{E_1(u)E_1(v_k)} \\ &\times \prod_{l=1}^n \left(\frac{\Theta_1(v'_l|q)}{\Theta'_1(0|q)^{\frac{1}{3}}} \right)^{2m+\frac{4n}{b^2}} \left(\frac{E_1(u+v'_l)}{E_1(u)E_1(v'_l)} \right)^{\frac{2}{b^2}} \cdot \prod_{i<j} \left| \frac{\Theta_1(v_i-v_j|q)}{\Theta'_1(0|q)^{\frac{1}{3}}} \right|^{-b^2} \prod_{i<j} \left| \frac{\Theta_1(v'_i-v'_j|q)}{\Theta'_1(0|q)^{\frac{1}{3}}} \right|^{-\frac{4}{b^2}} \\ &\times \prod_{i,j} \left(\frac{\Theta_1(v_i-v'_j|q)}{\Theta'_1(0|q)^{\frac{1}{3}}} \right)^{-2} q^{P^2} e^{\mp 2b^{-1}P(u+b^2(v_1+\dots+v_m)+2(v'_1+\dots+v'_n))} dv_1 \dots dv_m dv'_1 \dots dv'_n. \end{aligned} \quad (2.30)$$

Here,

$$E_1(u) = \frac{\Theta_1(u|q)}{\Theta'_1(0|q)}. \quad (2.31)$$

We estimate the behavior of the integral when $P \rightarrow \infty$ using the saddle point approximation. The saddle point equations are

$$\begin{aligned} b^2 \sum_{i \neq k} \frac{\Theta'_1(v_k - v_i|q)}{\Theta_1(v_k - v_i|q)} + 2 \sum_{j=1}^n \frac{\Theta'_1(v_k - v'_j|q)}{\Theta_1(v_k - v'_j|q)} - (mb^2 + 2n - 1) \frac{\Theta'_1(v_k|q)}{\Theta_1(v_k|q)} - \frac{\Theta'_1(u + v_k|q)}{\Theta_1(u + v_k|q)} &= \pm 2bP, \\ \frac{4}{b^2} \sum_{j \neq l} \frac{\Theta'_1(v'_l - v'_j|q)}{\Theta_1(v'_l - v'_j|q)} - 2 \sum_{i=1}^m \frac{\Theta'_1(v_i - v'_l|q)}{\Theta_1(v_i - v'_l|q)} - \left(2m + \frac{4n}{b^2} - \frac{2}{b^2} \right) \frac{\Theta'_1(v'_l|q)}{\Theta_1(v'_l|q)} - \frac{2}{b^2} \frac{\Theta'_1(u + v'_l|q)}{\Theta_1(u + v'_l|q)} &= \pm 4b^{-1}P. \end{aligned} \quad (2.32)$$

Since $P \rightarrow \infty$ on the right-hand side, the solutions are near zeroes of the denominators of the left-hand side. The saddle point that contributes the most is $v_k \approx 0$, $v'_l \approx 0$, since these are zeroes of $E_1(v_k)$ and $E_1(v'_l)$ in the denominator in the integrand. In the leading order in $1/P$, the position of the saddle point is of order $v_k, v'_l \sim 1/P$ and does not depend on u or q . Then, since we are not interested in the normalization factor, we just need to evaluate the integrand at the saddle point. We get

$$\begin{aligned} \Psi_P^\pm(u|q) &\sim q^{P^2} e^{\mp 2b^{-1}Pu} \Theta'_1(0|q)^{\frac{2n}{3}(1-\frac{2}{b^2})} \prod_{k=1}^m \Theta'_1(0|q)^{\frac{2}{3}(mb^2+2n)} \prod_{l=1}^n \Theta'_1(0|q)^{\frac{2}{3}(2m+\frac{4n}{b^2})} \\ &\times \prod_{i<j}^m \Theta'_1(0|q)^{-\frac{2}{3}b^2} \prod_{i<j}^n \Theta'_1(0|q)^{-\frac{8}{3b^2}} \prod_{i=1}^m \prod_{j=1}^n \Theta'_1(0|q)^{-\frac{4}{3}} = q^{P^2} e^{\mp 2b^{-1}Pu} \Theta'_1(0|q)^{\frac{((2m+1)b+4nb^{-1})^2}{12} - \frac{b^2}{12}}. \end{aligned} \quad (2.33)$$

Comparing this with (2.26), we see that the function $\mathfrak{f}$ correctly replicates the asymptotics of the exact conformal block.

The solutions for cases when the only momentum not equal to $ib/4$ is not P_2 are obtained from (2.30) by replacing all E_1 in the integrand in the following way:

$$P_1: E_1 \rightarrow E_2 = \frac{\Theta_2(u|q)}{\theta_2(q)}, \quad P_3: E_1 \rightarrow E_4 = \frac{\Theta_4(u|q)}{\theta_4(q)}, \quad P_4: E_1 \rightarrow E_3 = \frac{\Theta_3(u|q)}{\theta_3(q)}. \quad (2.34)$$

These replacements do not influence the leading order behavior of the integral, so we still get the same asymptotic, which agrees with (2.26).

3. AGT formula

AGT correspondence allows us to derive a formula for the five-point conformal blocks in the comb channel as a series expansion in y and x/y ; for reference, see [13]. We want to compare the formula with our five-point asymptotics $\mathfrak{f}$. To do that, we compare the logarithm of (2.19) with the logarithm of the AGT formula. The first few terms of the AGT series are

$$\begin{aligned} & \log \frac{\mathfrak{F}_{\text{AGT}}^{(b)}(\{P_k\}; P + \delta P, P|x, y)}{x^{\Delta(P+\delta P)-\Delta_1-\Delta_2} y^{\Delta(P)-\Delta_3-\Delta(P+\delta P)}} \\ &= \frac{(Q^2 + 4P^2 + 4P_4^2 - 4P_5^2)(Q^2 + 4P_3^2 - 8P\delta P - 4\delta P^2)}{8(Q^2 + 4P^2)} y + \frac{(Q^2 + 4(P + \delta P)^2 - 4P_1^2 + 4P_2^2)(Q^2 + 4P_3^2 + 8P\delta P + 4\delta P^2)}{8(Q^2 + 4(P + \delta P)^2)} \frac{x}{y} \\ &+ \frac{(Q^2 + 4P^2 + 4P_4^2 - 4P_5^2)(Q^2 + 4(P + \delta P)^2 + 4P_2^2 - 4P_1^2)(Q^2 + 4(P + \delta P)^2 + 4P^2 - 4P_3^2)}{16(Q^2 + 4P^2)(Q^2 + 4(P + \delta P)^2)} y \cdot \frac{x}{y} + O\left(y^2, \left(\frac{x}{y}\right)^2\right). \end{aligned} \quad (2.35)$$

In the first few orders, our asymptotic is

$$\begin{aligned} & \log \frac{\mathfrak{f}^{(b)}(\{P_k\}; P + \delta P, P|x, y)}{x^{\Delta(P+\delta P)-\Delta_1-\Delta_2} y^{\Delta(P)-\Delta_3-\Delta(P+\delta P)}} = \left(\frac{Q^2}{8} + \frac{1}{2}P_3^2 - P\delta P - \frac{1}{2}\delta P^2\right) y \\ &+ \left(\frac{Q^2}{8} + \frac{1}{2}P_3^2 + P\delta P + \frac{1}{2}\delta P^2\right) \frac{x}{y} + \left(\frac{Q^2}{16} + \frac{1}{4}P_3^2 - \frac{3}{8}P\delta P - \frac{1}{8}\delta P^2\right) y^2 \\ &+ \frac{1}{2} \left(\frac{Q^2}{8} + P^2 + (P + \delta P)^2 - P_1^2 + P_2^2 - \frac{1}{2}P_3^2 + P_4^2 - P_5^2\right) y \cdot \frac{x}{y} - \frac{P\delta P + \delta P^2}{8} y^2 \cdot \frac{y}{x} \\ &+ \left(\frac{Q^2}{16} + \frac{1}{4}P_3^2 + \frac{3}{8}P\delta P + \frac{1}{4}\delta P^2\right) \left(\frac{x}{y}\right)^2 + \frac{P\delta P}{8} y \\ &+ \frac{1}{64} \left(\frac{11Q^2}{4} + 13P^2 + 13P\delta P + 7\delta P^2 - 14P_1^2 + 18P_4^2 - 14P_5^2 + 18P_2^2 - 7P_3^2\right) y^2 \cdot \left(\frac{x}{y}\right)^2 \\ &+ O\left(y^3, \left(\frac{x}{y}\right)^3\right). \end{aligned} \quad (2.36)$$

While the series coefficients are different, if we expand (2.35) in $P \rightarrow \infty$ and keep the nonvanishing terms, they will coincide with (2.36). Higher order AGT series coefficients quickly become too big to write here and even to compute, but we checked that in the limit $P \rightarrow \infty$ the difference of logarithms is indeed zero up to terms including $y^3, (x/y)^3$.

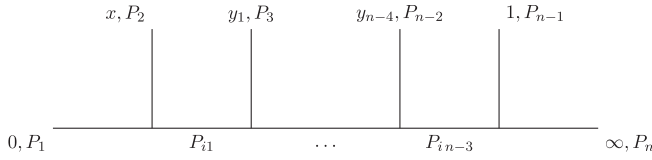
III. GENERALIZATION TO SPHERICAL n -POINT BLOCKS

Now we consider the same classical BPZ equation we had in Sec. II but for n -point conformal blocks:

$$\Psi'''(z) - \left[\frac{x(1-x)c_x}{z(1-z)(x-z)} + \sum_{k=1}^{n-4} \frac{y_k(1-y_k)c_{y_k}}{z(1-z)(y_k-z)} - Q(z, x, \{y_k\}) \right] \Psi(z) = 0, \quad (3.1)$$

where

$$Q(z, x, \{y_k\}) = \frac{\delta_1}{z^2} + \frac{\delta_2}{(z-x)^2} + \sum_{k=1}^{n-4} \frac{\delta_{k+2}}{(z-y_k)^2} + \frac{\delta_{n-1}}{(1-z)^2} + \frac{\sum_{i=1}^{n-1} \delta_i - \delta_n}{z(1-z)} \quad (3.2)$$


 FIG. 2. Comb channel for the n -point conformal block.

and

$$\begin{aligned} c_x(x, \{y_k\}) &= \frac{\partial \mathfrak{F}^{\text{cl}}(x, \{y_k\})}{\partial x}, \\ c_{y_k}(x, \{y_k\}) &= \frac{\partial \mathfrak{F}^{\text{cl}}(x, \{y_k\})}{\partial y_k}. \end{aligned} \quad (3.3)$$

We search for the solutions that satisfy the monodromy conditions

$$\begin{aligned} \text{Tr}M_{(y_k, \infty)} &= -2 \cosh 2\pi b P_{ik}, \\ \text{Tr}M_{(1, \infty)} &= -2 \cosh 2\pi b P_{in-3} \end{aligned} \quad (3.4)$$

for all $k = 1, \dots, n-4$. Such solutions correspond to conformal blocks in the comb channel (see Fig. 2).

Our goal is to obtain an asymptotic expression for the solutions in the limit where all bP_{ik} are simultaneously large compared to external dimensions δ_i . Applying the same procedure as in Sec. II, we get the following monodromy equations:

$$\begin{aligned} \frac{1}{2} \int_{\gamma_k} dt \sqrt{\frac{x(1-x)c_x}{t(1-t)(x-t)} + \sum_{k=1}^{n-4} \frac{y_k(1-y_k)c_{y_k}}{t(1-t)(y_k-t)}} \\ = \pi b(P_{ik} - P_{ik+1}) \equiv \pi b \delta P_k, \\ \int_1^\infty dt \sqrt{\frac{x(1-x)c_x}{t(1-t)(x-t)} + \sum_{k=1}^{n-4} \frac{y_k(1-y_k)c_{y_k}}{t(1-t)(y_k-t)}} \\ = \pi b P_{in-3} \equiv \pi b P. \end{aligned} \quad (3.5)$$

On the left-hand side, we are integrating a holomorphic differential on a hyperelliptic genus $g = n-3$ complex curve

$$w^2 = z(1-z)(x-z) \prod_{k=1}^{n-4} (y_k - z)(v_k - z). \quad (3.6)$$

Here, v_k are solutions for t of

$$\begin{aligned} x(1-x)c_x + \sum_{k=1}^{n-4} y_k(1-y_k)c_{y_k} \frac{x-t}{y_k-t} = 0, \\ |v_1| \leq \dots \leq |v_k|. \end{aligned} \quad (3.7)$$

Each contour γ_k encircles the interval (y_k, v_k) in the clockwise direction.

