

Color-kinematics and double-copy relations for self-dual solutions

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We clarify the relation between the classical double copy and the double copy for amplitudes in the setting of self-dual gauge and gravity theories. To this end we construct explicit all-order perturbative solutions in these theories and show that they are related by a version of color-kinematics duality. This relation can be expressed in a double-copy form and embodies the most general manifestation of the self-dual classical double copy. Our classical double-copy relations directly lead to known amplitude double-copy relations, both for Berends-Giele currents and on-shell amplitudes, through the perturbative expansion.

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I. INTRODUCTION

The discovery of the Kawai-Lewellen-Tye-like double-copy relations connecting amplitudes in gauge and gravity theories, and the underlying color-kinematics duality, has revolutionized our understanding of the on-shell structure of these theories (see [1] for a review and list of references). In a subsequent development, it was realized first in [2] that classical solutions of gauge and gravity theories are in some cases related by classical double-copy relations. We refer to [3] for a review and a guide to the extensive literature. While this classical double copy bears a formal similarity to the double-copy relations for amplitudes, the direct relation between the two is not well-understood in general, though it has been clarified in the case of radiation fields of point particles [4], see also [5–9].

A general and direct relation between classical solutions and tree-level amplitudes¹ exists in the form of the perturbative expansion [14–17], reviewed in [18]. More precisely, this expansion extracts from perturbative solutions to the equations of motion the Berends-Giele (BG) currents [19], which are amplitudes where one leg is off-shell. As a next step, on-shell amplitudes can be easily computed from the BG currents.

Motivated by these considerations, we aim in this work to use perturbative methods to clarify the relation between classical and amplitude double-copy relations. We focus on

the setting of self-dual Yang-Mills and gravity theories, which has the advantage that both theories, while nonlinear, are also integrable and therefore expected to be fully tractable. As stressed in [20], in the nonlinear setting the classical double copy should be viewed as a relation between the general perturbative structure of both theories, rather than a one-to-one map between solutions. In the self-dual sector of interest, an important step towards realizing this idea was taken in [21], where the kinematic algebra which underlies color-kinematics duality for self-dual theories was identified. The perturbative structure of classical double copy was further analyzed using homotopy algebraic techniques in [22–24], the latter of which deals with self-dual theories. Color-kinematics duality for self-dual theories was also analyzed at the level of actions in [25].

In this work we will build on [21] and, using recently developed techniques for solving BG recursion relations [26], derive a novel all-order perturbative solution to Plebanski's equation [27] for self-dual gravity. This solution can be written as a manifest double copy of the Bardeen-Cangemi [28,29] perturbative Yang-Mills solution. Our result implies,² through the use of the perturbative expansion, known double-copy relations between BG currents [30,31] and on-shell amplitudes in both theories.

II. SELF-DUAL YANG-MILLS AND GRAVITY

In this section we briefly review the self-dual Yang-Mills and Einstein equations, referring to [32] for more details. We will work in mostly minus signature and make use of

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¹Perturbative methods have recently been extended to loop amplitudes [10–12], see also [13].

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²The converse is not necessarily true, i.e., knowledge of the BG currents does not fully determine the all-order classical solution in general.

bispinor notation³ with

$$\sigma_{\alpha\dot{\alpha}}^\mu = \frac{1}{\sqrt{2}}(1, \sigma_i), \quad x^{\alpha\dot{\alpha}} = x^\mu \sigma_\mu^{\alpha\dot{\alpha}} \equiv \begin{pmatrix} u & w \\ \bar{w} & v \end{pmatrix}, \quad (1)$$

in terms of which the Minkowski metric reads

$$ds_M^2 = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} dx^{\alpha\dot{\alpha}} dx^{\beta\dot{\beta}} = 2(dudv - dw d\bar{w}). \quad (2)$$

We also use the spinor helicity formalism, where a massless on-shell momentum p with $p^2 = 0$ is decomposed as a product of spinors and where angle- and square-bracket inner products are introduced as

$$p_{\alpha\dot{\alpha}} = \mathfrak{p}_\alpha \tilde{\mathfrak{p}}_{\dot{\alpha}}, \quad [12] = \mathfrak{p}_1^\alpha \mathfrak{p}_{2\alpha}, \quad \langle 12 \rangle = \tilde{\mathfrak{p}}_1^{\dot{\alpha}} \tilde{\mathfrak{p}}_{2\dot{\alpha}}. \quad (3)$$

Polarization tensors for gluons and gravitons depend on an arbitrary massless reference momentum, reflecting gauge invariance. In what follows we will consider all polarizations tensors to involve the same reference momentum q , which we fix once and for all and decompose into spinors $\mathfrak{q}_\alpha, \tilde{\mathfrak{q}}_{\dot{\alpha}}$ as

$$q_{\alpha\dot{\alpha}} = \mathfrak{q}_\alpha \tilde{\mathfrak{q}}_{\dot{\alpha}}. \quad (4)$$

In terms of this reference momentum, the polarization tensors for a gluon resp. graviton with momentum $p \neq q$ are given by

$$\begin{aligned} e_+(p)_{\alpha\dot{\alpha}} &= \frac{\mathfrak{q}_\alpha \tilde{\mathfrak{p}}_{\dot{\alpha}}}{[qp]}, & e_{++}(p)_{\alpha\dot{\alpha}\beta\dot{\beta}} &= \frac{\mathfrak{q}_\alpha \mathfrak{q}_\beta \tilde{\mathfrak{p}}_{\dot{\alpha}} \tilde{\mathfrak{p}}_{\dot{\beta}}}{[qp]^2} \\ e_-(p)_{\alpha\dot{\alpha}} &= \frac{\mathfrak{p}_\alpha \tilde{\mathfrak{q}}_{\dot{\alpha}}}{\langle qp \rangle}, & e_{--}(p)_{\alpha\dot{\alpha}\beta\dot{\beta}} &= \frac{\mathfrak{p}_\alpha \mathfrak{p}_\beta \tilde{\mathfrak{q}}_{\dot{\alpha}} \tilde{\mathfrak{q}}_{\dot{\beta}}}{\langle qp \rangle^2}. \end{aligned} \quad (5)$$

We will now write the equations of motion for (anti-)self-dual Yang-Mills and gravity fields in a gauge which depends on the chosen reference spinor $\mathfrak{q}_\alpha$ (resp. $\tilde{\mathfrak{q}}_{\dot{\alpha}}$). We will restrict our attention to the self-dual sector, the anti-self-dual one being analogous. While it is of course possible to Lorentz rotate and rescale the reference spinor to a convenient value such as $\mathfrak{q}^\alpha = \delta_1^\alpha$, we will keep it general for easy comparison with the amplitude literature. We first introduce a spinor $\mathfrak{r}^\alpha$ so that $(\mathfrak{q}^\alpha, \mathfrak{r}^\alpha)$ form an orthonormal frame in the sense that

$$\mathfrak{q}^\alpha \mathfrak{r}^\beta - \mathfrak{q}^\beta \mathfrak{r}^\alpha = -\epsilon^{\alpha\beta}. \quad (6)$$

