

Probing doubly charged Higgs bosons with three-body associated production at future e^+e^- colliders

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We study the discovery prospects for a doubly charged Higgs boson $H^{\pm\pm}$ in the 2-Higgs doublet model with type-II seesaw at future e^+e^- colliders. Focusing on the three-body channels $e^+e^- \rightarrow H^{\pm\pm} H_1^\mp H_1^\mp$ and $e^+e^- \rightarrow H^{\pm\pm} H_1^\mp W^\mp$, we scan the model parameter space subject to theoretical consistency as well as current collider, flavor, and electroweak precision observables. We find that these $2 \rightarrow 3$ production modes can exceed the conventional pair production rate $e^+e^- \rightarrow H^{++}H^{--}$, followed by $H^{\pm\pm} \rightarrow H_1^\pm H_1^\pm$ and $H_1^\pm W^\pm$ decays, over wide regions, particularly above the $H^{\pm\pm} \rightarrow H_1^\pm H_1^\pm$ and $H^{\pm\pm} \rightarrow H_1^\pm W^\pm$ thresholds, reaching cross sections up to $\mathcal{O}(10^2)$ fb for $\sqrt{s} = 500\text{--}1500$ GeV. A detector-level analysis of the $4\ell + \cancel{E}_T$ signature, including dominant multiboson and top quark backgrounds, shows that discovery sensitivity is achievable for $\sqrt{s} = 1000\text{--}1500$ GeV with integrated luminosities in the few ab^{-1} range, even in the presence of realistic systematic uncertainties.

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I. INTRODUCTION

Despite its remarkable success, the standard model (SM) leaves a number of fundamental questions unanswered, including the nature of dark matter (DM), the origin of the baryon asymmetry of the Universe, and the stability of the electroweak (EW) scale [1–5]. These open issues have motivated extensive efforts to go beyond the SM (BSM), with a particularly well-motivated direction being an extended Higgs sector. In parallel, the observation of

neutrino oscillations established that neutrinos are massive and mix, thereby requiring new BSM dynamics [6–8]. In this respect, a broad class of possible explanations is provided by seesaw mechanisms, which generate tiny neutrino masses from the dimension-five Weinberg operator [9]. Besides accounting for neutrino masses and the lepton-mixing pattern, seesaw frameworks often predict lepton number–violating signatures and new states that can be searched for at collider experiments.

The Weinberg operator can be ultraviolet, completed through three canonical realizations [9]: the type-I seesaw, featuring an $SU(2)_L$ -singlet fermion [10–13], the type-II seesaw, based on an $SU(2)_L$ -triplet scalar [13–15], and the type-III seesaw, involving an $SU(2)_L$ -triplet fermion [16]. In the type-II case, light neutrino masses are induced once the neutral component of the triplet develops a vacuum expectation value (VEV), such that $m_\nu \propto v_t Y_\Delta$, where v_t denotes the triplet VEV and Y_Δ the corresponding Yukawa couplings.

A characteristic prediction of the type-II seesaw is the presence of a doubly charged Higgs boson, $H^{\pm\pm}$. The phenomenology of $H^{\pm\pm}$ states has been studied extensively at hadron and lepton colliders based on a variety of

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theoretical contexts [17–84]. Importantly, the collider signatures of $H^{\pm\pm}$ are largely controlled by v_t , which is simultaneously the key parameter governing neutrino-mass generation. For $v_t \ll 10^{-4}$ GeV, the doubly charged Higgs boson decays predominantly into same-sign dileptons, $H^{\pm\pm} \rightarrow \ell^\pm \ell^\pm$ ($\ell = e, \mu$) [17,84]. Conversely, for $v_t \gg 10^{-4}$ GeV, the decay into same-sign dibosons, $H^{\pm\pm} \rightarrow W^\pm W^\pm$, becomes dominant [17,84]. Consequently, present and future searches typically target both dilepton and diboson final states.

In this work we focus on the 2-Higgs doublet model with type-II seesaw (2HDMcT) [85–91]. Since mass generation in seesaw frameworks is intimately tied to EW symmetry breaking (EWSB), in close analogy to the Brout-Englert-Higgs mechanism, scalar-extended realizations are especially appealing and lead to a rich phenomenology. Relative to the standard 2HDM scalar sector, the 2HDMcT exhibits a broader spectrum and distinctive signatures, with $H^{\pm\pm}$ providing a particularly clean “smoking gun.” Furthermore, the 2HDMcT can be viewed as one of the simplest and most economical extensions of the SM that can simultaneously accommodate neutrino masses [84,92] and address the DM problem [93].

A doubly charged Higgs state is currently being intensively searched for by ATLAS and CMS Collaborations at the Large Hadron Collider (LHC), notably, through pair production $pp \rightarrow Z/\gamma \rightarrow H^{++}H^{--}$ followed by same-sign dilepton decays [86]. A key quantitative difference between the 2HDMcT and the minimal Higgs triplet model (HTM) emerges in certain cascade topologies. In the 2HDMcT the chain $pp \rightarrow Z/\gamma \rightarrow H_2^+ H_2^- \rightarrow H^{++}W^-H^{--}W^+ \rightarrow \ell^+ \ell^+ \ell^- \ell^- + 4j$ can proceed fully on-shell. This is because the additional scalars relax the oblique-parameter bounds and allow large mass splittings $\Delta m = m_{H^\pm} - m_{H^{\pm\pm}}$ reaching values of order $\mathcal{O}(m_W)$ for $H_2^\pm$. In the HTM, electroweak precision data confine $|\Delta m| \lesssim 40$ GeV [33,34,62], forcing off-shell W bosons¹ in the analogous sequence $pp \rightarrow Z/\gamma \rightarrow H^+H^- \rightarrow H^{++}W^{*-}H^{--}W^{*+}$ and severely suppressing its rate. The larger mass splitting in the 2HDMcT opens the on-shell decay channels $H^{\pm\pm} \rightarrow W^\pm H_1^\pm$ and $H^{\pm\pm} \rightarrow H^\pm H_1^\pm$, thereby rendering the associated production modes $H^{\pm\pm}H_1^\mp H_1^\mp$ and $H^{\pm\pm}H_1^\mp W^\mp$ promising discovery channels for $H^{\pm\pm}$ at current and future colliders. Beyond collider phenomenology, it has also been shown that interactions between the doublet and triplet fields can induce a strong first-order EW phase transition, thereby enabling EW baryogenesis [95]. Overall, compared to the HTM, the production and decay patterns of $H^{\pm\pm}$ in the 2HDMcT are significantly modified due to the presence of the second Higgs doublet (see Ref. [96]).

In this study, we investigate doubly charged Higgs production in association with a pair of singly charged

Higgs bosons, $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp$, as well as its production in association with a $W^\pm$ boson and a singly charged Higgs state, $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp W^\mp$, within the type-X 2HDMcT framework at future electron-positron colliders. Our numerical results are obtained from a detailed scan of its parameter space, subject to theoretical requirements (perturbative unitarity and vacuum stability) and current experimental constraints (including measurements of the SM-like Higgs boson, exclusion limits on additional Higgs states, electroweak precision observables (EWPOs), flavor bounds, and limits from lepton flavor-violating processes). We perform a full Monte Carlo (MC) simulation at detector level and quantify the sensitivity to these channels, at the, e.g., International Linear Collider (ILC) [97], for center-of-mass (CoM) energies of 500, 1000, and 1500 GeV.²

This paper is organized as follows. In Sec. II, we briefly review the 2HDMcT model. Section III summarizes the theoretical and experimental constraints imposed on the parameter space. Our computational procedure and numerical results are presented in Sec. IV. Finally, Sec. V is devoted to our conclusions.

II. 2HDMcT: BRIEF REVIEW

The 2HDMcT contains two Higgs doublets Φ_i ($i = 1, 2$) and one colorless Higgs field Δ transforming as a triplet under the $SU(2)_L$ gauge group with hypercharge $Y_\Delta = 2$. In this case, the most general gauge-invariant Lagrangian of the 2HDMcT is given by

$$\mathcal{L} = \sum_{i=1}^2 (D_\mu \Phi_i)^\dagger (D^\mu \Phi_i) + \text{Tr}(D_\mu \Delta)^\dagger (D^\mu \Delta) - V(\Phi_i, \Delta) + \mathcal{L}_{\text{Yukawa}}, \quad (1)$$

where the covariant derivatives are defined as

$$D_\mu \Phi_i = \partial_\mu \Phi_i + igT^a W_\mu^a \Phi_i + ig' B_\mu \Phi_i, \quad (2)$$

$$D_\mu \Delta = \partial_\mu \Delta + ig[T^a W_\mu^a, \Delta] + ig' \frac{Y_\Delta}{2} B_\mu \Delta, \quad (3)$$

with (W_μ^a, g) and (B_μ, g') denoting, respectively, the $SU(2)_L$ and $U(1)_Y$ (gauge fields, couplings), while $T^a \equiv \sigma^a/2$, where the σ^a ($a = 1, 2, 3$) are the Pauli matrices. In terms of the two $SU(2)_L$ Higgs doublets Φ_i and the triplet field Δ , the 2HDMcT Higgs potential is given by [85,86]

¹The sign of Δm can be positive or negative in the HTM (see Ref. [94]).

²The choice of the ILC is merely related to the use of a detector card specific to such a machine, while our results are equally applicable to other TeV scale collider setups, like the Compact Linear Collider [98].

$$V(\Phi_1, \Phi_2, \Delta)$$

$$\begin{aligned} = & m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - [m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{H.c.}] + \frac{\lambda_1}{2} (\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_2}{2} (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1) (\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_1) \\ & + \left\{ \frac{\lambda_5}{2} (\Phi_1^\dagger \Phi_2)^2 + [\beta_1 (\Phi_1^\dagger \Phi_1) + \beta_2 (\Phi_2^\dagger \Phi_2)] \Phi_1^\dagger \Phi_2 + \text{H.c.} \right\} + \lambda_6 \Phi_1^\dagger \Phi_1 \text{Tr}(\Delta^\dagger \Delta) + \lambda_7 \Phi_2^\dagger \Phi_2 \text{Tr}(\Delta^\dagger \Delta) + \lambda_8 \Phi_1^\dagger \Delta \Delta^\dagger \Phi_1 \\ & + \lambda_9 \Phi_2^\dagger \Delta \Delta^\dagger \Phi_2 + m_\Delta^2 \text{Tr}(\Delta^\dagger \Delta) + \bar{\lambda}_8 (\text{Tr} \Delta^\dagger \Delta)^2 + \bar{\lambda}_9 \text{Tr}(\Delta^\dagger \Delta)^2 + \{\mu_1 \Phi_1^T i \sigma^2 \Delta^\dagger \Phi_1 + \mu_2 \Phi_2^T i \sigma^2 \Delta^\dagger \Phi_2 \\ & + \mu_3 \Phi_1^T i \sigma^2 \Delta^\dagger \Phi_2 + \text{H.c.}\}, \end{aligned} \quad (4)$$

where Tr denotes the trace over 2×2 matrices. The triplet Δ and the Higgs doublets Φ_i are represented by

$$\Delta = \begin{pmatrix} \delta^+/\sqrt{2} & \delta^{++} \\ (v_t + \delta^0 + i\eta_0)/\sqrt{2} & -\delta^+/\sqrt{2} \end{pmatrix}, \quad (5)$$

$$\Phi_1 = \begin{pmatrix} \phi_1^+ \\ \phi_1^0 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \phi_2^+ \\ \phi_2^0 \end{pmatrix}, \quad (6)$$

with $\phi_1^0 = (v_1 + \rho_1 + i\eta_1)/\sqrt{2}$ and $\phi_2^0 = (v_2 + \rho_2 + i\eta_2)/\sqrt{2}$. After spontaneous EWSB, the Higgs doublet and triplet fields acquire their VEVs denoted by, respectively, v_1 , v_2 , and v_t , so that eleven physical Higgs states appear, namely, three CP -even neutral Higgs bosons (h_1, h_2, h_3), four singly charged Higgs bosons³ ($H_1^\pm, H_2^\pm$), two CP -odd neutral Higgs bosons (A_1, A_2), and, finally, two doubly charged Higgs bosons $H^{\pm\pm}$.

