Quantum contribution to domain wall tension from spectral methods

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In field theory, domain walls are constructed by embedding localized field configurations varying in one space dimension, such as the ϕ^4 kink, in two or three space dimensions. At the classical level, the kink mass straightforwardly turns into the energy per unit length or area, known as the domain wall tension. The quantum contribution to the tension is more difficult to compute, because the quantum fluctuations about the domain wall in the additional coordinates must be included. We show that spectral methods, making use of scattering data for the interaction of quantum fluctuations with the domain wall background, are an efficient way to compute the leading quantum correction to the domain wall tension. In particular we demonstrate that within this approach it is straightforward to pass from one renormalization scheme to another.

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I. INTRODUCTION

Nonlinear field theories in one space dimension with degenerate translationally invariant vacua typically allow for soliton (or solitary wave) solutions that mediate between various of these vacua as the spatial coordinate varies from negative to positive spatial infinity [1]. When such solitons are embedded in higher dimensions, they turn into domain walls that separate domains with different physical properties. In cosmology there may be domains with different vacuum expectation values of scalar fields like the Higgs boson [2,3]. Electric or magnetic materials may have regions with different polarization and/or magnetization [4,5], while in condensed matter or statistical physics, the domain walls may separate regimes of different phases [6]. For the many facets of domain walls in string theory, see, e.g., Chap. IV in Ref. [7]. The soliton energy becomes the energy per unit length or area when embedded in two or three space dimensions, respectively. These densities are frequently called tensions. Like the soliton energy, the tension has classical and quantum contributions.

The domain wall problem in cosmology is a particular motivation to compute quantum corrections to the tension. If the tension exceeds a certain limit, domain walls will dominate the Universe's energy and cause significant anisotropies [8]. Taking the mass of the scalar field fixed, the classical tension is proportional to the square of the vacuum expectation value of the scalar field and thus the domain wall problem sets an upper bound for this expectation value. The leading quantum correction to the tension only depends on the mass, so this correction will alter the bound. In particular, if the correction is negative, as is typical for kinks [9], this bound will be increased.

Here we will explore the vacuum polarization energy (VPE), the leading (one-loop) quantum correction, for n transverse coordinates. This approach will also allow us to regularize the ultraviolet divergent components by analytic continuation in n, and then renormalize them via standard counterterms. We will set up a general approach, which we will then apply to the kink and sine-Gordon solitons.

There have been previous studies of tensions of domain walls constructed from the kink and sine-Gordon solitons [10,11]. Those calculations involve multiplicative renormalizations of the classical soliton mass. This approach may fail when counterterm structures do not match the components of the classical mass, as is the case for the renormalization of the vacuum expectation value of the scalar field in the kink model. Our main focus will therefore be to show that spectral methods [12,13] can compute the tension in an efficient, constructive, and transparent manner, precisely implementing various renormalization conditions. In this way we will be able to examine the causes of the discrepancies in the earlier results, not only in magnitude but more importantly in sign. Since Ref. [11] has attributed this difference to the chosen renormalization scheme, it is important to analyze the VPE for domain walls using an approach that easily links different renormalization schemes.

In Sec. II we will briefly introduce the classical soliton models, and in Sec. III we introduce the interface formalism [14] to compute the VPE. In particular, we will consider the

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regularization prescription needed for finite results when $n = 2$, and also apply it to $n = 0$, 1 as a consistency check. In Sec. IV we will consider physically motivated renormalization schemes and compare to Refs. [10,11]. For the particular models investigated here, analytic results are available. In Sec. V we will use these results to express the VPE renormalized at an arbitrary scale, and we will give concluding remarks in Sec. VI.

II. THE MODELS

We start from scalar models defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 - U(\phi) \tag{1}$$

in $D = 1 + 1$ spacetime dimensions, with dot and prime denoting time and space derivatives of ϕ, respectively. The field potentials are

$$U_{\rm sG} = \frac{\mu^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}\phi}{\mu}\right)\right] \quad \text{and} \quad U_{\rm kink} = \frac{\lambda}{8}\left[\phi^2 - \frac{\mu^2}{\lambda}\right]^2 \tag{2}$$

for the sine-Gordon and kink models. Here μ is a mass parameter and λ is the interaction strength of the (classical) four-point function. These models have static, localized solutions [1]

$$\phi_{\rm cl,sG}(x) = \frac{4\mu}{\sqrt{\lambda}}\arctan(e^{-\mu x}) \quad \text{and} \quad \phi_{\rm cl,kink}(x) = \frac{\mu}{\sqrt{\lambda}}\tanh\left(\frac{\mu x}{2}\right) \tag{3}$$

that we will call solitons, even though strictly speaking $\phi_{\rm cl,kink}(x)$ is only a solitary wave. These solitons have classical energies

$$E_{\rm cl,sG} = 8\frac{\mu^3}{\lambda} \quad \text{and} \quad E_{\rm cl,kink} = \frac{2}{3}\frac{\mu^3}{\lambda}. \tag{4}$$

We will then embed these solitons into $D = (n + 1) + 1$ dimensions with $n = 0$, 1, 2, such that the configurations are translationally invariant in the n additional coordinates. The energies in Eq. (4) then turn into energies per unit length ($n = 1$) and area ($n = 2$). Note that λ has canonical energy dimension $2 - n$.

III. INTERFACE FORMALISM WITH TWO SUBTRACTIONS

Expanding the fields around the soliton as $\phi(t, x) = \phi_{\rm cl}(x) + \eta_k(x)e^{-i\omega t}$ with $\omega = \omega(k) = \sqrt{k^2 + \mu^2}$ turns the field equations into a scattering problem for the fluctuation $\eta_k(x)$. The respective potentials are

$$V_{\rm sG}(x) = -2\mu^2\,\mathrm{sech}^2\mu x \quad \text{and} \quad V_{\rm kink}(x) = -\frac{3\mu^2}{2}\,\mathrm{sech}^2\frac{\mu x}{2}. \tag{5}$$

From that scattering problem we determine the bound state solutions with energy eigenvalues $\omega_j = \sqrt{\mu^2 - \kappa_j^2}$ and phase shifts $\delta_\pm(k)$ in the symmetric and antisymmetric channels, which decouple because the potentials are invariant under $x \leftrightarrow -x$. The total phase shift is $\delta(k) = \delta_+(k) + \delta_-(k)$ and we denote the jth order of its Born expansion (power series in V) as $\delta_j(k)$.