We then have

³Spinor indices are raised and lowered according to

$$\mathfrak{p}_\alpha = \mathfrak{p}^\beta \epsilon_{\beta\alpha}, \quad \mathfrak{p}^\alpha = \epsilon^{\alpha\beta} \mathfrak{p}_\beta, \quad \epsilon_{12} = \epsilon^{12} = 1.$$

and similarly for dotted indices.

$$\mathfrak{q}^\alpha \partial_{\alpha\dot{\alpha}} = \frac{\partial}{\partial x^{\dot{\alpha}}} \quad \text{with} \quad x^{\dot{\alpha}} := \mathfrak{r}_\alpha x^{\alpha\dot{\alpha}}. \quad (7)$$

We make the following *Ansätze* for the Yang-Mills and gravitational fields in terms of “prepotential” functions Φ and Ψ :

$$A_{\alpha\dot{\alpha}} = g \mathfrak{q}_\alpha \partial_{\dot{\alpha}} \Phi \quad (8)$$

$$g_{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} + \kappa \mathfrak{q}_\alpha \mathfrak{q}_\beta \partial_{\dot{\alpha}} \partial_{\dot{\beta}} \Psi. \quad (9)$$

These *Ansätze* satisfy the Lorentz resp. transverse-traceless gauge conditions and solve the linearized self-duality conditions $\epsilon^{\dot{\alpha}\dot{\beta}} \partial_{[\alpha\dot{\alpha}} A_{\beta\dot{\beta}]} = \epsilon^{\dot{\alpha}\dot{\beta}} \partial_{[\alpha\dot{\alpha}} \Gamma_{\beta\dot{\beta}]}^\mu = 0$, provided that the prepotentials satisfy d'Alembert's equation $\partial^2 \Phi = \partial^2 \Psi = 0$, with $\partial^2 \equiv \partial^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}}$. Taking Φ, Ψ to be appropriately normalized plane waves $\Phi = J e^{ip \cdot x}$, $\Psi = \mathcal{J} e^{ip \cdot x}$ leads to positive helicity gluon resp. graviton fields, i.e.,

$$\begin{aligned} J &= -\frac{i}{[qp]^2} \Rightarrow A_{\alpha\dot{\alpha}} = g e_+(q; p)_{\alpha\dot{\alpha}} e^{ip \cdot x} \\ \mathcal{J} &= -\frac{1}{[qp]^4} \Rightarrow h_{\alpha\dot{\alpha}\beta\dot{\beta}} = \kappa e_{++}(q; p)_{\alpha\dot{\alpha}\beta\dot{\beta}} e^{ip \cdot x}. \end{aligned} \quad (10)$$

The full nonlinear self-dual equations reduce to

$$\partial^2 \Phi = g \epsilon^{\dot{\alpha}\dot{\beta}} [\partial_{\dot{\alpha}} \Phi, \partial_{\dot{\beta}} \Phi] \quad (11)$$

$$\partial^2 \Psi = \frac{\kappa}{2} \epsilon^{\dot{\alpha}\dot{\beta}} \{ \partial_{\dot{\alpha}} \Psi, \partial_{\dot{\beta}} \Psi \}, \quad (12)$$

where, in the second line, we have introduced a Poisson bracket as

$$\{F, G\} = \epsilon^{\dot{\alpha}\dot{\beta}} \partial_{\dot{\alpha}} F \partial_{\dot{\beta}} G. \quad (13)$$

The Eqs. (11) and (12) are equivalent to the standard self-dual Yang-Mills resp. gravity equations. For example, choosing $\mathfrak{q}^\alpha = \delta_1^\alpha$ leads to $x^1 = u, x^2 = w$ [cf. (1)] and the gravity equation becomes Plebanski's second heavenly equation [27],

$$(\partial_{uv} - \partial_{w\bar{w}}) \Psi = \frac{\kappa}{2} (\partial_u^2 \Psi \partial_w^2 \Psi - (\partial_{uw} \Psi)^2), \quad (14)$$

where $\partial_{uv} \equiv \partial_u \partial_v$. The Eqs. (11) and (12) can be suggestively rewritten in momentum space. To simplify the formulas we will treat the momentum dependence as a continuous index, writing, e.g., $f^k := f(k)$, $\delta_l^k := \delta(k-l)$ and adopting an Einstein summation convention

$$f^k g_k = \int d^4 k f(k) g(k). \quad (15)$$

Equations (11) and (12) then take the form

$$\Phi^k = 2g \frac{F_{k_1 k_2}^k}{s_{12}} \Phi^{k_1} \Phi^{k_2} \quad (16)$$

$$\Phi^{ak} = g \frac{f_{a_1 a_2}^a F_{k_1 k_2}^k}{s_{12}} \Phi^{a_1 k_1} \Phi^{a_2 k_2} \quad (17)$$

$$\Psi^k = \kappa \frac{\tilde{F}_{k_1 k_2} F_{k_1 k_2}^k}{2s_{12}} \Psi^{k_1} \Psi^{k_2}, \quad (18)$$

where $s_{ij} := (k_i + k_j)^2$. The first two equations are equivalent, but it will be convenient in what follows to separately consider the equations for the Lie-algebra valued scalar $\Phi^k = \Phi^{ak} T_a$ and for its color-components Φ^{ak} . The $f_{a_1 a_2}^a$ are the structure constants of the Yang-Mills gauge group, while the $F_{k_1 k_2}^k$ and $\tilde{F}_{k_1 k_2}$ arise from the Poisson bracket of plane waves:

$$\{e^{ik_1 \cdot x}, e^{ik_2 \cdot x}\} = \sum_k F_{k_1 k_2}^k e^{ik \cdot x}. \quad (19)$$

In particular,

$$F_{k_1 k_2}^k = \tilde{F}_{k_1 k_2} \delta_{k_1 + k_2}^k, \quad \tilde{F}_{k_1 k_2} = q^\alpha q^\beta \epsilon^{\dot{\alpha} \dot{\beta}} k_{1\alpha\dot{\alpha}} k_{2\beta\dot{\beta}}. \quad (20)$$

For on-shell momenta, $k_1^2 = k_2^2 = 0$, the $\tilde{F}_{k_1 k_2}$ reduce to

$$\tilde{F}_{k_1 k_2} = -[q1][q2]\langle 12 \rangle. \quad (21)$$

The $F_{k_1 k_2}^k$ are in fact structure constants of an infinite-dimensional “kinematic” Lie algebra [21], namely the infinite-dimensional algebra $sdiff_2$ of infinitesimal area-preserving diffeomorphisms. To see this, we consider the symplectic form corresponding to (13):

$$\Omega = -\frac{1}{2} \epsilon_{\dot{\alpha}\dot{\beta}} dx^{\dot{\alpha}} \wedge dx^{\dot{\beta}}. \quad (22)$$

Any function $F(x^{\dot{\alpha}})$ gives rise to a Hamiltonian vector field L_F through $i_{L_F} \Omega = dF$, which preserves the area form Ω in the sense that $\mathcal{L}_{L_F} \Omega = 0$. The Lie bracket of the associated vector fields is

$$[L_F, L_G] = -L_{\{F, G\}}. \quad (23)$$

Therefore, the $F_{k_1 k_2}^k$ are structure constants of $sdiff_2$ in a basis associated to exponential functions. In another basis, associated to monomials, the structure constants reduce to those of the algebra $w_{1+\infty}$ [33]. Indeed, defining

$$w_m^{(i)} = (x^1)^{m+i-1} (x^2)^{i-1} \quad (24)$$

we have

$$\{w_m^{(i)}, w_n^{(j)}\} = ((j-1)m - (i-1)n) w_{m+n}^{(i+j-2)}. \quad (25)$$

The interpretation of $F_{k_1 k_2}^k$ as kinematic structure constants makes it clear that the self-dual Yang-Mills and Einstein equations in the form (17) and (18) are related by a type of color-kinematics duality.