We consider the case where $\delta P_i/P \ll 1$ are small parameters of the same order $\delta P/P$. As in the five-point case, we expect $c_x \sim b^2 P^2$ and $c_{y_k}/c_x \sim \delta P_k/P$. Then, in the limit $\delta P_k \rightarrow 0$ the solution is $c_{y_k} \rightarrow 0$ and hence, $v_k \rightarrow y_k$, so the hyperelliptic curve degenerates. The first correction to v_k is

$$v_k = y_k - \frac{c_{y_k} y_k (1-y_k)}{c_x x (1-x)} (y_k - x) + O\left(\frac{\delta P}{P}\right)^2. \quad (3.8)$$

Using this, we compute the integrals over γ_k . In the leading order, only the terms with c_x and c_{y_k} contribute. At leading order, c_x is the same as before and

$$c_{y_k} = -\frac{\pi b^2 P \delta P_k}{\sqrt{y_k(1-y_k)(y_k-x)} K(x)} + O(\delta P^2). \quad (3.9)$$

Substituting this into the last equation in (3.5), we find first corrections to c_x ,

$$\begin{aligned} c_x &= \frac{\pi^2 b^2 P^2}{4x(1-x)K(x)^2} \\ &+ \sum_{k=1}^{n-4} \frac{(1-y_k)(xK(x) + (y_k-x)\Pi(y_k, x))}{x(1-x)\sqrt{y_k(1-y_k)(y_k-x)}K(x)^2} \\ &\times \pi b^2 P \delta P_k + O(\delta P^2). \end{aligned} \quad (3.10)$$

One could derive higher order corrections in $\delta P_k/P$.

We now turn to corrections from the external dimensions. Since they are of order 1, the terms of order δP^2 that we discarded should be of order less than 1. Hence, from now on we consider $\delta P_k \sim 1/P$. This corresponds to fixed differences of internal dimensions rather than momenta. The correction to c_{y_k} is again obtained from the integral over γ_k :

$$\begin{aligned} c_{y_k} &= -\frac{\pi b^2 P \delta P_k}{\sqrt{y_k(1-y_k)(y_k-x)} K(x)} \\ &+ \frac{(x-2y_k x + y_k(3y_k-2))}{2y_k(1-y_k)(y_k-x)} \delta_{k+2} + o(1). \end{aligned} \quad (3.11)$$

For c_x , we get

$$\begin{aligned}
c_x = & \frac{\pi^2 b^2 P^2}{4x(1-x)K(x)^2} + \sum_{k=1}^{n-4} \frac{(1-y_k)(xK(x) + (y_k-x)\Pi(y_k, x))}{x(1-x)\sqrt{y_k(1-y_k)(y_k-x)K(x)^2}} \pi b^2 P \delta P_k \\
& - \frac{E(x)\delta_1 + (E(x) - xK(x))\delta_2 + (E(x) - K(x))\delta_{n-1} + (E(x) - (1-x)K(x))\delta_n}{x(1-x)K(x)} \\
& + \frac{3E(x) + (x-2)K(x)}{4x(1-x)K(x)} + \sum_{k=1}^{n-4} \frac{2(y_k(1-x)K(x) - (y_k-x)E(x))}{4x(1-x)K(x)(y_k-x)} \delta_{k+2} + o(1).
\end{aligned} \tag{3.12}$$

From (3.12) and (3.11), we obtain the asymptotics of the conformal block:

$$\mathfrak{F}^{(b)}\left(\{P_k\}; P + \sum_{k=1}^{n-4} \delta P_k, \dots, P + \delta P_{n-4}, P|x, \{y_k\}\right) \sim \mathfrak{f}^{(b)}\left(\{P_k\}; P + \sum_{k=1}^{n-4} \delta P_k, \dots, P + \delta P_{n-4}, P|x, \{y_k\}\right) \tag{3.13}$$

with

$$\begin{aligned}
& \mathfrak{f}^{(b)}\left(\{P_k\}; P + \sum_{k=1}^{n-4} \delta P_k, \dots, P + \delta P_k, P|x, \{y_k\}\right) \\
& = (16q)^{P^2} e^{2i \sum_{k=1}^{n-4} P \delta P_k (u_k - i \log 2)} x^{\frac{Q^2}{4} - \Delta_1 - \Delta_2} \\
& \quad \times (1-x)^{\frac{Q^2}{4} - \Delta_2 - \Delta_{n-1}} \prod_{k=1}^{n-4} (y_k(1-y_k)(y_k-x))^{-\frac{1}{2}\Delta_{k+2}} \theta_3(q)^{3Q^2 - 2(2\Delta_1 + 2\Delta_2 + 2\Delta_{n-1} + 2\Delta_n + \sum_{k=1}^{n-4} \Delta_{k+2})}.
\end{aligned} \tag{3.14}$$

Here we denote $u_k \equiv u(y_k, x)$. As in the five-point case [see (2.21)], we see that the terms with $P \delta P_k$ and P^2 are multiplied by the corresponding entries of the period matrix of the degenerate curve [see (B14)]. We elaborate on this appearance of the period matrix in Sec. IV.

In terms of internal dimensions, this can be rewritten as

$$\mathfrak{F}^{(b)}(\{P_k\}; P_{i1}, \dots, P_{in-3}|x, \{y_k\}) \sim \mathfrak{f}_\Delta^{(b)}(\{P_k\}; P_{i1}, \dots, P_{in-3}|x, \{y_k\}) \tag{3.15}$$

for $P_{i1}, \dots, P_{in-3} \rightarrow \infty$ with $P_{ik}^2 - P_{ik+1}^2$ fixed and

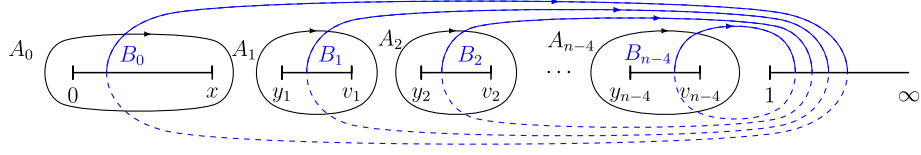
$$\begin{aligned}
& \mathfrak{f}_\Delta^{(b)}(\{P_k\}; P_{i1}, \dots, P_{in-3}|x, \{y_k\}) \\
& = (16q)^{P_{i1}^2} e^{2i \sum_{k=1}^{n-4} (P_{ik}^2 - P_{ik+1}^2)(u_k - \frac{\pi}{2}\tau + i \log 2)} x^{\frac{Q^2}{4} - \Delta_1 - \Delta_2} \\
& \quad \times (1-x)^{\frac{Q^2}{4} - \Delta_2 - \Delta_{n-1}} \prod_{k=1}^{n-4} (y_k(1-y_k)(y_k-x))^{-\frac{1}{2}\Delta_{k+2}} \theta_3(q)^{3Q^2 - 2(2\Delta_1 + 2\Delta_2 + 2\Delta_{n-1} + 2\Delta_n + \sum_{k=1}^{n-4} \Delta_{k+2})}.
\end{aligned} \tag{3.16}$$

Here we added the necessary terms of order δP_k^2 to obtain the correct limit $x, y_1, \dots, y_k \rightarrow 0$:

$$\mathfrak{f}_\Delta^{(b)}(\{P_k\}; P_{i1}, \dots, P_{in-3}|x, \{y_k\}) = x^{\Delta(P_{i1}) - \Delta_1 - \Delta_2} \prod_{k=1}^{n-4} y_k^{\Delta(P_{ik+1}) - \Delta(P_{ik}) - \Delta_{k+2}} \left(1 + O\left(y_{n-4}, \dots, \frac{x}{y_1}\right)\right). \tag{3.17}$$

IV. GEOMETRIC INTERPRETATION

In Secs. II and III, we saw that the $P \rightarrow \infty$ limit of the conformal block with $\delta P_k/P \ll 1$ fixed is related to the period matrix of the degenerate WKB curve. Here we derive the generalization of this relation for arbitrary $\delta P_k/P$, i.e., for the nondegenerate curve. The curve in question is


 FIG. 3. A and B periods on the hyperelliptic curve.

$$w^2 = z(1-z)(x-z) \prod_{k=1}^{n-4} (y_k - z)(v_k - z) \equiv R(z). \quad (4.1)$$

Here, v_k are roots of the polynomial $S(z)$:

$$\begin{aligned} S(z) &= x(1-x)c_x \prod_{k=1}^{n-4} (z - y_k) \\ &\quad + \sum_{k=1}^{n-4} y_k(1-y_k)c_{y_k} (z-x) \prod_{\substack{l=1 \\ l \neq k}}^{n-4} (z - y_l) \\ &= a^2 \prod_{k=1}^{n-4} (z - v_k). \end{aligned} \quad (4.2)$$

We assume that $|x| \leq |y_1| \leq |v_1| \leq \dots \leq 1$ and fix the $A_0, \dots, A_{n-4}$ and $B_0, \dots, B_{n-4}$ cycles as shown in Fig. 3. For convenience, we absorb the $\pi^2 b^2 P^2$ factor into c_x, c_{y_k} and temporarily redefine $\delta P_k / P \rightarrow \delta P_k$ so that the curve and c_x, c_{y_k} depend only on δP_k . Then the monodromy equations are

$$\begin{aligned} \frac{a}{2} \int_{A_0} \frac{dz}{w} \prod_{i=1}^{n-4} (z - v_i) &= 1 + \sum_{k=1}^{n-4} \delta P_k, \\ \frac{a}{2} \int_{A_k} \frac{dz}{w} \prod_{i=1}^{n-4} (z - v_i) &= \delta P_k, \end{aligned} \quad (4.3)$$

where $k = 1, \dots, n-4$. Let $\boldsymbol{\eta}$ be the $(g = n-3)$ -dimensional vector of holomorphic differentials:

$$\begin{aligned} \boldsymbol{\eta} &= \begin{pmatrix} \eta_0 \\ \dots \\ \eta_{n-4} \end{pmatrix} = \frac{\mathcal{U}(z)}{w} dz, \quad \mathcal{U}_0 = \prod_{k=1}^{n-4} (z - v_k), \\ \mathcal{U}_k &= \prod_{\substack{l=1 \\ l \neq k}}^{n-4} (z - v_l). \end{aligned} \quad (4.4)$$

Denote by $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ the A -period and B -period matrices of the $\boldsymbol{\eta}$ differentials:

$$\omega_{jk} = \frac{1}{2} \int_{A_k} \eta_j, \quad \omega'_{jk} = \frac{1}{2} \int_{B_k} \eta_j. \quad (4.5)$$

Then the period matrix of the curve is $\boldsymbol{\tau} = \boldsymbol{\omega}^{-1} \boldsymbol{\omega}'$. The key ingredient we need for the derivation is the following formula for the derivative of the period matrix [Eq. (4.17) in [24]; also see [25]]:

$$\frac{\partial \boldsymbol{\tau}}{\partial e_l} = \frac{i\pi}{R'(e_l)} \boldsymbol{\omega}^{-1} \mathcal{U}(e_l) \mathcal{U}(e_l)^T (\boldsymbol{\omega}^T)^{-1}. \quad (4.6)$$

Here, e_l are roots of the polynomial $R(z)$ that defines the curve. A crucial feature of this formula is that the right-hand side does not contain explicitly the B -period matrix $\boldsymbol{\omega}'$. We also note that (4.6) is invariant under the change of basis of the holomorphic differentials. Our goal is to prove the following:

$$\begin{aligned} \delta \mathbf{P}^T \frac{\partial \boldsymbol{\tau}}{\partial x} \delta \mathbf{P} &= -i\pi c_x, & \delta \mathbf{P}^T \frac{\partial \boldsymbol{\tau}}{\partial y_k} \delta \mathbf{P} &= -i\pi c_{y_k}, \\ \delta \mathbf{P}^T \frac{\partial \boldsymbol{\tau}}{\partial v_k} \delta \mathbf{P} &= 0, \end{aligned} \quad (4.7)$$

where $\delta \mathbf{P}^T = (1 + \sum_{k=1}^{n-4} \delta P_k \delta P_1 \dots \delta P_{n-4})$. To do this, we need to compute $\mathbf{x}^T = \delta \mathbf{P}^T \boldsymbol{\omega}^{-1}$. This vector is the solution of the system of equations $\delta \mathbf{P}^T = \mathbf{x}^T \boldsymbol{\omega}$. From (4.3), we know that the first row of the period matrix is $a^{-1} \delta \mathbf{P}^T$. Hence, the solution is just $\mathbf{x}^T = (a \ 0 \ \dots \ 0)$. Substituting this into (4.6), we get

$$\begin{aligned} \delta \mathbf{P}^T \frac{\partial \boldsymbol{\tau}}{\partial e_l} \delta \mathbf{P} &= \frac{i\pi}{R'(e_l)} \mathbf{x}^T \mathcal{U}(e_l) \mathcal{U}(e_l)^T \mathbf{x} = \frac{i\pi}{R'(e_l)} a^2 \mathcal{U}_0(e_l)^2 \\ &= \frac{i\pi}{a^2 R'(e_l)} S(e_l)^2. \end{aligned} \quad (4.8)$$

