

Virial theorem for rigidly rotating matter

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The scaling property of the thermodynamic free energy (Φ) of a system at global equilibrium has been examined using a real-time method known as the virial theorem. We demonstrate these scaling properties through a derived relation based on the general structure of equal-time commutators among Poincaré charges and their densities. This relation is applicable to any renormalizable fields with spin ≤ 1 , excluding gauge fields. In this particular study, we investigate a rigidly rotating solution ($\Omega = \text{const}$) at global equilibrium for massless fermionic matter. It has been shown that the applicability of a hydrodynamic description requires a hierarchy $\Omega R \gg \Omega \beta_0$, where R is the radius of the cylindrical-shaped rotating matter and $\beta_0 = 1/T_0$ is the inverse temperature on the rotation axis. Consequently, the thermodynamic free energy Φ depends on the angular velocity through the product ΩR , a dependency that extends to other thermodynamic variables as well. These findings are consistent with recent lattice QCD simulation results. Furthermore, we compute the moment of inertia for massless fermions and estimate the light-quark contribution to the total moment of inertia of the quark-gluon plasma produced in heavy-ion collisions.

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I. INTRODUCTION

The study of rotational phenomena in many-body systems has profound implications across diverse branches of physics [1–11]. In recent years, relativistic heavy-ion collision experiments at the RHIC and the LHC have emerged as a unique platform for probing quantum matter under extreme rotation [4,9–11]. These experiments reveal that the quark-gluon plasma (QGP) formed in off-central collisions acquires the largest vorticity, with angular velocities reaching $\Omega \sim 10^{22} \text{ s}^{-1}$ ($\approx 7 \text{ MeV}$)—the highest vorticity ever observed in a laboratory setting [12–15]. This vorticity manifests directly in the spin polarization of hyperons (e.g., Λ , $\bar{\Lambda}$), where the alignment of particle spins with the fluid’s rotational structure¹ [16] successfully explains experimental polarization data [11,12,17], linking spin and rotation in a way similar to the Barnett effects observed in its nonrelativistic counterpart [6,7]. Measurement of the global polarization of hyperons shows good agreement with hydrodynamic prediction [18–22], which implies that polarization phenomena are driven by the

collective properties of the system rather than by a specific hadron-dependent coupling. Given the effectiveness of the hydrodynamic description for the observables of heavy-ion collision in earlier studies [23–25], the inclusion of the rotation is a natural extension to understand the local properties of QGP in terms of local vortical structure [26–30].

In particular, the aspect of thermodynamic systems under rotation within the grand canonical ensemble formalism captures the effect of rotation on the conserved charges or thermodynamic properties [31–33]. Consequently, the impact of rotation on the equation of state reveals new features of the QCD phase structure [34–44]. The prevailing theoretical frameworks [20,45–48] consider a simple rigidly rotating thermal system, where each physical point in the matter rotates with the same angular velocity Ω around a fixed axis. Although the expanding fireball (along the beam axis) is not a rigid rotating system, the model of uniform rotating global equilibrium provides an understanding of global spin polarization of produced hadrons [13,15]. However, with the same simple model, there is a lack of consensus between the analytical and numerical results. For example, in several investigations [14,49], the effect of rotation is independent of the size of the system R and depends solely on some function of Ω . In some recent lattice calculations [50,51], it has been found that the effect of rotation on thermodynamic properties as a function of the angular velocity comes through the common product ΩR .

Specifically, the interplay between rotation and the finite size of the system introduces a natural causal boundary at a

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¹For the sake of simplicity, one may consider the structure to be analogous to a rigidly rotating cylinder.

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distance Ω^{-1} , requiring that any physical description satisfies the constraint $\Omega R \leq 1$, where R denotes the radial extent of the system. On the other hand, for a hydrodynamic description to remain valid, the system must satisfy $R \gg \lambda_{\text{th}}$ [52], where λ_{th} is the thermal de Broglie wavelength. This constraint ensures that collective behavior dominates over microscopic thermal fluctuations, maintaining the validity of the hydrodynamic formalism.² In this work, the dependence of the thermodynamic variables on system's size, similar to Refs. [38–41,44,51,53,54], has been motivated by the extensivity properties of the thermodynamic potential.

We use the formalism of statistical mechanics with the grand canonical density operator $\hat{\rho}$ to study the thermodynamic properties of a system at global equilibrium with the Poincaré charges $\hat{P}^\mu, \hat{J}^{\mu\nu}$ as well as some underlying matter charges $\hat{Q}_a$ [31–33] (also see Refs. [11,22,34,38,55–66] for recent developments). In thermodynamics, the description of a closed system is made by introducing extensive variables (e.g., volume V , total energy E , entropy S , etc.) to capture the large-scale properties of the matter, irrespective of the specific small-scale physics of the system. At equilibrium, the extensivity property of the logarithm of the partition function $\log Z$ (or equivalently, the thermodynamic potential) can be ensured by imposing the scaling property $\log Z \rightarrow \lambda^3 \log Z$ under $\mathbf{x} \rightarrow \lambda \mathbf{x}$, which also gives the thermodynamic consistency conditions [52,67–69]. Interestingly, a connection between this extensivity property and the virial theorem was established for both classical and quantum systems [70–75], by equating the hydrodynamic pressure (defined via a spacelike trace over ensemble-averaged energy-momentum tensor) with the thermodynamic pressure (defined via a derivative of thermodynamic potential with respect to volume). As a consequence of extensivity property of the thermodynamic potential, the intensivity of the thermodynamic pressure is guaranteed.

In this article, we present a field-theoretic version of the virial theorem [76–78] by invoking a spatial scaling transformation, in the line of Ref. [79]. Specifically, we apply this framework for rotating thermal systems, considering massless fermions to represent the light-quark contributions of the rotating QGP in a qualitative manner. The effect of rotation on thermodynamic pressure is expressed as a function of angular velocity Ω . Because of the intensive nature of thermodynamic pressure, the Ω dependency appears through dimensionless combinations ΩR and $\Omega \beta_0$, where $\beta_0 = \frac{1}{T_0}$ is the inverse of the temperature on the rotation axis. Recalling the stringent condition $R \gg \lambda_{\text{th}}$ [52,67,80], where the thermal wavelength goes as inverse temperature, i.e., $\lambda_{\text{th}} \sim \frac{1}{T_0}$ [52,81] for a massless (or strongly coupled) system,

we show that a separation in magnitude arises between these two dimensionless quantities, i.e., $\Omega R \gg \Omega \beta_0$. Consequently, the system size dependency of thermodynamic properties is pronounced through the domination of the common product ΩR .

This article is organized as follows: In Sec. II, a general description of the finite temperature grand canonical system with Poincaré charges and matter charges is discussed, and the definition of the partition function is reviewed. A field-theoretic counterpart of the virial theorem is reviewed through the introduction of the rigid scaling transformation and its action on the local operator in Sec. III. We also show that the general structure of the equal-time commutator (ETC) among different components of energy-momentum tensors, as well as the charge currents, leads to a general relation (36) without considering any particular renormalized matter fields of spin ≤ 1 (except gauge theories). Section IV is devoted to understanding this general relation with two specific solutions of the Killing equation at the global equilibrium, namely, nonrotating homogeneous and rotating inhomogeneous solutions. We show that the imposition of the extensivity of the thermodynamic potential gives us an insight into the effect of rotation on thermodynamic quantities, such as the moment of inertia. We further find the noninteracting fermionic system, which turns out to be positive and meets the criterion of thermodynamical stability. An estimation of the moment of inertia for the quark content in the QGP is done in Sec. V by considering appropriate values of physical parameters with a simple model of rigidly rotating massless fermions, which shows the dominant nature of the common product ΩR , which is similar to the observation of Refs. [38,44,50,51] in the slow rotation limit.

Notation and convention: Throughout the paper, we use the metric tensor that has the signature $(+, -, -, -)$. Three-vectors are denoted by bold fonts (e.g., $\mathbf{a}$) as well as the usual forms (e.g., $\vec{a}$). The scalar products for both three- and four-vectors are denoted by a dot, i.e., $a \cdot b = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}$ (or $= a^0 b^0 - \vec{a} \cdot \vec{b}$).