The vacuum polarization energy is the renormalized sum of the differences between the zero-point energies of the fluctuations in the presence and absence of the soliton. This sum contains bound and scattering state contributions. The latter is computed as the continuum integral over the energies $\omega(k)$ weighted by the change of the density of states induced by the soliton. The central idea of spectral methods is to write this change as the momentum derivative of the total phase shift [15]. In the n trivial transverse coordinates the wave functions are simple plane waves. Their momenta are integrated over in dimensional regularization, as given by the bosonic case of Eq. (7) in Ref. [14] for $m = 1$,

$$\mathcal{E}^{(n)} = -\frac{\Gamma(-\frac{n+1}{2})}{2(4\pi)^{\frac{n+1}{2}}}\left[\sum_j^{\rm b.s.}\left(\omega_j^{n+1} - \mu^{n+1} + \frac{n+1}{2}\kappa_j^2\mu^{n-1}\right) + \int_0^\infty \frac{dk}{\pi}\left(\omega^{n+1}(k) - \mu^{n+1} - \frac{n+1}{2}k^2\mu^{n-1}\right) \times \frac{d}{dk}(\delta(k) - \delta_1(k) - \delta_2(k))\right] + E_{\rm FD} + E_{\rm CT}. \tag{6}$$

In what follows we will refer to $E_{\rm FD} + E_{\rm CT}$ as the perturbative part of the VPE. The two subtractions from $\omega^{n+1}(k)$ are identities from (generalizations of) Levinson's theorem and avoid infrared singularities at $n = 0$ and anomalous contributions at $n = 1$ [16]. The second subtraction, in particular, causes the residue of the pole at $n = 1$ to vanish. The associated sum rule is [17]

$$\int_0^\infty \frac{dk}{\pi}k^2\frac{d}{dk}(\delta(k) - \delta_1(k)) = \sum_j^{\rm b.s.}\kappa_j^2.$$

The Born subtractions render the momentum integral finite and are added back as Feynman diagrams yielding $E_{\rm FD}$. Together with the counterterm contribution, $E_{\rm CT}$, they make a finite contribution to the VPE. Equation (6) has been numerically verified in Ref. [16] for potentials like those in Eq. (5). However, we want to employ the imaginary momentum formulation, which has since then been observed to be more efficient in many aspects [13]. This formulation starts from recognizing that, for real non-negative momenta

k, the phase shift is the negative phase of the Jost function $F(k)$. The phase of the Jost function is odd for real k while the magnitude is even; hence $\delta(k) = \frac{\mathrm{i}}{2}[\ln F(k) - \ln F(-k)]$. Most importantly, the Jost function is analytic for $\mathrm{Im}(k) \geq 0$ with simple zeros at $k = \mathrm{i}\kappa_j$ [18]. Thus the logarithmic derivative has poles with unit residue, which in the contour integral exactly cancel the explicit bound state contribution in Eq. (6) [19]. Finally, the Born subtractions ensure that the integral along the semicircle at $|k| \to \infty$ vanishes. The only contributions arise from nonanalytic pieces in Eq. (6) contained in $\omega^{\frac{n+1}{2}}(k)$, and thus it is sufficient to consider the integral in

$$\mathcal{E}^{(n)} = -\frac{\Gamma(-\frac{n+1}{2})}{2(4\pi)^{\frac{n+1}{2}}}\int_{-\infty}^{\infty} dk(k^2+m^2)^{\frac{n+1}{2}}\frac{\mathrm{i}}{2}\frac{d}{dk}[\ln F(k)]_p + E_{\mathrm{FD}} + E_{\mathrm{CT}} + \ldots, \tag{7}$$

where the ellipsis denotes bound state contributions, which eventually will cancel with corresponding poles. Here the subscript indicates the subtractions of the first p terms of the Born expansion. Also, we did not write pieces that are analytic for $\mathrm{Im}(k) \geq 0$ as they do not contribute to the contour integral. For $n = 0$ and $n = 2$, the Γ-function is regular and the relevant discontinuities for $k = \mathrm{i}t$ with $t > \mu$ are

$$[(\mathrm{i}t + 0^+)^2 + \mu^2]^{\frac{n+1}{2}} - [(\mathrm{i}t - 0^+)^2 + \mu^2]^{\frac{n+1}{2}} = (-1)^n \mathrm{i}[t^2 - \mu^2]^{\frac{n+1}{2}}. \tag{8}$$

For $n \approx 1$ we encounter a pole $\Gamma(-\frac{n+1}{2}) \approx \frac{2}{n-1} + \cdots$. Then

$$\Gamma\left(-\frac{n+1}{2}\right)\omega^{\frac{n+1}{2}} = \left[\frac{2}{n-1} + \ldots\right]\left[\omega^2\left(1 + \frac{n-1}{2}\ln\frac{\omega^2}{\mu^2}\right)\right]. \tag{9}$$

We have introduced μ as the scale in the logarithm for convenience. It is arbitrary because it adds an analytic piece in the contour integral.[1] The only nonzero contribution to the contour integral then arises from the discontinuity

$$\ln\left[\left(\mathrm{i}\frac{t}{\mu} + \epsilon\right)^2 + 1\right] - \ln\left[\left(\mathrm{i}\frac{t}{\mu} - \epsilon\right)^2 + 1\right] = 2\pi\mathrm{i}. \tag{10}$$

Putting pieces together we find, with $p = 2$,

$$\mathcal{E}^{(0)} = \int_0^\infty \frac{d\tau}{2\pi}[\nu(t) - \nu_1(t) - \nu_2(t)]_{t=\sqrt{\tau^2+\mu^2}} + E_{\mathrm{FD}} + E_{\mathrm{CT}},$$
$$\mathcal{E}^{(1)} = \int_0^\infty \frac{d\tau}{4\pi}\tau[\nu(t) - \nu_1(t) - \nu_2(t)]_{t=\sqrt{\tau^2+\mu^2}} + E_{\mathrm{FD}} + E_{\mathrm{CT}},$$
$$\mathcal{E}^{(2)} = \int_0^\infty \frac{d\tau}{4\pi^2}\tau^2[\nu(t) - \nu_1(t) - \nu_2(t)]_{t=\sqrt{\tau^2+\mu^2}} + E_{\mathrm{FD}} + E_{\mathrm{CT}}, \tag{11}$$

where $\nu(t) = \ln F(\mathrm{i}t)$ and the subscripts on ν denote the order of its Born series. Since the potentials, Eq. (5), do not contain the coupling constant λ, the mass parameter μ sets the scale for the τ-integrals.

For $n = 0$ and $n = 1$, only the tadpole diagram is ultraviolet divergent and only the $\mathcal{O}(V)$ subtractions are needed. Then the no-tadpole (NT) renormalization condition gives $(E_{\mathrm{FD}} + E_{\mathrm{CT}})_{\mathrm{NT}} = 0$ and the VPE becomes

$$\mathcal{E}^{(0)}_{\mathrm{NT}} = \int_0^\infty \frac{d\tau}{2\pi}[\nu(t) - \nu_1(t)]_{t=\sqrt{\tau^2+\mu^2}} \quad \text{and}$$
$$\mathcal{E}^{(1)}_{\mathrm{NT}} = \int_0^\infty \frac{d\tau}{4\pi}\tau[\nu(t) - \nu_1(t)]_{t=\sqrt{\tau^2+\mu^2}}. \tag{12}$$