Expanding the covariant derivative D_μ , and performing the usual transformations on the gauge and scalar fields to obtain the physical fields, one can identify the Higgs couplings of h_i to the massive gauge bosons $V = W, Z$, as given in Table I. Note that in our model, the triplet field Δ couples directly to the SM particles, so a new contribution will appear, and the two couplings $C_V^{h_i}$ ($V = W^\pm, Z$) differ from one to another by a factor of 2 associated to v_t .

A. The Yukawa sector

The Yukawa Lagrangian encompasses the entire Yukawa sector of the 2HDM along with an additional term $\mathcal{L}_{\text{Yukawa}}$. We list in Table II all the CP -even h_i ($i = 1, 2, 3$) and CP -odd A_j ($j = 1, 2$) Yukawa couplings for the type-X Yukawa texture in the model. The additional Yukawa term responsible for neutrino masses is given by

TABLE I. The normalized couplings of the neutral CP -even Higgs bosons (h_i) to the gauge bosons $V = W, Z$ in the 2HDMcT.

	$C_W^{h_i}$	$C_Z^{h_i}$
h_1	$\frac{v_1}{v} \mathcal{E}_{11} + \frac{v_2}{v} \mathcal{E}_{21} + 2 \frac{v_t}{v} \mathcal{E}_{31}$	$\frac{v_1}{v} \mathcal{E}_{11} + \frac{v_2}{v} \mathcal{E}_{21} + 4 \frac{v_t}{v} \mathcal{E}_{31}$
h_2	$\frac{v_1}{v} \mathcal{E}_{12} + \frac{v_2}{v} \mathcal{E}_{22} + 2 \frac{v_t}{v} \mathcal{E}_{32}$	$\frac{v_1}{v} \mathcal{E}_{12} + \frac{v_2}{v} \mathcal{E}_{22} + 4 \frac{v_t}{v} \mathcal{E}_{32}$
h_3	$\frac{v_1}{v} \mathcal{E}_{13} + \frac{v_2}{v} \mathcal{E}_{23} + 2 \frac{v_t}{v} \mathcal{E}_{33}$	$\frac{v_1}{v} \mathcal{E}_{13} + \frac{v_2}{v} \mathcal{E}_{23} + 4 \frac{v_t}{v} \mathcal{E}_{33}$

$$-\mathcal{L}_{\text{Yukawa}} = Y_{ij} L_i^T C i \sigma_2 \Delta L_j + \text{H.c.}, \quad (7)$$

where Y is a 3×3 complex symmetric matrix and $L = (\nu_L, l_L)^T$ denotes the $SU(2)_L$ doublets of left-handed leptons. This term, following EWSB, generates tiny (Majorana) masses for the neutrinos, which can be obtained from the Yukawa terms in Eq. (7) as

$$m_\nu = \sqrt{2} Y v_\Delta. \quad (8)$$

The 3×3 neutrino mass matrix, m_ν , can then be diagonalized through a unitary transformation involving the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, U_{PMNS} , according to

$$U_{\text{PMNS}}^T m_\nu U_{\text{PMNS}} = m_\nu^d = \text{diag}(m_1, m_2, m_3), \quad (9)$$

where m_1 , m_2 , and m_3 are the three neutrino mass eigenvalues. The matrix U_{PMNS} is conventionally parametrized in terms of three mixing angles ($\theta_{12}, \theta_{23}, \theta_{13}$), one Dirac CP -violating phase (δ), and two Majorana phases (ϕ_1, ϕ_2). It is given by

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{12}s_{13}s_{23}e^{i\delta} - c_{23}s_{12} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{23}s_{12}s_{13}e^{i\delta} - c_{12}s_{23} & c_{13}c_{23} \end{pmatrix} \cdot \text{diag}(e^{i\phi_1/2}, 1, e^{i\phi_2/2}). \quad (10)$$

³A detailed study of the singly charged Higgs boson at muon colliders was performed in Ref. [91].

TABLE II. The normalized Yukawa coupling coefficients of the neutral Higgs bosons (h_i and A_j) to the leptons (ℓ), up (U), and down quarks (D) in 2HDMcT. For the expressions of the coefficients $\mathcal{E}_{ij}$ and $\mathcal{O}_{ij}$, see Ref [86].

	$C_U^{h_1}$	$C_D^{h_1}$	$C_\ell^{h_1}$	$C_U^{h_2}$	$C_D^{h_2}$	$C_\ell^{h_2}$	$C_U^{h_3}$	$C_D^{h_3}$	$C_\ell^{h_3}$	$C_U^{A_1}$	$C_D^{A_1}$	$C_\ell^{A_1}$	$C_U^{A_2}$	$C_D^{A_2}$	$C_\ell^{A_2}$
Type-X	$\frac{\mathcal{E}_{12}}{s_\beta}$	$\frac{\mathcal{E}_{12}}{s_\beta}$	$\frac{\mathcal{E}_{11}}{c_\beta}$	$\frac{\mathcal{E}_{22}}{s_\beta}$	$\frac{\mathcal{E}_{22}}{s_\beta}$	$\frac{\mathcal{E}_{21}}{c_\beta}$	$\frac{\mathcal{E}_{32}}{s_\beta}$	$\frac{\mathcal{E}_{32}}{s_\beta}$	$\frac{\mathcal{E}_{31}}{c_\beta}$	$\frac{\mathcal{O}_{22}}{s_\beta}$	$\frac{\mathcal{O}_{22}}{s_\beta}$	$\frac{\mathcal{O}_{21}}{c_\beta}$	$\frac{\mathcal{O}_{32}}{s_\beta}$	$\frac{\mathcal{O}_{32}}{s_\beta}$	$\frac{\mathcal{O}_{31}}{c_\beta}$

Assuming a normal hierarchy (NH) of neutrino masses, $m_{\nu_1} < m_{\nu_2} < m_{\nu_3}$, all free parameters of the neutrino sector can be expressed in terms of the neutrino oscillation parameters and the mass of the lightest neutrino,

$$m_{\nu_1}, \theta_{12}, \theta_{13}, \theta_{23}, \Delta m_{21}^2, \Delta m_{31}^2, \delta, \phi_1, \phi_2. \quad (11)$$

The neutrino mass-squared differences in the NH are taken as $\Delta m_{31}^2 > 0$ and $\Delta m_{21}^2 > 0$, allowing one to determine the heavier neutrino masses through

$$m_{\nu_2} = \sqrt{m_{\nu_1}^2 + \Delta m_{21}^2}, \quad m_{\nu_3} = \sqrt{m_{\nu_1}^2 + \Delta m_{31}^2}. \quad (12)$$

In the case of an inverted hierarchy (IH), the lightest neutrino is m_{ν_3} ($m_{\nu_3} < m_{\nu_2} < m_{\nu_1}$), which sets the overall neutrino mass scale. The corresponding set of input parameters becomes

$$m_{\nu_3}, \theta_{12}, \theta_{13}, \theta_{23}, \Delta m_{21}^2, \Delta m_{32}^2, \delta, \phi_1, \phi_2, \quad (13)$$

with $\Delta m_{21}^2 > 0$ and $\Delta m_{32}^2 < 0$, with the remaining neutrino masses determined by

$$m_{\nu_1} = \sqrt{m_{\nu_3}^2 - \Delta m_{32}^2 - \Delta m_{21}^2}, \quad m_{\nu_2} = \sqrt{m_{\nu_3}^2 - \Delta m_{32}^2}. \quad (14)$$

The parameters $m_{\nu_3}, \Delta m_{21}^2, \Delta m_{32}^2, \theta_{12}, \theta_{13}, \theta_{23}$, and δ are largely fixed by the global fit of oscillation data [99], while the smallest neutrino mass $m_{\nu_{\min}} = m_{\nu_1}$ (in NH (IH)), and the two Majorana phases, ϕ_1, ϕ_2 , are not constrained by current oscillation data.

In the charged sector, the doubly charged eigenvalue $m_{H^{\pm\pm}}^2$, corresponding to the doubly charged eigenstate $H^{\pm\pm}$, can simply be determined by collecting all the coefficients of $\delta^{++}\delta^{--}$ in the Higgs potential. This gives

$$m_{H^{\pm\pm}}^2 = \frac{\sqrt{2}\mu_1 v_1^2 + \sqrt{2}\mu_3 v_1 v_2 + \sqrt{2}\mu_2 v_2^2 - \lambda_8 v_1^2 v_t - \lambda_9 v_2^2 v_t - 2\bar{\lambda}_9 v_t^3}{2v_t}. \quad (15)$$

III. CONSTRAINTS

The phenomenological analysis in 2HDMcT is performed via implementation of a full set of theoretical constraints [86,87] as well as Higgs exclusion limits from various experimental measurements at colliders, as follows:

- (i) *Unitarity*: the Higgs and Goldstone $2 \rightarrow 2$ scattering matrix elements must obey conservation of probability.
- (ii) *Perturbativity*: the quartic couplings of the Higgs potential are constrained by the following conditions: $|\lambda_i| < 8\pi$.
- (iii) *Vacuum stability*: boundedness from below arising from the positivity in any direction of the fields Φ_i and Δ is required.
- (iv) *EWPOs*: the oblique parameters S , T , and U [100,101] have been calculated in the 2HDMcT [87] and measured at the Large Electron-Positron (LEP) collider as well as Tevatron. In the light of the new Particle Data Group mass of the $W^\pm$ boson, they yield [102] (including correlation)

$$\hat{S}_0 = -0.05 \pm 0.07, \quad \hat{T}_0 = 0.00 \pm 0.06, \\ \rho_{ST} = 0.93,$$

for which we use the following χ_{ST}^2 test:

$$\frac{(S - \hat{S}_0)^2}{\sigma_S^2} + \frac{(T - \hat{T}_0)^2}{\sigma_T^2} - 2\rho_{ST} \frac{(S - \hat{S}_0)(T - \hat{T}_0)}{\sigma_S \sigma_T} \\ \leq R^2(1 - \rho_{ST}^2), \quad (16)$$

with $R^2 = 2.3$ and 5.99 corresponding to 68.3% and 95% confidence levels (CLs), respectively. Our numerical analysis is performed with χ_{ST}^2 at 95% CL.

- (v) *Colliders*: to further delimit the allowed parameter space, the HiggsTools package [103] is employed. This ensures that the allowed parameter regions align with the observed properties of the 125 GeV Higgs boson (via HiggsSignals [103–106]) and with the limits from searches for additional Higgs bosons at the LHC, Tevatron, and LEP (via HiggsBounds [103,107–110]).

TABLE III. Experimental result on the flavor observable $\bar{B} \rightarrow X_s \gamma$ at 95% CL.

Observable	Experimental result	95% CL
$\text{BR}(\bar{B} \rightarrow X_s \gamma)$ [87]	$(3.49 \pm 0.19) \times 10^{-4}$ [111]	$[3.11 \times 10^{-4}, 3.87 \times 10^{-4}]$

- (vi) *Flavor*: flavor constraints are also implemented in our analysis. We have used B -physics results, derived in [87], as well as the experimental data at 2σ CL [111] displayed in Table III.

Constraints specific to the parameter space of the 2HDMcT are as follows:

- (i) ρ *Parameter*: the presence of a scalar triplet modifies the ρ parameter as

$$\rho = \frac{v_0^2 + 2v_t^2}{v_0^2 + 4v_t^2} \approx 1 - 2\frac{v_t^2}{v_0^2} = 1 + \delta\rho. \quad (17)$$

In the SM, one has $\rho = 1$ at tree level. Any departure from this value is tightly constrained and the latest global fit to EWPOs [102] yields

$$\rho = 1.00031 \pm 0.00019, \quad (18)$$

which is about 1.6σ above the SM tree-level prediction. This small but nonzero deviation imposes an upper bound on the triplet VEV (v_t) in the 2HDMcT framework [52,62].