III. ALL-ORDER PERTURBATIVE SOLUTIONS

In this section we construct perturbative solutions to the self-dual equations (16)–(18). We expand the prepotentials as

$$\Phi = \sum_{n=1}^{\infty} g^{n-1} \Phi_{(n)}, \quad \Psi = \sum_{n=1}^{\infty} \left(\frac{\kappa}{2}\right)^{n-1} \Psi_{(n)}. \quad (26)$$

We write the n -th order solution as an integral over momenta of a kernel as follows:

$$\Phi_{(n)}^k = K_{k_1 \dots k_n}^k \Phi_{(1)}^{k_1} \dots \Phi_{(n)}^{k_n} \quad (27)$$

$$\Phi_{(n)}^{ak} = \frac{1}{n!} K_{a_1 \dots a_n, k_1 \dots k_n}^{ak} \Phi_{(1)}^{a_1 k_1} \dots \Phi_{(n)}^{a_n k_n} \quad (28)$$

$$\Psi_{(n)}^k = \frac{1}{n!} \mathcal{K}_{k_1 \dots k_n}^k \Psi_{(1)}^{k_1} \dots \Psi_{(n)}^{k_n}. \quad (29)$$

In the Yang-Mills case we have defined both a color-stripped kernel $K_{k_1 \dots k_n}^k$ and a color-dressed one $K_{a_1 \dots a_n, k_1 \dots k_n}^{ak}$, and for later convenience we have included combinatorial factors in the last two definitions. To further simplify the formulas we will use the shorthand notation

$$K^k(1 \dots n) := K_{k_1 \dots k_n}^k \quad (30)$$

$$K^{ak}(1 \dots n) := K_{a_1 \dots a_n, k_1 \dots k_n}^{ak} \quad (31)$$

$$\mathcal{K}^k(1 \dots n) := \mathcal{K}_{k_1 \dots k_n}^k. \quad (32)$$

We will also use straightforward generalizations of these definitions where the argument $1 \dots n$ is replaced by a general word (i.e., a sequence of positive integer letters), and is linearly extended to combinations of words P, Q , i.e., $K^k(aP + bQ) = aK^k(P) + bK^k(Q)$.

The color-stripped kernel K^k must satisfy nontrivial properties which guarantee that the RHS of (27), which formally belongs to the universal enveloping algebra, in fact reduces to an element of the Lie algebra. These are the shuffle symmetries [34,35]

$$K^k(P \sqcup Q) = 0, \quad (33)$$

where the shuffle product of two words P and Q is the sum over all permutations of PQ which preserve the orderings of both words, for example $12 \sqcup 3 = 123 + 132 + 312$.

Both the color-dressed kernel K^{ak} and the gravity kernel $\mathcal{K}^k$ are symmetric under permutations of the letters, i.e.,

$$K^{ak}(\sigma \cdot P) = K^{ak}(P), \mathcal{K}^k(\sigma \cdot P) = \mathcal{K}^k(P), \quad (34)$$

where $\sigma \cdot P$ is an arbitrary permutation of the word P .

Substituting (27)–(29) in the Eqs. (16)–(18) leads to quadratic recursion relations for the kernels:

$$K^k(P) = \frac{2}{s_P} \sum_{QR=P} F_{lm}^k K^l(Q) K^m(R) \quad (35)$$

$$K^{ak}(\mathcal{P}) = \frac{1}{s_{\mathcal{P}}} \sum_{Q \cup \mathcal{R} = \mathcal{P}} f_{bc}^a F_{lm}^k K^{bl}(Q) K^{cm}(\mathcal{R}) \quad (36)$$

$$\mathcal{K}^k(\mathcal{P}) = \frac{1}{s_{\mathcal{P}}} \sum_{Q \cup \mathcal{R} = \mathcal{P}} \tilde{F}_{lm} F_{lm}^k \mathcal{K}^l(Q) \mathcal{K}^m(\mathcal{R}), \quad (37)$$

where $s_P = (k_{P_1} + \dots k_{P_{|P|}})^2$ and $|P|$ denotes the length of the word P . The starting values for the above recursions are

$$K^k(1) = \mathcal{K}^k(1) = \delta_{k_1}^k, \quad K^{ak}(1) = \delta_{a_1}^a \delta_{k_1}^k. \quad (38)$$

In the above expressions, Latin letters $P, Q, \dots$ stand for arbitrary words while calligraphic ones $\mathcal{P}, \mathcal{Q}, \dots$ denote ordered words, whose letters are in increasing order. The sum over $QR = P$ runs over all deconcatenations of P into nonempty words, while the sum over $Q \cup \mathcal{R} = \mathcal{P}$ runs over all de-shufflings, i.e., ways to distribute the letters of $\mathcal{P}$ into nonempty ordered words $\mathcal{Q}$ and $\mathcal{R}$. Some examples are

$$\begin{aligned} P = 132 &\Rightarrow (Q, R) = (1, 32), (13, 2) \\ \mathcal{P} = 123 &\Rightarrow (\mathcal{Q}, \mathcal{R}) = (12, 3), (13, 2), (23, 1), \\ &\quad (1, 23), (2, 13), (3, 12). \end{aligned}$$

The sums of the type $Q \cup \mathcal{R} = \mathcal{P}$ in the recursions for K^{ak} and $\mathcal{K}^k$ arise from averaging over the permutation group, guaranteeing that the kernels are fully symmetric [18].