From (4.2), we see that

$$S(x) = x(1-x)c_x \prod_{l=1}^{n-4} (x - y_l),$$

$$S(y_k) = y_k(1-y_k)c_{y_k} \prod_{\substack{l=1 \\ l \neq k}}^{n-4} (y_k - y_l), \quad S(v_k) = 0. \quad (4.9)$$

Finally,

$$\begin{aligned}
a^2 R'(x) &= a^2 \underbrace{\prod_{l=1}^{n-4} (x - v_l)}_{S(x)} \times x(x-1) \underbrace{\prod_{l=1}^{n-4} (x - y_l)}_{-S(x)/c_k} = -\frac{S(x)^2}{c_x}, \\
a^2 R'(y_k) &= a^2 (y_k - v_k) \prod_{\substack{l=1 \\ l \neq k}}^{n-4} (y_k - v_l) \times y_k(y_k - 1)(y_k - x) \prod_{\substack{l=1 \\ l \neq k}}^{n-4} (y_k - y_l) = -\frac{S(y_k)^2}{c_{y_k}}.
\end{aligned} \tag{4.10}$$

This gives us (4.7). Returning to our original notation, we get

$$i\pi b^2 P^2 \delta P^T \frac{\partial \tau}{\partial x} \delta P = c_x, \quad i\pi b^2 P^2 \delta P^T \frac{\partial \tau}{\partial y_k} \delta P = c_{y_k}, \quad \delta P^T \frac{\partial \tau}{\partial v_k} \delta P = 0. \tag{4.11}$$

From these equations, we get the following result for the asymptotics of the conformal block with zero external dimensions:

$$\mathfrak{F}^{(b)}\left(\left\{\frac{iQ}{2}\right\}; P + \sum_{k=1}^{n-4} \delta P_k, \dots, P + \delta P_{n-4}, P \middle| x, \{y_k\}\right) = e^{i\pi P^2 \delta P^T \tau \delta P} \left(1 + O\left(\frac{1}{P}\right)\right). \tag{4.12}$$

This formula for $\mathfrak{F}$ is very beautiful, but not useful in practice: The period matrix is a complicated function of v_k , which in turn is determined by $\delta P_k/P$ in a complicated way. It is only analytically tractable in degeneration limits, which we consider in other parts of the work.

This result is remarkably similar in spirit to the result of Zamolodchikov on conformal blocks in the Ashkin-Teller model [26]. However, in Zamolodchikov's result the roots of the polynomial that defines the curve include only the points of operator insertions, in contrast to the WKB curve that has additional branching points v_k .

V. APPLICATIONS

A. Δ recursion for five-point blocks

1. Derivation of the recursion

Perhaps the main application of the large- P asymptotics of the four-point conformal block is the derivation of Zamolodchikov's recursion relations. In order to do the same thing for the five-point block, we should divide it by large- P asymptotics to obtain a meromorphic function without essential singularity at infinity. If we tried to use (2.19) we would run into problems: The function $\mathfrak{f}$ in (2.19) involves terms with δP^2 , which are not expressed uniquely and analytically in terms of the internal dimensions. Moreover, the conformal blocks in the residues also would not be analytic functions of $\Delta(P + \delta P)$ and $\Delta(P)$. In order to avoid these problems, we consider the conformal block with fixed difference of internal dimensions $\delta\Delta = P_{i1}^2 - P_{i2}^2$, instead of δP . This means that

$$\delta P \sim \frac{1}{P}, \tag{5.1}$$

and we should drop the unwanted terms with δP^2 . If we define

$$\mathfrak{F}^{(b)}(\{P_k\}; P_{i1}, P_{i2} | x, y) = \mathfrak{f}_{\Delta}^{(b)}(\{P_k\}; P_{i1}, P_{i2} | x, y) \mathfrak{H}^{(b)}(\{P_k\}; P_{i1}, P_{i2} | u, q), \tag{5.2}$$

where

$$\begin{aligned}
\mathfrak{f}_{\Delta}^{(b)}(\{P_k\}; P_{i1}, P_{i2} | x, y) &= (16q)^{P_{i1}^2} e^{2i(P_{i1}^2 - P_{i2}^2)(u - \frac{\pi}{2}\tau + i \log 2)} x^{\frac{Q^2}{4} - \Delta_1 - \Delta_2} (1-x)^{\frac{Q^2}{4} - \Delta_2 - \Delta_4} \\
&\times (y(1-y)(y-x))^{-\frac{1}{2}\Delta_3} \theta_3(q)^{3Q^2 - 2(2\Delta_1 + 2\Delta_2 + 2\Delta_3 + 2\Delta_4 + \Delta_3)},
\end{aligned} \tag{5.3}$$

we have

$$\begin{aligned}
&\lim_{\substack{P_{i1}, P_{i2} \rightarrow \infty \\ P_{i1}^2 - P_{i2}^2 \text{ fixed}}} \mathfrak{H}^{(b)}(\{P_k\}; P_{i1}, P_{i2} | x, y) \\
&= \lim_{\substack{P_{i1}, P_{i2} \rightarrow \infty \\ P_{i1}^2 - P_{i2}^2 \text{ fixed}}} \frac{\mathfrak{F}^{(b)}(\{P_k\}; P_{i1}, P_{i2} | x, y)}{\mathfrak{f}_{\Delta}^{(b)}(\{P_k\}; P_{i1}, P_{i2} | x, y)} = 1.
\end{aligned} \tag{5.4}$$

Now we consider the $\mathfrak{H}$ block with internal dimensions $\Delta_{i1} = \Delta + \delta\Delta$ and $\Delta_{i2} = \Delta$ as a meromorphic function of Δ . It has two series of simple poles at $\Delta = \Delta(P_{m,n})$ and $\Delta = \Delta(P_{m,n}) - \delta\Delta$ that correspond to Δ_{i1} or Δ_{i2} being degenerate.² It is well known [27] that the residues at the poles are proportional to conformal blocks with the corresponding internal dimension changed to $\Delta(P_{m,-n})$.

²We consider such $\delta\Delta$ that $\Delta(P_{m,n}) - \delta\Delta$ is not degenerate and then analytically continue the result to other $\delta\Delta$.

For example, for the pole at $\Delta = \Delta(P_{m,n})$ the second internal dimension of the residue block would be $\Delta(P_{m,-n})$, while in the first internal dimension, we just take the limit $\Delta \rightarrow \Delta(P_{m,n})$. The blocks we get here are

$$\begin{aligned}\Delta = \Delta(P_{m,n}) : \mathfrak{S}^{(b)}\left(\{P_k\}; \sqrt{P_{m,n}^2 + P_{i1}^2 - P_{i2}^2}, P_{m,-n}|u, q\right), \\ \Delta = \Delta(P_{m,n}) - \delta\Delta : \mathfrak{S}^{(b)}\left(\{P_k\}; P_{m,-n}, \sqrt{P_{m,n}^2 - P_{i1}^2 + P_{i2}^2}|u, q\right).\end{aligned}\quad (5.5)$$

The proportionality coefficient in the residue that corresponds to degenerate Δ_{ik} may be computed by considering the inverse of the corresponding Shapovalov matrix. The residue only depends on dimensions of two fields, whose OPE resulted in $V_{\Delta_{ik}}$ and two fields in the OPE, where $V_{\Delta_{ik}}$ itself is involved. So these coefficients for an n -point block are the same as for the four-point block, except for the inserted momenta. We use the results of [18,27] to get the following residues for the $\mathfrak{S}$ conformal block³:

$$\begin{aligned}\Delta = \Delta(P_{m,n}) : R_{m,n}\left(P_3, \sqrt{P_{m,n}^2 + P_{i1}^2 - P_{i2}^2}, P_4, P_5\right) \mathfrak{S}^{(b)}\left(\{P_k\}; \sqrt{P_{m,n}^2 + P_{i1}^2 - P_{i2}^2}, P_{m,-n}|x, y\right), \\ \Delta = \Delta(P_{m,n}) - \delta\Delta : R_{m,n}\left(P_1, P_2, \sqrt{P_{m,n}^2 - P_{i1}^2 + P_{i2}^2}, P_3\right) \mathfrak{S}^{(b)}\left(\{P_k\}; P_{m,-n}, \sqrt{P_{m,n}^2 - P_{i1}^2 + P_{i2}^2}|x, y\right).\end{aligned}\quad (5.6)$$

Here,

$$R_{m,n}(P_1, P_2, P_3, P_4) = -\frac{1}{2} \prod_{p,q} (P_1 + P_2 - P_{p,q})(P_1 - P_2 - P_{p,q})(P_3 + P_4 - P_{p,q})(P_3 - P_4 - P_{p,q}) \prod_{k,l} (2P_{k,l})^{-1} \quad (5.7)$$

and

$$\begin{aligned}p \in \{-m+1, -m+3, \dots, m-3, m-1\}, \quad q \in \{-n+1, -n+3, \dots, n-3, n-1\}, \\ k \in \{-m+1, -m+2, \dots, m-1, m\}, \quad l \in \{-n+1, -n+2, \dots, n-1, n\}, \quad (k, l) \notin \{(0,0), (m,n)\}.\end{aligned}\quad (5.8)$$

Note that despite the presence of square roots, $R_{m,n}$ depend analytically on P_{i1} and P_{i2} . For example,

$$\begin{aligned}R_{m,n}\left(P_3, \sqrt{P_{m,n}^2 + P_{i1}^2 - P_{i2}^2}, P_4, P_5\right) \\ = -\frac{1}{2} \prod_{p,q} \left((P_3 - P_{p,q})^2 - (P_{m,n}^2 + P_{i1}^2 - P_{i2}^2)\right) (P_4 + P_5 - P_{p,q})(P_4 - P_5 - P_{p,q}) \prod_{k,l} (2P_{k,l})^{-1}.\end{aligned}\quad (5.9)$$

Finally, for the $\mathfrak{S}$ block we get an additional coordinate prefactor that corresponds to the ratio of the asymptotics $\mathfrak{f}_\Delta$,

$$\begin{aligned}\Delta = \Delta(P_{m,n}) : \frac{\mathfrak{f}_\Delta^{(b)}\left(\{P_k\}; \sqrt{P_{m,n}^2 + P_{i1}^2 - P_{i2}^2}, P_{m,-n}|x, y\right)}{\mathfrak{f}_\Delta^{(b)}\left(\{P_k\}; \sqrt{P_{m,n}^2 + P_{i1}^2 - P_{i2}^2}, P_{m,n}|x, y\right)} = e^{2imn(\frac{\xi}{2}\tau - u - i \log 2)}, \\ \Delta = \Delta(P_{m,n}) - \delta\Delta : \frac{\mathfrak{f}_\Delta^{(b)}\left(\{P_k\}; P_{m,-n}, \sqrt{P_{m,n}^2 - P_{i1}^2 + P_{i2}^2}|x, y\right)}{\mathfrak{f}_\Delta^{(b)}\left(\{P_k\}; P_{m,n}, \sqrt{P_{m,n}^2 - P_{i1}^2 + P_{i2}^2}|x, y\right)} = e^{2imn(u - i \log 2)}.\end{aligned}\quad (5.10)$$

³We note that there is an alternative way to derive the residues. One could consider the analytic structure of the OPE and demand that certain unwanted poles cancel out. Then the residue of the conformal block is expressed through the ratio of structure constants in Liouville theory and the constant that comes from higher equations of motion in Liouville CFT. For details, see [5].