For the sine-Gordon and kink models we have analytic expressions for ν and ν_1 (but not ν_2), cf. Sec. V. We can then use Eq. (12) for $n = 0$ and $n = 1$. In the former case we get the well-known quantum corrections [1]

$$\mathcal{E}^{(0)}_{\mathrm{NT,sG}} = -\frac{\mu}{\pi} \approx -0.31831\mu \quad \text{and}$$
$$\mathcal{E}^{(0)}_{\mathrm{NT,kink}} = \mu\left(\frac{1}{4\sqrt{3}} - \frac{3}{2\pi}\right) \approx -0.33313\mu, \tag{13}$$

while in the latter case we obtain [20]

$$\mathcal{E}^{(1)}_{\mathrm{NT,sG}} = -\frac{\mu^2}{4\pi} \approx -0.079577\mu^2 \quad \text{and}$$
$$\mathcal{E}^{(1)}_{\mathrm{NT,kink}} = \frac{3\mu^2}{32\pi}(\ln 3 - 4) \approx -0.086582\mu^2. \tag{14}$$

The numerical computation of $\nu(t)$ and in particular $\nu_2(t)$ starts from the Jost solution $f(k,x)$ to the wave equation for the fluctuations. This solution behaves asymptotically like an outgoing plane wave. Parametrizing $f(k,x) = g(k,x)\mathrm{e}^{\mathrm{i}kx}$ and continuing $k = \mathrm{i}t$, the factor function is subject to the wave equation

$$g''(\mathrm{i}t, x) = 2tg'(\mathrm{i}t, x) + V(x)g(\mathrm{i}t, x) \tag{15}$$

with the boundary condition $\lim_{x\to\infty} g(\mathrm{i}t, x) = 1$. For spatially symmetric potentials, the scatting solutions are linear combinations of $f(k,x)$ and $f(-k,x)$ such that either the wave function or its derivative vanish at $x = 0$. The coefficients are the Jost functions $F_-(\mp k)$ and $F_+(\mp k)$, respectively. We then get [21]

[1] It also vanishes by the sum rule for the scattering data. This is the same argument because the sum rule arises from $F(k)$ being analytic.

$$\nu(t) = \lim_{x\to 0} \ln\left[g(\mathrm{i}t,x)\left(g(\mathrm{i}t,x) - \frac{1}{t}g'(\mathrm{i}t,x)\right)\right]. \quad (16)$$

We expand $g = 1 + g_1 + g_2 + \cdots$, where g_p is $\mathcal{O}(V^p)$. These functions vanish at spatial infinity and solve the differential equations

$$g_1''(\mathrm{i}t,x) = 2tg_1'(\mathrm{i}t,x) + V(x) \quad \text{and}$$
$$g_2''(\mathrm{i}t,x) = 2tg_2'(\mathrm{i}t,x) + V(x)g_1(\mathrm{i}t,x), \quad (17)$$

which lead to

$$\nu_1(t) + \nu_2(t) = \lim_{x\to 0}\left[2(g_1(\mathrm{i}t,x) + g_2(\mathrm{i}t,x)) - \frac{1}{t}(g_1'(\mathrm{i}t,x) + g_2'(\mathrm{i}t,x)) - \frac{1}{2}g_2^2(\mathrm{i}t,x) - \frac{1}{2}\left(g_2(\mathrm{i}t,x) - \frac{1}{t}g_2'(\mathrm{i}t,x)\right)^2\right]. \quad (18)$$

This completes the scattering part of the calculation. The perturbative part is most straightforwardly obtained from the one-loop part of the effective action $\mathcal{A}_{\text{eff}} \sim \frac{\mathrm{i}}{2}\mathrm{TrLog}[\partial^2 + \mu^2 - \mathrm{i}\epsilon + V]$. The $\mathcal{O}(V)$ contribution is fully canceled in the no-tadpole scheme, while the second-order contribution

$$\mathcal{A}_{\text{eff}} = -\frac{\mathrm{i}}{4}\mathrm{Tr}[(\partial^2 + \mu^2 - \mathrm{i}\epsilon)^{-1}V(\partial^2 + \mu^2 - \mathrm{i}\epsilon)^{-1}V] + \mathcal{O}(V^3) \quad (19)$$

contains the relevant piece to be added back in. Standard techniques yield for $n = 0$, 1

$$\mathcal{A}_{\text{eff}} = \frac{1}{2^n \cdot 16\pi}\int \frac{d^{n+2}p}{(2\pi)^{n+2}}\int_0^1 d\alpha \frac{|\tilde{V}(p)|^2}{[\mu^2 - \alpha(1-\alpha)p^2]^{1-\frac{n}{2}}} + \mathcal{O}(V^3), \quad (20)$$

where $\tilde{V}(p)$ is the Fourier transform of the potential in Eq. (15). For static potentials, only spacelike Fourier momenta contribute and the Feynman parameter integrals can be computed without any further restrictions, giving

$$E_{\text{FD}} = -\frac{\mu^{n+1}}{2^n \cdot 4\pi^2}\int_0^\infty ds\, v^2(s) I_n(s), \quad (21)$$

where $v(s) = \int_0^\infty dx\, \cos(sx)V(x)|_{\mu=1}$,

$$I_0(s) = -2\frac{\ln\left[1 + \frac{s^2}{2} - \frac{s}{2}\sqrt{4+s^2}\right]}{s\sqrt{4+s^2}}, \quad \text{and}$$
$$I_1(s) = \frac{2}{s}\arctan\left(\frac{s}{2}\right). \quad (22)$$

TABLE I. The VPE for $n = 0$ (left) and $n = 1$ (right) numerically computed according to Eq. (11) in the no-tadpole scheme. The entry labeled $\int d\tau\ldots$ denotes the integral contribution in Eq. (11), E_{FD} is from Eq. (21), and $\mathcal{E}_{\text{NT}}$ is their sum. For convenience we have set $\mu = 1$.

$n = 0$	Kink	sG
$\int d\tau\ldots$	−0.21728	−0.22210
E_{FD}	−0.11584	−0.09621
$\mathcal{E}_{\text{NT}}^{(0)}$	−0.33312	−0.31831

$n = 1$	Kink	sG
$\int d\tau\ldots$	−0.027806	−0.029196
E_{FD}	−0.058776	−0.050381
$\mathcal{E}_{\text{NT}}^{(1)}$	−0.086582	−0.079576

Numerical results from this formalism for the kink and sine-Gordon models are listed in Table I. As expected, we find perfect agreement with the exact results in Eqs. (13) and (14). This verifies the VPE calculation with imaginary momenta and two subtractions, which will become compulsory for $n = 2$. Not only are Eq. (11) much more compact than what can be extracted from Eq. (6), they also avoid the need to compute the bound state energies (although they are known analytically for the two examples considered). The formalism can be directly generalized to any spatially symmetric potential. If this symmetry is absent, one has to follow the treatment of Appendix B in Ref. [21].

We have also numerically computed the VPE according to Eq. (12) and indeed obtained agreement with the analytic results listed in Eqs. (13) and (14). However, in that approach the numerical cutoff on the τ integral had taken an order of magnitude larger, because $\nu(t) - \nu_1(t)$ approaches zero significantly more slowly than $\nu(t) - \nu_1(t) - \nu_2(t)$ when $t \to \infty$.