The solution of the recursion (35) for the color-stripped Yang-Mills kernel K^k was found in [28,29] and takes the beautiful and simple form

$$K^k(1\dots n) = \frac{\delta_{1+\dots+n}^k}{S_{12} S_{23} \dots S_{n-1,n}}, \quad (39)$$

where we have defined

$$S_{12} := \frac{s_{12}}{2\tilde{F}_{12}} = -\frac{[12]}{[q1][q2]}. \quad (40)$$

To verify that this satisfies (35) one makes use of the identity

$$2 \sum_{m=1}^{n-1} \tilde{F}_{1+\dots+m, m+1+\dots+n} S_{m, m+1} = s_{1\dots n}. \quad (41)$$

Furthermore, we have used that $\Phi_{(1)}^k$ in (27)–(29) has support on the mass-shell $k^2 = 0$, so that the momenta k_i can be treated as on-shell momenta and the expression (21) for $\tilde{F}_{ij}$ can be used. One can also verify that the result (39) satisfies the shuffle symmetry (33), making use of the properties

$$S_{ij} + S_{ji} = 0, \quad S_{ij} + S_{jk} + S_{ki} = 0. \quad (42)$$

Our goal will be to derive, in analogy with (39), similarly explicit solutions to the recursions (36) and (37) for the color-dressed Yang-Mills kernel K^{ak} and the gravity kernel $\mathcal{K}^k$. It is instructive to first work them out for $n = 2, 3$. One finds that they can be written suggestively as

$$\begin{aligned} K^{ak}(12) &= f_{a_1 a_2}^a K^k(12) \\ K^{ak}(123) &= f_{a_1 a_2}^b f_{ba_3}^a K^k(123) + f_{a_1 a_3}^b f_{ba_2}^a K^k(132) \\ \mathcal{K}^k(12) &= F_{k_1 k_2}^k \tilde{\mathcal{K}}(12) \end{aligned} \quad (43)$$

$$\mathcal{K}^k(123) = F_{k_1 k_2}^l F_{lk_3}^k \tilde{\mathcal{K}}(123) + F_{k_1 k_3}^l F_{lk_2}^k \tilde{\mathcal{K}}(132), \quad (44)$$

where, as in (20), a tilde on a quantity means that we strip off an overall momentum-conserving delta function, e.g., $K^k(1\dots n) = \tilde{K}(1\dots n) \delta_{1+\dots+n}^k$. In (44) we used the Jacobi identities for the gauge and kinematic structure constants f_{bc}^a resp. F_{lm}^k . As it turns out, the pattern of (44) extends to all orders, and the explicit solutions for the kernels are

$$\begin{aligned} K^{ak}(1\dots n) &= \sum_{P \in S(2, \dots, n)} c^a(\mathcal{E}[1P]) K^k(1P) \\ \mathcal{K}^k(1\dots n) &= \sum_{P \in S(2, \dots, n)} n^k(\mathcal{E}[1P]) \tilde{\mathcal{K}}(1P). \end{aligned} \quad (45)$$

Here, the sum runs over all permutations of the letters $2, \dots, n$ and $c^a(\Gamma)$ and $n^k(\Gamma)$ are color resp. kinematic group factors associated to a Lie monomial Γ , i.e., a fully bracketed expression of integer-valued letters. The definition $c^a(\Gamma)$ and $n^k(\Gamma)$ becomes clear upon considering some examples:

$$\begin{aligned} c^a([1, 2], 3) &= f_{a_1 a_2}^b f_{ba_3}^a \\ n^k([1, 2], 3) &= F_{k_1 k_2}^l F_{lk_3}^k \\ c^a([1, 2], [3, 4]) &= f_{a_1 a_2}^b f_{a_3 a_4}^c f_{bc}^a \\ n^k([1, 2], [3, 4]) &= F_{k_1 k_2}^l F_{k_3 k_4}^m F_{lm}^k. \end{aligned}$$

Finally, the notation $\mathcal{E}[1\dots n]$ stands for the left-bracketed Lie monomial

$$\ell[1\dots n] := [\dots[[1, 2], 3]\dots n]. \quad (46)$$

More explicitly the group factors in (45) are given by

$$\begin{aligned} c^a(\ell[1\dots n]) &= f_{12}^{b_1} f_{b_1 3}^{b_2} \dots f_{b_{n-2} n}^a \\ n^k(\ell[1\dots n]) &= F_{12}^{l_1} F_{l_1 3}^{l_2} \dots F_{l_{n-2} n}^k \\ &= \delta_{1+\dots+n}^k \sum_{\{a_i\}, a_i \leq i} \tilde{F}_{12} \tilde{F}_{a_2 3} \dots \tilde{F}_{a_{n-1} n} \end{aligned}$$

Note in particular that the kinematic factors $n^k(\ell[1\dots n])$ are local, i.e., without poles in the momenta, since the $\tilde{F}_{ij}$ are local, see (21). From this and the K^k momentum dependence one sees that the momentum integrals in (27)–(29) do not suffer from soft divergences. In the Appendix we prove that the expressions (45) solve the recursions (36) and (37). The proof relies on elementary properties of Lie polynomials which are conveniently reviewed in [26]. The result (45) shows that the general perturbative solution to the self-dual Yang-Mills and gravity equations can be written in a BCJ-like form and are related by color-kinematics duality.

IV. DOUBLE-COPY FORM

In this section we rewrite our formula for the self-dual gravity solution in (45) in a manifestly double-copy form. To achieve this, we first express (45) in terms of combinatorial objects which were introduced in [26,36,37] (see also [31,38]) to solve recursion relations for Berends-Giele currents using the theory of Lie polynomials. Having obtained the double-copy form of the general classical solution, it is straightforward to derive from it double-copy relations for tree-level amplitudes, as will be discussed in the next section.

We start by introducing the following hierarchy of combinatorial objects:

$$b = \sum_{\Gamma} \frac{\Gamma \otimes \Gamma}{s_{\Gamma}} \quad (47)$$

$$b(P) = \sum_{\Gamma} \frac{(P, \Gamma) \Gamma}{s_{\Gamma}} \quad (48)$$

$$b(P|Q) = \sum_{\Gamma} \frac{(P, \Gamma)(Q, \Gamma)}{s_{\Gamma}} \quad (49)$$

Here, the sum runs over all Lie monomials Γ , and the inner product of words is defined as $(P, Q) = \delta_{P, Q}$ and extended to combinations of words by linearity.

Let us first discuss how these objects encode the combinatorial information contained in the integration kernels introduced in the previous section. We start with

the object $b(P)$ in (48), which can be seen to generate the binary tree graphs arising in the diagrammatic solution of the quadratic recursion (35). In particular it can be shown [26] that $b(q)$ satisfies the recursion relation

$$b(P) = \sum_{QR=P} [b(Q), b(R)], \quad b(i) = 1. \quad (50)$$

Comparing to (35) shows that $b(P)$ is related to the color-stripped kernel as

$$K^k(P) = 2^{|P|-1} n^k(b(P)). \quad (51)$$

Now let us discuss the meaning of b in (47). A basis of Lie monomials is furnished by the left-bracketed monomials (46), in terms of which any Lie polynomial of length n can be expanded as

$$\Gamma = \sum_{P \in S(2, \dots, n)} (1P, \Gamma) \ell[1P]. \quad (52)$$

We see from this that our expressions (45) can be written as

$$K^{ak}(1\dots n) = 2^{n-1} (c^a \otimes n^k)(b), \quad (53)$$

$$\mathcal{K}^k(1\dots n) = 2^{n-1} (\tilde{n} \otimes n^k)(b), \quad (54)$$

where we have defined, e.g., $(c^a \otimes n^k)(\Gamma_1 \otimes \Gamma_2) = c^a(\Gamma_1) n^k(\Gamma_2)$. This shows that b is the basic combinatorial object underlying the color-dressed and gravity kernels. The object $b(P|Q)$ in (49) can be shown to govern the color-stripped kernel of a biadjoint scalar theory, which we will not discuss here.