- (ii) *Lepton flavor violation (LFV)*: from the Yukawa interaction shown in Eq. (7), LFV decays like $\mu \rightarrow e\gamma$ at loop level and $\mu \rightarrow 3e$ at tree level can be possible. The branching ratios (BRs) in the 2HDMcT can be calculated as [86,112–114]

$$\text{BR}(\mu \rightarrow e\gamma) = 384\pi^2 |A_R|^2, \quad (19)$$

$$A_R = \frac{-q_e |(h^\dagger h)_{e\mu}|}{48\sqrt{2}\pi^2 G_F} \left(\frac{C_{23}}{m_{H^\pm}^2} + \frac{C_{33}}{m_{H^\pm}^2} + \frac{8}{m_{H^{\pm\pm}}^2} \right), \quad (20)$$

$$\text{BR}(\mu \rightarrow 3e) = \frac{|h_{ee}|^2 |h_{\mu e}|^2}{4G_F^2 m_{H^{\pm\pm}}^4}, \quad (21)$$

with

$$h_{ij} = \frac{m_{ij}}{\sqrt{2}v_t} = \frac{1}{\sqrt{2}v_t} (U_{\text{PMNS}} \text{diag}(m_1, m_2, m_3) U_{\text{PMNS}}^T)_{ij}. \quad (22)$$

Here, α denotes the electromagnetic fine-structure constant and G_F the Fermi constant, while C_{23} and C_{33} are elements of the charged-scalar rotation matrix. The explicit expressions for C_{23} and C_{33} , as well as for the other rotation matrix elements, can be found in Ref. [86]. The upper bounds on the BRs of the above processes are 1.5×10^{-13} for $\mu \rightarrow e\gamma$ [115] and 1.0×10^{-12} for $\mu \rightarrow 3e$ [116].

- (iii) *Neutrino oscillation experiments*: the experimental constraints on the neutrino oscillation parameters [99],

$$\Delta m_{21}^2 = 7.49_{-0.19}^{+0.19} \times 10^{-5} \text{ eV}^2, \quad (23)$$

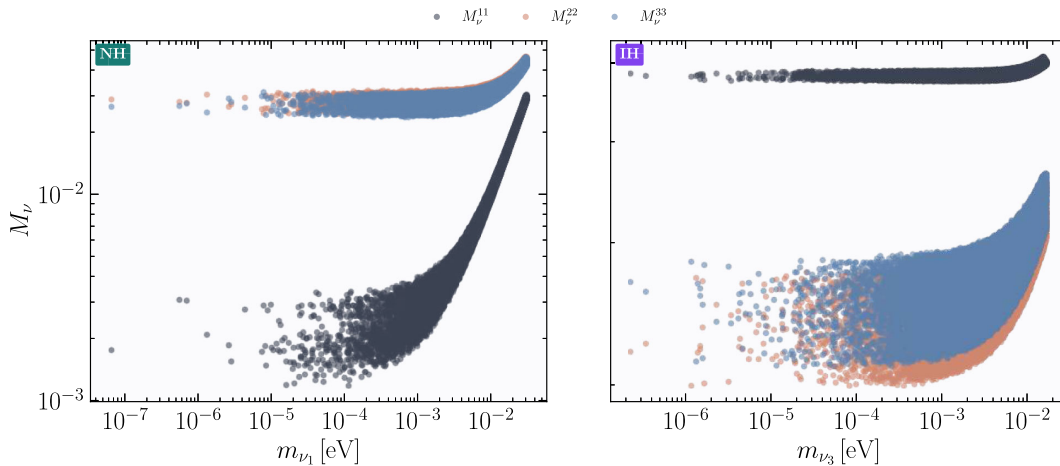


FIG. 1. Constraints on the diagonal entries of the neutrino mass matrix M_ν as functions of the lightest neutrino mass, m_{ν_1} for the NH (left) and m_{ν_3} for the IH (right). The phases are fixed to $\Phi_1 = 0$ and $\Phi_2 = 0$.

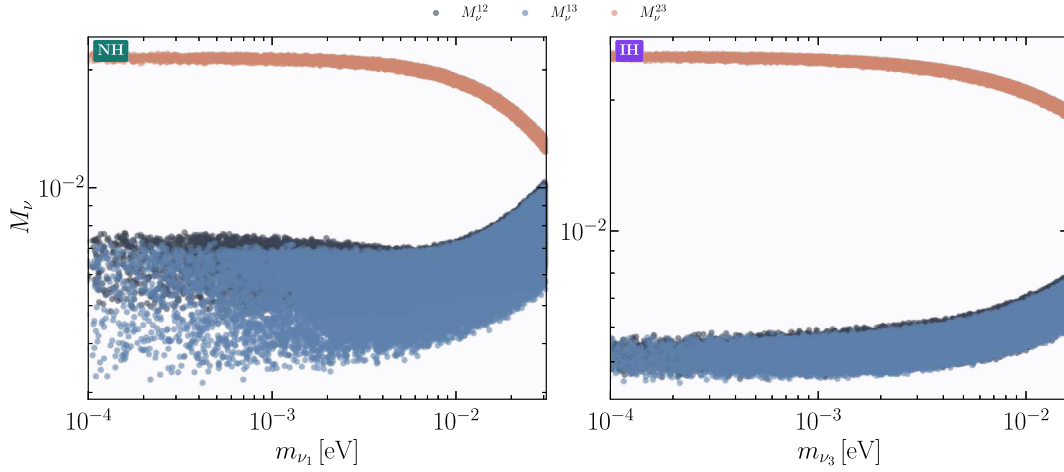


FIG. 2. Constraints on the off-diagonal elements' entries of the neutrino mass matrix M_ν as functions of the lightest neutrino mass, m_{ν_1} for the NH (left) and m_{ν_3} for the IH (right). The phases are fixed to $\Phi_1 = 0$ and $\Phi_2 = 0$.

$$\sin^2\theta_{12} = 0.308^{+0.012}_{-0.011}, \quad (24)$$

$$\sin^2\theta_{13}: \begin{cases} \sin^2\theta_{13} = 0.02215^{+0.00056}_{-0.00058} & (\text{NH}), \\ \sin^2\theta_{13} = 0.02231^{+0.00056}_{-0.00056} & (\text{IH}), \end{cases} \quad (25)$$

$$\sin^2\theta_{23}: \begin{cases} \sin^2\theta_{23} = 0.470^{+0.017}_{-0.013} & (\text{NH}), \\ \sin^2\theta_{23} = 0.550^{+0.012}_{-0.015} & (\text{IH}), \end{cases} \quad (26)$$

$$|\Delta m_{3l}^2|: \begin{cases} \Delta m_{3l}^2 = 2.513^{+0.021}_{-0.019} \times 10^{-3} \text{ eV}^2 & (\text{NH}), \\ \Delta m_{3l}^2 = -2.484^{+0.020}_{-0.020} \times 10^{-3} \text{ eV}^2 & (\text{IH}). \end{cases} \quad (27)$$

Also, cosmological observations require the sum of neutrino masses to be [102]

$$\sum_{\nu=1}^3 m_\nu < 0.12 \text{ eV}. \quad (28)$$

The above experimental constraints affect the allowed values for the neutrino mass matrix m_ν , as seen in Figs. 1 and 2, as a function of the lightest neutrino mass. For illustration we fix the phases to $\Phi_1 = 0$ and $\Phi_2 = 0$. In fact, these results directly reflect the structure of the neutrino mass matrix and the observed mixing pattern. In the case of NH (left panel), the diagonal elements satisfy

$$M_\nu^{11} < M_\nu^{22}, M_\nu^{33}, \quad (29)$$

whereas for IH (right panel) one finds

$$M_\nu^{11} > M_\nu^{22}, M_\nu^{33}. \quad (30)$$

Among the off-diagonal elements, M_ν^{23} attains the largest values in both hierarchies, reflecting the large atmospheric mixing angle, as illustrated in Fig. 2.

IV. COMPUTATIONAL PROCEDURE AND RESULTS

To identify the phenomenologically viable regions of the 2HDMcT we scan its parameter space while enforcing all theoretical and experimental constraints introduced earlier. The full set of independent input parameters considered in our analysis is

$$\mathcal{P}_I = \{\alpha_1, \alpha_2, \alpha_3, m_{h_1}, m_{h_2}, \lambda_1, \lambda_3, \lambda_4, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \bar{\lambda}_8, \bar{\lambda}_9, \mu_1, v_t, \tan\beta\}. \quad (31)$$

These parameters are varied within the following ranges:

$$\begin{aligned} m_{h_1} &= 125.09 \text{ GeV}, & m_{h_1} &\leq m_{h_2} \leq m_{h_3} \leq 1000 \text{ GeV}, \\ -\frac{\pi}{2} &\leq \alpha_1 \leq \frac{\pi}{2}, & -0.1 &\leq \alpha_{2,3} \leq 0.1, \\ 0.5 &\leq \tan\beta \leq 120, & -10^2 &\leq \mu_1 \leq 10^2, \\ 0 &\leq v_t \leq 2 \text{ GeV}, & -8\pi &\leq \lambda_i, \bar{\lambda}_i \leq 8\pi, \end{aligned} \quad (32)$$

with $\tan\beta = v_2/v_1$. Hereafter, the lightest CP -even state h_1 is taken to represent the observed Higgs boson, and its mass is fixed to 125 GeV [117,118]. Each parameter point generated during the scan is subjected to all relevant constraints and only the parameter points that satisfy these are subsequently passed to FormCalc [119–121], which we employ to compute the cross sections of the production channels considered in this work. This procedure ensures that all reported results correspond exclusively to parameter configurations that are theoretically consistent and experimentally allowed.

As intimated, here, we investigate the electron-positron production of a doubly charged Higgs boson in association with a pair of singly charged Higgs bosons ($e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp$), as well as in association with a

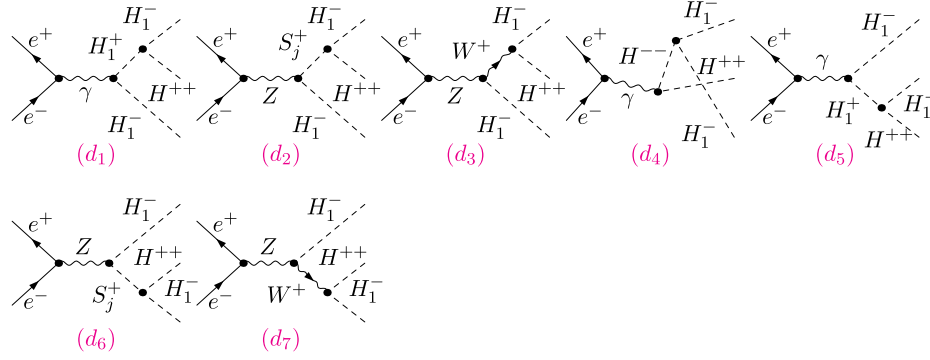


FIG. 3. Tree-level generic Feynman diagrams for $e^-e^+ \rightarrow H^{++}H_1^-H_1^-$ are shown in $(d_{1,\dots,7})$. For all diagrams $S_j^\pm$ refers to $H_1^\pm$ or $H_2^\pm$.

singly charged Higgs boson and a $W^\pm$ boson ($e^+e^- \rightarrow H^{\pm\pm}H_1^\mp W^\mp$). The corresponding Feynman diagrams are displayed in Figs. 3 and 4, respectively.

For the process $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp$, the amplitude receives its dominant contributions from s -channel topologies. The s -channel diagrams [Fig. 3 ($d_{1,\dots,7}$)] are mediated by a γ or a Z boson.