Combining all of the above and applying the Liouville theorem, we get

$$\begin{aligned} & \mathfrak{H}^{(b)}(\{P_k\}; P_{i1}, P_{i2}|u, q) \\ &= 1 + \sum_{m,n=1}^{\infty} (4qe^{-2iu})^{mn} \frac{R_{m,n}\left(P_3, \sqrt{P_{m,n}^2 + P_{i1}^2 - P_{i2}^2}, P_4, P_5\right)}{P_{i2}^2 - P_{m,n}^2} \mathfrak{H}^{(b)}\left(\{P_k\}; \sqrt{P_{m,n}^2 + P_{i1}^2 - P_{i2}^2}, P_{m,-n}|u, q\right) \\ &+ \sum_{m,n=1}^{\infty} (4e^{2iu})^{mn} \frac{R_{m,n}\left(P_1, P_2, \sqrt{P_{m,n}^2 - P_{i1}^2 + P_{i2}^2}, P_3\right)}{P_{i1}^2 - P_{m,n}^2} \mathfrak{H}^{(b)}\left(\{P_k\}; P_{m,-n}, \sqrt{P_{m,n}^2 - P_{i1}^2 + P_{i2}^2}|u, q\right). \end{aligned} \quad (5.11)$$

2. Analytic checks

The recursion we derived is valid for arbitrary conformal blocks. In particular, it is valid in the exactly solvable case $P_1 = P_2 = P_4 = P_5 = ib/4$, $P_3 = P_{1,2}$, where we should get

$$\mathfrak{H}^{(b)}\left(\frac{ib}{4}, \frac{ib}{4}, P_{1,2}, \frac{ib}{4}, \frac{ib}{4}; P \pm \frac{i}{2b}, P|u, q\right) = \left(\frac{e^{-2iu} 2i\Theta_1'(0|q)}{4 \Theta_1(2u|q)}\right)^{-\frac{1}{4b^2}}. \quad (5.12)$$

Note that from (A3) it follows that if we expand the right-hand side in q at each term q^M , the powers of e^{iu} exponents would be at least $-2M$, which agrees with the form of the series we get from the recursion. However, from the recursion it is not obvious even that this function does not depend on P . Since $\delta\Delta = 2P\delta P + \delta P^2$ depends on P , the $R_{m,n}$ and conformal blocks in the residues also depend on P . Moreover, P_{i1}^2 and P_{i2}^2 also depend on P , so there appear to be poles in P .

One can see that in the first step of the recursion the poles are canceled by $R_{m,n}$. For terms with P_{i2}^2 in the denominator, the $R_{m,n}$ contains the following factors:

$$\begin{aligned} & +: P_{m,n}^2 + \left(P + \frac{i}{2b}\right)^2 - P^2 - (P_{1,2} + P_{m-1,n-1})^2, \\ & -: P_{m,n}^2 + \left(P - \frac{i}{2b}\right)^2 - P^2 - (P_{1,2} + P_{m-1,n-3})^2, \end{aligned}$$

which cancel the pole at $P = P_{m,n}$ for the two choices of signs. Note that because $P_1 = P_2 = P_4 = P_5 = ib/4$, only $R_{m,n}$ with even n are nonzero; hence, there is always a factor with $q = n - 3$. The same factors also cancel the poles for terms with P_{i1}^2 in the denominators.

In the second step of the recursion, this simple logic fails. Consider, for example, the conformal block $\mathfrak{H}^{(b)}(\{P_k\}; \sqrt{P_{m,n}^2 + (P + i/2b)^2 - P^2}, P_{m,-n}|u, q)$ that appears in the residue in (5.11). After applying the recursion formula (5.11) for it, again consider the term with

$$P_{m,n}^2 + \left(P + \frac{i}{2b}\right)^2 - P^2 - P_{r,s}^2$$

in the denominator. It is multiplied by $R_{r,s}$, which is a product of

$$P_{r,s}^2 - \left(P_{m,n}^2 + \left(P + \frac{i}{2b}\right)^2 - P^2\right) + P_{m,-n}^2 - (P_{1,2} + P_{p,q})^2,$$

so for them to cancel the pole, we need

$$P_{1,2} + P_{p,q} = \pm P_{m,-n}.$$

We already mentioned that n is even. Since s is also even, q is odd. Thus, such p and q do not exist, and such a simple cancellation of the pole does not occur. Nevertheless, we checked up to terms $q^M e^{2i(N-M)u}$ with $N, M \leq 10$ that the recursion formula reproduces (5.12).

We also checked up to terms including y^3 , $(x/y)^3$ that the recursion reproduces the AGT series.

3. Numerical checks of the elliptic recursion

Random block. We compare elliptic recursion and AGT series expansion for a conformal block with “random” parameters:

$$b = \frac{1}{e}, \quad \{P_1, P_2, P_3, P_4, P_5\} = \left\{\frac{1}{3}, \frac{i}{4}, \frac{2}{7}, i\sqrt{\frac{3}{2}}, \frac{e}{\pi}\right\},$$

$$y = 0.6.$$

In the AGT method, we expand up to tenth order in y and x/y . In Figs. 4 and 5, we compare the recursion truncated at different order in elliptic variables with AGT for different x on the real line and some directions in the complex plane, fixing intermediate momenta $P_{i1,2}$, as well as for fixed

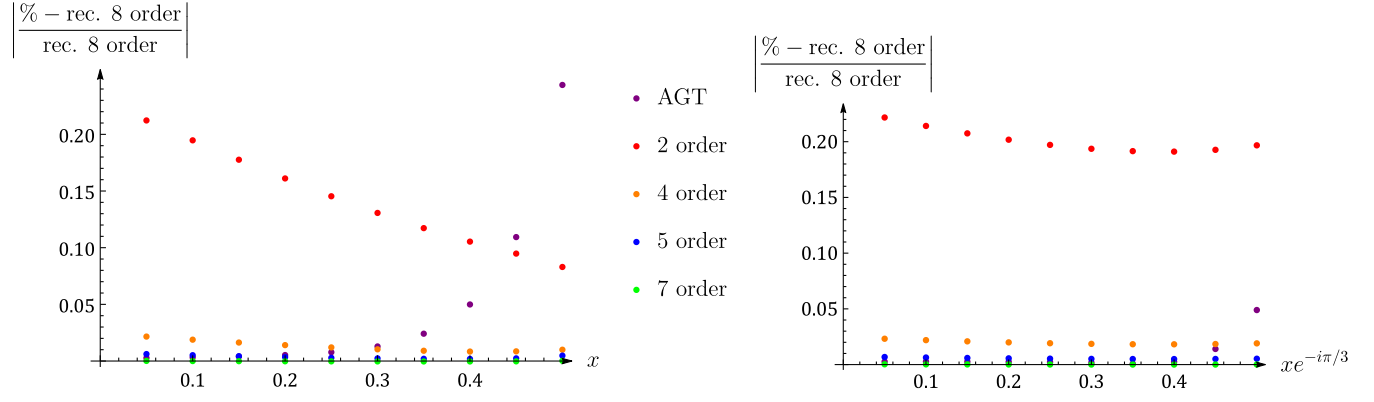


FIG. 4. Relative difference between elliptic recursion at eighth order and other methods with fixed intermediate momenta $P_{i1} = 1, P_{i2} = 1.5$, for real x (left) and $\text{Arg } x = \pi/3$ (right).

P_{i2}, x and different P_{i1} . For the plots in Fig. 4, the difference between AGT results and elliptic recursion quickly grows when we approach the boundary of the supposed region of convergence $|x| = |y|$, while the recursion behaves in a more stable way: The relative difference between seventh and eighth order is already $\sim 10^{-4}$. In any case, the recursion, even up to eighth order, is much faster than computing the AGT series (at least in our implementation).

Crossing symmetry for the Liouville correlator. As a complementary check of elliptic recursion [VA](#), we use it to evaluate the probe five-point Liouville correlator and check its behavior under some projective transformations. For transformations $z \rightarrow 1 - z$ and $z \rightarrow \frac{z-x}{1-x}$, the correlator should behave as⁴

$$\begin{aligned}
 & \overbrace{\langle V_{P_1}(0) V_{P_2}(x) V_{P_3}(y) V_{P_4}(1) V_{P_5}(\infty) \rangle}^{(1)} \\
 &= \overbrace{\langle V_{P_4}(0) V_{P_2}(1-y) V_{P_3}(1-x) V_{P_1}(1) V_{P_5}(\infty) \rangle}^{(2)} \\
 &= |1-x|^{-2(\Delta_1+\Delta_2+\Delta_3+\Delta_4-\Delta_5)} \underbrace{\left\langle V_{P_2}(0) V_{P_1}\left(\frac{-x}{1-x}\right) V_{P_3}\left(\frac{y-x}{1-x}\right) V_{P_4}(1) V_{P_5}(\infty) \right\rangle}_{(3)}. \tag{5.13}
 \end{aligned}$$

As an example, consider $\{P_1, \dots, P_5\} = \{0.3, 0.4, 0.6, 0.5, 0.25\}$ and the central charge $b = \sqrt{\frac{2}{11}}$. We use the

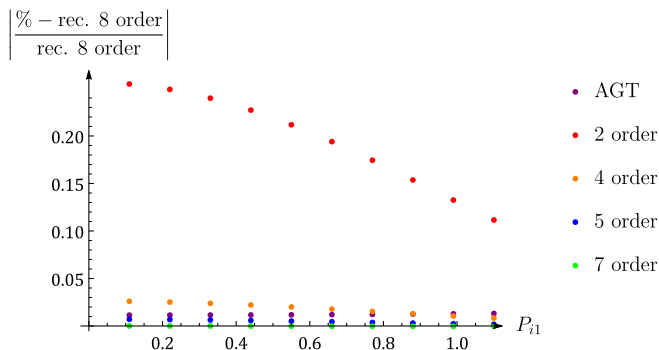


FIG. 5. Relative difference between elliptic recursion for eighth order and other methods for fixed $x = 0.3$ and $P_{i2} = 1.5$ as a function of P_{i1} .

normalization conventions and the formula for structure constants of [\[9\]](#). For such a central charge, the Liouville structure constants can be expressed in terms of the Barnes G-function (again, see [\[9\]](#)), which has a standard implementation in Wolfram *Mathematica* and can be computed efficiently. The integral over intermediate momenta is calculated by approximating it with the Riemann sum with step 0.05 for each momenta. In [Table I](#), the numerical results for correlators (1), (2), (3) for different values of (x, y) are presented. While accuracy somewhat decreases

⁴Note that the second transformation $\frac{z-x}{1-x}$ is, in fact, an exact symmetry of our elliptic block, as can be seen from the recursion [\(5.11\)](#). Under this transformation, $\tau(x) \rightarrow \tau(x) + 1$, $u(y, x) \rightarrow u(y, x) - \frac{\pi}{2}$, so $q e^{-2iu}$ is invariant and e^{2iu} changes sign. The extra sign is compensated by permuting P_1, P_2 in $R_{m,n}(P_1, P_2, \dots)$. Thus, in the cases where $|\frac{x}{1-x}| < |\frac{y-x}{1-x}|$, the symmetry under this transformation is trivial. A similar statement would be valid for the transformation $z \rightarrow \frac{z}{z-1}$.

TABLE I. Checks of crossing symmetry. (1), (2), and (3) denote the values of correlators in (5.13).

(x, y)	Rec. order	(1)	(2)	(3)	$ 1-x ^{-2}(\sum \Delta_i - 2\Delta_5)$
(0.3, 0.7)	5	3.494×10^{-8}	3.495×10^{-8}	3.246×10^{-10}	107.64
(0.35, 0.85)	6	1.282×10^{-7}	1.291×10^{-7}	4.507×10^{-10}	284.55
(0.3, 0.4)	6	6.565×10^{-7}	6.549×10^{-7}	6.102×10^{-9}	107.64
$(0.2(1+i), 0.7)$	5	1.644×10^{-8}	1.652×10^{-8}	1.31×10^{-9}	12.55

when $x \rightarrow y$ or $y \rightarrow 1$, we see that crossing symmetry can be established with good precision, using reasonable recursion depth.