To write the gravity kernel in a double-copied form, we first perform a basis expansion of the Lie polynomial $b(P)$ in (51):

$$K^k(1P) = 2^{|P|} \sum_{Q \in S(2, \dots, n)} b(1P|1Q) n^k(\ell[1Q]). \quad (55)$$

As shown in [39], this relation can be inverted to obtain the kinematic numerator $n^p(\ell[1Q])$ by means of the Kawai-Lewellen-Tye (KLT) matrix $S(P|Q)_1$ [40,41]:

$$n^k(\ell[1P]) = 2^{-|P|} \sum_{Q \in S(2, \dots, n)} S(P|Q)_1 K^k(1Q). \quad (56)$$

Substituting in (45) we obtain a double-copy form for the gravity kernel, expressed in terms of two copies of the color-stripped Yang-Mills kernel and the KLT matrix:

$$\tilde{\mathcal{K}}(1\dots n) = 2^{1-n} \sum_{P, Q \in S(2, \dots, n)} \tilde{K}(1P) S(P|Q)_1 \tilde{K}(1Q). \quad (57)$$

To make the expression fully explicit, we recall from (39) and (40) that

$$\tilde{K}(1\dots n) = (-1)^{n-1} \frac{[q1][q2]^2\dots[q(n-1)]^2[qn]}{[12][23]\dots[(n-1)n]} \quad (58)$$

and that the KLT matrix is given by [41]

$$\begin{aligned} \tilde{K}(123) &= \frac{1}{4} (\tilde{K}(123)\tilde{K}(123)(s_{12}s_{13} + s_{12}s_{23}) + \tilde{K}(123)\tilde{K}(132)s_{12}s_{13} + \tilde{K}(132)\tilde{K}(123)s_{12}s_{13} \\ &\quad + \tilde{K}(132)\tilde{K}(132)(s_{13}s_{12} + s_{13}s_{23})) \\ &= \frac{1}{4} ((\tilde{K}(123))^2 s_{12}s_{23} + (\tilde{K}(312))^2 s_{31}s_{12} + (\tilde{K}(231))^2 s_{23}s_{31}), \end{aligned} \quad (60)$$

where in the last identity we used the shuffle symmetry $\tilde{K}(12 \sqcup 3) = 0$. It's straightforward to show that this last expression is equivalent to (44).

V. AMPLITUDE DOUBLE-COPY RELATIONS

Having derived that perturbative classical solutions exhibit a double-copy structure, we now directly link this to double-copy relations for amplitudes. The cleanest link between equations of motion and tree-level amplitudes is through the perturbiner expansion [14–17] (see [18,42] for a modern perspective), which yields the BG currents of any field theory from a perturbative solution to its equations of motion. The BG currents are tree-level form factors, i.e., amplitudes in which one particle is off-shell while all the others are on-shell, from which on-shell amplitudes are easily derived. We now show that our above results lead directly to known double-copy relations both for BG currents and on-shell amplitudes.

The perturbiner expansions for the self-dual Yang-Mills and gravity prepotentials take the form

$$\begin{aligned} \Phi^k &= \sum_P g^{|P|-1} J_P T^P \delta_{p_P}^k \\ &= \sum_i J_i T^{a_i} \delta_{p_i}^k + g \sum_{i,j} J_{ij} T^{a_i} T^{a_j} \delta_{p_{ij}}^k + \dots \end{aligned} \quad (61)$$

$$\begin{aligned} \Psi^k &= \sum_{\mathcal{P}} \left(\frac{\kappa}{2}\right)^{|\mathcal{P}|-1} \mathcal{J}_{\mathcal{P}} \delta_{p_{\mathcal{P}}}^k \\ &= \sum_i \mathcal{J}_i \delta_{p_i}^k + \frac{\kappa}{2} \sum_{i,j} \mathcal{J}_{ij} \delta_{p_{ij}}^k + \dots \end{aligned} \quad (62)$$

Here, the momenta p_i are on-shell, $p_i^2 = 0$, and the coefficients $J_i, J_{ij}, \mathcal{J}_i, \mathcal{J}_{ij}, \dots$ are assumed to be nilpotent, $J_i^2 = J_{ij}^2 = \mathcal{J}_i^2 = \mathcal{J}_{ij}^2 = \dots = 0$, in order to omit terms which do not contribute to the BG currents. The expansions (62) give rise to perturbiner expansions of the gauge potential and the metric through the relations (8) and (9):

$$S[i_1 \dots i_k | j_1 \dots j_k]_1 = \prod_{t=1}^k \left(s_{i_t,1} + \sum_{q>k} \theta(i_t, i_q) s_{i_t, i_q} \right), \quad (59)$$

where $\theta(i_t, i_q)$ is 0 if i_t sequentially comes before i_q in $\{j_1, \dots, j_k\}$, and 1 otherwise.

For example, for $n = 3$ the formula (57) yields

$$A_{\alpha\alpha}^k = \sum_P g^{|P|} A_{\alpha\alpha}^P T^P \delta_{p_P}^k, \quad h_{\alpha\alpha\beta\beta}^k = \sum_{\mathcal{P}} \frac{\kappa^{|\mathcal{P}|}}{2^{|\mathcal{P}|-1}} h_{\alpha\alpha\beta\beta}^{\mathcal{P}} \delta_{p_{\mathcal{P}}}^k, \quad (63)$$

with

$$A_{\alpha\alpha}^P = v_{\alpha\alpha}^P J_P, \quad h_{\alpha\alpha\beta\beta}^P = v_{\alpha\alpha}^P v_{\beta\beta}^P \mathcal{J}_P \quad (64)$$

and where

$$v_{\alpha\alpha}^P = i q_{\alpha} \sum_{i=1}^{|P|} [q P_i] \tilde{\mathfrak{p}}_{\alpha}^{P_i}. \quad (65)$$

Evaluating $\Phi_{(n)}^k, \Psi_{(n)}^k$ in (27) and (29) with the first order parts given by

$$\Phi_{(1)}^k = \sum_i J_i T^{a_i} \delta_{p_i}^k, \quad \Psi_{(1)}^k = \sum_i \mathcal{J}_i \delta_{p_i}^k \quad (66)$$

we readily find the currents in terms of the integration kernels

$$J_{i_1 \dots i_n} = J_{i_1} \dots J_{i_n} \tilde{K}_{k_{i_1} \dots k_{i_n}} \quad (67)$$

$$\mathcal{J}_{i_1 \dots i_n} = \mathcal{J}_{i_1} \dots \mathcal{J}_{i_n} \tilde{\mathcal{K}}_{k_{i_1} \dots k_{i_n}}. \quad (68)$$

We should remark that, while the kernels $\tilde{K}_{k_1 \dots k_n}$ and $\tilde{\mathcal{K}}_{k_1 \dots k_n}$ uniquely determine the currents, the converse is not true: due to the nilpotency of the J_i and $\mathcal{J}_i$ we could add arbitrary ‘contact terms’ with delta-function support on the locus where some of the momenta being equal, without changing the currents. In the self-dual case at hand, we have demonstrated that such contact terms are in fact absent and the kernels are simple analytic continuations of the currents; it would be interesting to understand when this is the case in general.