For the associated production channel $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp W^\mp$, the amplitude arises from a broader set of gauge and Higgs interactions, as illustrated in Fig. 4. In this case, the diagrams may be grouped into s -channel and t -channel classes. Specifically, the s -channel contributions [Fig. 4 ($d_{1,\dots,7}$)] proceed through the exchange of a γ or a Z boson. These include configurations where the intermediate vector boson subsequently splits into a $W^\pm H_1^\mp$. The t -channel contributions [Fig. 4 (d_8)] arise from neutrino exchange and are characteristic of processes that produce a final-state $W^\pm$ boson, for which emission can take place either from an internal neutrino or Higgs propagator.

A. Relevant couplings

The production mechanisms for $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp$ and $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp W^\mp$ are controlled by a set of gauge and Higgs interactions characteristic of the 2HDMcT framework. Here, we briefly summarize the most important

couplings that govern the size and behavior of the amplitudes:

- (a) *Gauge interactions.* The doubly charged Higgs boson couples to the EW gauge bosons through its $SU(2)_L$ -triplet nature. The most relevant vertices are as follows:
 - (i) $H^{++}H_1^{--}\gamma$ and $H^{++}H_1^{--}Z$: these arise from covariant-derivative interactions [see Eqs. (2) and (3)] and play a central role in the s -channel production via photon and Z exchange.
 - (ii) $H^{++}H_1^-W^-$: a gauge interaction proportional to the EW coupling g . This vertex contributes to both the $e^-e^+ \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp$ and $e^-e^+ \rightarrow H^{\pm\pm}H_1^\mp W^\mp$ processes.
 - (iii) $W^-\nu_e e^-$: the charged-current interaction, responsible for neutrino-mediated t -channel diagrams in both processes. This vertex is essential for the process $e^-e^+ \rightarrow H^{\pm\pm}H_1^\mp W^\mp$.
- (b) *Higgs interactions.* The extended Higgs sector of the 2HDMcT gives rise to multiple trilinear and quartic interactions involving both charged and neutral Higgs states [see the Higgs potential in Eq. (4)]. Both processes are directly sensitive to the detail of the scalar potential through diagrams like d_2 and others. The most relevant interactions are listed below:

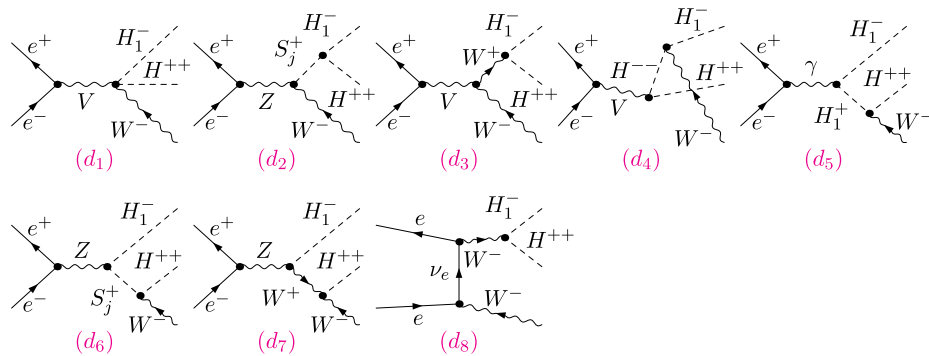
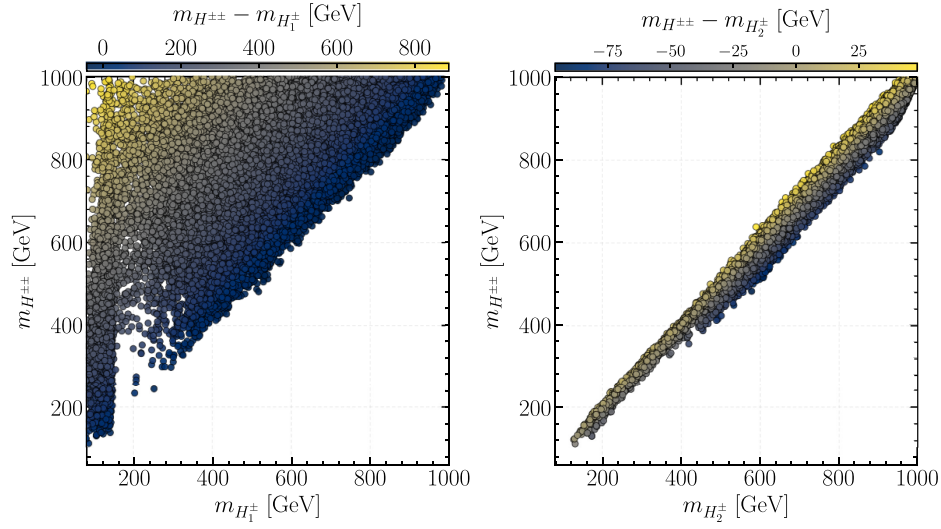
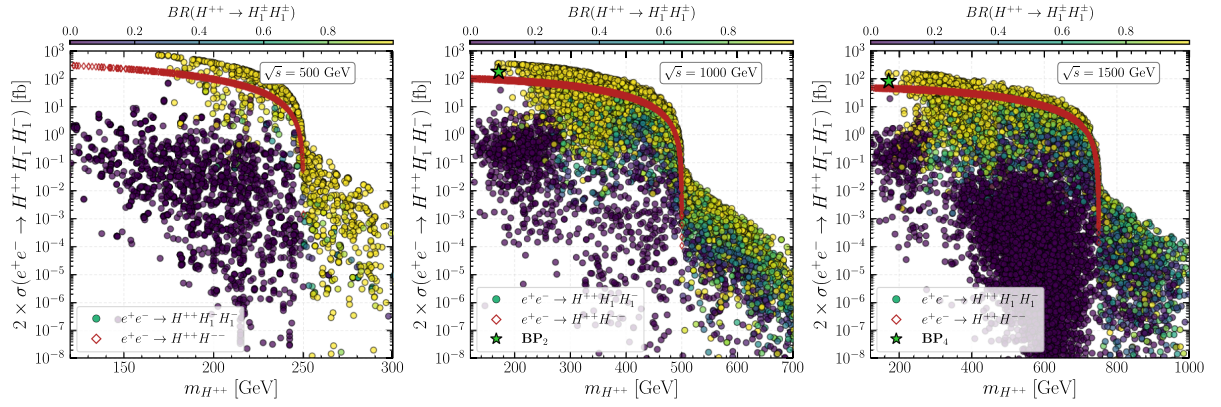


FIG. 4. Tree-level generic Feynman diagrams for $e^-e^+ \rightarrow H^{++}H_1^-W^-$ are shown in $(d_{1,\dots,8})$. For all diagrams $S_j^\pm$ refers to $H_1^\pm$ or $H_2^\pm$. Furthermore, $V = \gamma, Z$.

FIG. 5. Mass splitting between $H^{\pm\pm}$ and $H_1^\pm$ (left), and between $H^{\pm\pm}$ and $H_2^\pm$ (right).FIG. 6. Production cross sections for $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp$ as a function of $m_{H^{++}}$ at $\sqrt{s} = 500, 1000$, and 1500 GeV.

- (i) $H^{--}H_1^-H_1^-$: a trilinear scalar coupling generated by the triplet-doublet mixing. This vertex contributes to both the $e^-e^+ \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp$ and $e^-e^+ \rightarrow H^{\pm\pm}H_1^\mp W^\mp$ processes.
- (ii) $H^{++}H_2^+H_1^-$: a trilinear scalar coupling generated by the triplet-doublet mixing. It appears in diagrams mediated by a Z boson, such as Fig. 3 (d_6) and Fig. 4 (d_2). This vertex also

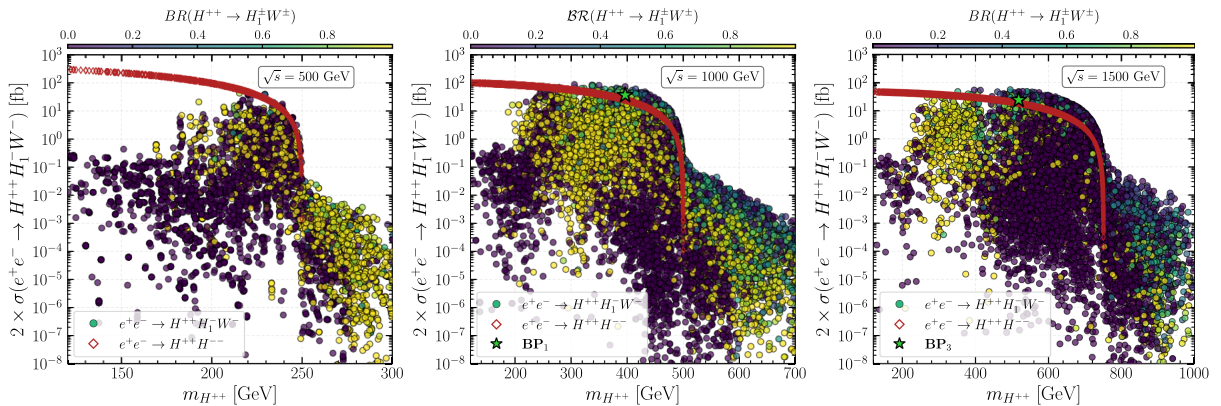
FIG. 7. Production cross sections for $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp W^\mp$ as a function of $m_{H^{++}}$ at $\sqrt{s} = 500, 1000$, and 1500 GeV.

TABLE IV. BPs used for the signal-to-background analysis.

	m_{h_1}	m_{h_2}	λ_1	λ_3	λ_4	λ_6	λ_7	λ_8	λ_9	$\bar{\lambda}_8$	$\bar{\lambda}_9$	$\tan\beta$	α_1	α_2	α_3	v_t	σ (fb)
BP1	125.09	164.40	0.7043	-6.22×10^{-2}	0.5547	0.3290	0.0862	-0.1013	0.2678	0.2703	0.8361	118.71	-1.5636	-3.68×10^{-4}	-4.28×10^{-2}	0.4271	36.58
BP2	125.09	145.30	0.5678	-4.03×10^{-2}	0.2655	0.7227	7.99×10^{-2}	-0.5346	7.68×10^{-2}	2.0904	0.3526	82.41	-1.5615	-2.96×10^{-4}	-3.30×10^{-2}	5.88×10^{-2}	177.02
BP3	125.09	266.81	0.1888	1.9616	-2.74×10^{-2}	7.0303	0.9439	-8.54×10^{-2}	0.9211	-0.7390	1.8138	115.93	-1.5628	-1.40×10^{-4}	-6.13×10^{-2}	0.4143	24.00
BP4	125.09	145.30	0.5678	-4.03×10^{-2}	0.2655	0.7227	0.0799	-0.5346	0.0768	2.0904	0.3526	92.58	-1.5615	-2.96×10^{-4}	-3.30×10^{-2}	0.0588	81.83

enters both the $e^-e^+ \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp$ and $e^-e^+ \rightarrow H^{\pm\pm}H_1^\mp W^\mp$ processes.

- (iii) $h_k H^{++}H^{--}$ and $h_k H_i^- H_j^+$: couplings of the neutral CP -even Higgs bosons ($k = 1, 2, 3$) to singly charged ($i, j = 1, 2$) and doubly charged Higgs states. These contribute to the s -channel diagrams, which are suppressed due to the small electron Yukawa coupling.

Overall, the competition between gauge and scalar interactions sets the diagram hierarchy and dictates the cross section behavior within the allowed 2HDMcT parameter space.