Having established the two-subtraction formalism, we may now proceed to the case with two transverse coordinates, $n \to 2$, i.e., $D = 3 + 1$. The momentum integral in Eq. (11) only requires a slight modification of the $n = 0$ and $n = 1$ cases. The essential novelty is the second-order term of the effective action, Eq. (20), since the first-order term is still absent in the no-tadpole scheme. In dimensional regularization the second-order term reads

$$\mathcal{A}_{\text{eff}} = \frac{\Lambda^{2-n}}{4(4\pi)^{\frac{n}{2}+1}}\Gamma\left(1-\frac{n}{2}\right)\int\frac{d^{n+2}p}{(2\pi)^{n+2}}\int_0^1 d\alpha\,\frac{|\tilde{V}(p)|^2}{[\mu^2-\alpha(1-\alpha)p^2]^{1-\frac{n}{2}}}+\mathcal{O}(V^3)$$
$$= \frac{1}{64\pi^2}\int\frac{d^4p}{(2\pi)^4}|\tilde{V}(p)|^2\left[C_n-\int_0^1 d\alpha\,\ln\left(1-\alpha(1-\alpha)\frac{p^2}{\mu^2}\right)\right]+\mathcal{O}(V^3) \tag{23}$$

for $n \lesssim 2$, where $C_n = \frac{2}{2-n} - \gamma + \ln\frac{4\pi\Lambda^2}{\mu^2}$ and we have dropped terms of order $n-2$ in the second equation. We have introduced the scale Λ such that the loop integral has the same dimensions as for $n=2$. Since $\int\frac{d^4p}{(2\pi)^4}|\tilde{V}(p)|^2 = \int d^4x V^2(x)$ we can cancel the ultraviolet divergence in C_n by adding a Lagrangian counterterm $\mathcal{L}_{\text{CT}} \propto C_n V^2(x)$, which is the $\overline{\text{MS}}$ renormalization scheme. For the kink model $V^2 \propto (\phi^2 - v^2)^2$, and this counterterm renormalizes the coupling constant λ. For the sine-Gordon model, V^2 is not part of the original Lagrangian; the model is not renormalizable in $D = 3+1$ dimensions since the only available nonderivative counterterm is proportional to V [22]. As a matter of completeness we can nevertheless compute the correspondingly subtracted quantity by omitting the C_n piece. For brevity and convenience we call this treatment of the sine-Gordon model an $\overline{\text{MS}}$ scheme as well. Carrying out the Feynman parameter integral and substituting the soliton profile yields

$$(E_{\text{FD}} + E_{\text{CT}})_{\overline{\text{MS}}} = \frac{\mu^3}{8\pi^3}\int_0^\infty ds\, v^2(s)[I_2(s)-1] \quad \text{with}$$
$$I_2(s) = \frac{1}{s}\sqrt{4+s^2}\,\text{arsinh}\left(\frac{s}{2}\right) \tag{24}$$

for the perturbative part of the VPE per unit area. A consistency check is that the expression in square brackets in Eq. (24) vanishes at $s=0$ in the $\overline{\text{MS}}$ scheme. Our numerical results are presented in Table II. Within this renormalization scheme, the perturbative part of the VPE tends to slightly reduce the magnitude of the quantum correction.

TABLE II. The VPE for $n=2$ numerically computed according to Eq. (11), which implements the $\overline{\text{MS}}$-no-tadpole scheme for the kink model. For the sine-Gordon model, which is not renormalizable in this dimension, it implements the analogous subtractions. The entry labeled $\int d\tau\ldots$ denotes the integral contribution in Eq. (11), $E_{\text{FD}} + E_{\text{CT}}$ is from Eq. (24), and $\mathcal{E}^{(2)}_{\overline{\text{MS}}}$ is their sum. For convenience we have set $\mu = 1$.

	Kink	sG
$\int d\tau\ldots$	-6.0403×10^{-3}	-6.5534×10^{-3}
$E_{\text{FD}} + E_{\text{CT}}$	0.2974×10^{-3}	0.9246×10^{-3}
$\mathcal{E}^{(2)}_{\overline{\text{MS}}}$	-5.7429×10^{-3}	-5.6288×10^{-3}

IV. PHYSICALLY MOTIVATED SCHEMES FOR KINK

In Ref. [10] the surface tensions for the kink were computed for several physically motivated renormalization schemes. Those authors were able to express both the phase shift and the counterterm coefficients in terms of hypergeometric functions of the number of transverse dimensions, which is specific to the kink model.

Different schemes have different conditions on the coefficients in the counterterm Lagrangian, but the momentum integrals in Eq. (11) are unaffected. This is the great advantage of our approach: we only need to adjust $E_{\text{FD}} + E_{\text{CT}}$. The authors of Ref. [10] consider four different sets of renormalization conditions that they label MR, OS, ORS, and ZM. All four implement a no-tadpole scheme, but differ by the conditions on the two-point function for the quantum fluctuations about the translationally invariant vacuum.

The minimal renormalization (MR) is the pure no-tadpole scheme. It is only applicable for $n=0$ and $n=1$, and corresponds to our results in Table I.

The general counterterm Lagrangian in the kink model is

$$\mathcal{L}_{\text{CT}} = \frac{c_0}{2}\partial_\mu\phi\partial^\mu\phi + \frac{c_1}{8}(\phi^2-v^2)^2 + \frac{c_2}{2}(\phi^2-v^2), \tag{25}$$

with $v^2 = \frac{\mu^2}{\lambda}$. Note that c_0 is finite at one-loop order for $n \le 2$. In the no-tadpole scheme, we ignore both the $\mathcal{O}(V)$ term in the expansion of $\mathcal{A}_{\text{eff}}$ and the c_2 term above. To determine c_0 and c_1, the authors of [10] imposed conditions on the polarization functions for fluctuation about the translationally invariant vacuum, $\phi(x) = v + h(x)$. With $V = 3\mu\sqrt{\lambda}h + \cdots$ we need the $\mathcal{O}(V^2)$ quantum contribution, since this term contains the only $\mathcal{O}(h^2)$ contribution in the no-tadpole scheme. We renormalize by adding the $\mathcal{O}(h^2)$ pieces of $\int d^Dx\mathcal{L}_{\text{CT}}$,

$$\mathcal{A}_{\text{ren,eff}} = \mu^2\lambda\int\frac{d^{n+2}p}{(2\pi)^{n+2}}|\tilde{h}(p)|^2\Pi_h(p^2) + \mathcal{O}(h^3), \tag{26}$$

with the polarization function

$$\Pi_h(p^2) = -\frac{9\mathrm{i}}{4}\int \frac{d^{n+2}l}{(2\pi)^{n+2}}\int_0^1 \frac{d\alpha}{[l^2-\mu^2+\mathrm{i}\epsilon+\alpha(1-\alpha)p^2]^2} + \frac{c_0}{2\mu^2\lambda}p^2 + \frac{c_1}{2\lambda^2}, \tag{27}$$

where the h subscript denotes the insertion of this field at the vertices of the two-point function. The on-shell scheme (OS) sets $c_0 = 0$ and determines c_1 from $\Pi_h(\mu^2) = 0$. In our formulation, this corresponds to

$$(E_{\mathrm{FD}} + E_{\mathrm{CT}})_{\mathrm{OS}} = K_n\mu^{n+1}\int_0^\infty ds\, v^2(s)[I_n(s) - I_n], \tag{28}$$

with the coefficients $K_0 = -\frac{1}{4\pi}$, $K_1 = -\frac{1}{8\pi^2}$, and $K_2 = \frac{1}{8\pi^3}$ read off from Eqs. (22) and (24). The subtractions are $I_0 = \frac{2\pi}{3\sqrt{3}}$, $I_1 = \ln 3$, and $I_2 = \frac{\pi}{2\sqrt{3}}$. By construction, $I_n(\mathrm{i}) = I_n$.