B. Δ recursion for N -point blocks

Similar to what we did for the five-point case, we now use the asymptotics of the N -point block to write a recursion relation. To this end, we define the elliptic block:

$$\begin{aligned} & \mathfrak{F}^{(b)}(\{P_k\}; \{P_{ik}\}|x, \{y_k\}) \\ &= \mathfrak{f}_\Delta^{(b)}(\{P_k\}; \{P_{ik}\}|x, \{y_k\}) \mathfrak{H}_\Delta^{(b)}(\{P_k\}; \{P_{ik}\}|\{u_k\}, q). \end{aligned} \quad (5.14)$$

Here we again denote $u_k \equiv u(y_k, x)$. We then have

$$\lim_{\substack{P_{i1}, \dots, P_{iN-3} \rightarrow \infty \\ P_{ik}^2 - P_{ik+1}^2 \text{ fixed}}} \mathfrak{H}_\Delta^{(b)}(\{P_k\}; \{P_{ik}\}|\{u_k\}, q) = 1. \quad (5.15)$$

The $\mathfrak{H}$ block with internal dimensions $\Delta_{iN-3} = \Delta$, $\Delta_{iN-4} = \Delta + \delta\Delta_{N-4}$, ..., $\Delta_{i1} = \Delta + \sum_{l=1}^{N-4} \delta\Delta_l$ has a regular point at infinity as a function of Δ (with $\delta\Delta_l = P_{il}^2 - P_{il+1}^2$ fixed). The poles of $\mathfrak{H}$ form $N-3$ series, where

each series corresponds to some internal dimension being degenerate. We denote the conformal block that appears in the residue when $\Delta = \Delta(P_{m,n}) - \sum_{l=k}^{N-4} \delta\Delta_l$, i.e., when Δ_{ik} is degenerate, as $\mathfrak{H}_k^{m,n}(\{P_l\}, \{P_{il}\}|\{u_l\}, q)$. It has the following internal momenta:

$$\begin{aligned} 1 \leq l < k: P_{il} &= \sqrt{P_{m,n}^2 + \sum_{j=l}^{k-1} \delta\Delta_j}, \\ l = k: P_{il} &= P_{m,-n}, \\ N-3 \geq l > k: P_{il} &= \sqrt{P_{m,n}^2 - \sum_{j=k}^{l-1} \delta\Delta_j}. \end{aligned} \quad (5.16)$$

The corresponding ratio of the asymptotics is

$$\begin{aligned} k = 1: & e^{2imn(u_1 - i \log 2)}, \\ k = 2, \dots, N-4: & e^{2imn(u_k - u_{k+1})}, \\ k = N-3: & e^{-2imn(u_{N-4} - \frac{\pi}{2}\tau + i \log 2)}. \end{aligned} \quad (5.17)$$

Then, applying the Liouville theorem, we get

$$\begin{aligned} & \mathfrak{H}_\Delta^{(b)}(\{P_k\}; \{P_{ik}\}|\{u_k\}, q) \\ &= 1 + \sum_{m,n=1}^{\infty} e^{2imn(\frac{\pi}{2}\tau - u_{N-4} - i \log 2)} \frac{R_{m,n} \left(P_{N-2}, \sqrt{P_{m,n}^2 + \delta\Delta_{N-4}}, P_{N-1}, P_N \right)}{P_{iN-3}^2 - P_{m,n}^2} \mathfrak{H}_{N-3}^{m,n}(\{P_l\}, \{P_{il}\}|\{u_l\}, q) \\ &+ \sum_{k=2}^{N-4} \sum_{m,n=1}^{\infty} e^{2imn(u_k - u_{k+1})} \frac{R_{m,n} \left(\sqrt{P_{m,n}^2 + \delta\Delta_{k-1}}, P_{k+1}, \sqrt{P_{m,n}^2 - \delta\Delta_k}, P_{k+2} \right)}{P_{ik}^2 - P_{m,n}^2} \mathfrak{H}_k^{m,n}(\{P_l\}, \{P_{il}\}|\{u_l\}, q) \\ &+ \sum_{m,n=1}^{\infty} e^{2imn(u_1 - i \log 2)} \frac{R_{m,n} \left(P_1, P_2, \sqrt{P_{m,n}^2 - \delta\Delta_1}, P_3 \right)}{P_{i1}^2 - P_{m,n}^2} \mathfrak{H}_1^{m,n}(\{P_l\}, \{P_{il}\}|\{u_l\}, q). \end{aligned} \quad (5.18)$$

Here, as in the five-point case, we used the results of [18] to write the coefficients $R_{m,n}$ in the residues.

C. Amplitudes in Liouville gravity

1. Setting up the problem

As mentioned in the Introduction, we consider string theories with the world sheet CFT consisting of three sectors: BC system of BRST ghosts, Liouville CFT, and some other CFT as “matter.” We will concentrate on the Virasoro minimal model as matter (“minimal string theory”). We will denote kinematical quantities (like the central charge, dimensions, etc.) related to the matter sector by a hat, e.g., $\hat{c}$, $\hat{b}$, $\hat{\Delta}(\hat{P})$. A solution for the central charge constraint ($\hat{c} + c = 26$) is $\hat{b} = ib$. Also, the primary fields in the matter sector will be denoted as $\Phi_{\hat{P}}$ to distinguish them from Liouville primaries V_P .

We will only recap general information relevant to define what we are computing. For a more thorough introduction, see other works, e.g., [28].

The world sheet theory has a BRST charge, the cohomology of which defines physical operators. Their spectrum depends on the matter theory and in general is nonempty for all ghost numbers. We will consider two types of operators in this work:

- (1) The “tachyon” operators of ghost number 2 built by combining primary fields from the Liouville and matter sector such that their total conformal dimension is $(1, 1)$ and dressing additionally with C ghosts

$$\mathcal{T}_P = C\bar{C}V_P\Phi_{iP}.$$

We have solved the dimension constraint explicitly: $\Delta(P) + \hat{\Delta}(iP) = 1$. Such operators exist for all studied choices of matter theory and have been thoroughly examined the most.

- (2) The “ground ring” operators of ghost number zero exist if the matter theory has degenerate primary fields $\Phi_{m,n}$. Ground ring operators have the general form

$$O_{m,n} = H_{m,n}\bar{H}_{m,n}V_{m,n}\Phi_{m,n},$$

where $H_{m,n} = L_{-1}^{mn} + \dots$ is a polynomial of degree $mn - 1$ built from Virasoro modes L_{-k}, M_{-k} (in the Liouville and matter sector, respectively) and ghost

modes B_k, C_k acting on $V\Phi$. No general formula is known, but in specific examples, the requirement of BRST closedness allows us to find them explicitly [29]. The main example is

$$H_{2,1} = L_{-1} - M_{-1} + b^2 B_{-2} C_1.$$

Interpreted as a field, this ground ring operator

$$O_{2,1}(z) = (\partial_L - \partial_M + b^2 :BC:(z))(\overline{\dots})V_{2,1}\Phi_{2,1}(z), \quad (5.19)$$

where $\partial_{L/M}$ are derivatives that act only on the Liouville/matter field in the product.

Correlators of n physical operators on some surface of genus g give rise to closed differential forms on the moduli space of complex curves $\mathcal{M}_{g,n}$, which can then be integrated to obtain string amplitudes. The general prescription to construct the integrand is given, e.g., in [30,31]. The degree of the form depends on the total ghost number: If we take n tachyon operators, the result of this construction is a top degree $(6g - 6 + 2n)$ -form, so it gives a number after integrating over all moduli space. For example, in the case of genus zero, the amplitude can be computed as the integral over $(n - 3)$ cross ratios that parametrize $\mathcal{M}_{0,n}$,

$$\int \prod_{i=1}^{n-3} d^2 z_i \left\langle \mathcal{T}_{P_{n-2}}(0) \mathcal{T}_{P_{n-1}}(1) \mathcal{T}_{P_n}(\infty) \prod_{i=1}^{n-3} V_{P_i} \Phi_{iP_i}(z_i) \right\rangle. \quad (5.20)$$

However, if we try to incorporate k ground ring operators in the construction, the degree of the form drops by $2k$. On the sphere, the correlator of $n - 3$ ground ring operators and three tachyons is well defined, being a closed 0-form on $\mathcal{M}_{0,n}$, i.e., a number. But, for example, the correlator of one ground ring operator $O_{2,1}$ and four tachyons on $\mathcal{M}_{0,5}$ gives a “ground ring 2-form”:

$$\begin{aligned} & \langle \mathcal{T}_{P_1}(0) O_{2,1}(y) (V_{P_2} \Phi_{iP_2})(x) \mathcal{T}_{P_4}(1) \mathcal{T}_{P_5}(\infty) \rangle dx \wedge d\bar{x} \\ & - b^2 \langle \mathcal{T}_{P_1}(0) B\bar{H}_{2,1} V_{2,1} \Phi_{2,1}(y) (C V_{P_2} \Phi_{iP_2})(x) \mathcal{T}_{P_4}(1) \mathcal{T}_{P_5}(\infty) \rangle dy \wedge d\bar{x} \\ & - b^2 \langle \mathcal{T}_{P_1}(0) \bar{B}H_{2,1} V_{2,1} \Phi_{2,1}(y) (\bar{C} V_{P_2} \Phi_{iP_2})(x) \mathcal{T}_{P_4}(1) \mathcal{T}_{P_5}(\infty) \rangle dx \wedge d\bar{y} \\ & - b^4 \langle \mathcal{T}_{P_1}(0) B\bar{B}V_{2,1} \Phi_{2,1}(y) \mathcal{T}_{P_2}(x) \mathcal{T}_{P_4}(1) \mathcal{T}_{P_5}(\infty) \rangle dy \wedge d\bar{y}. \end{aligned}$$

The fact that the form is closed can be understood using the identities for BRST descendants

$$\begin{aligned}\partial H_{2,1} V_{2,1} \Phi_{2,1} &= \mathcal{Q}(b^2 B V_{2,1} \Phi_{2,1}), \\ \partial(CV_P \Phi_{iP}) &= \mathcal{Q}(V_P \Phi_{iP}),\end{aligned}$$

and moving the action of BRST charge from one operator to the other inside the correlator. Equivalently, one can just use conformal Ward identities and the BPZ equation. After choosing a 2-cycle, we can integrate a 2-form over it [e.g., over $(y, \bar{y})$ at fixed x and vice versa] to obtain a “string amplitude with ground ring.”

In the next subsection, we will test the applicability of our expressions for five-point conformal blocks in numerical studies of such amplitudes. The reason we introduce the ground ring into the picture is that, while still probing the higher-point kinematics, such amplitudes are expected to be easier to compute than purely tachyonic ones. Namely, they involve only integrals over simple submanifolds of $\mathcal{M}_{0,5}$, and due to the presence of degenerate fields, the CFT correlators that we need to compute have a simpler structure than a general five-point function. Another motivation is to test a recent proposal on how ground ring operators (or something closely related) can emerge in a matrix model dual for the minimal string [32].

2. Computation of the integral over d^2x

Here we illustrate how one can use the elliptic recursion of Sec. VA for numerical computations of the integral of a “ground ring 2-form” over the position x of one of the tachyons. We consider as an example the case of the $(2, 9)$ minimal model ($b = \sqrt{2/9}$) as the matter theory, with external tachyon momenta

$$\{P_1, P_2, P_4, P_5\} = \{P_{-3,1}, P_{-2,1}, P_{-3,1}, P_{-3,1}\}. \quad (5.21)$$

These Liouville fields “dress” minimal model primaries $\Phi_{3,1}, \Phi_{2,1}, \Phi_{3,1}, \Phi_{3,1}$, respectively. The amplitude is of the form

$$\begin{aligned}A &= \int d^2x \hat{H}_{2,1} \hat{H}_{2,1} \langle \Phi_{3,1}(0) \Phi_{2,1}(x) \Phi_{2,1}(y) \Phi_{3,1}(1) \Phi_{3,1}(\infty) \rangle \\ &\quad \times \langle V_{-2,1}(0) V_{-2,1}(x) V_{2,1}(y) V_{-3,1}(1) V_{-3,1}(\infty) \rangle,\end{aligned}$$

where $\hat{H}_{2,1} = \partial_y^L - \partial_y^M + b^2 \frac{2y-1}{y(1-y)}$ because the ghost correlator is

$$\begin{aligned}\langle C(0) B(y) C(x) C(1) C(\infty) \rangle &= \frac{x(1-x)}{y(1-y)(y-x)}, \\ \langle C(0) : BC : (y) C(1) C(\infty) \rangle &= \frac{2y-1}{y(1-y)}.\end{aligned}$$

A subgroup $\mathcal{S}$ of projective transformations of order 6 generated by $\mathcal{A}: z \rightarrow 1/z$ and $\mathcal{B}: z \rightarrow 1-z$ can be used to divide the x plane into six regions; see Fig. 6(a) images of the fundamental domain $\mathbf{G} = \{\text{Re } x < 1/2; |1-x| < 1\}$.