B. Cross sections

As previously mentioned, the mass splitting between the doubly charged Higgs state and the singly charged ones plays a role in evading EWPO constraints in the 2HDMcT, hence, we present it in Fig. 5. From here, it is clear that the mass difference $m_{H^{++}} - m_{H_1^+}$ can be quite significant, whereas $m_{H^{++}}$ and $m_{H_2^+}$ are nearly degenerate. In fact, such a mass gap is also important in driving a hierarchy between the production rates of $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp$ relative to those for $e^+e^- \rightarrow H^{++}H^{--}$ followed by $H^{\pm\pm} \rightarrow H_1^\pm H_1^\pm$ decays (and consequently also in the case of $H^{\pm\pm} \rightarrow H_1^\pm W^\pm$ transitions).

In Fig. 6, we present the cross section σ for the process $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp$ at $\sqrt{s} = 500, 1000$, and 1500 GeV. This production channel is primarily governed by the trilinear scalar vertex $H^{++}H_1^-H_1^-$ and the gauge-Higgs vertex $H^{++}H_1^-W^-$, which are proportional to the couplings $\lambda_{H^{++}H_1^-H_1^-}$ and C_{23} , respectively. As a result, the majority of the contributing diagrams depend sensitively on these interactions. The clear message stemming from this figure is that the $H^{\pm\pm}H_1^\mp H_1^\mp$ final state is largely driven by $H^{++}H^{--}$ production (the empty red diamond curves), followed by $H^{\pm\pm} \rightarrow H_1^\pm H_1^\pm$, yet, the contribution from all other diagrams can be significant not just when the latter decay is forbidden ($m_{H^{--}} < 2m_{H_1^-}$) but also when it is not, thus pointing to their significance (in part also aided by the off-shellness effect in such a decay), in fact, raising the $\sigma(e^+e^- \rightarrow H^{++}H^{--}) \times \text{BR}(H^{\pm\pm} \rightarrow H_1^\pm H_1^\pm)$ rates by more than a factor of 2 in the $m_{H^{--}} > 2m_{H_1^-}$ region. Such an effect is visible at all energies considered, altogether

yielding excess cross sections (i.e., well beyond those from on-shell pair production and decay) already for $m_{H^{++}}$ values around 180 GeV.

In a similar manner, Fig. 7 displays the cross sections for the process $\sigma(e^+e^- \rightarrow H^{\pm\pm}H_1^\mp W^\mp)$ at $\sqrt{s} = 500, 1000$, and 1500 GeV. This channel is predominantly controlled by the trilinear scalar interaction $H^{++}H_1^-H_1^-$ and the gauge-scalar interaction $H^{++}H_1^-W^-$, which are governed by the couplings $\lambda_{H^{++}H_1^-H_1^-}$ and C_{23} , respectively. Only the diagrams (d_1) and (d_7) are independent of these two vertices. Consequently, the majority of the diagrams contributing to this process exhibit pronounced sensitivity to the values of these two couplings. The cross sections arising from the $2 \rightarrow 3$ production mode can again exceed those obtained from the $2 \rightarrow 2$ production followed by decay, however, this occurs only for significantly larger values and within a narrower mass range of the doubly charged Higgs boson (and only at $\sqrt{s} = 1000$ and 1500 GeV).

Therefore, we define four benchmark points (BPs) for which the discussed $2 \rightarrow 3$ body cross section dominates the $2 \rightarrow 2$ body one (times BR), as exemplified by the green stars in the center and right panels of Figs. 6 and 7. The corresponding input parameter values are given in Table IV. We will use these in the forthcoming MC analysis, which we will limit to the CoM energies of 1000 and 1500 GeV.

C. Signal-to-background analysis

To quantify the observability of the doubly charged Higgs boson $H^{\pm\pm}$ predicted in the 2HDMcT with the type-X Yukawa texture,⁴ crucially, in the region dominated by the discussed $2 \rightarrow 3$ body processes, we develop search strategies aimed at maximizing the separation between the ensuing signals and SM backgrounds. The analysis is based on a state-of-the-art simulation chain that accounts for matrix element generation, resonance decays, parton showering, hadronization, heavy flavor decays, jet reconstruction, and detector effects. As a representative case study, we focus on the clean multilepton signature $\ell^+\ell^+\ell^-\ell^- + E_T$ (henceforth, $4\ell + E_T$ for short), which we will prove to provide strong signal sensitivity once the

⁴Note that the production cross sections are largely independent of the Yukawa texture, as they are governed primarily by gauge interactions. The dependence on the type-X structure enters through the decays of the singly charged Higgs states.

dominant backgrounds are efficiently suppressed. For each collider energy considered in this work, we select a representative BP from the scan, reported in Table IV. The production cross sections discussed above indicate that the signal rates are potentially accessible, provided that a dedicated background rejection strategy is implemented.

- (a) *MC event generation and detector simulation.* Signal events are generated at parton level with MadGraph5_aMC_v3.4.2 [122,123], interfaced to Pythia-8.20 [124] for parton showering, hadronization, and heavy flavor decays. Jets are clustered using FastJet [125]. Detector effects are modeled with Delphes-3.4.5 [126], using the default Delphes_Card_ILCgen⁵ detector card, which implements the anti- k_t algorithm [127]. Finally, both the signals and background events are analyzed with MadAnalysis5 [128,129].
- (b) *Signal channels and decay topologies.* The $4\ell + \cancel{E}_T$ signature is obtained from two different production and decay patterns. For the first channel, we consider $H^{\pm\pm}H_1^\mp W^\mp$ production followed by the cascade $H^{\pm\pm} \rightarrow H_1^\pm W^\pm$ and $H_1^\pm \rightarrow \tau_\ell \nu_\tau$ with leptonic $W^\pm$ decays,

$$\begin{aligned} e^+e^- \rightarrow H^{\pm\pm}H_1^\mp W^\mp &\rightarrow H_1^\pm W^\pm H_1^\mp W^\mp \\ &\rightarrow \tau_\ell^+ \ell^+ \tau_\ell^- \ell^- + \cancel{E}_T \\ &\rightarrow 4\ell + \cancel{E}_T. \end{aligned}$$

The second channel is $H^{\pm\pm}H_1^\mp H_1^\mp$ production with $H^{\pm\pm} \rightarrow H_1^\pm H_1^\pm$ and leptonic τ decays,

$$\begin{aligned} e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp &\rightarrow H_1^\pm H_1^\pm H_1^\mp H_1^\mp \\ &\rightarrow \tau_\ell^+ \tau_\ell^+ \tau_\ell^- \tau_\ell^- + \cancel{E}_T \\ &\rightarrow 4\ell + \cancel{E}_T. \end{aligned}$$

Here $\tau^\pm$ denotes the charged τ lepton, τ_ℓ indicates its leptonic decay mode, and $\ell = e, \mu$ light charged leptons. The missing transverse energy $\cancel{E}_T$ originates from neutrinos in τ and $W^\pm$ decays.

- (c) *SM backgrounds.* The dominant SM backgrounds to the $4\ell + \cancel{E}_T$ signature arise from multiboson and top-associated production: W^+W^-Z , ZZZ , $ZZ\gamma$, $t\bar{t}Z$, $t\bar{t}\gamma$, and $W^+W^-W^+W^-$. We assume semileptonic top decays and leptonic $W^\pm$ decays throughout. The Z boson is taken to decay into $\ell^+\ell^-$ or $\nu_\ell\bar{\nu}_\ell$, while $\gamma \rightarrow \ell^+\ell^-$ is imposed when relevant. The background channels are generated according to the following list:
- (i) $e^+e^- \rightarrow W^-W^+Z, (W^- \rightarrow \ell^-\bar{\nu}_\ell), (W^+ \rightarrow \ell^+\nu_\ell), (Z \rightarrow \ell^+\ell^-)$.
 - (ii) $e^+e^- \rightarrow ZZZ, (Z \rightarrow \ell^+\ell^-), (Z \rightarrow \ell^+\ell^-), (Z \rightarrow \nu_\ell\bar{\nu}_\ell)$.
 - (iii) $e^+e^- \rightarrow ZZ\gamma, (Z \rightarrow \ell^+\ell^-), (\gamma \rightarrow \ell^+\ell^-), (Z \rightarrow \nu_\ell\bar{\nu}_\ell)$.
 - (iv) $e^+e^- \rightarrow t\bar{t}Z, (t \rightarrow W^+b, W^+ \rightarrow \ell^+\nu_\ell), (\bar{t} \rightarrow W^-\bar{b}, W^- \rightarrow \ell^-\bar{\nu}_\ell), (Z \rightarrow \ell^+\ell^-)$.
 - (v) $e^+e^- \rightarrow t\bar{t}\gamma, (t \rightarrow W^+b, W^+ \rightarrow \ell^+\nu_\ell), (\bar{t} \rightarrow W^-\bar{b}, W^- \rightarrow \ell^-\bar{\nu}_\ell), (\gamma \rightarrow \ell^+\ell^-)$.
 - (vi) $e^+e^- \rightarrow W^+W^-W^+W^-, (W^- \rightarrow \ell^-\bar{\nu}_\ell), (W^+ \rightarrow \ell^+\nu_\ell), (W^- \rightarrow \ell^-\bar{\nu}_\ell), (W^+ \rightarrow \ell^+\nu_\ell)$.
- (d) *Object preselection.* Events are required to satisfy the baseline acceptance and isolation criteria

$$|\eta^{j,\ell}| < 2.5, \quad \Delta R^{\ell\ell,jj} \geq 0.5, \quad (33)$$

ensuring reconstructed leptons and jets remain within the fiducial region and are well separated.

- (e) *Significance definition.* The signal sensitivity $\mathcal{Z}$ is quantified using the median discovery significance of Ref. [130]. For signal and background yields s and b , respectively, and a fractional systematic uncertainty δ on the background, we compute

$$\mathcal{Z} = \sqrt{2 \left[(s+b) \ln \left(\frac{(s+b)(1+\delta^2 b)}{b + \delta^2 b(s+b)} \right) - \frac{1}{\delta^2} \ln \left(1 + \delta^2 \frac{s}{1 + \delta^2 b} \right) \right]}. \quad (34)$$

1. Processes at $\sqrt{s}=1000$ GeV

We illustrate in Fig. 8 the kinematic distributions for the $4\ell + \cancel{E}_T$ final state at $\sqrt{s} = 1000$ GeV. These include the pseudorapidity $\eta(\ell_1)$ of the leading lepton (left panel), the transverse momentum $p_T(\ell_1)$ (middle panel), and the missing transverse energy $\cancel{E}_T$ (right panel).

$e^+e^- \rightarrow H^{\pm\pm}H_1^\mp W^\mp \rightarrow 4\ell + \cancel{E}_T$ (BP1) Guided by these distributions, we construct a sequential cut strategy summarized in Table V. We first suppress top-induced backgrounds by rejecting events with more than one b -tagged jet, $N(b) \leq 1$, which removes about 78% of the $t\bar{t}Z/\gamma$ contributions. We then restrict the leading-lepton pseudorapidity to the central region, $-0.9 < \eta(\ell_1) < 0.9$, eliminating most of the remaining $t\bar{t}Z/\gamma$ events and about 50% of W^+W^-Z , while retaining $\sim 87.5\%$ of the signal. An upper bound $P_T(\ell_1) < 300$ GeV further reduces the

⁵<https://github.com/iLCSoft/ILCDelphes>

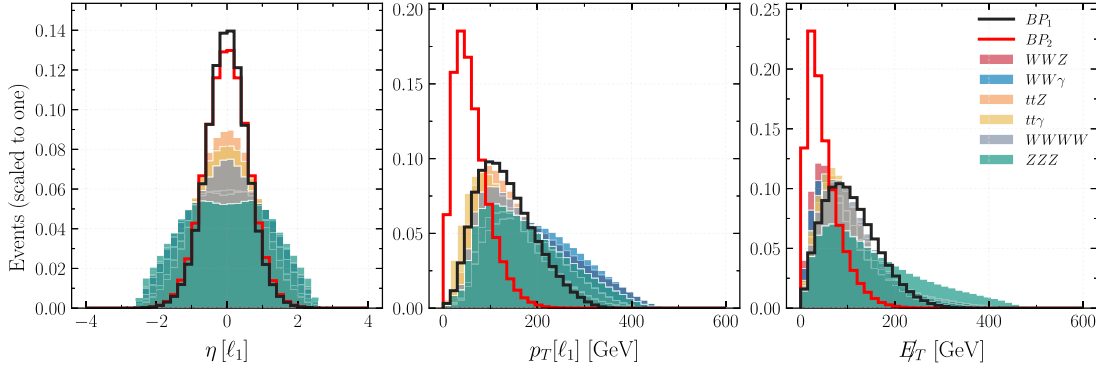


FIG. 8. Kinematic distributions for the $4\ell + E_T$ final state at $\sqrt{s} = 1000$ GeV. The pseudorapidity $\eta(\ell_1)$ of the leading lepton (left), the transverse momentum $p_T(\ell_1)$ (middle), and the missing transverse energy E_T (right).