The on-shell-residue (OSR) scheme augments the OS conditions by requiring that the residue of the propagator does not have any quantum correction. This condition changes the above c_1 by $-c_0\lambda$, and extracts c_0 from $\frac{\partial\Pi_h(\mu^2)}{\partial\mu^2} = 0$. In total, the constants added to the above are

$$(E_{\mathrm{FD}} + E_{\mathrm{CT}})_{\mathrm{OSR}} = (E_{\mathrm{FD}} + E_{\mathrm{CT}})_{\mathrm{OS}} + \Delta\tilde{\mathcal{E}}^{(n)}, \tag{29}$$

where $\Delta\tilde{\mathcal{E}}^{(0)} = \frac{\sqrt{3}\pi-9}{18\pi}\mu$, $\Delta\tilde{\mathcal{E}}^{(1)} = \frac{1}{16\pi}(3\ln 3 - 4)\mu^2$, and $\Delta\tilde{\mathcal{E}}^{(2)} = \frac{3\sqrt{3}-2\pi}{16\sqrt{3}\pi}\mu^3$.

Finally, the zero mass (ZM) condition requires that $\Pi_h(p^2) = \sum_{l=2} a_l(p^2)^l$. This condition determines c_1 and c_0 from $\Pi_h(0)$ and $\frac{\partial\Pi_h(p^2)}{\partial p^2}|_{p^2=0}$, respectively. The perturbative part of the VPE is then

TABLE III. Domain wall tension for various renormalization schemes described in the text for $\mu = 1$.

n	$\mathcal{E}^{(n)}_{\mathrm{OS}}$	$\mathcal{E}^{(n)}_{\mathrm{OSR}}$	$\mathcal{E}^{(n)}_{\mathrm{ZM}}$
0	−0.18878	−0.25171	−0.23365
1	-2.1013×10^{-2}	-3.5022×10^{-2}	-3.1872×10^{-2}
2	-3.9742×10^{-3}	-7.9485×10^{-3}	-7.3260×10^{-3}

$$(E_{\mathrm{FD}} + E_{\mathrm{CT}})_{\mathrm{ZM}} = K_n\mu^{n+1}\int_0^\infty ds\, v^2(s)[I_n(s) - 1] + \Delta\tilde{\mathcal{E}}_0^{(n)}, \tag{30}$$

with $\Delta\tilde{\mathcal{E}}_0^{(0)} = -\frac{1}{16\pi}\mu$, $\Delta\tilde{\mathcal{E}}_0^{(1)} = -\frac{1}{64\pi}\mu^2$, and $\Delta\tilde{\mathcal{E}}_0^{(2)} = -\frac{1}{64\pi^2}\mu^3$. It is also possible to derive an analytic expression for the integral $\int_0^\infty ds\, v^2(s)$ and combine those pieces with $\Delta\tilde{\mathcal{E}}^{(n)}$ or $\Delta\tilde{\mathcal{E}}_0^{(n)}$. We will use this approach in Sec. V, but here we prefer the above formulation because $I_n(\mathrm{i}) = I_n$ and $I_n(0) = 1$. Our numerical results are listed in Table III. In all cases considered we agree with the results presented in Ref. [10].

The renormalization scheme of Ref. [11] only includes counterterms that are compulsory for $n = 2$:

$$\mathcal{L}_{\mathrm{CT}} = \frac{c_1}{8}(\phi^2 - v^2)^2 + \frac{c_2}{2}(\phi^2 - v^2). \tag{31}$$

That scheme has a condition on the three-point function, so we expand the quantum correction to the effective action up to third order in V to collect all contributions $\mathcal{O}(h^3)$ that build the three-point function. With no contribution linear in h (since it would lead to a quantum correction for the VEV), this expansion yields

$$\mathcal{A}_{\mathrm{ren,eff}} \sim \int \frac{d^{n+2}p}{(2\pi)^{n+2}}|\tilde{h}(p)|^2\Pi_h(p^2) + \int \frac{d^{n+2}p}{(2\pi)^{n+2}}\int \frac{d^{n+2}q}{(2\pi)^{n+2}}\tilde{h}(p)\tilde{h}(q)\tilde{h}(-p-q)\Gamma_3(p,q) + \mathcal{O}(h^4). \tag{32}$$

Without the no-tadpole condition, the polarization function

$$-\frac{3\mu^2\lambda}{4(4\pi)^{\frac{n}{2}+1}}\Gamma\left(-\frac{n}{2}\right)(\mu^2)^{\frac{n}{2}-1} + \frac{9\mu^2\lambda}{4(4\pi)^{\frac{n}{2}+1}}\Gamma\left(1-\frac{n}{2}\right)\int_0^1 d\alpha[\mu^2 - \alpha(1-\alpha)p^2]^{\frac{n}{2}-1} + \frac{c_2}{2} + \frac{c_1}{2}\frac{\mu^2}{\lambda} \tag{33}$$

has two ultraviolet divergent contributions, one from $\mathcal{O}(V)$ and another one from $\mathcal{O}(V^2)$. The (amputated) three-point function is

$$\Gamma_3(p,q) = \frac{9}{4}\frac{\mu\lambda^{3/2}}{(4\pi)^{\frac{n}{2}+1}}\Gamma\left(1-\frac{n}{2}\right)\int_0^1 d\alpha[\mu^2 - \alpha(1-\alpha)p^2]^{\frac{n}{2}-1} - \frac{9}{2}\mu^3\sqrt{\lambda}^3\mathrm{i}\int \frac{d^{n+2}l}{(2\pi)^{n+2}}\frac{1}{l^2-\mu^2+\mathrm{i}\epsilon}\frac{1}{(l-p)^2-\mu^2+\mathrm{i}\epsilon}\frac{1}{(l-q)^2-\mu^2+\mathrm{i}\epsilon} + \frac{c_1}{2}\frac{\mu}{\sqrt{\lambda}}. \tag{34}$$