The canonically normalized BG currents are obtained from (68) by treating J_i and $\mathcal{J}_i$ now as c-numbers and substituting the appropriate values (10), i.e.,

$$J_i = -\frac{i}{[qi]^2}, \quad \mathcal{J}_i = -\frac{1}{[qi]^4}. \quad (69)$$

Upon making this substitution the BG currents compute the off-shell tree-level amplitudes

$$\langle 1^+ \dots n^+ A_{\alpha\dot{\alpha}}(k) \rangle_{\text{tree}} = A_{\alpha\dot{\alpha}}^{1\dots n} \delta_{1\dots n}^k, \quad (70)$$

$$\langle 1^{++} \dots n^{++} h_{\alpha\dot{\alpha}\beta\dot{\beta}}(k) \rangle_{\text{tree}} = h_{\alpha\dot{\alpha}\beta\dot{\beta}}^{1\dots n} \delta_{1\dots n}^k. \quad (71)$$

From (57) we obtain a double-copy relation for BG currents:

$$h_{\alpha\dot{\alpha}\beta\dot{\beta}}^{1\dots n} = 2^{1-n} \sum_{P,Q \in S(2,\dots,n)} A_{\alpha\dot{\alpha}}^{1P} S[P|Q]_1 A_{\beta\dot{\beta}}^{1Q}. \quad (72)$$

This relation was conjectured in [30] and recently proven in [31] using results from [38]. Once again we can obtain a more explicit formula by substituting the Yang-Mills currents:

$$A_{\alpha\dot{\alpha}}^{1\dots n} = \frac{i^{n+1} \mathbf{q}_\alpha \sum_{i=1}^n [qi] \tilde{\mathbf{p}}_\alpha^i}{[12][23] \dots [(n-1)n][q1][qn]}. \quad (73)$$

We note that this expression agrees with the one found in [15] using somewhat different methods.

Let us now discuss the double-copy relations for on-shell amplitudes which follow from the above. Equation (71) implies that from the self-dual BG currents we can compute the on-shell amplitudes:

$$\langle 1^+ \dots n^+ (n+1)^\pm \rangle_{\text{tree}} = \lim_{s_{1\dots n} \rightarrow 0} s_{1\dots n} A_{\alpha\dot{\alpha}}^{1\dots n} e_{\pm}^{\alpha\dot{\alpha}}(p_{n+1}) \delta_{1\dots n}^{n+1}$$

$$\langle 1^{++} \dots n^{++} (n+1)^{\pm\pm} \rangle_{\text{tree}} = \lim_{s_{1\dots n} \rightarrow 0} s_{1\dots n} h_{\alpha\dot{\alpha}\beta\dot{\beta}}^{1\dots n} e_{\pm\pm}^{\alpha\dot{\alpha}\beta\dot{\beta}}(p_{n+1}) \delta_{1\dots n}^{n+1}.$$

These can be evaluated using (5), (72), and (73). For $n > 2$, both the Yang-Mills and gravity amplitudes vanish due to the absence of a pole in $s_{1\dots n}$ in the combinations on the RHS. For $n = 2$ however we find the standard nonzero results

$$\langle 1^{+2+3-} \rangle = i \frac{\langle 12 \rangle [q3]^2}{[q1][q2]} = -i \frac{\langle 12 \rangle^3}{\langle 13 \rangle \langle 23 \rangle} \quad (74)$$

$$\langle 1^{++2++3--} \rangle = \langle 1^{+2+3-} \rangle^2. \quad (75)$$

In the first line we have used momentum-conservation relations such as $[q3]\langle 23 \rangle = -[q1]\langle 21 \rangle$, while the second line readily follows from (72) and the fact that $S(2|2)_1 = s_{12}$.

To summarize, though the self-dual classical double copy (45) leads to infinitely many nontrivial double-copy relations between BG currents (72), the three-point double-copy relation (75) is the only nonvanishing on-shell amplitude relation which is directly encoded in it. Nevertheless, the appearance of the KLT matrix, which governs double-copy relations of general amplitudes, in (45) suggests that self-dual solutions ‘know’ about more general on-shell amplitudes than (75). Indeed, more general amplitudes can be obtained by perturbing around (45) away from self-duality [43].

VI. DISCUSSION AND OUTLOOK

In this letter we have derived color-kinematics duality (45) and double-copy relations (57) in the perturbative structure of self-dual solutions. Let us first comment on the

connection with the more standard viewpoint of the classical double copy as an explicit map between gauge and gravity solutions. A canonical map between solutions exists only in the case where the gauge theory solution lives in a $u(1)$ subalgebra (and therefore linearizes), $\Phi = \Phi_{(1)} c^a T_a$. Then the unique double-copied solution is obtained from the linearized starting point $\Psi_{(1)} = \Phi_{(1)}$. We note that this gravity solution doesn’t linearize in general. Most of the literature focuses on special examples where the gravity solution itself also linearizes, such as the pp-wave [2] $\Psi_{(1)} = \Phi_{(1)} \sim e^{ipx}$ and the Eguchi-Hanson [44] $\Psi_{(1)} = \Phi_{(1)} \sim \frac{1}{x^2}$ solutions. For genuinely nonabelian Yang-Mills solutions, there is in general no unique double-copied gravity solution, since we can pick the component $\Phi_{(1)}^a$ along any color direction T_a to be the gravity perturbation $\Psi_{(1)}$.

In spite of this, one may wonder whether a truly nonabelian double copy might be defined for certain subclasses of Yang-Mills solutions. For this it would be necessary to embed their (kinematic $\oplus$ gauge) structure in the gravitational kinematic algebra $sdiff_2$. For example, if the gauge group is $SU(2)$ the generators can be embedded in the kinematic algebra as (complex) combinations of $w_1^{(2)}, w_0^{(2)}, w_{-1}^{(2)}$ given in (25). A Yang-Mills solution with first order components $\Phi_{(1)}^m$ can then be tentatively be double copied to a gravity solution with first order part $\Psi_{(1)} = \Phi_{(1)}^m w_m^{(2)}$. For this to make sense we have to further impose $\partial_{\alpha\dot{\alpha}} \Phi_{(1)}^m \partial^{\alpha\dot{\alpha}} w_m^{(2)} = 0$. At second order, the structure of the Yang-Mills (YM) solution (11) is reproduced if we

impose $\{\partial_{\dot{\alpha}}\Psi_{(1)}, \partial^{\dot{\alpha}}\Psi_{(1)}\} = 2\{\Phi_{(1)}^m, \Phi_{(1)}^n\}\{w_m^{(2)}, w_n^{(2)}\}$, and so on. It would be interesting to explore if this procedure can lead to nontrivial solutions.