W^+W^-Z background by 64% with only a modest loss in signal efficiency. Finally, requiring $E_T < 300$ GeV yields an additional suppression of about 85% of the top backgrounds and 65% of W^+W^-Z . The cumulative impact of these requirements on the final cross sections is reported in Table IX.

$e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp \rightarrow 4\ell + E_T$ (BP2) An analogous optimization is performed for this channel, with the selection requirements listed in Table VI. After the b -jet requirement $N(b) \leq 1$, about 78% of the $t\bar{t}Z$ and $t\bar{t}\gamma$ backgrounds are rejected. Constraining the leading-lepton pseudorapidity to $-1.6 < \eta(\ell_1) < 1.6$ suppresses the bulk of the top-induced contributions and about 18% of W^+W^-Z , while retaining 97.6% of the signal. A tighter transverse-momentum cut $P_T(\ell_1) < 200$ GeV reduces the remaining W^+W^-Z contribution by 60% with a signal efficiency of 97.3%. The requirement $E_T < 200$ GeV further improves rejection, removing roughly 86% of $t\bar{t}Z/\gamma$ and 65% of W^+W^-Z . The full cut-flow is shown in Table X.

2. Processes at $\sqrt{s} = 1500$ GeV

The corresponding kinematic distributions at $\sqrt{s} = 1500$ GeV are presented in Fig. 9. As before, the pseudorapidity $\eta(\ell_1)$ of the leading lepton (left panel), the transverse momentum $p_T(\ell_1)$ (middle panel), and the missing transverse energy E_T (right panel) are shown.

TABLE V. Selection criteria applied in the signal-to-background analysis of the process $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp \rightarrow 4l + E_T$ at $\sqrt{s} = 1000$ GeV for BP1.

Cuts	Definition
Trigger	$N(b) \leq 1$
Cut-1	$-0.9 < \eta[l_1] < 0.9$
Cut-2	$P_T[l_1] < 300$ GeV
Cut-3	$E_T < 300$ GeV

$e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp \rightarrow 4\ell + E_T$ (BP3) Based on these distributions, we define the optimized selection summarized in Table VII. The requirement $N(b) \leq 1$ removes about 80% of the $t\bar{t}Z$ and $t\bar{t}\gamma$ backgrounds. Restricting the leading-lepton pseudorapidity to $-0.9 < \eta(\ell_1) < 0.9$ suppresses roughly 88% of the remaining top backgrounds and about 59% of W^+W^-Z , while maintaining 89.2% of the signal. We then impose $P_T(\ell_1) < 400$ GeV, which reduces the remaining EW backgrounds substantially, followed by $E_T < 400$ GeV, which eliminates about 90% of the top-induced backgrounds and about 79% of W^+W^-Z . The corresponding cut-flow is given in Table XI.

$e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp \rightarrow 4\ell + E_T$ (BP4) For this topology, the optimized selection strategy is summarized in Table VIII. After applying $N(b) \leq 1$, nearly 79% of the $t\bar{t}Z$ and $t\bar{t}\gamma$ backgrounds are removed. Requiring $-1.4 < \eta(\ell_1) < 1.4$ further suppresses the remaining top backgrounds and about 20% of the W^+W^-Z/γ contributions, while keeping the signal efficiency close to 94%. Imposing $P_T(\ell_1) < 300$ GeV reduces the W^+W^-Z background by 71%, with 93.7% of the signal retained. Finally, $E_T < 300$ GeV yields an additional rejection of about 90% of $t\bar{t}Z/\gamma$ and 75% of W^+W^-Z . The resulting cross section evolution through the cut-flow is reported in Table XII.

3. Summary of sensitivities

In Table XIII we present the discovery significance $\mathcal{Z}$ for all BPs, explicitly indicating the associated production

TABLE VI. Selection criteria applied in the signal-to-background analysis of the process $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp \rightarrow 4l + E_T$ at $\sqrt{s} = 1000$ GeV for BP2.

Cuts	Definition
Trigger	$N(b) \leq 1$
Cut-1	$-1.6 < \eta[l_1] < 1.6$
Cut-2	$P_T[l_1] < 200$ GeV
Cut-3	$E_T < 200$ GeV

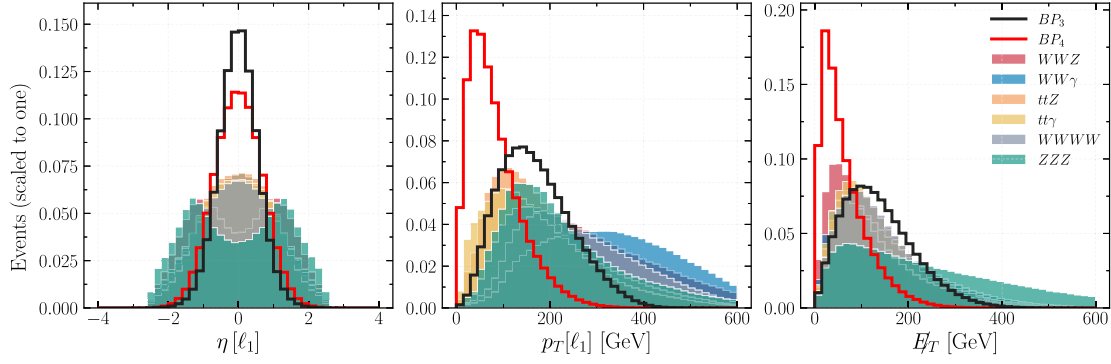


FIG. 9. Kinematic distributions for the $4\ell + E_T$ final state at $\sqrt{s} = 1500$ GeV. The pseudorapidity $\eta(\ell_1)$ of the leading lepton (left), the transverse momentum $p_T(\ell_1)$ (middle), and the missing transverse energy E_T (right).

processes. Systematic uncertainties of $\delta = 5\%$ and 10% are included for $\sqrt{s} = 1000$ and 1500 GeV, and the projected sensitivities are shown for integrated luminosities of $\mathcal{L} = 500, 1000$, and 1500 fb^{-1} . The results demonstrate

that, already at $\mathcal{L} = 500 \text{ fb}^{-1}$, discovery-level significance can be achieved for $e^+e^- \rightarrow H^{++}H_1^-H_1^- \rightarrow 4\ell + E_T$, whereas $e^+e^- \rightarrow H^{++}H_1^-W^- \rightarrow 4\ell + E_T$ typically requires higher luminosity to reach a comparable sensitivity.

TABLE VII. Selection criteria applied in the signal-to-background analysis of the process $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp W^\mp \rightarrow 4l + E_T$ at $\sqrt{s} = 1500$ GeV for BP3.

Cuts	Definition
Trigger	$N(b) \leq 1$
Cut-1	$-0.9 < \eta[l_1] < 0.9$
Cut-2	$P_T[l_1] < 400 \text{ GeV}$
Cut-3	$E_T < 400 \text{ GeV}$

TABLE VIII. Selection criteria applied in the signal-to-background analysis of the process $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp \rightarrow 4l + E_T$ at $\sqrt{s} = 1500$ GeV for BP4.

Cuts	Definition
Trigger	$N(b) \leq 1$
Cut-1	$-1.4 < \eta[l_1] < 1.4$
Cut-2	$P_T[l_1] < 300 \text{ GeV}$
Cut-3	$E_T < 300 \text{ GeV}$

TABLE IX. Cut flow of the cross sections (in fb) for the signal and SM backgrounds at $\sqrt{s} = 1000$ GeV using BP1.

Cuts	Signal	Backgrounds					
	BP1	W^+W^-Z	$W^+W^-\gamma$	$t\bar{t}Z$	$t\bar{t}\gamma$	$W^+W^-W^+W^-$	ZZZ
Basic cut	1.00×10^{-1}	1.22×10^{-1}	1.98×10^{-2}	1.07×10^{-2}	2.08×10^{-3}	1.26×10^{-3}	1.04×10^{-3}
Tagger	9.99×10^{-2}	1.221×10^{-1}	1.979×10^{-2}	2.44×10^{-3}	4.58×10^{-4}	1.26×10^{-3}	1.04×10^{-3}
Cut-1	8.75×10^{-2}	6.34×10^{-2}	1.03×10^{-2}	1.71×10^{-3}	3.04×10^{-4}	7.52×10^{-4}	4.96×10^{-4}
Cut-2	8.68×10^{-2}	4.48×10^{-2}	7.04×10^{-3}	1.60×10^{-3}	2.83×10^{-4}	6.77×10^{-4}	3.83×10^{-4}
Cut-3	8.66×10^{-2}	4.27×10^{-2}	6.88×10^{-3}	1.59×10^{-3}	2.81×10^{-4}	6.73×10^{-4}	3.14×10^{-4}
Total efficiencies (%)	86.6	34.9	34.7	14.8	13.5	53.3	30.3

TABLE X. Cut flow of the cross sections (in fb) for the signal and SM backgrounds at $\sqrt{s} = 1000$ GeV using BP2.

Cuts	Signal	Backgrounds					
	BP2	W^+W^-Z	$W^+W^-\gamma$	$t\bar{t}Z$	$t\bar{t}\gamma$	$W^+W^-W^+W^-$	ZZZ
Basic cut	7.94×10^{-1}	1.22×10^{-1}	1.98×10^{-2}	1.07×10^{-2}	2.08×10^{-3}	1.26×10^{-3}	1.04×10^{-3}
Tagger	7.943×10^{-1}	1.221×10^{-1}	1.979×10^{-2}	2.44×10^{-3}	4.58×10^{-4}	1.26×10^{-3}	1.04×10^{-3}
Cut-1	7.76×10^{-1}	1.01×10^{-1}	1.65×10^{-2}	2.28×10^{-3}	4.20×10^{-4}	1.10×10^{-3}	8.25×10^{-4}
Cut-2	7.74×10^{-1}	4.86×10^{-2}	7.08×10^{-3}	1.69×10^{-3}	3.15×10^{-4}	7.42×10^{-4}	4.55×10^{-4}
Cut-3	7.72×10^{-1}	4.30×10^{-2}	6.54×10^{-3}	1.56×10^{-3}	2.94×10^{-4}	7.11×10^{-4}	3.19×10^{-4}
Total efficiencies (%)	97.2	35.1	33.0	14.5	14.1	56.3	30.8

TABLE XI. Cut flow of the cross sections (in fb) for the signal and SM backgrounds at $\sqrt{s} = 1500$ GeV using BP3.