II. FINITE TEMPERATURE QFT WITH GLOBAL CHARGES

In Minkowski space, the grand canonical density operator at global equilibrium is expressed as

$$\hat{\rho} = \frac{1}{Z} \exp \left[-b_\mu \hat{P}^\mu + \frac{1}{2} \omega_{\mu\nu} \hat{J}^{\mu\nu} + \sum_a \alpha_a \hat{Q}_a \right], \quad (1)$$

where $\hat{P}^\mu, \hat{J}^{\mu\nu}$ represents the ten conserved mechanical integrals of motion: the momentum four-vector and angular momentum, respectively. Here, the operator $\hat{Q}_a$ denotes additional conserved charges, with different charges labeled by the index a . The Lagrange multipliers $b_\mu, \omega_{\mu\nu}$, and α_a are associated with these conserved quantities. The first two

²These constraints dictate the regimes in which a quasiparticle-based description remains valid and provide a fundamental limit on the role of rotation in thermal QCD systems.

constitute the Killing field $\beta_\mu(x)$ as

$$\beta_\mu(x) = b_\mu + \omega_{\mu\nu}x^\nu, \quad (2)$$

and $\alpha_a (= \text{const})$ having the homogeneous profile are the solution of the global equilibrium conditions

$$\partial_\mu \beta_\nu(x) + \partial_\nu \beta_\mu(x) = 0 \quad \text{and} \quad \partial_\mu \alpha_a = 0, \quad (3)$$

correspondingly. Here, $\beta_\mu(x)$ and α_a help define the proper comoving is temperature $T(x) = (\sqrt{\beta_\mu \beta^\mu})^{-1}$ and the comoving chemical potential $\mu_a(x) = \alpha_a T(x)$, respectively.

Here, the partition function

$$Z = \text{Tr} \left[\exp \left(-b_\mu \hat{P}^\mu + \frac{1}{2} \omega_{\mu\nu} \hat{J}^{\mu\nu} + \sum_a \alpha_a \hat{Q}_a \right) \right] \quad (4)$$

is defined for a finite system size, ensuring proper normalization of the density operator. Moreover, the extensive nature of $\log(Z)$ helps to define the thermodynamic pressure (also called the kinetic pressure) [75,79]. With the assistance of this density operator, one can define the ensemble average of an operator $\hat{O}(x)$ as

$$\langle \hat{O}(x) \rangle = \text{Tr}(\hat{\rho} \hat{O}(x)), \quad (5)$$

where the trace is defined for the finite volume.

III. VIRIAL THEOREM

For both classical and quantum systems in equilibrium, the virial theorem [75,77–79] establishes an equivalence between the hydrostatic pressure, defined by the spacelike trace of the energy-momentum tensor, and the thermodynamic pressure, derived from the partition function. The field-theoretic version of this theorem has been discussed in Ref. [79] and will be employed in this section to explain the system in global equilibrium with the density operator from Eq. (1).

A. Dilatation as pseudogauge transformation

In the context of field theory, the virial theorem can be proved by involving a spatial dilatation, which scales the spacetime coordinate and fields as

$$t \rightarrow t, \quad \mathbf{x} \rightarrow e^\lambda \mathbf{x}, \quad \Phi(x) \rightarrow e^{\Delta\lambda} \Phi(t, e^\lambda \mathbf{x}), \quad (6)$$

where Δ is the scale dimension of the field considered. The infinitesimal form of this transformation gives

$$\mathbf{x} \rightarrow (1 + \lambda)\mathbf{x}, \quad \Phi(x) \rightarrow (1 + \lambda(\Delta + \mathbf{x} \cdot \partial))\Phi(x), \quad (7)$$

so the Noether charge [82], which generates the spatial dilatation, can be written as

$$\hat{D}(t) = \int d^3 \mathbf{x} [x_i \hat{T}_B^{0i}(x) + \hat{V}^0(x)]. \quad (8)$$

Here, $\hat{T}_B^{\mu\nu}$ is the Belinfante energy-momentum tensor, and $\hat{V}^0$ is the time component of the virial field that vanishes for spin $\frac{1}{2}$ field and the renormalizable massive spin 1 field. The virial field has a nontrivial structure for the scalar and gauge theories. However, in cases other than gauge theories,³ the dilatation generator Eq. (8) can be reexpressed for all renormalizable theories as

$$\hat{D}(t) = \int d^3 \mathbf{x} x_i \hat{T}^{0i}(x), \quad (9)$$

in terms of the “improved” energy-momentum tensor [83–85], which is defined as

$$\hat{T}^{\mu\nu} = \hat{T}_B^{\mu\nu} - \frac{1}{6} \sum_a (\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \phi_a^2. \quad (10)$$

In the previous equation, the difference between the Belinfante and the improved energy-momentum tensor occurs only for the scalar fields (ϕ_a).

As mentioned in the previous section, we assume that the fluid enjoys time, space, and Lorentz (boost + rotation) invariance as well as global $U(1)_a$ symmetry invariance generated by the corresponding charges, $\hat{P}^0$ (or Hamiltonian $\hat{H}$), $\hat{P}^i$, $\hat{J}^{\mu\nu}$ ($\hat{J}^{0i} + \hat{J}^{ij}$), and $\hat{Q}_a$, respectively. We adopt a static coordinate system (x^0, x^i) or $(t, \mathbf{x})$, namely, the laboratory frame that possesses the Killing vector ∂_0 . Relative to this static coordinate, the fluid flows with four velocity $u_\mu(x) = \frac{1}{\sqrt{\beta^2}} \beta_\mu(x)$, where $\beta_\mu(x)$ defines a timelike Killing vector that helps to define thermodynamics consistently. Thus, in the static frame, the spacetime symmetry generators are these Hermitian operators $\hat{P}_\mu = i\partial_\mu$, and $\hat{J}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ corresponding to the spacetime translation and the Lorentz transformation in differential representation, respectively. On the other hand, the diffeomorphism and global $U(1)$ invariance of the action relate these generators to the tensor densities⁴

$$\hat{P}^\mu = \int d^3 \mathbf{x} \hat{T}_B^{0\mu}(x), \quad (11)$$

³In gauge theories, the virial field $\hat{V}^\mu$ depends on the choice of gauge. However, this gauge dependence does not survive when taking the grand canonical expectation over physical states [79]. These subtleties are not explored in this article; a generalization of the analysis is left for future work.

⁴Here, we adopt the Belinfante pseudogauge, in which the total angular momentum $J^{\mu\nu}$ receives contributions solely from the orbital part due to the vanishing spin current. Subsequently, in Eqs. (30) and (31), the conserved charges are defined using the improved energy-momentum tensor $\hat{T}^{\mu\nu}(x)$.

$$\begin{aligned}\hat{J}^{\mu\nu} &= \int d^3\mathbf{x} \hat{J}_B^{0\mu\nu}(x) \\ &= \int d^3\mathbf{x} (x^\mu \hat{T}_B^{0\nu}(x) - x^\nu \hat{T}_B^{0\mu}(x)),\end{aligned}\quad (12)$$

$$\hat{Q}_a = \int d^3\mathbf{x} \hat{J}_a^0(x). \quad (13)$$

Here, the angular momentum tensor density is defined by $\hat{J}_B^{\mu\alpha\beta} = (x^\alpha \hat{T}_B^{\mu\beta} - x^\beta \hat{T}_B^{\mu\alpha})$, and $\hat{J}_a^\mu$ is the conserved current density associated with the $U(1)_a$ symmetry. While the global Poincaré charges in Eqs. (11) and (12) are defined in terms of the Belinfante energy-momentum tensor $\hat{T}_B^{\mu\nu}(x)$, they can be reexpressed with some new energy-momentum tensor $\hat{T}'^{\mu\nu}(x)$, provided the latter satisfies

$$\begin{aligned}\hat{T}'^{\mu\nu}(x) &= \hat{T}_B^{\mu\nu}(x) + \frac{1}{2} \partial_\lambda (\hat{\Phi}^{\lambda,\mu\nu} + \hat{\Phi}^{\nu,\mu\lambda} + \hat{\Phi}^{\mu,\nu\lambda}) \\ &\quad + \frac{1}{2} \partial_\lambda \partial_\rho \hat{\Psi}^{\lambda\rho\mu\nu}.\end{aligned}\quad (14)$$

To keep the global angular momentum tensor $\hat{J}^{\mu\nu}$ unaltered, we have to simultaneously modify the angular momentum tensor as

$$\hat{J}'^{\mu\alpha\beta} = x^\alpha \hat{T}'^{\mu\beta} - x^\beta \hat{T}'^{\mu\alpha} + \hat{S}'^{\mu,\alpha\beta}, \quad (15)$$

where

$$\hat{S}'^{\lambda,\mu\nu} = -\hat{\Phi}^{\lambda,\mu\nu} + \partial_\rho \hat{Z}^{\mu\nu,\lambda\rho}. \quad (16)$$