$O_{2,1}$ and $\mathcal{T}_{k,1}$ are primary operators of dimension zero, so the correlator transforms in a simple way under the action of $\mathcal{S}$, and we can reduce the integration over d^2x to six integrals over $\mathbf{G}$:

$$\begin{aligned}A &= a(y) + a\left(\frac{1}{y}\right) + a(1-y) + a\left(\frac{y-1}{y}\right) + a\left(\frac{1}{1-y}\right) + a\left(\frac{y}{y-1}\right), \\ a(y) &:= \int_{\mathbf{G}} d^2x \langle \mathcal{T}_{P_1}(0) (V_{P_2} \Phi_{iP_2})(x) O_{2,1}(y) \mathcal{T}_{P_3}(1) \mathcal{T}_{P_4}(\infty) \rangle.\end{aligned} \quad (5.22)$$

Finally, in every term one can decompose the correlators in conformal blocks. E.g., let us expand in the comb channel $(0, x, y, 1, \infty)$; the contribution to the amplitude then takes the form

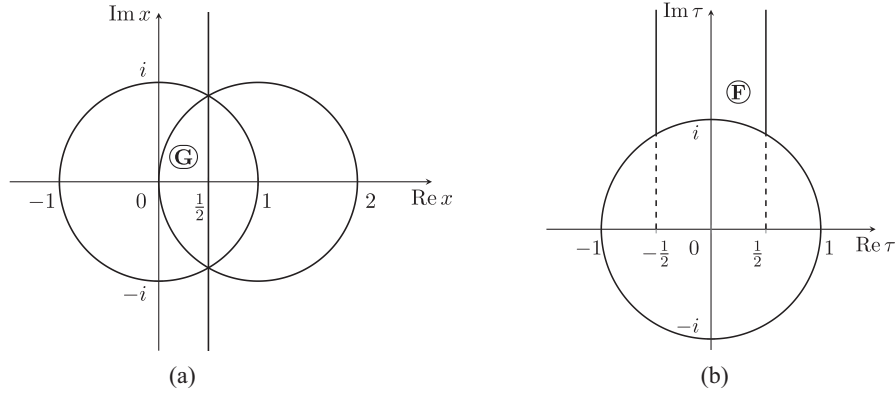
$$\begin{aligned}a(y) &= \sum_{(kl)} r_{kl} \sum_{\epsilon=\pm} \int \frac{dP_{i1}}{4\pi} r_{\epsilon}(P_{i1}) \\ &\quad \times \underbrace{\int_{\mathbf{G}} d^2x |\hat{H}_{2,1} \mathfrak{F}^{(b)}(x, y) \mathfrak{F}^{(ib)}(x, y)|^2}_{I_{\epsilon, (kl)}(P_{i1}, y)},\end{aligned} \quad (5.23)$$

where we omit most of the parameters of the blocks for brevity:

$$\begin{aligned}\mathfrak{F}^{(b)}(x, y) &:= \mathfrak{F}^{(b)}(P_1, P_2, P_{2,1}, P_4, P_5; P_{i1}, P_{i2}|x, y), \\ \mathfrak{F}^{(ib)}(x, y) &:= \mathfrak{F}^{(ib)}(iP_1, iP_2, \hat{P}_{2,1}, iP_4, iP_5; \hat{P}_{i1}, \hat{P}_{i2}|x, y).\end{aligned}$$

r_{kl} and $r_{\epsilon}(P_{i1})$ are products of structure constants in matter and Liouville theory, respectively. They are indexed by intermediate channels (k, l) on the minimal model side such that $\hat{P}_{i1} = \hat{P}_{k,1}$, $\hat{P}_{i2} = \hat{P}_{l,1}$, and a sign $\epsilon = \pm$ on the Liouville side such that $P_{i2} = P_{i1} + \epsilon \frac{ib}{2}$. In our example, (k, l) take the values $(4, 5), (4, 3), (2, 3), (2, 1)$.

For demonstrative purposes, we will concentrate on studying the integral $I_{\epsilon, (kl)}(P, y)$ over moduli space as a function of the intermediate dimensions. We will take $y = 1/2$: This choice is the most symmetric, since its orbit under $\mathcal{S}$ consists only of three different points


 FIG. 6. (a) Images of the fundamental domain $\mathbf{G}$. (b) Image $\mathbf{F}$ of domain $\mathbf{G}$.

instead of six. Also, it is easy to check that two choices for $\epsilon = \pm$ differ by complex conjugation, so we take $\epsilon = +$. Computing its values on a fine enough lattice in P -space, one can multiply it by the structure constants, which are smooth functions of P_{i1} and approximate the remaining integral by the Riemann sum. The numerical computation of the structure constants is a simpler problem, where many effective approaches were developed, such as the infinite product formula [33] and the

aforementioned expression as a finite product of Barnes G -functions for rational b^2 .

Under the elliptic map $x \rightarrow \tau(x) = i \frac{K(1-x)}{K(x)}$, region $\mathbf{G}$ maps 1:1 to a standard “keyhole” region $\mathbf{F}$ on the τ upper half plane $\{\text{Im } \tau > 0, |\text{Re } \tau| < 1/2, |\tau| > 1\}$ [see Fig. 6(b)], which is convenient to integrate over when using elliptic variables.

The result of action of $\hat{H}_{2,1}$ on a product of conformal blocks given by (5.2) simplifies to

$$\begin{aligned}
 dx H_{2,1} \mathfrak{F}^{(b)}(x, y) \mathfrak{F}^{(ib)}(x, y) &= d\tau \pi 16^{P_{i1}^2 + \hat{P}_{i1}^2} q^{P_{i2}^2 + \hat{P}_{i2}^2} e^{2i(u + i \log 2)(P_{i1}^2 + \hat{P}_{i1}^2 - P_{i2}^2 - \hat{P}_{i2}^2)} \\
 &\times \left[\frac{i}{2} (\mathfrak{F}^{(ib)}(u, q) \partial_u \mathfrak{F}^{(b)}(u, q) - \mathfrak{F}^{(b)}(u, q) \partial_u \mathfrak{F}^{(ib)}(u, q)) \right. \\
 &+ \left(\frac{b^2 \theta_3(q)^2}{4} \sqrt{y(1-y)(y-x)} \left(\frac{3}{y-x} + \frac{2y-1}{y(1-y)} \right) \right) \mathfrak{F}^{(b)}(u, q) \mathfrak{F}^{(ib)}(u, q) \\
 &\left. - (P_{i1}^2 - P_{i2}^2 + \hat{P}_{i2}^2 - \hat{P}_{i1}^2) \mathfrak{F}^{(b)}(u, q) \mathfrak{F}^{(ib)}(u, q) \right], \quad (5.24)
 \end{aligned}$$

where we used that

$$dx = \pi x(1-x) \theta_3^4(q) d\tau.$$

Interesting simplification occurs also if we act with $H_{2,1}$ on the WKB answer (2.19) that accounts for δP^2 corrections; in the considered case, $\delta P^2 = -\delta \hat{P}^2 = -\frac{b^2}{4}$, so one gets

$$\begin{aligned}
 dx H_{2,1} \mathfrak{f}^{(b)}(x, y) \mathfrak{f}^{(ib)}(x, y) &= d\tau \frac{i\pi}{2} (16q)^{P_{i2}^2 + \hat{P}_{i2}^2} e^{4(iu + \log 2)(P_{i2} \delta P + \hat{P}_{i2} \delta \hat{P})} \\
 &\times \left(2b^2 \frac{\Theta_1'(u, q)}{\Theta_1(u, q)} + 4i(P_{i2} \delta P - \hat{P}_{i2} \delta \hat{P}) \right). \quad (5.25)
 \end{aligned}$$

The prefactors dependent on external dimensions almost cancel the Jacobian $x \rightarrow \tau$. As an aside, a similar simplified expression exists for a different component of the ground ring 2-form, which we do not consider in this paper:

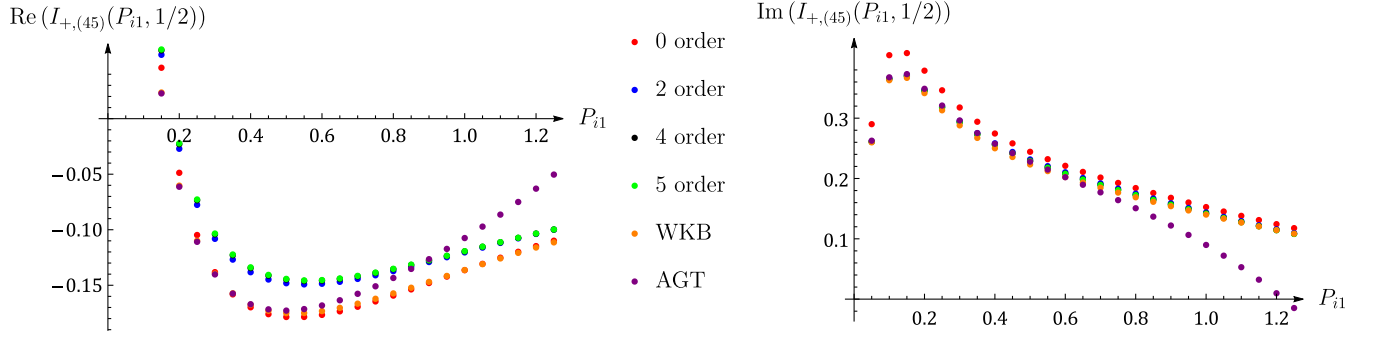
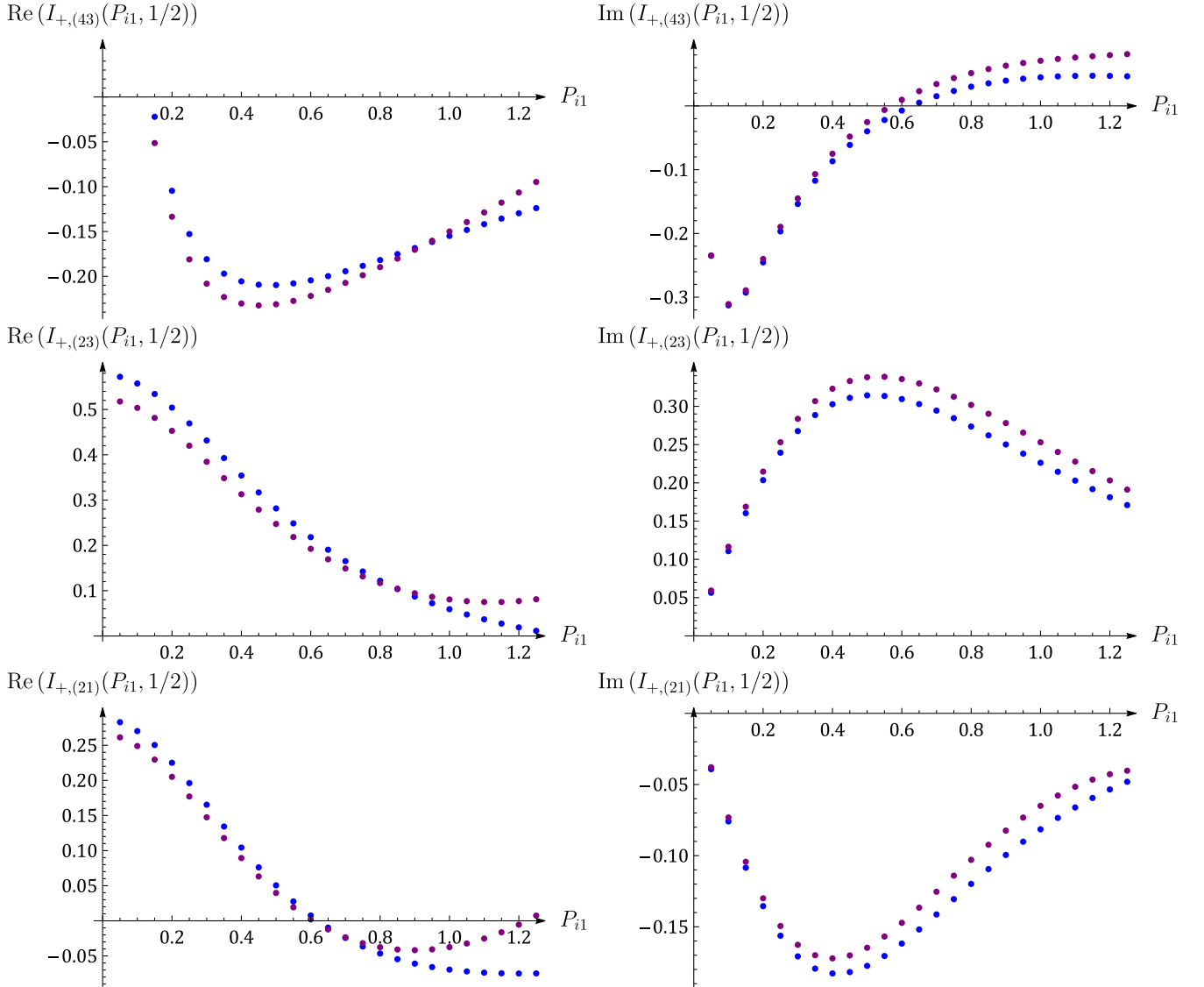
FIG. 7. Real and imaginary parts of $I_{+, (45)}(P_{i1}, 1/2)$ computed using different approaches.FIG. 8. Real and imaginary parts of $I_{+, (kl)}(P_{i1}, 1/2)$ for different (kl) computed using recursion to fifth order (blue) and AGT expansion (purple).