Cuts	Signal	Backgrounds					
	BP3	W^+W^-Z	$W^+W^-\gamma$	$t\bar{t}Z$	$t\bar{t}\gamma$	$W^+W^-W^+W^-$	ZZZ
Basic cut	5.87×10^{-2}	8.30×10^{-2}	1.86×10^{-2}	1.46×10^{-3}	3.03×10^{-4}	1.66×10^{-3}	8.09×10^{-4}
Tagger	5.873×10^{-2}	8.30×10^{-2}	1.858×10^{-2}	1.46×10^{-3}	3.03×10^{-4}	1.66×10^{-3}	8.09×10^{-4}
Cut-1	5.25×10^{-2}	3.47×10^{-2}	1.08×10^{-2}	8.72×10^{-4}	1.77×10^{-4}	9.33×10^{-4}	2.81×10^{-4}
Cut-2	5.23×10^{-2}	1.89×10^{-2}	5.12×10^{-3}	7.44×10^{-4}	1.39×10^{-4}	6.99×10^{-4}	1.96×10^{-4}
Cut-3	5.22×10^{-2}	1.79×10^{-2}	4.91×10^{-3}	7.30×10^{-4}	1.37×10^{-4}	6.91×10^{-4}	1.35×10^{-4}
Total efficiencies (%)	88.7	21.7	26.4	10.2	9.5	41.5	16.7

TABLE XII. Cut flow of the cross sections (in fb) for the signal and SM backgrounds at $\sqrt{s} = 1500$ GeV using BP4.

Cuts	Signal	Backgrounds					
	BP4	W^+W^-Z	$W^+W^-\gamma$	$t\bar{t}Z$	$t\bar{t}\gamma$	$W^+W^-W^+W^-$	ZZZ
Basic cut	1.947×10^{-1}	8.30×10^{-2}	1.858×10^{-2}	1.46×10^{-3}	3.03×10^{-4}	1.66×10^{-3}	8.09×10^{-4}
Tagger	1.947×10^{-1}	8.30×10^{-2}	1.858×10^{-2}	1.46×10^{-3}	3.03×10^{-4}	1.66×10^{-3}	8.09×10^{-4}
Cut-1	1.83×10^{-1}	5.79×10^{-2}	1.49×10^{-2}	1.21×10^{-3}	2.45×10^{-4}	1.30×10^{-3}	5.02×10^{-4}
Cut-2	1.83×10^{-1}	2.44×10^{-2}	4.45×10^{-3}	9.20×10^{-4}	1.65×10^{-4}	7.66×10^{-4}	3.09×10^{-4}
Cut-3	1.83×10^{-1}	2.19×10^{-2}	4.03×10^{-3}	8.53×10^{-4}	1.55×10^{-4}	7.37×10^{-4}	1.87×10^{-4}
Total efficiencies (%)	93.6	26.4	21.6	12.0	10.8	44.3	23.0

TABLE XIII. Significances $\mathcal{Z}$ for all BPs. Systematic uncertainties $\delta = 5\%$ and 10% are included at energies $\sqrt{s} = 1000$ and 1500 GeV for integrated luminosities of $\mathcal{L} = 500, 1000$, and 1500 fb^{-1} . The underlined results refer to the case of a clear discovery.

BP	Process	$\sqrt{s}$ [GeV]	δ [%]	$\mathcal{L} = 500 \text{ fb}^{-1}$	$\mathcal{L} = 1000 \text{ fb}^{-1}$	$\mathcal{L} = 1500 \text{ fb}^{-1}$
BP1	$e^+e^- \rightarrow H^{\pm\pm}H_1^\mp W^\mp \rightarrow 4l + \cancel{E}_T$	1000	5	4.65	<u>6.03</u>	<u>6.86</u>
			10	3.72	4.32	4.60
BP2	$e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp \rightarrow 4l + \cancel{E}_T$	1000	5	25.62	<u>32.15</u>	<u>35.84</u>
			10	<u>19.14</u>	<u>21.59</u>	<u>22.68</u>
BP3	$e^+e^- \rightarrow H^{\pm\pm}H_1^\mp W^\mp \rightarrow 4l + \cancel{E}_T$	1500	5	3.36	4.46	<u>5.15</u>
			10	2.82	3.38	3.66
BP4	$e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp \rightarrow 4l + \cancel{E}_T$	1500	5	<u>9.71</u>	<u>12.71</u>	<u>14.56</u>
			10	7.93	9.34	10.02

As expected, increasing δ induces a mild reduction in $\mathcal{Z}$, while the sensitivity degrades more noticeably at larger $\sqrt{s}$ due to the reduced signal cross sections. We emphasize that systematic uncertainties are treated here in a simplified manner by assuming a flat fractional uncertainty on the total background [130].

V. CONCLUSION

We have investigated the discovery prospects for a doubly charged Higgs boson in the type-X configuration of the 2HDMcT framework at future e^+e^- colliders, concentrating on the $2 \rightarrow 3$ body production channels $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp$ and $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp W^\mp$. We have done so by carrying out a detailed detector-level MC

simulation of the ensuing $4\ell + \cancel{E}_T$ signal ($\ell = e, \mu$) and dominant SM backgrounds. The BPs considered were required to satisfy theoretical requirements of self-consistency and to comply with current experimental limits.

We have found that these production modes can substantially exceed the yield of the conventional ($2 \rightarrow 2$ body) pair production process $e^+e^- \rightarrow H^{++}H^{--}$ followed by the decays $H^{\pm\pm} \rightarrow H_1^\pm H_1^\pm$ and $H^{\pm\pm} \rightarrow H_1^\pm W^\pm$, even when the latter are kinematically open, over sizable regions of the parameter space of our BSM scenario.

In short, our results establish $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp H_1^\mp$ and $e^+e^- \rightarrow H^{\pm\pm}H_1^\mp W^\mp$ as powerful and, for some parameter configurations, leading probes of $H^{\pm\pm}$ states at lepton colliders, as our sophisticated signal-to-background study

has demonstrated that a 5σ discovery is achievable through a very clean BSM signature for representative BPs at promoted CoM energies and integrated luminosities of a future e^+e^- collider operating at the TeV scale. Furthermore, beam polarization is expected to enhance the signal cross sections and thereby improve the overall discovery potential.

We therefore encourage experimental analyses to depart from mainstream approaches leveraging $H^{\pm\pm}$ pair production and decay to test triplet Higgs models in e^+e^- settings. In fact, we finally emphasize that our choice of BSM scenario is by no means restrictive, as it was used for quantitatively illustrative purposes of the advocated new discovery channels.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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- [1] F. Zwicky, *Helv. Phys. Acta* **6**, 110 (1933).
 - [2] V. C. Rubin and W. K. Ford, Jr., *Astrophys. J.* **159**, 379 (1970).
 - [3] M. J. G. Veltman, *Acta Phys. Pol. B* **12**, 437 (1981).
 - [4] A. G. Riess *et al.* (Supernova Search Team), *Astron. J.* **116**, 1009 (1998).
 - [5] A. Crivellin and B. Mellado, *Nat. Rev. Phys.* **6**, 294 (2024).
 - [6] T. Kajita, *Rev. Mod. Phys.* **88**, 030501 (2016).
 - [7] A. B. McDonald, *Rev. Mod. Phys.* **88**, 030502 (2016).
 - [8] Y. Cai, T. Han, T. Li, and R. Ruiz, *Front. Phys.* **6**, 40 (2018).
 - [9] S. Weinberg, *Phys. Rev. Lett.* **43**, 1566 (1979).
 - [10] P. Minkowski, *Phys. Lett. B* **67**, 421 (1977).
 - [11] M. Gell-Mann, P. Ramond, and R. Slansky, *Conf. Proc. C* **790927**, 315 (1979).
 - [12] R. N. Mohapatra and G. Senjanovic, *Phys. Rev. Lett.* **44**, 912 (1980).
 - [13] J. Schechter and J. W. F. Valle, *Phys. Rev. D* **22**, 2227 (1980).
 - [14] T. P. Cheng and L. F. Li, *Phys. Rev. D* **22**, 2860 (1980).
 - [15] R. N. Mohapatra and G. Senjanovic, *Phys. Rev. D* **23**, 165 (1981).
 - [16] R. Foot, H. Lew, X. G. He, and G. C. Joshi, *Z. Phys. C* **44**, 441 (1989).
 - [17] T. Han, B. Mukhopadhyaya, Z. Si, and K. Wang, *Phys. Rev. D* **76**, 075013 (2007).
 - [18] A. G. Akeroyd, M. Aoki, and H. Sugiyama, *Phys. Rev. D* **77**, 075010 (2008).
 - [19] A. G. Akeroyd and C. W. Chiang, *Phys. Rev. D* **80**, 113010 (2009).
 - [20] A. G. Akeroyd, C. W. Chiang, and N. Gaur, *J. High Energy Phys.* **11** (2010) 005.
 - [21] F. del Aguila and J. A. Aguilar-Saavedra, *Nucl. Phys. B* **813**, 22 (2009).
 - [22] M. Aoki, S. Kanemura, and K. Yagyu, *Phys. Rev. D* **85**, 055007 (2012).
 - [23] C.-W. Chiang, T. Nomura, and K. Tsumura, *Phys. Rev. D* **85**, 095023 (2012).
 - [24] H. Sugiyama, K. Tsumura, and H. Yokoya, *Phys. Lett. B* **717**, 229 (2012).
 - [25] A. G. Akeroyd, S. Moretti, and H. Sugiyama, *Phys. Rev. D* **85**, 055026 (2012).
 - [26] E. J. Chun and P. Sharma, *J. High Energy Phys.* **08** (2012) 162.
 - [27] F. del Águila and M. Chala, *J. High Energy Phys.* **03** (2014) 027.
 - [28] S. Kanemura, K. Yagyu, and H. Yokoya, *Phys. Lett. B* **726**, 316 (2013).
 - [29] S. Kanemura, M. Kikuchi, K. Yagyu, and H. Yokoya, *Phys. Rev. D* **90**, 115018 (2014).
 - [30] S. Kanemura, M. Kikuchi, H. Yokoya, and K. Yagyu, *Prog. Theor. Exp. Phys.* **2015**, 51B02 (2015).
 - [31] E. J. Chun and P. Sharma, *Phys. Lett. B* **728**, 256 (2014).
 - [32] A. G. Akeroyd and M. Aoki, *Phys. Rev. D* **72**, 035011 (2005).
 - [33] A. Melfo, M. Nemevsek, F. Nesti, G. Senjanovic, and Y. Zhang, *Phys. Rev. D* **85**, 055018 (2012).
 - [34] A. Arhrib, R. Benbrik, M. Chabab, G. Moulataka, M. C. Peyranere, L. Rahili, and J. Ramadan, *Phys. Rev. D* **84**, 095005 (2011).
 - [35] K. Huitu, J. Maalampi, A. Pietila, and M. Raidal, *Nucl. Phys. B* **487**, 27 (1997).
 - [36] J. F. Gunion, C. Loomis, and K. T. Pitts, *eConf C* **960625**, LTH096 (1996).
 - [37] S. Chakrabarti, D. Choudhury, R. M. Godbole, and B. Mukhopadhyaya, *Phys. Lett. B* **434**, 347 (1998).
 - [38] M. Muhlleitner and M. Spira, *Phys. Rev. D* **68**, 117701 (2003).