Here, these additional boundary terms $\hat{\Phi}^{\lambda,\mu\nu}$, $\hat{\Psi}^{\lambda\rho\mu\nu}$, and $\hat{Z}^{\mu\nu,\alpha\beta}$ are known as superpotential; they possess the following symmetries with respect to the exchange of the indices:

$$\begin{aligned}\hat{\Phi}^{\lambda,\mu\nu} &= -\hat{\Phi}^{\lambda,\nu\mu}, \\ \hat{\Psi}^{\lambda\rho\mu\nu} &= \frac{1}{3!} \hat{\Psi}^{[\lambda\rho\mu]\nu},\end{aligned}\quad (17)$$

and

$$\hat{Z}^{\mu\nu,\lambda\rho} = -\hat{Z}^{\nu\mu,\lambda\rho} = -\hat{Z}^{\mu\nu,\rho\lambda}. \quad (18)$$

These symmetry properties are crucial for ensuring the conservation of the local energy-momentum (through $\partial_\mu \hat{T}'^{\mu\nu} = 0$) and the local angular momentum (through $\partial_\lambda \hat{J}'^{\lambda\mu\nu} = 0$), provided that the corresponding conservation laws were satisfied in the Belinfante choice (i.e., $\partial_\mu \hat{T}_B^{\mu\nu} = 0$ and $\partial_\lambda \hat{J}_B^{\lambda,\alpha\beta} = 0$). Despite all these symmetry constraints of Eqs. (17) and (18) imposed on the indices of the superpotentials, there is an arbitrariness in the choice of tensor densities ($\hat{T}'^{\mu\nu}(x)$, $\hat{J}'^{\lambda\mu\nu}(x)$) with the same total Poincaré charges $\hat{P}^\mu$, $\hat{J}^{\mu\nu}$, which is known as “pseudogauge freedom”

in the literature [86–91]. Consequently, the improved energy-momentum tensor defined in Eq. (10) can be interpreted as a pseudogauge transformation as given in Eqs. (14) and (16), with the choice of $\hat{\Phi}^{\mu,\alpha\beta}(x) = 0$, $\hat{Z}^{\lambda\rho,\alpha\beta}(x) = 0$ but nonzero $\hat{\Psi}^{\mu\nu\alpha\beta}(x)$ [83,90,92].

The rigid scaling transformation introduces the dilatation $\hat{D}(t)$ operator in Eq. (8), which transforms an operator $\hat{O}(t)$ by

$$\hat{O}(t) \rightarrow \hat{O}(t) + i\lambda[\hat{D}(t), \hat{O}(t)]. \quad (19)$$

To analyze the scaling properties of the thermal system, one needs to understand the action of the dilation on the density matrix $\hat{\rho}$. Thus, it is crucial to know the ETC relations between the dilatation generator $\hat{D}$ and other symmetry generators. Using the differential representation of dilatation generator $\hat{D} = ix^i \partial_i$, it can be shown that the only nonvanishing commutators are

$$[\hat{D}(t), \hat{P}_i] = -i\hat{P}_i, \quad (20)$$

$$\begin{aligned}[\hat{D}(t), \hat{J}_{0i}] &= -i(x_i \hat{P}_0 + x_0 \hat{P}_i), \\ &= -i(2x_0 \hat{P}_i - \hat{J}_{0i}).\end{aligned}\quad (21)$$

Alternatively, these ETCs can be obtained by using representations given in Eqs. (8), (9), and (11)–(13), which involve ETC of the tensor fields ($\hat{T}^{0\nu}(x)$, $\hat{J}^0(x)$, etc.). The explicit computation of the ETC between dilatation generator $\hat{D}(t)$ and the current density $\hat{J}_a^0(x)$ associated with the a -type charge has been performed in Refs. [93–97], yielding

$$i[\hat{D}(t), \hat{J}_a^0(x)] = (3 + x^i \partial_i) \hat{J}_a^0(x). \quad (22)$$

This relation shows that the charge density has a scale dimension of three.⁵

Furthermore, the ETCs between the dilatation generator and the spacetime generators involve the commutators of energy-momentum tensors; these have been discussed in a general form in Refs. [79,93,98–101]. In our particular context, the following commutators are required [100,101]⁶:

$$\begin{aligned}i[\hat{T}^{00}(t, \mathbf{x}), \hat{T}^{0i}(t, \mathbf{y})] &= [\hat{T}^{ji}(t, \mathbf{x}) - g^{ji} \hat{T}^{00}(t, \mathbf{y})] \\ &\quad \times \partial_j \delta^3(\mathbf{x} - \mathbf{y}) + \hat{\tau}^{00,0i}(x, y),\end{aligned}\quad (23)$$

$$\begin{aligned}i[\hat{T}^{0i}(t, \mathbf{x}), \hat{T}^{0j}(t, \mathbf{y})] &= [\hat{T}^{0j}(t, \mathbf{x}) \partial^i + \hat{T}^{0i}(t, \mathbf{y}) \partial^j] \\ &\quad \times \delta^3(\mathbf{x} - \mathbf{y}) + \hat{\tau}^{0i,0j}(x, y).\end{aligned}\quad (24)$$

⁵The extra term in Eq. (10) does not contribute to this commutator, if $\hat{J}_a^0$ is a matter current or corresponding to global gauge transformations.

⁶These equal-time commutators hold for any energy-momentum tensor $\hat{T}'^{\mu\nu}$, which are connected to the Belinfante $\hat{T}_B^{\mu\nu}$ via pseudogauge transformation as given in Eq. (14).

The term on the right-hand side containing $\partial_i \delta^3(\mathbf{x} - \mathbf{y})$ is essential to ensure the Poincaré algebra is generated by $\hat{P}^\mu = \int d^3\mathbf{x} \hat{T}^{0\mu}(x)$ and $\hat{J}^{\mu\nu} = \int d^3\mathbf{x} (x^\mu \hat{T}^{0\nu}(x) - x^\nu \hat{T}^{0\mu}(x))$. The last term of Eqs. (23) and (24), also known as the Schwinger term, contains terms involving higher-order spatial derivatives of the $\delta(\mathbf{x} - \mathbf{y})$ functions. Hence, the following integrals must vanish:

$$\int d^3\mathbf{x} \hat{\tau}^{00,0i}(x) = 0, \quad \int d^3\mathbf{x} \hat{\tau}^{0i,0j}(x) = 0, \quad (25)$$

and

$$\int d^3\mathbf{x} \hat{\tau}^{00,0i}(x) = 0, \quad \int d^3\mathbf{x} \hat{\tau}^{0i,0j}(x) = 0. \quad (26)$$

Employing these commutator relations for nongauge theories, one can arrive at

$$i[\hat{D}(t), \hat{T}^{00}(t, \mathbf{x})] = (3 + x^i \partial_i) \hat{T}^{00}(x) - \hat{T}_i^i(x), \quad (27)$$

$$i[\hat{D}(t), \hat{T}^{0i}(t, \mathbf{x})] = (4 + x^i \partial_i) \hat{T}^{0i}(x), \quad (28)$$

which shows that the tensor component of the improved energy-momentum tensor $\hat{T}^{00}(x)$ has no definite scale dimension, whereas $\hat{T}^{0i}(x)$ clearly has a scale dimension of 4. This construction will be useful for analyzing the extensive properties of thermodynamical free energy, as described in the next subsection.