TABLE II. Values of the integrals $I_{+, (kl)}(P_{i1}, 1/2)$ for different minimal model intermediate momenta and $P_{i2} = P_{i1} + \frac{ib}{2}$.

P	0.2	0.4	0.6	0.8	1.0
$(k, l) = (4, 5)$					
WKB (2.19)	$-0.060 + 0.343i$	$-0.167 + 0.251i$	$-0.173 + 0.203i$	$-0.157 + 0.170i$	$-0.136 + 0.141i$
Rec. second order	$-0.027 + 0.348i$	$-0.138 + 0.259i$	$-0.148 + 0.211i$	$-0.137 + 0.176i$	$-0.120 + 0.145i$
Rec. fourth order	$-0.023 + 0.347i$	$-0.134 + 0.257i$	$-0.145 + 0.209i$	$-0.135 + 0.174i$	$-0.119 + 0.143i$
Rec. fifth order	$-0.022 + 0.346i$	$-0.133 + 0.257i$	$-0.145 + 0.209i$	$-0.135 + 0.173i$	$-0.119 + 0.143i$
$(k, l) = (4, 3)$					
Rec. fourth order	$-0.105 - 0.245i$	$-0.206 - 0.086i$	$-0.204 - 0.006i$	$-0.182 + 0.031i$	$-0.155 + 0.046i$
Rec. fifth order	$-0.104 - 0.245i$	$-0.205 - 0.086i$	$-0.204 - 0.006i$	$-0.181 + 0.031i$	$-0.154 + 0.046i$
$(k, l) = (2, 3)$					
Rec. fourth order	$0.504 + 0.204i$	$0.354 + 0.304i$	$0.219 + 0.311i$	$0.123 + 0.275i$	$0.060 + 0.227i$
Rec. fifth order	$0.505 + 0.204i$	$0.355 + 0.303i$	$0.219 + 0.310i$	$0.123 + 0.274i$	$0.060 + 0.227i$
$(k, l) = (2, 1)$					
Rec. fourth order	$0.225 - 0.135i$	$0.105 - 0.182i$	$0.008 - 0.161i$	$-0.046 - 0.120i$	$-0.069 - 0.081i$
Rec. fifth order	$0.226 - 0.135i$	$0.105 - 0.182i$	$0.008 - 0.161i$	$-0.046 - 0.120i$	$-0.069 - 0.081i$

$$\begin{aligned}
& dy \langle C(0)B(y)C(x)C(1)C(\infty) \rangle \mathfrak{f}^{(b)}(x, y) \mathfrak{f}^{(ib)}(x, y) \\
&= dy \langle C(0)B(y)C(x)C(1)C(\infty) \rangle \mathfrak{f}_\Delta^{(b)}(x, y) \mathfrak{f}_\Delta^{(ib)}(x, y) \\
&= du(-2i)(16q)^{P_{i2}^2 + \hat{P}_{i2}^2} e^{4(iu + \log 2)(P_{i2}\delta P + \hat{P}_{i2}\delta \hat{P})}. \quad (5.26)
\end{aligned}$$

Now we present the results of the computation. The plots in Figs. 7 and 8 illustrate $I_{+, (kl)}(P, 1/2)$ calculated using recursion VA truncated to some level in elliptic variables qe^{-2iu} and e^{2iu} , the WKB asymptotic (2.19), and the AGT series expansion.⁵ In Table II, the results of the calculation for a few values of P_{i1} using different methods are provided. We compare all available methods for the channel $(k, l) = (4, 5)$, which is expected to give the largest contribution to $a(1/2)$. Two subtle points about this computation are as follows:

- (i) To compute the integral over $d^2\tau$, we use the standard numerical integration method in Wolfram *Mathematica* for a subdomain of $\mathbf{F}$ $\{0 < \text{Im } \tau < \Lambda, |\text{Re } \tau| < 1/2, |\tau| > 1\}$ with sufficiently large Λ . The integral over the remaining part with $\text{Im } \tau > \Lambda$ is taken by truncating to the leading power in the q expansion of the integrand and integrating this

⁵We use the AGT series expansion as follows: We compute numerically the blocks and their derivatives as series in x/y and y up to order $n = 9$ in both variables (for higher orders, the computation time quickly grows and numerical mistakes are accumulated), then we rewrite blocks as a series in q and y and integrate their product over $d^2\tau$. We only expand the integrand up to $q^{\lfloor n/2 \rfloor} = q^4$ because the coefficients of the series in q are themselves series in y , and we need enough terms for these coefficients to converge. This method is expected to have low accuracy.

analytically. This is enough for the desired accuracy, as can be checked by varying Λ and checking that the sum of two contributions remains the same.

- (ii) Since the function $u(y, x)$ is multivalued in the region $x \in \mathbf{G}$, it is important to check that its values on the correct sheet are used in the calculation. The correct branch of $u(y, x)$ is defined by the requirement $\overline{u(y, x)} = -u(\bar{y}, \bar{x})$.

We see that the recursion method stabilizes quickly: The results for fourth and fifth order are almost indistinguishable, and second order is quite close (difference of order 2%). The imaginary part of the WKB answer agrees well with the recursion, but the real part agrees better with the recursion at 0 order and AGT answers at small momenta. The discrepancy between WKB and high-order recursion is of order 10%–15%. The AGT results start to deviate from other methods significantly at $P_{i1} \sim 0.8$.

VI. CONCLUSION

We finish by listing some possible directions for future work:

- (i) While higher genus blocks are likely to be untractable using this method, it is interesting to study the case of spherical blocks in other channels (for a five-point block, only comb channel blocks exist, but starting with six points, other options appear, such as the trifundamental channel). It is tempting to conjecture that the geometric interpretation given in Sec. IV still holds, and conformal blocks in the limit of large intermediate dimensions are computed using the period matrix of the WKB curve. To this

end, one can investigate how the large- P limit of the crossing matrix that relates blocks in different channels agrees with this proposal: It is known that, e.g., the modular transformation in this limit is simply a Fourier transform [34].

- (ii) After successfully testing our elliptic recursion in the context of Liouville gravity, it would be nice to finish the calculation of the amplitude with the ground ring. To compute other contributions to the amplitude [such as $a(2)$], one needs to go away from the standard comb channel kinematic domain $0 < |x| < |y| < 1$. It would be most convenient to just analytically continue the blocks outside of this domain without changing the expansion channel, using our elliptic representation, but in practice, elliptic functions such as $u(y, x)$ behave in a very weird way when evaluated, e.g., for $|y| > 1$. This issue needs to be addressed. It is also interesting to apply elliptic blocks for more complicated cases, such as five-point tachyon amplitudes in MS or VMS.
- (iii) For four-point amplitudes, elliptic recursion allows us to prove an important analyticity property of the amplitudes in MS and VMS [32]—lack of discontinuities associated with “descendant on-shell poles.” This is an important part of the “analytic bootstrap” argument presented in [5] that fixes the possible expression for the four-point amplitude in VMS. Whether the same can be proven for higher-point amplitudes remains to be understood. Perhaps the elliptic recursion will be useful in this context as well.

ACKNOWLEDGMENTS

We are grateful to Alexey Litvinov, Andrei Marshakov, and Vladimir Belavin for fruitful discussions on this and related topics and interest in this work. This work was

supported by the Basis Foundation. The work performed at Landau Institute has been supported by the Russian Science Foundation under Grant No. 23-12-00333.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: USEFUL FORMULAS

An alternative definition of $u(y, x)$ is

$$u(y, x) = \frac{\pi}{2} - \frac{\pi}{4K(x)} \int_0^{\frac{y}{x}} \frac{dt}{\sqrt{t(1-t)(1-xt)}}. \quad (\text{A1})$$

The derivatives of $u(y, x)$ are

$$\begin{aligned} \frac{\partial u(y, x)}{\partial y} &= \frac{i\pi}{4\sqrt{y(1-y)(y-x)K(x)}}, \\ \frac{\partial u(y, x)}{\partial x} &= \frac{\pi}{4i} \frac{(1-y)(xK(x) + (y-x)\Pi(y, x))}{x(1-x)\sqrt{y(1-y)(y-x)K(x)^2}}. \end{aligned} \quad (\text{A2})$$

Some properties of the theta functions are as follows:

$$\Theta_1(u|q) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} \sin((2n+1)u), \quad (\text{A3})$$

$$\Theta_1(u + \pi(m + n\tau)|q) = (-1)^{m+n} q^{-n^2} e^{-2inu} \Theta_1(u|q), \quad (\text{A4})$$

$$\Theta_1(2u|q) = \frac{2\Theta_1(u|q)\Theta_2(u|q)\Theta_3(u|q)\Theta_4(u|q)}{\Theta_1'(0|q)}, \quad (\text{A5})$$

$$\Theta_1'(0|q) = \theta_2(q)\theta_3(q)\theta_4(q), \quad (\text{A6})$$

$$\frac{\partial}{\partial x} \log \Theta_3(u(y, x)|q(x)) = \frac{yE(x)K(x) + (1-y)(xK(x)^2 - 2xK(x)\Pi(y, x) - (y-x)\Pi(y, x)^2)}{4yx(1-x)K(x)^2}, \quad (\text{A7})$$

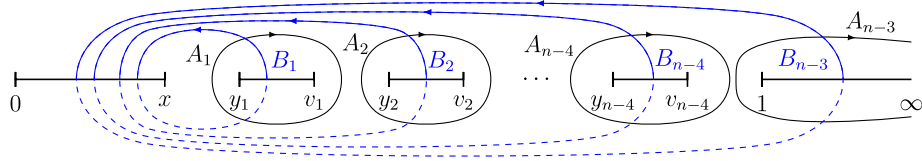
$$x = \frac{\theta_2(q)^4}{\theta_3(q)^4}, \quad y = \frac{\theta_2(q)^2 \Theta_2(u|q)^2}{\theta_3(q)^2 \Theta_3(u|q)^2}, \quad (\text{A8})$$

$$1-x = \frac{\theta_4(q)^4}{\theta_3(q)^4}, \quad 1-y = \frac{\theta_4(q)^2 \Theta_4(u|q)^2}{\theta_3(q)^2 \Theta_3(u|q)^2}, \quad y-x = -\frac{\theta_2(q)^2 \theta_4(q)^2 \Theta_1(u|q)^2}{\theta_3(q)^4 \Theta_3(u|q)^2}. \quad (\text{A9})$$

APPENDIX B: PERIODS OF A DEGENERATE HYPERELLIPTIC CURVE

Consider a genus $g = n - 3$ hyperelliptic curve

$$w^2 = z(1-z)(x-z) \prod_{k=1}^{n-4} (z-y_k)(z-v_k). \quad (\text{B1})$$


 FIG. 9. A and B periods on the hyperelliptic curve.