It has contributions from $\mathcal{O}(V^2)$ and from $\mathcal{O}(V^3)$. In this language the renormalization conditions of Ref. [11] are

$$\Pi_h(\mu^2) = 0 \quad \text{and} \quad \Gamma_3(0,0) = 0, \tag{35}$$

which we refer to as the MG_3 scheme since it is specified via the mass and the three-point Green's function. In general, the three-point function is a complicated object to calculate [23]; fortunately not so for vanishing external momenta. Since there are two Born subtractions in Eq. (11), we can read off the Feynman diagram contributions to be added back from the quantum contribution to the effective action that is second order in V

$$\begin{aligned}\mathcal{A}_{\text{ren,eff}} = &\frac{\mu^{n-2}}{4}\frac{\Gamma(1-\frac{n}{2})}{(4\pi)^{\frac{n}{2}+1}}\int\frac{d^{n+2}p}{(2\pi)^D}|\tilde{V}(p)|^2\int_0^1 d\alpha\left\{\left[1-\alpha(1-\alpha)\frac{p^2}{\mu^2}\right]^{\frac{n}{2}-1}-1\right\}\\ &+\int d^{n+2}x\left[\frac{\bar{c}_1}{8}(\phi^2-v^2)^2+\frac{\bar{c}_2}{2}(\phi^2-v^2)\right]+\mathcal{O}(V^3)\end{aligned} \tag{36}$$

with

$$\begin{aligned}\bar{c}_1 &= \frac{9\lambda^2\mu^{n-2}}{2(4\pi)^{\frac{n}{2}+1}}\Gamma\left(2-\frac{n}{2}\right) \quad \text{and}\\ \bar{c}_2 &= \frac{9}{2}\frac{\lambda\mu^n}{(4\pi)^{\frac{n}{2}+1}}\left\{\Gamma\left(1-\frac{n}{2}\right)\left[1-\int_0^1 d\alpha(1-\alpha(1-\alpha))^{\frac{n}{2}-1}\right]-\Gamma\left(2-\frac{n}{2}\right)\right\}.\end{aligned} \tag{37}$$

The term with $\Gamma(1-\frac{n}{2})$ in $\bar{c}_2$ cancels the Feynman parameter integral for $p^2=\mu^2$ in Eq. (36) at $\mathcal{O}(h^2)$. For $n=0$ and $n=1$, we may omit the "1" in both of these terms, but we keep it to illustrate that both terms are finite as $n\to 2$. The terms with $\Gamma(2-\frac{n}{2})$ in $\bar{c}_1$ and $\bar{c}_2$ cancel the contributions $\mathcal{O}(h^2)$ in the counterterms, so we have implemented $\Pi_h(\mu^2)=0$. Finally, we note that the $\Gamma(2-\frac{n}{2})$ term in $\bar{c}_1$ cancels the h^3 term originating from $\mathcal{O}(V^3)$ in $\mathcal{A}_{\text{eff}}$ for zero external momenta. Since the curly bracket in Eq. (36) vanishes for $p^2=0$, we indeed have $\Gamma_3(0,0)=0$.

The momentum integral in Eq. (36) leads to the Feynman diagrams that contribute to the VPE as in Eq. (28) with $I_n=1$, while the finite counterterms give

$$\begin{aligned}(\Delta E_{\text{CT}})_{\text{MG}_3} &= -\frac{\bar{c}_1}{4}\int_0^\infty dx[v^2-\phi^2]^2+\bar{c}_2\int_0^\infty dx[v^2-\phi^2]\\ &= -\frac{\bar{c}_1}{3}\frac{\mu^3}{\lambda^2}+2\bar{c}_2\frac{\mu}{\lambda}.\end{aligned} \tag{38}$$

TABLE IV. Results for the renormalization scheme of Ref. [11] with $\mu=1$. The last column lists the VPE prediction from that reference. Here $\bar{c}_1$ and $\bar{c}_2$ are calculated with $\lambda=1$, while the λ dependence cancels in the remaining quantities.

n	$\bar{c}_1$	$\bar{c}_2$	ΔE_{CT}	$\mathcal{E}^{(n)}_{\text{MG}_3}$	Ref. [11]
0	$\frac{9}{8\pi}$	$-\frac{\sqrt{3}}{4}$	−0.98539	−1.19915	0.38856
1	$\frac{9}{32\pi}$	$\frac{9}{32\pi}(1-2\ln(3))$	−0.24420	−0.27110	0.12189
2	$\frac{9}{32\pi^2}$	$\frac{9}{32\pi^2}(\frac{\pi}{\sqrt{3}}-3)$	−0.07710	−0.08285	0.04110

Note that ΔE_{CT} will have an overall factor of μ^{n+1} and no λ dependence. Results are presented in Table IV. Obviously there are significant discrepancies, both in sign and magnitude, compared to the results of Ref. [11], which does not substitute the soliton into the counterterm Lagrangian. Rather the multiplicative renormalization of μ and λ is substituted into the expression for the classical energy. However, the (finite) $\bar{c}_2$-type counterterm is not part of the classical Lagrangian; it enters only via $v^2 \longrightarrow v^2+\Delta v^2$ at $\mathcal{O}(\Delta v^2)$. We note that Ref. [10] applies a similar multiplicative procedure for the finite wave function renormalization without explicitly introducing the c_0 type counterterm listed in Eq. (25). Doing so does not cause a problem in that case, however, because $\int dx\,\phi'^2(x) = 2\int dx\,U(\phi(x)) = \frac{1}{9\lambda}\int dx\,V^2(\phi(x))$ for a soliton solution according to Derrick's theorem [24]. That is, for the soliton the spatial integral of the c_0-type counterterm is that of the c_1-type counterterm.

As our general calculation shows, the spectral method approach is straightforwardly applicable even in cases where there is no analytic expression for the fluctuation potential and/or the Jost function. For the kink and sine-Gordon models, however, we can use exact results to avoid the need for numerical simulations, as we will describe in the following section.

V. GENERAL RENORMALIZATION SCHEME IN DIMENSIONAL REGULARIZATION

In the preceding calculation, we have subtracted the full diagram contributions and added them back in combination with renormalization counterterms. By using dimensional

regularization, however, we can shortcut this process by subtracting the counterterm directly. (For the tadpole graph, these subtractions are identical because the diagram is local.) This approach allows us to carry out the full calculation analytically, at a general renormalization scale M, with $M = \mu$ and $M = 0$ corresponding to the OS and $\overline{\text{MS}}$ or ZM schemes above.

For general transverse dimension n, closing the contour for the integral in Eq. (7) and using

$$\Omega^{n+1}(\mathrm{i}^{n+1} - (-\mathrm{i})^{n+1}) = 2\mathrm{i}\Omega^{n+1}\sin\frac{(n+1)\pi}{2}, \tag{39}$$

where $\Omega = \sqrt{t^2 - \mu^2}$, along with

$$\sin\pi z = -\frac{\pi}{\Gamma(z+1)\Gamma(-z)} \tag{40}$$

yields [25]

$$\mathcal{E}^{(n)}_{\text{NT}} = -\frac{1}{2(4\pi)^{\frac{n+1}{2}}\Gamma(\frac{n+3}{2})}\int_\mu^\infty (t^2-\mu^2)^{\frac{n+1}{2}}\frac{\partial}{\partial t}[\nu(t) - \nu_1(t)]dt \tag{41}$$

for the VPE in the no-tadpole formulation with one subtraction, where $\nu(t)$ is given by Eq. (16). As before, this expression denotes the VPE per L^n, which is the generalized volume of the trivial coordinates.