B. Scaling at global equilibrium

Based on the commutators provided in Eqs. (20) and (21), and the definition of the expectation value at global equilibrium from Eq. (5), the following identity can be derived:

$$\begin{aligned} & \left\langle i \left[\hat{D}(t), -b_\mu \hat{P}^\mu + \frac{1}{2} \omega_{\mu\nu} \hat{J}^{\mu\nu} + \sum_a \alpha_a \hat{Q}_a \right] \right\rangle \\ &= -\omega_{0i} \langle \hat{J}^{0i} \rangle - (b_i + 2\omega_{i0} x^0) \langle \hat{P}^i \rangle. \end{aligned} \quad (29)$$

Concurrently, one can express the Poincaré charges in terms of the improved energy-momentum tensor $\hat{T}^{\mu\nu}(x)$ as follows:

$$\hat{P}^\mu = \int d^3\mathbf{x} T^{0\mu}(x), \quad (30)$$

$$\hat{J}^{\mu\nu} = \int d^3\mathbf{x} (x^\mu \hat{T}^{0\nu}(x) - x^\nu \hat{T}^{0\mu}(x)) \quad (31)$$

and the matter charges as given in Eq. (13). This aids in expressing the left-hand side of Eq. (29) in the following form by using Eqs. (22), (27), and (28),

$$\begin{aligned} & \left\langle i \left[\hat{D}(t), -b_\mu \hat{P}^\mu + \frac{1}{2} \omega_{\mu\nu} \hat{J}^{\mu\nu} + \sum_a \alpha_a \hat{Q}_a \right] \right\rangle \\ &= \int d^3\mathbf{x} \langle -b_0 (\mathcal{D}(\mathbf{x}) \hat{T}^{00}(x) - \hat{T}_k^k(x)) \\ &\quad - b_i (\mathcal{D}(\mathbf{x}) \hat{T}^{0i}(x) + \hat{T}^{0i}(x)) + \omega_{0i} (2x^0 \hat{T}^{0i}(x) \\ &\quad + (\mathcal{D}(\mathbf{x}) - 1)(x^0 \hat{T}^{0i}(x) - x^i \hat{T}^{00}(x)) + x^i \hat{T}_k^k(x) \\ &\quad + \frac{\omega_{ij}}{2} (\mathcal{D}(\mathbf{x}) (x^i \hat{T}^{0j}(x) - x^j \hat{T}^{0i}(x))) \\ &\quad + \alpha_a (\mathcal{D}(\mathbf{x}) \hat{J}_a^0(x)) \rangle. \end{aligned} \quad (32)$$

Here, the differential operator $\mathcal{D}(\mathbf{x}) = (3 + x^i \partial_i)$ appears due to the action of the dilatation generator on each local operator $\hat{O}(x)$, which helps to write each integral as

$$\int d^3\mathbf{x} \mathcal{D}(\mathbf{x}) \hat{O}(x) = \frac{\partial}{\partial \lambda} \left(\int d^3\mathbf{x} e^{3\lambda} \hat{O}(t, e^\lambda \mathbf{x}) \right) \Big|_{\lambda \rightarrow 0}. \quad (33)$$

Hence, the integrands in the last equation that appear with $\mathcal{D}(x)$ can be rewritten as

$$\begin{aligned} & \left\langle \int d^3\mathbf{x} \mathcal{D}(\mathbf{x}) \left(-b_0 \hat{T}^{00}(x) - b_i \hat{T}^{0i}(x) + \omega_{0i} (x^0 \hat{T}^{0i}(x) - x^i \hat{T}^{00}(x)) + \frac{\omega_{ij}}{2} (x^i \hat{T}^{0j}(x) - x^j \hat{T}^{0i}(x)) + \alpha_a \hat{J}_a^0(x) \right) \right\rangle \\ &= \left\langle \frac{\partial}{\partial \lambda} \left(\int d^3\mathbf{x} e^{3\lambda} \left(-b_\mu \hat{T}^{0\mu}(t, e^\lambda \mathbf{x}) + \omega_{0i} (x^0 \hat{T}^{0i}(t, e^\lambda \mathbf{x}) - x^i \hat{T}^{00}(t, e^\lambda \mathbf{x})) \right. \right. \right. \\ &\quad \left. \left. + \frac{\omega_{ij}}{2} e^\lambda (x^i \hat{T}^{0j}(t, e^\lambda \mathbf{x}) - x^j \hat{T}^{0i}(t, e^\lambda \mathbf{x})) + \alpha_a \hat{J}_a^0(t, e^\lambda \mathbf{x}) \right) \right) \Big|_{\lambda \rightarrow 0} \right\rangle \\ &= \partial_\lambda \log Z(\lambda) \Big|_{\lambda \rightarrow 0}, \end{aligned} \quad (34)$$

where

$$\begin{aligned} \log Z(\lambda) &= \int d^3\mathbf{x} e^{3\lambda} \left(-b_\mu \hat{T}^{0\mu}(t, e^\lambda \mathbf{x}) + \omega_{0i} (x^0 \hat{T}^{0i}(t, e^\lambda \mathbf{x}) - x^i \hat{T}^{00}(t, e^\lambda \mathbf{x})) + \frac{\omega_{ij}}{2} e^\lambda (x^i \hat{T}^{0j}(t, e^\lambda \mathbf{x}) \right. \\ &\quad \left. - x^j \hat{T}^{0i}(t, e^\lambda \mathbf{x})) + \alpha_a \hat{J}_a^0(t, e^\lambda \mathbf{x}) \right) \end{aligned} \quad (35)$$

is the logarithm of the scaled partition function. Here, $Z(\lambda)$ can be obtained from the definition (4) by scaling each of the extensive quantities appearing in the argument of Z (for example, $V \rightarrow e^{3\lambda}V$ or $L \rightarrow e^\lambda L$). Consequently, with the help of the expressions in Eqs. (29) and (32), one shall arrive at

$$\int d^3\mathbf{x} (b_0 + \omega_{0i}x^i) \langle \hat{T}_k^k(x) \rangle = -\partial_\lambda \log Z(\lambda)|_{\lambda \rightarrow 0}. \quad (36)$$

This shows that the scaling property of the thermodynamic partition function is encoded in the general identity given in Eq. (36), which holds for a system in global equilibrium possessing the maximal number of Poincaré charges and various matter charges, within renormalizable nongauge field theories with $\text{spin} \leq 1$. The implications of this identity, in the context of different solutions to the global Killing condition (3), are discussed in the next section.

IV. EXTENSIVITY PROPERTY OF $\log Z$

The global equilibrium state in Minkowski spacetime possesses ten constants of motion, encoded in the Killing fields given in Eq. (2), corresponding to the Poincaré symmetry group. We begin by considering a familiar form of the density operator, which can be recovered from Eq. (1) by choosing $b_\mu = \frac{1}{T_0}(1, \mathbf{0}) = \beta_0 \delta_\mu^0$ and $\omega_{\mu\nu} = 0$, which is

$$\hat{\rho} = \frac{1}{Z} \exp \left[-\beta_0 \hat{H} + \sum_a \alpha_a \hat{Q}_a \right]. \quad (37)$$

Here, $\hat{H}$ is the Hamiltonian of the equilibrated system, which measures the energy as seen by an observer moving along the Killing vector ∂_0 in the static coordinate frame. With this Killing vector, we obtain the homogeneous proper temperature $T_0 = \beta_0^{-1}$. Therefore, the ensemble average of any local operator as given in Eq. (5) becomes position independent, due to the translational invariance of the density operator (37).

Following the right-hand side of Eq. (36), and with homogeneous thermodynamic parameters, we arrive at

$$\int d^3\mathbf{x} \beta_0 \langle \hat{T}_k^k(x) \rangle = V \beta_0 \langle \hat{T}_k^k(x) \rangle. \quad (38)$$

Here, the left-hand side of Eq. (36) provides insight into the scaling property of the logarithm of the partition function, $\log Z$. Numerous studies [25,68,70,102,103] have shown that robustness of extensivity ensures writing $\log Z(\lambda) = \log Z(\beta_0, \alpha_a, V e^{3\lambda}) = e^{3\lambda} \log Z$, which leads to the identity

$$\begin{aligned} \beta_0 \langle \hat{T}_k^k(x) \rangle &= -\frac{1}{V} \partial_\lambda \log Z(\beta_0, \alpha_a, V e^{3\lambda})|_{\lambda \rightarrow 0} \\ &= -3 \frac{\partial}{\partial V} (\log Z(\beta_0, \alpha_a, V)). \end{aligned} \quad (39)$$

The last relation allows us to identify the isotropic hydrodynamic pressure defined through the spatial trace of the energy-momentum tensor $\langle \hat{T}_k^k(x) \rangle = 3P_0$ in the rest frame of the fluid, provided a suitable energy-momentum tensor has been chosen. On the other hand, the thermodynamic pressure—an intensive variable—is defined as follows [70,71]:

$$\mathcal{P}(\beta_0, \alpha_a) = -\partial_V \Phi(\beta_0, \alpha_a, V), \quad (40)$$

where $\Phi(\beta_0, \alpha_a, V)$ is the thermodynamic potential defined through the partition function

$$\begin{aligned} \Phi(\beta_0, \alpha_a, V) &= -T_0 \log Z(\beta_0, \alpha_a, V), \\ &= -\mathcal{P}(\beta_0, \alpha_a) V. \end{aligned} \quad (41)$$

Hence, Eq. (39) equates hydrodynamic and thermodynamic pressures, $P_0 = \mathcal{P}(\beta_0, \alpha_a)$, as a consequence of the extensivity of the thermodynamic potential; an identity also known as the virial theorem [75,79].