Choose cycles $A_1, \dots, A_{n-3}$ and $B_1, \dots, B_{n-3}$ as shown on Fig. 9. The intersection numbers of the cycles are

$$A_i \cap B_j = \delta_{ij}, \quad A_i \cap A_j = B_i \cap B_j = 0. \quad (\text{B2})$$

On a genus g curve, there is a g -dimensional space of holomorphic differentials with the following basis:

$$\eta_{n-3} = \frac{dz}{w} \prod_{l=1}^{n-4} (z - y_l), \quad \eta_k = \frac{dz}{w} \prod_{\substack{l=1 \\ l \neq k}}^{n-4} (z - y_l),$$

for $k = 1, \dots, n-4$. (B3)

When $v_k \rightarrow y_k$, the curve degenerates, and the basis becomes

$$\eta_{n-3} = \frac{dz}{\sqrt{z(1-z)(x-z)}},$$

$$\eta_k = \frac{dz}{\sqrt{z(1-z)(x-z)}} \frac{1}{z - y_k}, \quad \text{for } k = 1, \dots, n-4. \quad (\text{B4})$$

The η_{n-3} is the unique (up to normalization) holomorphic differential on the degenerate curve, and η_k have simple poles at the two points of the curve with $z = y_k$. Such differentials are called Abelian differentials of the third kind. They can be expressed in terms of theta functions up to terms with holomorphic differentials:

$$\eta_k = \lambda_k \eta_{n-3} + \frac{dz}{\sqrt{y_k(1-y_k)(x-y_k)}} \times \frac{\partial}{\partial z} \log \frac{\Theta_1(u(z, x) - u(y_k, x)|q)}{\Theta_1(u(z, x) - \pi + u(y_k, x)|q)}. \quad (\text{B5})$$

Here the zeroes of the theta functions correspond to the two poles of the differential.

There is another basis of differentials, which are normalized with respect to the A periods:

$$\int_{A_j} \omega_k = \delta_{jk}. \quad (\text{B6})$$

Using this basis, one defines the period matrix τ of the curve

$$\tau_{jk} = \int_{B_j} \omega_k. \quad (\text{B7})$$

If the basis of cycles satisfies (B2), the matrix satisfies Riemann bilinear relations; i.e., it is symmetric and its imaginary part is positive definite.

For the η basis, we have

$$\int_{A_{n-3}} \eta_{n-3} = 4K(x), \quad \int_{A_j} \eta_{n-3} = 0,$$

$$\int_{A_j} \eta_k = -2\pi i \frac{1}{\sqrt{y_k(1-y_k)(x-y_k)}} \delta_{jk} \quad (\text{B8})$$

for $j, k = 1, \dots, n-4$. Using the periodic properties of $\Theta_1(u|q)$ and the fact that $\Theta_1(-u|q) = -\Theta_1(u|q)$, one can see that

$$\begin{aligned} & \int_{A_{n-3}} (\eta_k - \lambda_k \eta_{n-3}) \\ &= \frac{2}{\sqrt{y_k(1-y_k)(x-y_k)}} \\ & \times \log \frac{\Theta_1(\frac{\pi}{2}\tau + \frac{\pi}{2} - u(y_k, x)|q) \Theta_1(\frac{\pi}{2}\tau - \pi + u(y_k, x)|q)}{\Theta_1(\frac{\pi}{2}\tau - \frac{\pi}{2} + u(y_k, x)|q) \Theta_1(\frac{\pi}{2}\tau - u(y_k, x)|q)} = 0. \end{aligned} \quad (\text{B9})$$

Thus,

$$\omega_{n-3} = \frac{1}{4K(x)} \frac{dz}{\sqrt{z(1-z)(x-z)}},$$

$$\omega_k = -\frac{dz}{2\pi i} \frac{\partial}{\partial z} \log \frac{\Theta_1(u(z, x) - u(y_k, x)|q)}{\Theta_1(u(z, x) - \pi + u(y_k, x)|q)} \quad (\text{B10})$$

for $k = 1, \dots, n-4$. Now we can compute the period matrix. First of all, we see that the integrals of η_k over B_j diverge when $n-3 > j \geq k$, because the contour of integration B_j passes through the pole of η_k . It follows that the diagonal entries of the period matrix are infinite, except for the last one:

$$\begin{aligned}\tau_{n-3n-3} &= -\frac{1}{2K(x)} \int_x^1 \frac{dz}{\sqrt{z(1-z)(x-z)}} \\ &= \frac{iK(1-x)}{K(x)} = \tau(x), \quad \tau_{kk} = \infty, \quad k < n-3.\end{aligned}\quad (\text{B11})$$

The rest of the last column of the period matrix is

$$\begin{aligned}\tau_{jn-3} &= \int_{B_j} \omega_{n-3} = -\frac{1}{2K(x)} \int_x^{y_j} \frac{dz}{\sqrt{z(1-z)(x-z)}} \\ &= \frac{2}{\pi} \int_x^{y_j} dz \frac{\partial}{\partial z} u(z, x) = \frac{2}{\pi} u(y_j, x).\end{aligned}\quad (\text{B12})$$

The remaining entries in the last row are

$$\begin{aligned}\int_{B_{n-3}} \omega_k &= \frac{1}{2\pi i} \log \frac{\Theta_1(\frac{\pi}{2}\tau - u(y_k, x)|q)^2 \Theta_1(u(y_k, x) - \pi|q)^2}{\Theta_1(\frac{\pi}{2}\tau - \pi + u(y_k, x)|q)^2 \Theta_1(-u(y_k, x)|q)^2} \\ &= \frac{2}{\pi} u(y_k, x).\end{aligned}\quad (\text{B13})$$

As expected, they are the same as in the last column, since the period matrix is symmetric.

Thus, for the degenerate curve, we have the following period matrix:

$$\tau(\{y_k\}, x) = \begin{pmatrix} B & \frac{2}{\pi} \mathbf{u} \\ \frac{2}{\pi} \mathbf{u}^T & \tau(x) \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u(y_1, x) \\ \vdots \\ u(y_{n-4}, x) \end{pmatrix}, \quad (\text{B14})$$

where B is an $(n-4) \times (n-4)$ matrix with ∞ on the diagonal. Off-diagonal elements of B are

$$B_{jk} = \frac{1}{2\pi i} \log \frac{\Theta_1(u(y_k, x) - u(y_j, x)|q)^2}{\Theta_1(u(y_k, x) + u(y_j, x)|q)^2}. \quad (\text{B15})$$

Also, we see that the diagonal elements of the period matrix diverge in the following way:

$$\begin{aligned}B_{kk} &= -2 \int_x^{y_k-\varepsilon} \omega_k \sim \frac{1}{\pi i} \log(\varepsilon u'(y_k, x)) \\ &\quad + \frac{1}{\pi i} \log \frac{\Theta_1'(0|q)}{\Theta_1(2u(y_k, x)|q)}.\end{aligned}\quad (\text{B16})$$

-
- [1] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, *Nucl. Phys.* **B241**, 333 (1984).
- [2] V. Rosenhaus, Multipoint conformal blocks in the comb channel, *J. High Energy Phys.* **02** (2019) 142.
- [3] M. Ammon, J. Hollweck, T. Hössel, and K. Wölfl, Conformal blocks in two and four dimensions from oscillator representations, *J. High Energy Phys.* **05** (2025) 091.
- [4] K. B. Alkalaev and V. A. Belavin, From global to heavy-light: 5-point conformal blocks, *J. High Energy Phys.* **03** (2016) 184.
- [5] D. Khromov and A. Litvinov, On correlation numbers $V_{0,4}$ and $V_{1,1}$ in Virasoro minimal string theory, *J. High Energy Phys.* **04** (2026) 126.
- [6] V. Belavin, Torus amplitudes in minimal Liouville gravity and matrix models, *Phys. Lett. B* **698**, 86 (2011).
- [7] K. Aleshkin and V. Belavin, On the construction of the correlation numbers in minimal Liouville gravity, *J. High Energy Phys.* **11** (2016) 142.
- [8] V. A. Rodriguez, M. Usatyuk, and Z.-Y. Wang, ADE minimal strings and multi-matrix duals, *arXiv:2511.21851*.
- [9] S. Collier, L. Eberhardt, B. Mühlmann, and V. A. Rodriguez, The Virasoro minimal string, *SciPost Phys.* **16**, 057 (2024).
- [10] S. Collier, L. Eberhardt, B. Mühlmann, and V. A. Rodriguez, The complex Liouville string: The worldsheet, *SciPost Phys.* **19**, 033 (2025).
- [11] B. Balthazar, V. A. Rodriguez, and X. Yin, The $c = 1$ string theory S-matrix revisited, *J. High Energy Phys.* **04** (2019) 145.
- [12] L. F. Alday, D. Gaiotto, and Y. Tachikawa, Liouville correlation functions from four-dimensional gauge theories, *Lett. Math. Phys.* **91**, 167 (2010).
- [13] V. A. Alba, V. A. Fateev, A. V. Litvinov, and G. M. Tarnopolskiy, On combinatorial expansion of the conformal blocks arising from AGT conjecture, *Lett. Math. Phys.* **98**, 33 (2011).
- [14] A. B. Zamolodchikov, Conformal symmetry in two-dimensional space: Recursion representation of conformal block, *Theor. Math. Phys.* **73**, 1088 (1987).
- [15] J. Maldacena, D. Simmons-Duffin, and A. Zhiboedov, Looking for a bulk point, *J. High Energy Phys.* **01** (2017) 013.
- [16] L. Hadasz, Z. Jaskolski, and P. Suchanek, Elliptic recurrence representation of the $N = 1$ Neveu-Schwarz blocks, *Nucl. Phys.* **B798**, 363 (2008).
- [17] L. Hadasz, Z. Jaskolski, and P. Suchanek, Recursive representation of the torus 1-point conformal block, *J. High Energy Phys.* **01** (2010) 063.
- [18] M. Cho, S. Collier, and X. Yin, Recursive representations of arbitrary virasoro conformal blocks, *J. High Energy Phys.* **04** (2019) 018.
- [19] M. Beşken, S. Datta, and P. Kraus, Semi-classical virasoro blocks: Proof of exponentiation, *J. High Energy Phys.* **01** (2020) 109.

- [20] M. Gerbershagen and J. Hollweck, Exponentiation of higher-point and higher-genus virasoro conformal blocks in the semiclassical limit, [arXiv:2606.18345](#).
- [21] K. Iwaki, Les Houches lectures on exact WKB analysis and Painlevé equations, [arXiv:2512.17599](#).
- [22] T. Koike, On the exact WKB analysis of second order linear ordinary differential equations with simple poles, *Publ. RIMS* **36** (04, 2000).
- [23] V. A. Fateev, A. V. Litvinov, A. Neveu, and E. Onofri, A differential equation for a four-point correlation function in Liouville field theory and elliptic four-point conformal blocks, *J. Phys. A* **42**, 304011 (2009).
- [24] V. Enolski and P. Richter, Periods of hyperelliptic integrals expressed in terms of θ -constants by means of thomae formulae, *Phil. Trans. R. Soc. A* **366**, 1005 (2007).
- [25] J. Thomae, Beitrag zur bestimmung von ... $(0, 0, \dots, 0)$ durch die klassenmoduln algebraischer functionen, *J. Reine Angew. Math.* **71**, 201 (1869), <https://eudml.org/doc/148112>.
- [26] A. Zamolodchikov, Conformal scalar field on the hyperelliptic curve and critical ashkin-teller multipoint correlation functions, *Nucl. Phys.* **B285**, 481 (1987).
- [27] A. B. Zamolodchikov, Conformal symmetry in two dimensions: An explicit recurrence formula for the conformal partial wave amplitude, *Commun. Math. Phys.* **96**, 419 (1984).
- [28] A. A. Belavin and A. B. Zamolodchikov, Integrals over moduli spaces, ground ring, and four-point function in minimal Liouville gravity, *Theor. Math. Phys.* **147**, 729 (2006).
- [29] C. Imbimbo, S. Mahapatra, and S. Mukhi, Construction of physical states of non-trivial ghost number in $c < 1$ string theory, *Nucl. Phys.* **B375**, 399 (1992).
- [30] J. Polchinski, *String Theory*, Cambridge Monographs on Mathematical Physics Vol. 1 (Cambridge University Press, Cambridge, England, 1998).
- [31] A. Sen and B. Zwiebach, String field theory: A review, [arXiv:2405.19421](#).
- [32] A. Artemev, x-y swap for a $(2, 2p+1)$ minimal string, *Phys. Rev. D* **112**, 046019 (2025).
- [33] S. Ribault and R. Santachiara, Liouville theory with a central charge less than one, *J. High Energy Phys.* **08** (2015) 109.
- [34] N. Nemkov, S-duality as Fourier transform for arbitrary ϵ_1, ϵ_2 , *J. Phys. A* **47**, 105401 (2014).