Let us comment on some further open issues. Though the self-dual theories we have considered are integrable, this structure was not used directly in our approach. It would therefore be interesting to make contact with integrability techniques such as Bethe *Ansätze* as well as with the twistor space construction for self-dual gravity solutions [45] which underly the form of the self-dual perturbation solution presented in [17]. Finally, it would be of obvious interest to extend the present work beyond the self-dual sector.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: PROOF OF THE SOLUTION TO THE RECURSION RELATIONS

In this appendix, we will prove that the expression (45) for the self-dual gravity kernel,

$$\mathcal{K}^k(1\dots n) = \sum_{P \in S(2, \dots, n)} n^k(\mathcal{E}[1P])\tilde{K}(1P), \quad (\text{A1})$$

solves the recursion (37),

$$\mathcal{K}^k(1\dots n) = \frac{1}{S(1\dots n)} \sum_{Q \cup \mathcal{R} = 1\dots n} \tilde{F}_{lm} F_{lm}^k \mathcal{K}^l(Q) \mathcal{K}^m(\mathcal{R}). \quad (\text{A2})$$

The proof that the color-dressed Yang-Mills kernel in (45) solves (36) is completely analogous under replacing kinematic structure constants with gauge ones. We will work by induction in n . For $n = 2$, the solution of the recursion (43) agrees with (A1). Assuming that $\mathcal{K}^k(1\dots m)$ given by (A1) is correct for all $m < n$, we write the RHS of (A2) as

$$\frac{1}{S(1\dots n)} \sum_{1\dots n = Q \cup \mathcal{R}} \tilde{F}_{lm} F_{lm}^k \mathcal{K}^l(Q) \mathcal{K}^m(\mathcal{R}) = n^k(\Lambda), \quad (\text{A3})$$

where Λ is the following Lie polynomial with momentum-dependent coefficients

$$\Lambda = \frac{1}{S(1\dots n)} \sum_{Q \cup \mathcal{R} = 1\dots n} \sum_{\substack{Q \in S(Q_2, \dots, Q_{|Q|}) \\ R \in S(\mathcal{R}_2, \dots, \mathcal{R}_{|\mathcal{R}|})}} [\mathcal{E}[Q_1 Q], \mathcal{E}[\mathcal{R}_1 R]] \tilde{F}_{lm} K^l(Q_1 Q) K^m(\mathcal{R}_1 R). \quad (\text{A4})$$

Next, we expand Λ in the basis of left-bracketed monomials as

$$\Lambda = \sum_{P \in S(2, \dots, n)} (1P, \Lambda) \mathcal{E}[1, P], \quad (\text{A5})$$

where the inner product of words is defined as $(P, Q) \equiv \delta_{P, Q}$ and is extended to Lie polynomials by linearity. To prove (A1) it remains to show that $(1P, \Lambda) = \tilde{K}(1P)$, which proceeds in the following steps:

$$\begin{aligned} (1P, \Lambda) &= \frac{1}{S(1\dots n)} \sum_{XY=1P} \sum_{Q \cup \mathcal{R}=1\dots n} \sum_{\substack{Q \in S(Q_2, \dots, Q_{|Q|}) \\ R \in S(\mathcal{R}_2, \dots, \mathcal{R}_{|\mathcal{R}|})}} ((X, \mathcal{E}[Q_1 Q])(Y, \mathcal{E}[\mathcal{R}_1 R]) - (Y, \mathcal{E}[Q_1 Q])(X, \mathcal{E}[\mathcal{R}_1 R])) \\ &\quad \tilde{F}_{lm} K^l(Q_1 Q) K^m(\mathcal{R}_1 R) \\ &= \frac{2}{S(1\dots n)} \sum_{XY=1P} \tilde{F}_{lm} K^l(X) K^m \left(\sum_{R \in S(Y \setminus Y_{\min})} (Y, \mathcal{E}[Y_{\min} R]) Y_{\min} R \right) \\ &= \frac{2}{S(1\dots n)} \sum_{1P=XY} \tilde{F}_{lm} K^l(X) K^m(Y) \\ &= \tilde{K}(1P) \quad \blacksquare \end{aligned}$$

In the first equality, we have used the identity [see Eq. (A.10) in [26]]

$$(Z, [\Gamma_1, \Gamma_2]) = \sum_{XY=Z} ((X, \Gamma_1)(Y, \Gamma_2) - (X, \Gamma_2)(Y, \Gamma_1)). \quad (\text{A6})$$

In the second equality, we have used that the first letter of X is 1, so that $(X, \ell[\mathcal{Q}_1 Q])$ is nonvanishing only when $\mathcal{Q}_1 = 1$ (since $\mathcal{Q}_1 < \mathcal{Q}_i, \forall i$). Furthermore, from the identity $(1Z, \ell[1T]) = \delta_{Z,T}$ we see that the $(X, \ell[\mathcal{Q}_1 Q])$ $(Y, \ell[\mathcal{R}_1 R])$ terms in the sum are nonvanishing only when $\mathcal{Q}_1 Q = X$ and when $\mathcal{R}$ is obtained from Y by reordering its letters in increasing order. We denoted by $Y_{\min}$ the smallest

letter in Y . The $(Y, \ell[\mathcal{Q}_1 Q])(X, \ell[\mathcal{R}_1 R])$ terms give the same contribution leading to the factor of 2. In the third equality we have used that [see Eq. (2.10) in [26]]

$$\sum_{T \in S(2, \dots, n)} (Y, \ell[1T]) 1T = Y + \text{complete shuffles}, \quad (\text{A7})$$

(where the omitted terms are combinations of complete shuffles $A \sqcup B$), as well as the shuffle symmetry of K^l (33). Finally, in the last equality we used the recursion relation for $\tilde{K}$ which follows from (35).