We now turn to the global thermodynamics of a system undergoing pure rotation at constant angular velocity Ω along the z axis, characterized by $b_\mu = (\beta_0, \mathbf{0})$ and $\omega_{\mu\nu} = \beta_0 \Omega (g_{\mu x} g_{\nu y} - g_{\mu y} g_{\nu x})$. The corresponding equilibrium density operator is given by

$$\hat{\rho} = \frac{1}{Z} \exp \left[-\beta_0 \hat{H} + \beta_0 \Omega \hat{J}^{xy} + \sum_a \alpha_a \hat{Q}_a \right], \quad (42)$$

which has captured recent interest, as described in Refs. [11,20–22,33,70,104]. Moreover, the Killing vector $\beta_\mu(x)$ [61] does not lead to a homogeneous thermodynamic equilibrium, because temperature and the chemical potential become position dependent. Specifically, they become a function of the radial coordinate $r = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$, due to the angular velocity along the z axis, which can be written as

$$T(x) = \Gamma(r) T_0, \quad \mu_a(x) = \Gamma(r) \mu_0, \quad (43)$$

with a constant ratio of chemical potential to temperature throughout the system, given by $\alpha_a = \frac{\mu_0}{T_0}$. Here, $T_0 = \beta_0^{-1}$ and μ_0 denote the temperature and chemical potential on the axis of rotation $r = 0$, while both diverge at radius $r = \Omega^{-1}$ due to the Lorentz factor $\Gamma(r) = \frac{1}{\sqrt{1 - \Omega^2 r^2}}$.

Recalling Eq. (36) in the context of pure rotation, we can express it in the following form:

$$\int d^3\mathbf{x} \langle \hat{T}_k^k(x) \rangle = -T_0 \partial_\lambda \log Z(\lambda) \Big|_{\lambda \rightarrow 0}. \quad (44)$$

In contrast to the case of homogeneous equilibrium, $\langle \hat{T}_k^k(x) \rangle$ cannot be taken outside the integral due to its explicit coordinate dependence [105,106]. Owing to the inhomogeneous profile of the thermodynamic parameters

as given in Eq. (43), the tensor densities become radially coordinate-dependent thermodynamic quantities. This can give rise to a scenario in which the volume-averaged tensor densities acquire an explicit dependence on the system size in the direction transverse to the rotation axis, denoted by R , the radius of the cylindrical rotating matter.

In equilibrium, the condition of thermodynamic consistency entails extensivity, which follows from the assumption that thermodynamic variables [e.g., $O(x)$] vary over length scales much larger than the equilibrium correlation length. That is, $\lambda_{\text{th}} \partial O \ll L \partial O$, where L and λ_{th} correspond to the macroscopic scale ($\sim$ system size) and microscopic length scales (e.g., thermal de Broglie wavelength), respectively, thus establishing the scale hierarchy required for the applicability of hydrodynamic expansion [67,107]. For rigid rotation, the matter fields must be confined in the transverse direction relative to the rotation axis to ensure causality. Consequently, the radial extent R must satisfy the condition $\Omega R \leq 1$. No such constraint applies to the extension L_z of matter along the rotation axis, and it is assumed that the system is sufficiently extended in the longitudinal direction to meet the hydrodynamic hierarchy,

i.e., $L_z \gg \lambda_{\text{th}}$. Similarly, along the radial direction, the hydrodynamic scale separation is maintained by imposing the condition $\lambda_{\text{th}} \ll R$ [52,67,80]—which excludes scenarios where the radial extent of the system is $\lesssim \lambda_{\text{th}}$ —a situation that typically arises under conditions of very fast rotation and low temperatures.

At this point, we focus on evaluating the left-hand side of Eq. (44), examining whether the volume-integrated space-like trace of the energy-momentum tensor can be expressed in terms of the extensive variable L_z (or $V = \pi R^2 L_z$) and R , along with the intensive thermodynamic parameters of the system. In this work, we consider a rotating gas of massless fermions. The computation is based on the statistically averaged energy-momentum tensor, using the expressions derived in Refs. [105,106], where the results are provided in polar coordinates $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \rightarrow (r, \phi, \mathbf{z})$,

$$\begin{aligned} \langle \hat{T}^{rr}(x) \rangle &= \Gamma^4(r) \left[P_0 + \frac{\Omega^2}{24} B_0 \left(\frac{4}{3} \Gamma^2(r) - \frac{1}{3} \right) \right] \\ &= \langle \hat{T}^{zz}(x) \rangle, \end{aligned} \quad (45)$$

$$\langle \hat{T}^{\phi\phi}(x) \rangle = r^{-2} \Gamma^4(r) \left[P_0 (4\Gamma^2(r) - 3) + \frac{\Omega^2}{24} B_0 (8\Gamma^4(r) - 8\Gamma^2(r) + 1) \right], \quad (46)$$

$$\langle \hat{T}^{tt}(x) \rangle = \Gamma^6(r) \Omega \left[4P_0 + B_0 \frac{2\Omega^2}{9} \left(\frac{3}{2} \Gamma^2(r) - \frac{1}{2} \right) \right], \quad (47)$$

$$\langle \hat{T}^{tt}(x) \rangle = \Gamma^4(r) \left[P_0 (4\Gamma^2(r) - 1) + \frac{\Omega^2}{8} B_0 \left(\frac{8}{3} \Gamma^4(r) - \frac{16}{9} \Gamma^2(r) + \frac{1}{9} \right) \right]. \quad (48)$$

Here, the divergent vacuum expectation value has been subtracted and the renormalized expressions for the averaged energy-momentum tensor components are presented in Eqs. (45) and (46). In the limit $\Omega \rightarrow 0$, one recovers the known result for the nonrotating, massless fermion case. The expression $P_0(T_0, \alpha_a) = g T_0^4 \left(\frac{7\pi^2}{360} + \frac{\alpha_a^2}{12} + \frac{\alpha_a^4}{24\pi^2} \right)$ can be identified as the isotropic pressure in the nonrotating limit, where g denotes the degeneracy factor of the fermion particles. Additionally, we observe that the thermodynamic function $B_0(T_0, \alpha_a) = g T_0^2 \left(1 + \frac{3}{\pi^2} \alpha_a^2 \right)$ has a magnitude comparable to $\sim \beta_0^2 P_0$. Substituting the expressions (45) and (46) into the left-hand side of Eq. (44), we arrive at

$$\begin{aligned} \int d^3 \mathbf{x} \langle \hat{T}_k^k(x) \rangle &= - \int d^3 \mathbf{x} (\langle \hat{T}^{rr}(x) \rangle + \langle \hat{T}^{zz}(x) \rangle + r^2 \langle \hat{T}^{\phi\phi}(x) \rangle), \\ &= -3VP_0(T_0, \alpha_a) \frac{(1 - \frac{\Omega^2 R^2}{3})}{(1 - \Omega^2 R^2)^2} - \frac{V\Omega^2 B_0}{216(1 - \Omega^2 R^2)^3} (3R^4 \Omega^4 - 6R^2 \Omega^2 + 27). \end{aligned} \quad (49)$$

Compared to the nonrotating case, the final result acquires a dependence on the angular velocity Ω through the appearance of the common dimensionless combination ΩR . Moreover, within the hydrodynamic regime, the magnitudes of the first and second terms in the previous expression differ significantly. This is because, in massless theories

(or strongly coupled systems), the thermal de Broglie wavelength is $\lambda_{\text{th}} \sim \frac{1}{T(r)} \Gamma(r) = \frac{1}{T_0}$,⁷ where $\Gamma(r)$ appears as a consequence of the Lorentz transformation from the static coordinate frame and the local fluid rest frame [52,81]. Therefore, to ensure the validity of a hydrodynamic description, a stringent scale hierarchy must be imposed by the condition $R \gg \frac{1}{T_0}$ (or, equivalently, $\Omega R \gg \frac{\Omega}{T_0}$). This implies that the terms involved with ΩR dominate over those involved with $\beta_0 \Omega$ in thermodynamic quantities. As a result, the second term in Eqs. (45)–(49) can be neglected compared to the first term because $\Omega^2 B_0 \sim (\Omega \beta_0)^2 P_0 \ll P_0$.