We can write both scattering potentials in the general form

$$V(x) = -\frac{\ell+1}{\ell}\mu^2\,\mathrm{sech}^2\frac{\mu x}{\ell}, \tag{42}$$

with $\ell = 1$ for the sine-Gordon soliton and $\ell = 2$ for the kink. These are exactly solvable Pöschl-Teller potentials, with

$$g^{\text{sG}}(k,x) = \frac{k + \mathrm{i}\mu\tanh\mu x}{k+\mathrm{i}\mu},$$
$$g^{\text{kink}}(k,x) = \frac{1}{k+\mathrm{i}\mu}\frac{1}{k+\mathrm{i}\frac{\mu}{2}}\left(\frac{\mu^2}{4} + k^2 + \frac{3}{2}\mathrm{i}\mu k\tanh\frac{\mu x}{2} - \frac{3}{4}\mu^2\tanh^2\frac{\mu x}{2}\right). \tag{43}$$

Similarly, the first Born approximation is given by [16]

$$\nu_1(t) = 2g_1(\mathrm{i}t,0) - \frac{1}{t}g_1'(\mathrm{i}t,0) \tag{44}$$

with

$$g_1(k,x) = \frac{\mathrm{i}}{2k}\int_x^\infty (1 - e^{2\mathrm{i}k(y-x)})V(y)dy$$
$$= \mathrm{i}(\ell+1)\left[\ell\pi e^{-2\mathrm{i}kx}\,\mathrm{csch}\frac{\ell k\pi}{\mu} - \frac{\mu}{k}\,{}_2F_1\left(1, \frac{\mathrm{i}\ell k}{\mu}, 1 + \frac{\mathrm{i}\ell k}{\mu}, -e^{\frac{2\mu x}{\ell}}\right)\right]. \tag{45}$$

Since $V(-x) = V(x)$ this gives

$$\nu_1(t) = \frac{\langle V\rangle}{2t} = -(\ell+1)\frac{\mu}{t}, \tag{46}$$

written in terms of the average value of the potential per unit length or area,

$$\langle V(x)\rangle = \int_{-\infty}^\infty V(x)dx = -2\mu(\ell+1). \tag{47}$$

For $n = 0$ and $n = 1$, subtracting the tadpole graph is sufficient to renormalize the theory. For these cases we have the standard results given in Eqs. (13) and (14). The key observation for other renormalization schemes is that we can write the loop integral in the polarization function, Eq. (27), for $n < 2$ as [25]

$$\Pi_V(M^2) = -\frac{\mathrm{i}}{4}\int_0^1 d\alpha \int \frac{dE}{2\pi}\frac{d^{n+1}q}{(2\pi)^{n+1}}\frac{1}{(E^2 - q^2 - \mu^2 + M^2\alpha(1-\alpha) + \mathrm{i}\epsilon)^2}$$
$$= \frac{1}{2(4\pi)^{\frac{n+1}{2}}\Gamma(\frac{n+1}{2})}\int_0^\infty \frac{q^n}{\omega(4\omega^2 - M^2)}dq \quad \text{with} \quad \omega = \sqrt{q^2+\mu^2}$$
$$= -\frac{1}{2(4\pi)^{\frac{n+1}{2}}\Gamma(\frac{n+3}{2})}\int_\mu^\infty (t^2-\mu^2)^{\frac{n+1}{2}}\frac{\partial}{\partial t}\left(\frac{1}{2t}\cdot\frac{1}{4t^2 - M^2}\right)dt, \tag{48}$$

in terms of the arbitrary renormalization scale $M^2 < 4\mu^2$. Here the vacuum polarization is written in term of vertices with with potential V, and we have first carried out the integrals over the loop energy E and the Feynman parameter α, and then shifted the integration variable to match the branch cut along the imaginary axis, cf. Sec. III. The expansion of the effective action, Eq. (20), shows that this integral times the average value

$$\langle V^2(x)\rangle = \int_{-\infty}^{\infty} V^2(x)dx = \frac{4\mu^3(1+\ell)^2}{3\ell} \tag{49}$$

gives the counterterm contribution to the energy (which is minus the spatial integral of the Lagrangian) subject to the condition that the renormalized polarization function vanishes at $p^2 = M^2$.

Since the imaginary momentum integral in Eq. (48) is now of exactly the form as in Eq. (41), we can write the full renormalized energy per unit area as

$$\mathcal{E}_M^{(n)} = -\frac{1}{2(4\pi)^{\frac{n+1}{2}}\Gamma(\frac{n+3}{2})}\int_\mu^\infty (t^2-\mu^2)^{\frac{n+1}{2}}\frac{\partial}{\partial t}\left[\nu(t) - \frac{\langle V\rangle}{2t} + \frac{\langle V^2\rangle}{2t}\cdot\frac{1}{4t^2-M^2}\right]dt. \tag{50}$$

This expression is valid for any $n < 2$. By incorporating the counterterm directly into the momentum integral, we avoid the need to compute the full Feynman diagram, requiring instead only the local quantities $\langle V\rangle$ and $\langle V^2\rangle$. Most importantly, the limit $n \to 2$ is finite, giving

$$\mathcal{E}_M^{(2)} = -\frac{1}{12\pi^2}\int_\mu^\infty (t^2-\mu^2)^{\frac{3}{2}}\frac{\partial}{\partial t}\left[\nu(t) + (\ell+1)\frac{\mu}{t} + \frac{2\mu^3(1+\ell)^2}{3\ell t}\cdot\frac{1}{4t^2-M^2}\right]dt. \tag{51}$$

Carrying out the integral in both models, we obtain

$$\mathcal{E}_{M,\mathrm{sG}}^{(2)} = \frac{\mu^3}{6\pi^2}\left(\frac{2}{3} - \frac{\sqrt{4\mu^2-M^2}}{M}\arcsin\frac{M}{2\mu}\right),$$
$$\mathcal{E}_{M,\mathrm{kink}}^{(2)} = \frac{3\mu^3}{16\pi^2}\left(1 - \frac{\pi}{6\sqrt{3}} - \frac{\sqrt{4\mu^2-M^2}}{M}\arcsin\frac{M}{2\mu}\right) \tag{52}$$

for the energy per unit area. For $M = 0$, this result reproduces the VPE presented in Table II. For the kink, $M = \mu$ agrees with the $n = 2$ result in the OS scheme, as shown in Table III. For the sine-Gordon solution, this calculation is academic because V^2 is not part of the classical Lagrangian.