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- [1] Z. Bern, J. J. Carrasco, M. Chiodaroli, H. Johansson, and R. Roiban, The duality between color and kinematics and its applications, *J. Phys. A* **57**, 333002 (2024).
 - [2] R. Monteiro, D. O'Connell, and C. D. White, Black holes and the double copy, *J. High Energy Phys.* **12** (2014) 056.
 - [3] C. D. White, *The Classical Double Copy* (World Scientific, Singapore, 2024).
 - [4] A. Luna, R. Monteiro, I. Nicholson, D. O'Connell, and C. D. White, The double copy: Bremsstrahlung and accelerating black holes, *J. High Energy Phys.* **06** (2016) 023.
 - [5] W. D. Goldberger and A. K. Ridgway, Radiation and the classical double copy for color charges, *Phys. Rev. D* **95**, 125010 (2017).
 - [6] Y. F. Bautista and A. Guevara, From scattering amplitudes to classical physics: Universality, double copy and soft theorems, [arXiv:1903.12419](#).
 - [7] C.-H. Shen, Gravitational radiation from color-kinematics duality, *J. High Energy Phys.* **11** (2018) 162.
 - [8] A. P. V. and A. Manu, Classical double copy from color kinematics duality: A proof in the soft limit, *Phys. Rev. D* **101**, 046014 (2020).
 - [9] A. Guevara, Gravity from a color symmetry II: Celestial color kinematics for mass and spin, [arXiv:2412.19641](#).
 - [10] K. Lee, Quantum off-shell recursion relation, *J. High Energy Phys.* **05** (2022) 051.
 - [11] H. Gomez, R. Lipinski Jusinkas, C. Lopez-Arcos, and A. Quintero Velez, One-loop off-shell amplitudes from classical equations of motion, *Phys. Rev. Lett.* **130**, 081601 (2023).
 - [12] H. Gomez, R. Lipinski Jusinkas, C. Lopez-Arcos, and A. Quintero Velez, One-loop N -point correlators in pure gravity, *Phys. Rev. Lett.* **134**, 111602 (2025).
 - [13] P. Chattopadhyay and K. Krasnov, One-loop same helicity YM amplitudes from BG currents, *J. High Energy Phys.* **03** (2022) 191.
 - [14] A. A. Rosly and K. G. Selivanov, What we think about multiparticle amplitudes, [arXiv:hep-th/9610070](#).
 - [15] A. A. Rosly and K. G. Selivanov, On amplitudes in selfdual sector of Yang-Mills theory, *Phys. Lett. B* **399**, 135 (1997).
 - [16] K. G. Selivanov, SD perturbator in Yang-Mills + gravity, *Phys. Lett. B* **420**, 274 (1998).
 - [17] A. A. Rosly and K. G. Selivanov, Gravitational SD perturbator, [arXiv:hep-th/9710196](#).
 - [18] S. Mizera and B. Skrzypek, Perturbator methods for effective field theories and the double copy, *J. High Energy Phys.* **10** (2018) 018.
 - [19] F. A. Berends and W. T. Giele, Recursive calculations for processes with n gluons, *Nucl. Phys.* **B306**, 759 (1988).
 - [20] A. Luna, R. Monteiro, I. Nicholson, A. Ochirov, D. O'Connell, N. Westerberg, and C. D. White, Perturbative spacetimes from Yang-Mills theory, *J. High Energy Phys.* **04** (2017) 069.
 - [21] R. Monteiro and D. O'Connell, The kinematic algebra from the self-dual sector, *J. High Energy Phys.* **07** (2011) 007.
 - [22] R. Bonezzi, F. Diaz-Jaramillo, and O. Hohm, The gauge structure of double field theory follows from Yang-Mills theory, *Phys. Rev. D* **106**, 026004 (2022).
 - [23] R. Bonezzi, C. Chiafrino, F. Diaz-Jaramillo, and O. Hohm, Gauge invariant double copy of Yang-Mills theory: The quartic theory, *Phys. Rev. D* **107**, 126015 (2023).
 - [24] R. Bonezzi, F. Diaz-Jaramillo, and S. Nagy, Gauge independent kinematic algebra of self-dual Yang-Mills theory, *Phys. Rev. D* **108**, 065007 (2023).
 - [25] L. Borsten, B. Jurco, H. Kim, T. Macrelli, C. Saemann, and M. Wolf, Double-copying self-dual Yang-Mills theory to self-dual gravity on twistor space, *J. High Energy Phys.* **11** (2023) 172.
 - [26] H. Frost, C. R. Mafra, and L. Mason, A Lie bracket for the momentum kernel, *Commun. Math. Phys.* **402**, 1307 (2023).
 - [27] J. F. Plebanski, Some solutions of complex Einstein equations, *J. Math. Phys. (N.Y.)* **16**, 2395 (1975).
 - [28] W. A. Bardeen, Self-dual Yang-Mills theory, integrability and multiparton amplitudes, *Prog. Theor. Phys. Suppl.* **123**, 1 (1996).
 - [29] D. Cangemi, Self-dual Yang-Mills theory and one loop like—helicity QCD multi—gluon amplitudes, *Nucl. Phys.* **B484**, 521 (1997).
 - [30] K. Cho, K. Kim, and K. Lee, The off-shell recursion for gravity and the classical double copy for currents, *J. High Energy Phys.* **01** (2022) 186.

- [31] D. H. Correa, C. Lopez-Arcos, and A. Quintero Velez, Tree- and one-loop-level double copy for the (anti)self-dual sectors of Yang-Mills and gravity, *Phys. Rev. D* **111**, 065001 (2025).
- [32] K. Krasnov, Self-dual gravity, *Classical Quantum Gravity* **34**, 095001 (2017).
- [33] I. Bakas, The large n limit of extended conformal symmetries, *Phys. Lett. B* **228**, 57 (1989).
- [34] F. A. Berends and W. T. Giele, Multiple soft gluon radiation in parton processes, *Nucl. Phys.* **B313**, 595 (1989).
- [35] S. Lee, C. R. Mafra, and O. Schlotterer, Non-linear gauge transformations in $D = 10$ SYM theory and the BCJ duality, *J. High Energy Phys.* **03** (2016) 090.
- [36] C. R. Mafra and O. Schlotterer, Berends-Giele recursions and the BCJ duality in superspace and components, *J. High Energy Phys.* **03** (2016) 097.
- [37] C. R. Mafra, Planar binary trees in scattering amplitudes, in *Algebraic Combinatorics, Resurgence, Moulds and Applications (CARMA)* (2020), pp. 349–365, 10.4171/205-1/6.
- [38] V. G. Escudero, C. Lopez-Arcos, and A. Quintero Velez, Homotopy double copy and the Kawai–Lewellen–Tye relations for the non-Abelian and tensor Navier–Stokes equations, *J. Math. Phys. (N.Y.)* **64**, 032304 (2023).
- [39] F. Cachazo, S. He, and E. Y. Yuan, Scattering of massless particles: Scalars, gluons and gravitons, *J. High Energy Phys.* **07** (2014) 033.
- [40] Z. Bern, L. J. Dixon, M. Perelstein, and J. S. Rozowsky, Multileg one loop gravity amplitudes from gauge theory, *Nucl. Phys.* **B546**, 423 (1999).
- [41] N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng, and T. Sondergaard, Gravity and Yang-Mills amplitude relations, *Phys. Rev. D* **82**, 107702 (2010).
- [42] H. Gomez and R. L. Jusinkas, Multiparticle solutions to Einstein’s equations, *Phys. Rev. Lett.* **127**, 181603 (2021).
- [43] N. Miller, Proof of the graviton MHV formula using Plebanski’s second heavenly equation, [arXiv:2408.11139](https://arxiv.org/abs/2408.11139).
- [44] D. S. Berman, E. Chacón, A. Luna, and C. D. White, The self-dual classical double copy, and the Eguchi-Hanson instanton, *J. High Energy Phys.* **01** (2019) 107.
- [45] R. Penrose, Nonlinear gravitons and curved twistor theory, *Gen. Relativ. Gravit.* **7**, 31 (1976).