Revisiting the left-hand side of Eq. (44), we observe that the derivative with respect to the scaling parameter λ receives contributions from each extensive variable present in the argument of $\log Z$, provided that they are attributed to the scaling of the thermodynamic potential; that is,

$$\begin{aligned} \partial_\lambda \log Z(\lambda)|_{\lambda \rightarrow 0} &= L_z \partial_{L_z} (\log Z)|_R + R \partial_R (\log Z)|_{L_z} \\ &= 3V \partial_V (\log Z)|_{\Omega R} + \Omega R \partial_{\Omega R} (\log Z)|_V. \end{aligned} \quad (50)$$

It is important to note that the dependence R appears in the expression on the right-hand side of Eq. (49) through the dimensionless combination ΩR . Therefore, an ansatz for $\log Z$ can be proposed, incorporating the extensivity property

$$\log Z = \beta_0 \mathcal{P}(T_0, \alpha_a, \Omega R) V. \quad (51)$$

Here, the minus sign is introduced by convention. However, the precise functional form of $\mathcal{P}$ depends on the specific thermodynamic system under consideration. Nevertheless,

⁷A parallel understanding of this condition follows from quasiparticle kinematics, by requiring that the distance Δ between two successive collisions in the rotating fluid be significantly smaller than the system size, i.e., $\Delta \ll R$. The connection between Δ and the mean free path $\lambda_f(T(r), \alpha_a)$ is established via a Lorentz transformation from the static laboratory frame to the fluid rest frame, giving

$$\Delta = \lambda_f(T(r), \alpha_a) \Gamma(r) \left(1 + \frac{v}{v_p^\parallel} \right),$$

where v is the fluid velocity and $v_p^\parallel$ is the component of the quasiparticle velocity parallel to v . Since $\lambda_f(T(r), \alpha_a)$ depends on the local temperature $T(r)$, conformal invariance implies $\lambda_f(T(r), \alpha_a) = \lambda_f(T_0, \alpha_a)/\Gamma(r)$, so that

$$\Delta = \lambda_f(T_0, \alpha_a) \left(1 + \frac{v}{v_p^\parallel} \right).$$

As the factor $(1 + v/v_p^\parallel)$ lies between 1 and 2, the scale hierarchy is equivalently expressed as $\Delta \sim \lambda_f(T_0, \alpha_a) \ll R$. More generally, the mean free path λ_f can be replaced by a microscopic correlation length, such as the thermal de Broglie wavelength λ_{th} .

the extensivity of $\log Z$ is due to the scaling of volume V as well as through dependency of transverse size R . This is further supported by the expression in Eq. (49), where the dependence on R (as well as on Ω) appears exclusively through the dimensionless common product ΩR . Therefore, the average pressure $\mathcal{P}(T_0, \alpha_a, \Omega, R)$ can be better expressed as $\mathcal{P}(T_0, \alpha_a, \Omega R)$, and differentiation over $\log Z(\lambda)$ on the left-hand side of the Eq. (44) can be written as

$$\partial_\lambda \log Z(\lambda)|_{\lambda \rightarrow 0} = \beta_0 V (3 + \Omega R \partial_{\Omega R}) \mathcal{P}(\beta_0, \alpha_a, \Omega R). \quad (52)$$

Substituting the above expression and Eq. (49) into Eq. (44), one can derive the expression for $\mathcal{P}(T_0, \alpha_a, \Omega R)$ in the following form:

$$\mathcal{P}(T_0, \alpha_a, \Omega R) = \frac{P_0(T_0, \alpha_a)}{(1 - \Omega^2 R^2)} + \frac{B_0 \Omega^2 (3 - R^2 \Omega^2)}{72(1 - R^2 \Omega^2)^2}, \quad (53)$$

which, upon using Eq. (A5), gives the thermodynamic potential in the form⁸

$$\Phi(T_0, \alpha_a, \Omega R) = -V \left(\frac{P_0(T_0, \alpha_a)}{(1 - \Omega^2 R^2)} + \frac{B_0 \Omega^2 (3 - R^2 \Omega^2)}{72(1 - R^2 \Omega^2)^2} \right). \quad (54)$$

Using the hierarchy of hydrodynamic scales $R \gg \beta_0$, one obtains, at leading order, an expression that matches Ref. [52], namely,

$$\Phi(T_0, \alpha_a, \Omega R, V) = -V \frac{P_0(T_0, \alpha_a)}{(1 - \Omega^2 R^2)} + \mathcal{O}(\Omega^2 \beta_0^2), \quad (55)$$

and, similarly, the corresponding leading-order contribution to the pressure can be expressed as $\mathcal{P}(T_0, \alpha_a, \Omega R) = \mathcal{P}^0(T_0, \alpha_a, \Omega R) + \mathcal{O}(\Omega^2 \beta_0^2)$, where the leading-order pressure is given by

$$\mathcal{P}^0(T_0, \alpha_a, \Omega R) = \frac{P_0(T_0, \alpha_a)}{(1 - \Omega^2 R^2)}, \quad (56)$$

which also captures the pressure anisotropy induced by rotation, as follows:

⁸The dependency on the system size in the effective average pressure can occur with nonuniform pressure which has a non-zero gradient to oppose the external force; for instance, for an isothermal atmosphere, the pressure decreases exponentially with altitude z in the presence of constant gravitational acceleration g , expressed as $P(z) = P_0 e^{-mgz/\beta}$, where β , m , and g are the inverse of temperature and the mass of the molecules of each strata of the atmosphere. Where averaged pressure $\mathcal{P} = \frac{P_0}{mgh\beta} (1 - e^{-mgh\beta})$ for an air column with vertical extent h . Here, the vertical extent appears through the combination $mgh\beta$ in volume-averaged pressure $\mathcal{P}$.

$$\begin{aligned}\mathcal{P}_\perp &= -\frac{R}{2V}\partial_R\Phi = \frac{\mathcal{P}^0(T_0, \alpha_a, \Omega R)}{(1 - \Omega^2 R^2)} + \mathcal{O}(\Omega^2 \beta_0^2), \\ \mathcal{P}_\parallel &= -\frac{L_z}{V}\partial_{L_z}\Phi = \mathcal{P}^0(T_0, \alpha_a, \Omega R) + \mathcal{O}(\Omega^2 \beta_0^2).\end{aligned}\quad (57)$$

Here, $\mathcal{P}_\perp$ and $\mathcal{P}_\parallel$ are the pressure perpendicular and parallel to the rotation axis, respectively. So, this helps us to understand that the volume-averaged hydrodynamic pressure as expressed on the left-hand side of Eq. (44) can be written as $\mathcal{P}_\parallel + 2\mathcal{P}_\perp$. Following Eq. (55), one observes that the dependence on Ω (or R) enters the thermodynamic potential Φ through the common product ΩR . This structure carries over to all thermodynamic quantities, as required by thermodynamic consistency. For example, the thermodynamic quantities such as the entropy S , total angular momentum $\vec{J}$ along the rotation axis $\hat{n} = \frac{\vec{Q}}{\Omega}$, and the conserved charges Q_a can be extracted from the thermodynamic potential (see the Appendix). For the present, the total angular momentum carried by this rigidly rotating matter has been calculated as⁹

$$\begin{aligned}\langle \hat{J}^{xy} \rangle &= -T_0 \partial_\Omega \log Z \Big|_{T_0, \alpha_a} \\ &= V \frac{2\Omega R^2 P_0(T_0, \alpha_a)}{(1 - \Omega^2 R^2)^2} + V \frac{B_0 \Omega (3 + R^2 \Omega^2)}{36(1 - R^2 \Omega^2)^3},\end{aligned}\quad (58)$$

which allows us to compute the isothermal moment of inertia along the principal axis of rotation, taken to be $\hat{n} = \frac{\vec{Q}}{\Omega}$ (assumed, without loss of generality, to lie along the z axis), given by

$$\begin{aligned}I(\Omega) &= \frac{\langle \hat{J}^{xy} \rangle}{\Omega}, \\ &= V \left(\frac{2R^2 P_0(T_0, \alpha_a)}{(1 - \Omega^2 R^2)^2} + \frac{B_0(3 + R^2 \Omega^2)}{36(1 - R^2 \Omega^2)^3} \right).\end{aligned}\quad (59)$$