For consistency, we can also implement the same renormalization scheme for the cases of $n = 0$ and $n = 1$. In this case the second-order counterterm simply makes a finite correction to the energy, implementing the on-shell renormalization condition for the scattering amplitude. These corrections are

$$\Delta\mathcal{E}_M^{(0)} = \frac{\mu}{\pi}\frac{(\ell+1)^2}{3\ell}\frac{\mu^2}{M\sqrt{4\mu^2-M^2}}\arcsin\frac{M}{2\mu} \tag{53}$$

for $n = 0$ and

$$\Delta\mathcal{E}_M^{(1)} = \frac{\mu^2}{\pi}\frac{(\ell+1)^2}{12\ell}\frac{\mu}{M}\operatorname{arctanh}\frac{M}{2\mu} \tag{54}$$

for $n = 1$. Adding $\Delta\mathcal{E}_M$ to the no-tadpole results from Eqs. (13) and (14) agrees with the OS results in Table III.

Similarly, we can also compute the finite correction obtained by introducing a wave function renormalization counterterm $(\partial_\mu\phi)^2$: We expand the polarization function in

a Taylor series at the renormalization scale M as

$$\Pi_V(p^2) = \Pi(M^2) + (p^2 - M^2)\frac{\partial}{\partial M^2}\Pi_V(M^2) + \ldots, \quad (55)$$

and set renormalization conditions to cancel the leading constant and linear terms in this expansion. The subtraction of the constant term has already been implemented above. For the linear term, the counterterm to cancel the p^2 contribution contains $\phi'(x)^2$, while the counterterm to cancel the M^2 contribution has $V(x)^2$, and the renormalization condition applies only to the portion of the contribution quadratic in the deviation of ϕ from its vacuum expectation value.

For the sine-Gordon model, the vertex interaction $U''(\phi) = -m^2\cos\phi$ has no term linear in ϕ, so wave function renormalization is absent in that case. For the kink, the linear term in the vertex interaction $U''(\phi) = \frac{3\lambda}{2}(\phi^2 - v^2)$ is $3\mu\sqrt{\lambda}h$. For the soliton $h(x) = v(\tanh\frac{\mu x}{2} - 1)$ we thus have

$$\Delta\tilde{\mathcal{E}}_M^{(n)} = (9\mu^2\lambda\langle h'(x)^2\rangle + M^2\langle V(x)^2\rangle)\frac{\partial}{\partial M^2}\Pi_V(M^2). \quad (56)$$

The contribution from the $(\partial_\mu\phi)^2$ counterterm is proportional to

$$\langle h'(x)^2\rangle = \int_{-\infty}^{\infty} h'(x)^2 dx = \frac{2\mu^3}{3\lambda}, \quad (57)$$

while the coefficient of the additional contribution proportional to $\langle V(x)^2\rangle$ is determined by the condition that the combined correction to the inverse propagator that is quadratic in h should be proportional to $p^2 - M^2$, with $V(x) = 3\mu\sqrt{\lambda}h(x) + \mathcal{O}(h(x)^2)$. Using

$$\frac{\partial}{\partial M^2}\Pi_V(M^2) = \frac{1}{2(4\pi)^{\frac{n+1}{2}}\Gamma(\frac{n+3}{2})}\int_\mu^\infty (t^2-\mu^2)^{\frac{n+1}{2}}\frac{\partial}{\partial t} \times \left[\frac{1}{2t}\frac{\partial}{\partial M^2}\left(\frac{1}{4t^2 - M^2}\right)\right]dt, \quad (58)$$

we obtain the additional contributions

$$\Delta\tilde{\mathcal{E}}^{(0)}_{M,\text{kink}} = \frac{3\mu^3}{4\pi M^3}(M^2+\mu^2)\frac{\frac{4\mu^2-2M^2}{\sqrt{4\mu^2-M^2}}\arcsin\frac{M}{2\mu} - M}{4\mu^2 - M^2},$$
$$\Delta\tilde{\mathcal{E}}^{(1)}_{M,\text{kink}} = \frac{3\mu^3}{16\pi M^3}(M^2+\mu^2)\left(\text{artanh}\frac{M}{2\mu} - \frac{2\mu M}{4\mu^2 - M^2}\right),$$
$$\Delta\tilde{\mathcal{E}}^{(2)}_{M,\text{kink}} = \frac{3\mu^3}{32\pi^2 M^3}(M^2+\mu^2)\left(M - \frac{4\mu^2\arcsin\frac{M}{2\mu}}{\sqrt{4\mu^2 - M^2}}\right). \quad (59)$$

For $M = \mu$ and $M \to 0$, all of these results agree with those obtained numerically above for the ORS and ZM schemes, respectively.

VI. CONCLUSIONS

Using spectral methods, we have analyzed the one-loop quantum corrections to the tensions of domain walls constructed from kink and sine-Gordon solitons. These are stationary solutions in one space dimension, which become domain walls when embedded in two or three dimensions. The whole configuration is translationally invariant in the additional coordinates and the tension is the energy per unit length or area, respectively.

In spectral methods, the contribution of the continuum modes is computed from scattering data for the interaction of the quantum fluctuations with the potential induced by the domain wall. A key feature of the approach is the equivalence between the Born expansion for scattering data and the expansion of the effective action in powers of the potential. The former is subtracted from the integrand in the continuum integral, and this subtraction is compensated for by adding back the latter at the corresponding order. The combination of the latter expansion with the counterterms renders the tension finite. For total dimensions $D < 3 + 1$, it suffices to only consider the first order of these expansions, which is simple. However, for $D = 3 + 1$, a second-order subtraction is necessary to produce finite results. This subtraction can also be computed in $D = 1 + 1$ and $D = 2 + 1$, where it is finite and provides a check of our approach, and also makes it possible to implement more general renormalization schemes. We can then extend all of these calculations to $D = 3 + 1$ for the case of the kink (the sine-Gordon model is not strictly renormalizable in $D = 3 + 1$). In all cases that we evaluated, the quantum correction for the tension turned out to be negative.

For specific schemes, we have compared our calculations to results obtained previously using different methods. While we agree with Ref. [10], we observe large differences compared to Ref. [11]. We attribute this discrepancy to the multiplicative renormalization of the classical mass in Ref. [11] which does not incorporate the renormalization of the ultraviolet divergence in the first order of the expansions mentioned above. More generally, we show how the spectral approach provides a constructive, transparent, and straightforward tool for the computation of the leading quantum contribution to tensions.

Finally, we have used dimensional regularization in the transverse dimensions to directly implement all the necessary counterterms at an arbitrary energy scale within the integral over continuum scattering modes. This approach avoids the need for explicit computation of the second-order contributions and produces analytic results.

Note added. Recently, a study appeared [26] that addressed the discrepancy alluded to in Table IV. Those authors link that discrepancy to the different points around which the fields are expanded in the computation of Feynman diagrams in the perturbative sector. In our approach, we adhere to the rule that the one-loop effective action is to be computed by expanding around the classical vacuum. The ambiguity arising from the expansion about different vacua is absent in commonly imposed renormalization schemes that leave the vacuum expectation values of scalar fields unchanged.

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DATA AVAILABILITY

The data supporting this study's findings are available within the article.

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