- [39] E. J. Chun, K. Y. Lee, and S. C. Park, *Phys. Lett. B* **566**, 142 (2003), *J. High Energy Phys.* 04 (2012) 136.
- [40] A. G. Akeroyd and H. Sugiyama, *Phys. Rev. D* **84**, 035010 (2011).
- [41] B. Dutta, R. Eusebi, Y. Gao, T. Ghosh, and T. Kamon, *Phys. Rev. D* **90**, 055015 (2014).
- [42] Z. Kang, J. Li, T. Li, Y. Liu, and G. Z. Ning, *Eur. Phys. J. C* **75**, 574 (2015).
- [43] Z. L. Han, R. Ding, and Y. Liao, *Phys. Rev. D* **91**, 093006 (2015).
- [44] Z. L. Han, R. Ding, and Y. Liao, *Phys. Rev. D* **92**, 033014 (2015).
- [45] M. Mitra, S. Niyogi, and M. Spannowsky, *Phys. Rev. D* **95**, 035042 (2017).
- [46] D. K. Ghosh, N. Ghosh, I. Saha, and A. Shaw, *Phys. Rev. D* **97**, 115022 (2018).
- [47] S. Antusch, O. Fischer, A. Hammad, and C. Scherb, *J. High Energy Phys.* 02 (2019) 157.
- [48] P. S. Bhupal Dev and Y. Zhang, *J. High Energy Phys.* 10 (2018) 199.
- [49] T. B. de Melo, F. S. Queiroz, and Y. Villamizar, *Int. J. Mod. Phys. A* **34**, 1950157 (2019).
- [50] R. Primulando, J. Julio, and P. Uttayarat, *J. High Energy Phys.* 08 (2019) 024.
- [51] E. J. Chun, S. Khan, S. Mandal, M. Mitra, and S. Shil, *Phys. Rev. D* **101**, 075008 (2020).
- [52] R. Padhan, D. Das, M. Mitra, and A. Kumar Nayak, *Phys. Rev. D* **101**, 075050 (2020).
- [53] B. Fuks, M. Nemevšek, and R. Ruiz, *Phys. Rev. D* **101**, 075022 (2020).
- [54] O. Cakir, *New J. Phys.* **8**, 145 (2006).
- [55] T. Nomura, H. Okada, and H. Yokoya, *Nucl. Phys.* **B929**, 193 (2018).
- [56] S. Blunier, G. Cottin, M. A. Díaz, and B. Koch, *Phys. Rev. D* **95**, 075038 (2017).
- [57] A. Crivellin, M. Ghezzi, L. Panizzi, G. M. Pruna, and A. Signer, *Phys. Rev. D* **99**, 035004 (2019).
- [58] P. Agrawal, M. Mitra, S. Niyogi, S. Shil, and M. Spannowsky, *Phys. Rev. D* **98**, 015024 (2018).
- [59] S. Ashanujjaman, K. Ghosh, and K. Huitu, *Phys. Rev. D* **106**, 075028 (2022).
- [60] R. Ruiz, *J. High Energy Phys.* 10 (2022) 200.
- [61] C. W. Chiang, K. Enomoto, and M. Y. Liao, *Phys. Rev. D* **112**, 115003 (2025).
- [62] S. Ashanujjaman and K. Ghosh, *J. High Energy Phys.* 03 (2022) 195.
- [63] S. Ashanujjaman, K. Ghosh, and R. Sahu, *Phys. Rev. D* **107**, 015018 (2023).
- [64] P. S. B. Dev, S. Khan, M. Mitra, and S. K. Rai, *Phys. Rev. D* **99**, 115015 (2019).
- [65] X. H. Yang and Z. J. Yang, *Chin. Phys. C* **46**, 063107 (2022).
- [66] F. F. Deppisch, P. S. Bhupal Dev, and A. Pilaftsis, *New J. Phys.* **17**, 075019 (2015).
- [67] T. Li, C. Y. Yao, and M. Yuan, *J. High Energy Phys.* 03 (2023) 137.
- [68] S. P. Maharathy and M. Mitra, *Phys. Lett. B* **844**, 138105 (2023).
- [69] A. Jueid, T. A. Chowdhury, S. Nasri, and S. Saad, *Phys. Rev. D* **109**, 075011 (2024).
- [70] M. Belfkir, T. A. Chowdhury, and S. Nasri, *Phys. Lett. B* **852**, 138605 (2024).
- [71] J. C. Jia, Z. L. Han, F. Huang, Y. Jin, and H. Li, *Phys. Rev. D* **111**, 015009 (2025).
- [72] S. Chatrchyan *et al.* (CMS Collaboration), *Eur. Phys. J. C* **72**, 2189 (2012).
- [73] V. Khachatryan *et al.* (CMS Collaboration), *Phys. Rev. Lett.* **114**, 051801 (2015).
- [74] P. S. Bhupal Dev, D. K. Ghosh, N. Okada, and I. Saha, *J. High Energy Phys.* 03 (2013) 150; 05 (2013) 049(E).
- [75] G. Aad *et al.* (ATLAS Collaboration), *Eur. Phys. J. C* **72**, 2244 (2012).
- [76] G. Aad *et al.* (ATLAS Collaboration), *J. High Energy Phys.* 03 (2015) 041.
- [77] G. Aad *et al.* (ATLAS Collaboration), *J. High Energy Phys.* 08 (2015) 138.
- [78] M. Aaboud *et al.* (ATLAS Collaboration), *Eur. Phys. J. C* **78**, 199 (2018).
- [79] A. M. Sirunyan *et al.* (CMS Collaboration), *Phys. Rev. Lett.* **120**, 081801 (2018).
- [80] G. Aad *et al.* (ATLAS Collaboration), *J. High Energy Phys.* 06 (2021) 146.
- [81] G. Aad *et al.* (ATLAS Collaboration), *J. High Energy Phys.* 04 (2024) 026.
- [82] G. Aad *et al.* (ATLAS Collaboration), *Phys. Lett. B* **860**, 139137 (2025).
- [83] L. Guedes, G. Hoff, F. S. Queiroz, Y. M. Oviedo-Torres, and Y. Villamizar, *arXiv:2601.00083*.
- [84] P. Fileviez Perez, T. Han, G.-y. Huang, T. Li, and K. Wang, *Phys. Rev. D* **78**, 015018 (2008).
- [85] C. H. Chen and T. Nomura, *Phys. Rev. D* **90**, 075008 (2014).
- [86] B. A. Ouazghour, A. Arhrib, R. Benbrik, M. Chabab, and L. Rahili, *Phys. Rev. D* **100**, 035031 (2019).
- [87] B. A. Ouazghour and M. Chabab, *Phys. Lett. B* **846**, 138241 (2023).
- [88] B. A. Ouazghour, M. Chabab, and K. Goure, *Eur. Phys. J. C* **84**, 879 (2024).
- [89] B. Ait Ouazghour, M. Chabab, and K. Goure, *Springer Proc. Phys.* **425**, 137 (2025).
- [90] B. Ait-Ouazghour, M. Chabab, and K. Goure, *arXiv:2410.11140*.
- [91] B. Ait-Ouazghour, A. Arhrib, M. Chabab, and K. Goure, *Phys. Rev. D* **113**, 075006 (2026).
- [92] S. F. King, *J. Phys. G* **42**, 123001 (2015).
- [93] C. H. Chen and T. Nomura, *J. High Energy Phys.* 09 (2014) 120.
- [94] S. Ashanujjaman, P. S. B. Dev, J. Huang, and S. Zhou, *Phys. Rev. D* **113**, L051704 (2026).
- [95] M. J. Ramsey-Musolf, *J. High Energy Phys.* 09 (2020) 179.
- [96] C. H. Chen and T. Nomura, *Phys. Rev. D* **91**, 035023 (2015).
- [97] A. Aryshev *et al.* (ILC International Development Team), *arXiv:2203.07622*.
- [98] L. Linssen, A. Miyamoto, M. Stanitzki, and H. Weerts, *arXiv:1202.5940*.
- [99] I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler, J. P. Pinheiro, and T. Schwetz, *J. High Energy Phys.* 12 (2024) 216.

- [100] M.E. Peskin and T. Takeuchi, *Phys. Rev. D* **46**, 381 (1992).
- [101] W. Grimus, L. Lavoura, O. M. Ogreid, and P. Osland, *Nucl. Phys.* **B801**, 81 (2008).
- [102] S. Navas *et al.* (Particle Data Group), *Phys. Rev. D* **110**, 030001 (2024).
- [103] H. Bahl, T. Biekötter, S. Heinemeyer, C. Li, S. Paasch, G. Weiglein, and J. Wittbrodt, *Comput. Phys. Commun.* **291**, 108803 (2023).
- [104] P. Bechtle, S. Heinemeyer, O. Stål, T. Stefaniak, and G. Weiglein, *Eur. Phys. J. C* **74**, 2711 (2014).
- [105] P. Bechtle, S. Heinemeyer, O. Stål, T. Stefaniak, and G. Weiglein, *J. High Energy Phys.* **11** (2014) 039.
- [106] P. Bechtle, S. Heinemeyer, T. Klingl, T. Stefaniak, G. Weiglein, and J. Wittbrodt, *Eur. Phys. J. C* **81**, 145 (2021).
- [107] P. Bechtle, O. Brein, S. Heinemeyer, G. Weiglein, and K. E. Williams, *Comput. Phys. Commun.* **181**, 138 (2010).
- [108] P. Bechtle, O. Brein, S. Heinemeyer, G. Weiglein, and K. E. Williams, *Comput. Phys. Commun.* **182**, 2605 (2011).
- [109] P. Bechtle, O. Brein, S. Heinemeyer, O. Stål, T. Stefaniak, G. Weiglein, and K. E. Williams, *Eur. Phys. J. C* **74**, 2693 (2014).
- [110] P. Bechtle, D. Dercks, S. Heinemeyer, T. Klingl, T. Stefaniak, G. Weiglein, and J. Wittbrodt, *Eur. Phys. J. C* **80**, 1211 (2020).
- [111] S. Banerjee *et al.* (Heavy Flavor Averaging Group (HFLAV)), *Phys. Rev. D* **113**, 012008 (2026).
- [112] Y. Kuno and Y. Okada, *Rev. Mod. Phys.* **73**, 151 (2001).
- [113] L. Lavoura, *Eur. Phys. J. C* **29**, 191 (2003).
- [114] A. G. Akeroyd, M. Aoki, and H. Sugiyama, *Phys. Rev. D* **79**, 113010 (2009).
- [115] K. Afanaciev *et al.* (MEG II Collaboration), *Eur. Phys. J. C* **85**, 1177 (2025).
- [116] U. Bellgardt *et al.* (SINDRUM Collaboration), *Nucl. Phys.* **B299**, 1 (1988).
- [117] G. Aad *et al.* (ATLAS Collaboration), *Phys. Lett. B* **716**, 1 (2012).
- [118] S. Chatrchyan *et al.* (CMS Collaboration), *Phys. Lett. B* **716**, 30 (2012).
- [119] T. Hahn and C. Schappacher, *Comput. Phys. Commun.* **143**, 54 (2002).
- [120] T. Hahn and M. Perez-Victoria, *Comput. Phys. Commun.* **118**, 153 (1999).
- [121] J. Kublbeck, M. Bohm, and A. Denner, *Comput. Phys. Commun.* **60**, 165 (1990).
- [122] J. Alwall, R. Frederix, S. Frixione, V. Hirschi, F. Maltoni, O. Mattelaer, H. S. Shao, T. Stelzer, P. Torrielli, and M. Zaro, *J. High Energy Phys.* **07** (2014) 079.
- [123] K. Hagiwara, T. Li, K. Mawatari, and J. Nakamura, *Eur. Phys. J. C* **73**, 2489 (2013).
- [124] T. Sjostrand, S. Mrenna, and P. Z. Skands, *Comput. Phys. Commun.* **178**, 852 (2008).
- [125] M. Cacciari, G. P. Salam, and G. Soyez, *Eur. Phys. J. C* **72**, 1896 (2012).
- [126] J. de Favereau, C. Delaere, P. Demin, A. Giammanco, V. Lemaître, A. Mertens, and M. Selvaggi (DELPHES 3 Collaboration), *J. High Energy Phys.* **02** (2014) 057.
- [127] M. Cacciari, G. P. Salam, and G. Soyez, *J. High Energy Phys.* **04** (2008) 063.
- [128] E. Conte, B. Fuks, and G. Serret, *Comput. Phys. Commun.* **184**, 222 (2013).
- [129] E. Conte and B. Fuks, *J. Phys. Conf. Ser.* **523**, 012032 (2014).
- [130] G. Cowan, K. Cranmer, E. Gross, and O. Vitells, *Eur. Phys. J. C* **71**, 1554 (2011); **73**, 2501(E) (2013).