Using the hierarchy of hydrodynamic scales $R \gg \beta_0$, we obtain the leading-order contribution

$$I(\Omega) = \frac{2P_0 R^2}{(1 - R^2 \Omega^2)^2} + \mathcal{O}(\beta^2 \Omega^2). \quad (60)$$

Moreover, we can also define the isothermal differential moment of inertia, $\tilde{I} = \frac{\partial \langle \hat{J}^{xy} \rangle}{\partial \Omega}$, whose positivity ensures the stability of the rotating system. Using Eq. (58), one can show that its leading-order contribution is given by

⁹By using the expectation value of the (t, ϕ) component of the energy-momentum tensor as expressed in Eq. (47), one can calculate the angular momentum as $\vec{J} = \int d^3\mathbf{x} \langle \hat{T}^{t\phi}(x) \rangle$, which is consistent with the expression (58) in the hydrodynamic regime.

$$\begin{aligned}\tilde{I}(\Omega) &= \partial_\Omega \langle \hat{J}^{xy} \rangle \\ &= V \frac{2R^2 P_0(T_0, \alpha_a)}{(1 - \Omega^2 R^2)^3} (1 + 3\Omega^2 R^2) + \mathcal{O}(\Omega^2 \beta_0^2),\end{aligned}\quad (61)$$

which also helps identify how the difference between the pressures along the radial direction and parallel to the rotation axis is related to the rotational effect, namely, $\mathcal{P}_\perp - \mathcal{P}_\parallel = \tilde{I} \frac{\Omega^2}{2V} + \mathcal{O}(\beta_0^2 \Omega^2)$.

Furthermore, in the static limit $\lim_{\Omega \rightarrow 0}$, both types of moment of inertia defined in Eqs. (59) and (61) converge to the same expression, i.e., $\lim_{\Omega \rightarrow 0} I(\Omega) = \lim_{\Omega \rightarrow 0} \tilde{I}(\Omega) = I_s$, which is expressed as

$$I_s = 2VP_0(T_0, \alpha_a)R^2 + \frac{VB_0}{12}. \quad (62)$$

This result is also consistent with previous studies [52,108]. One can identify that, at the hydrodynamic scale, the term proportional to B_0 arises from a quantum correction due to the spin contribution to the moment of inertia [108], which is suppressed. Here, both isothermal moments of inertia remain positive within the range $\Omega R \leq 1$, ensuring rotational stability for the thermal system [70,109,109]. It is also worth noting that the moment of inertia $I(\Omega)$ at nonvanishing Ω is larger than its static value I_s , as rotation tends to redistribute matter radially outward. In comparison to Refs. [44,48,51,53,110], one can expand the thermodynamic potential under the assumption of slow rotation, i.e., $\Omega R \ll 1$, yielding

$$\begin{aligned}\Phi(T_0, \alpha_a, \Omega R, V) &= -VP_0(1 + \Omega^2 R^2) + \mathcal{O}(\Omega^4 R^4), \\ &= \Phi(T_0, \alpha_a, V) - I_s \frac{\Omega^2}{2}.\end{aligned}\quad (63)$$

From the last expression, one can identify the moment of inertia as I_s , which remains independent of the angular velocity Ω , which implies the system undergoes no shape deformation.

On a final note, this work has demonstrated how the principle of extensivity governs the structure of the thermodynamic potential in homogeneous and inhomogeneous settings, thereby ensuring consistency with the thermodynamic limit. In the presence of rigid rotation, the thermodynamic pressure $\mathcal{P}(\beta_0, \alpha_a, \Omega R)$ depends on the angular velocity Ω only through the dimensionless combination ΩR and remains finite as long as the system size satisfies $R < \Omega^{-1}$.

It is worth noting that, in general, the Ω dependence of the pressure may enter through either Ω/T_0 or ΩR , as seen from the right-hand side of Eq. (49). However, within the hydrodynamic regime, the hierarchy $\Omega R \gg \Omega/T_0$ must be maintained, thus justifying the retention of only the contributions involving ΩR in thermodynamic quantities.

V. CONCLUSIONS

Here, an alternative real-time method has been described, based on the ETC relations between the dilatation generator and both the global charges and their associated currents at global equilibrium, as discussed in Sec. III. It is also known that the definitions of Poincaré charges given in Eqs. (11) and (12) are independent of the particular choice of the energy-momentum tensor, provided the different forms are related by the pseudogauge transformations defined in Eqs. (14) and (16). In addition, note that, apart from the Schwinger terms in the ETCs, the remaining terms contain a single spatial derivative on the $\delta(\mathbf{x} - \mathbf{y})$ functions. These terms are invariant under pseudogauge transformations, and these structures are not affected by the particular choice of the matter field with spin ≤ 1 [79,93,99–101]. From the general structure of ETCs, we derive Eq. (36), which captures the scaling behavior of $\log(Z)$ in theories of renormalizable matter fields with spin ≤ 1 , excluding gauge fields. The exclusion of the gauge field occurs due to the particular choice of the dilatation generator $\hat{D}(t)$ through Eq. (9) instead of Eq. (8), which involves the improved energy-momentum tensor $\hat{T}^{\mu\nu}$ to investigate the scaling properties (here it is a rigid scaling transformation) of the systems considered [82–85]. In our calculation, we have considered fermionic particles of spin 1/2. Thus, following the definition of the dilatation generator, Eq. (9) has been used to compute thermodynamic variables, e.g., moments of inertia.

With homogeneous thermodynamic parameters, it has been shown that the thermodynamic potential Φ does not have any dependence on the size of the system other than volume term V . In the case of rigid rotation, the dependency on the radial extent R appears through the common product ΩR [see Eq. (55)]. In recent results from rotating $SU(3)$ lattice gauge theory [44,48,51,53,110] as well as some thermal field theory calculations [38,50,111], the dependence of the transverse size is pronounced through the presence of a common product ΩR . However, in some theoretical approaches, the dependence on angular frequency Ω arises from a different common product $\frac{\Omega}{T_0}$ [14,49]. Here, it has been shown that the condition for the description of the applicability of hydrodynamics can be ensured by the hierarchy $R \gg \frac{1}{T_0}$, which enforces the dominance of ΩR terms in thermodynamic quantities. Consequently, the term containing $B_0 \Omega^2$ ($\sim P_0 \beta_0^2 \Omega^2$) in Eqs. (45) and (46) can be neglected compared to the magnitude of P_0 . This has been taken care of to express the intensive pressure $\mathcal{P}$ as given in Eq. (55) from Eq. (53), along with the extensivity properties of $\log Z$.

Equation (60) of this study can be useful in understanding the contribution of quarks to the moment of inertia of QGP produced in heavy-ion collisions, which has the greatest vorticity of any fluid produced in a laboratory [12–15]. As

an example, for a vortical QGP medium, one may consider the following values of the physical parameters: $T_0 \sim 200$ and $\Omega \sim 10^{22} \text{ s}^{-1} \approx 7 \text{ MeV}$. In particular, for Au + Au collisions, one may also consider $\mu_0 \sim 0.5 \text{ fm}^{-1} c$, with a system size $R \sim 14 \text{ fm}$ [13]. Using this set of parameters, the condition for the validity of hydrodynamics in such systems is found to be valid as the hierarchy $\Omega^2 R^2 \sim 300 \Omega^2 \beta_0^2 \gg \Omega^2 \beta_0^2$ is maintained. Hence, the dependency on the angular frequency Ω in the thermodynamic variables from the contribution of $\Omega \beta_0$ is suppressed. Moreover, with this consideration, the moment of inertia has been computed for rigidly rotating massless fermionic matter, as given in Eq. (60). The qualitative estimation of the light-quark contribution to the QGP, $I \sim 4.15 I_s$ at $T_0 \geq T_c$ and $\Omega R \sim 0.5$, where the values of the static limit of moment of inertia are $I_s \approx 1.08 \times 10^3$ and $I_s \approx 1.62 \times 10^3 \text{ MeV}^{-1}$ for $N_f = 2$ (or $g = 4$) and $N_f = 3$ (or $g = 6$), respectively. The computed moment of inertia agrees with previously reported results [52,108]. However, it differs from the value obtained in [112], where the moment of inertia is associated with the axial current density and is pseudogauge dependent. In global equilibrium, the result should not depend on the pseudogauge, since the density matrix is pseudogauge invariant, as it is constructed from total conserved charges as shown in Eq. (42). In our approach, the moment of inertia is unaffected by ambiguities related to the separation of spin and orbital components because it is derived directly from the thermodynamic potential. Moreover, the moment of inertia associated with a local current is more appropriately interpreted as the vortical conductivity of the axial current, as discussed in [108].

Along these lines, an interesting extension would be to establish an analytical estimation of the moment of inertia for gluon fields using the real-time method described in Sec. III. An investigation has recently been conducted from the first-principle lattice calculation [48,53] with the help of imaginary-time formalism, and an interesting result was derived for QGP having a negative moment of inertia below a particular temperature $\sim 1.5 T_c$. However, the approach in Sec. III excludes the gauge field. It can be generalized for the gauge field by introducing the appropriate virial field in Eq. (8). For example, the $SU(N)$ Lagrangian for the non-Abelian gauge field A_a^μ yields $V^\mu = 2\eta A_a^\mu \partial \cdot A_a - \partial^\mu (\bar{\xi}_a \xi_a) - g f_{abc} \bar{\xi}_a A_b^\mu \xi_c$, where η , g , and ξ_a are the gauge parameter, coupling constant, and ghost field, respectively. The presence of this virial field in the dilatation generator $\hat{D}(t)$ in Eq. (8) modifies the ETC given in Eqs. (27) and (28) only by the BRST (Carlo Becchi, Alain Rouet, Raymond Stora and Igor Tyutin) transformed operator [79,113,114]. However, this extra BRS-transformed part of an operator vanishes on the grand canonical averages, which involves the projection of the physical states [115,116]. So, in the context of extracting thermodynamic quantities for gauge theory, the improved tensor $\hat{T}^{\mu\nu}$ can be replaced by the

Belinfante tensor $\hat{T}_B^{\mu\nu}$ in Eq. (36) because it is indistinguishable from the physical thermal states. A separate study with the gauge field will be reported elsewhere.

The present approach relies on the extensivity of $\log Z$, although this scaling may be altered by quantum corrections. In particular, conformal (trace) anomalies [117,118] are expected to modify the formalism, and exploring this extension constitutes an interesting avenue for future investigation.

Moreover, in the context of (non)relativistic many-body systems, spin hydrodynamics has been kept alive due to several studies [119–126], in particular, recent studies of the spin polarization of hadrons produced from QGP [11,19]; and in condensed matter, the spin voltage arises due to the vorticity of the flowing spin liquids [121] drawing attention to theoretical investigations (see Refs. [15,18,22,55,90,91,124,124,127–160]). In addition, the emergence of the hydrodynamic regime in graphene [161] could be helpful to understand the magnetovortical effect by establishing a tabletop experiment with a proper initial inhomogeneous temperature profile as given in Eq. (43) and allows for the conversion of the orbital angular momentum to the spin angular momentum via the spin orbital effect, as derived in [162]. In this article, the study of the rotating system in global equilibrium shows that the total angular momentum can be computed as given in Eq. (58) along the principal axis of the system. However, if shape deformation leads to a change (slowly) of the principal axis that gives the spin polarization in the presence of the spin-orbit coupling, the system becomes magnetized. Also, the rigid rotation of the matter discards the effect of shear and bulk viscosity, but with the non-equilibrium statistical method, one can take into account the off-equilibrium contribution in the density matrix $\hat{\rho}$ with hydrodynamic gradient expansions, which may lead to some effect on spin transport. Both of these aspects will be explored in forthcoming studies.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: THERMODYNAMIC RELATIONS

In this appendix, thermodynamics-consistent relations have been derived using the density operators (1), starting with the definition of entropy S as

$$S = -\langle \log \hat{\rho} \rangle, \\ = b_\mu \langle \hat{P}^\mu \rangle - \frac{1}{2} \omega_{\mu\nu} \langle \hat{J}^{\mu\nu} \rangle - \alpha_a \langle \hat{Q}_a \rangle + \log Z. \quad (\text{A1})$$

Here, b_μ , $\omega_{\mu\nu}$, and α_a are the Lagrange multipliers for the corresponding global charges $\hat{P}^\mu$, $\hat{J}^{\mu\nu}$, and $\hat{Q}_a$, respectively, and Z is the partition function defined in Eq. (4). Thus, one can obtain

$$\delta S = b_\mu \langle \delta \hat{P}^\mu \rangle - \frac{1}{2} \omega_{\mu\nu} \langle \delta \hat{J}^{\mu\nu} \rangle - \alpha_a \langle \delta \hat{Q}_a \rangle, \quad (\text{A2})$$

which is the first law of thermodynamics.

Turning to solutions of the global equilibrium condition (3), one can identify $b_0 = \frac{1}{T_0} = \beta_0$, $b_i = \frac{\mathbf{V}}{T_0}$, where $\omega_{\mu\nu}$ becomes $\frac{\mathbf{a}}{T_0} (\delta_0^\mu \delta_\nu^\nu - \delta_\nu^\mu \delta_0^\nu)$ and $\frac{\Omega}{T_0} (\delta_\mathbf{x}^\mu \delta_\mathbf{y}^\nu - \delta_\mathbf{y}^\mu \delta_\mathbf{x}^\nu)$ if the fluid is accelerated and rotating along the $\mathbf{z}$ axis, where T_0 , $\mathbf{V}$, $\mathbf{a}$, and Ω are temperature, center-of-mass velocity, acceleration, and angular velocity, respectively. However, $\mu_a = T_0 \alpha_a$ is the chemical potential for each corresponding conserved matter charge.

More specifically, if we rewrite Eq. (A1) for the rigid rotation solution with the Killing vector

$$b_\mu = (\beta_0, \mathbf{0}) \quad \text{and} \quad \omega_{\mu\nu} = \beta_0 \Omega (g_{\mu x} g_{\nu y} - g_{\mu y} g_{\nu x}), \quad (\text{A3})$$

which helps to rewrite Eq. (A1) in this context as

$$T_0 S = E - \vec{\Omega} \cdot \vec{J} - \mu_a Q_a + T_0 \log Z. \quad (\text{A4})$$

Here, $E = \langle P^0 \rangle$, $\vec{J} = \langle \hat{J} \rangle$, and $Q_a = \langle \hat{Q}_a \rangle$ are the energy, angular momentum along the principal axis, and the values of charges, respectively, computed from the average of the grand canonical ensemble at global equilibrium with the density operator $\hat{\rho}$. Define

$$\Phi = -T_0 \log Z. \quad (\text{A5})$$

The last equality of the last expression is the consequence of the extensivity property of the thermodynamic potential Φ under the scaling $\mathbf{x} \rightarrow e^\lambda \mathbf{x}$ as $\Phi \rightarrow e^{3\lambda} \Phi$, thus $\mathcal{P}$ is an intensive variable [see Eq. (51)] [33,52]. Applying the first law of thermodynamics (A2), one may arrive at

$$\delta \Phi = -\delta T_0 S - \delta \vec{\Omega} \cdot \vec{J} - \delta \mu_a Q_a, \quad (\text{A6})$$

which defines the thermodynamics variables entropy S , angular momentum $\vec{J}$, and charges Q_a as

$$S = - \left(\frac{\partial \Phi}{\partial T_0} \right)_{\vec{\Omega}, \mu_a}, \quad (\text{A7})$$

$$\vec{J} = -\left(\frac{\partial\Phi}{\partial\vec{\Omega}}\right)_{T_0, \mu_a}, \quad (\text{A8})$$

and

$$Q_a = -\left(\frac{\partial\Phi}{\partial\mu_a}\right)_{T_0, \vec{\Omega}}. \quad (\text{A9})$$

Using these definitions for a given thermodynamic potential Φ , Eq. (55) can be shown as

$$S = V \frac{\partial P_0}{\partial T_0} \frac{(1 - \frac{\Omega^2 R^2}{3})}{(1 - \Omega^2 R^2)^2} + \mathcal{O}(\beta^2 \Omega^2), \quad (\text{A10})$$

$$Q_a = V \frac{\partial P_0}{\partial \mu_a} \frac{(1 - \frac{\Omega^2 R^2}{3})}{(1 - \Omega^2 R^2)^2} + \mathcal{O}(\beta^2 \Omega^2), \quad (\text{A11})$$

and the angular momentum along the principle axis $\vec{J}$ has been calculated in Eq. (58).

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