

Imaginary scaling invariance of the one-loop effective potential

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Recently, a hitherto unknown class of renormalization group stable relations between parameters of bosonic field theories has been identified and dubbed as the r_0 or “GOOFy” symmetries. Here, one-loop properties of the r_0 invariant two-Higgs doublet model and a minimal symmetric model are discussed. It is concluded that the symmetry is present at the one-loop provided the UV cutoff squared transforms nontrivially under r_0 . The minimal model requires the presence of two real fields.

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I. INTRODUCTION

Investigating symmetries of the two-Higgs doublet model (2HDM) extension of the Standard Model (SM) of electro-weak interactions, the authors of Ref. [1] have recently noticed the existence of a new class of relations between parameters of the 2HDM scalar potential which are stable under the renormalization group equations (RGE) up to an infinite order of perturbation expansion. It turned out the relations are not implied by hitherto known symmetries of the 2HDM, so a new class of symmetries has been hypothesized in Ref. [1]. The symmetry transformations, dubbed as the r_0 or GOOFy¹ symmetries, include mapping real bosonic fields (scalars and vectors) into purely imaginary ones. Even if the tree-level potential and Lagrangian density have been shown to be invariant under the r_0 transformation, the invariance could be broken at higher orders in perturbation theory. In order

to investigate if the new transformations constitute genuine symmetries, we shall investigate properties of the one-loop effective potential of the 2HDM. In addition, we introduce, present, and discuss in detail a minimal r_0 -invariant scalar field model [2].

The idea of GOOFy symmetries has been picked up by Haber and Ferreira and applied to complexified GOOFy-symmetric models in Ref. [3]. Then Trautner developed the idea of GOOFy symmetry for generic scalar field theories in [4], by considering noninvariance of kinetic terms. The new symmetries could be naturally applied to the hierarchy problem; that direction has been picked up in Refs. [5,6]. Interesting discussions of the r_0 symmetry at the one-loop level has recently appeared in Ref. [7].

It should be noted that using the term “symmetries” to describe these findings is still open to debate. On the one hand, the procedure discovered in [1] has allowed the identification of hitherto unknown RGE-protected, to all orders, relations between parameters, which is certainly the hallmark of a symmetry. On the other hand, the process that seems to lead to those parameter regions involves imaginary scalings of both real components of scalar doublets (and gauge fields) and spacetime coordinates, which is unlike any symmetry used in quantum field theory. In [4] a possible interpretation involving spurions was proposed, and in [3] RGE-stable relations found through the GOOFy paradigm in a two real scalar model were interpreted as the consequence of a regular symmetry imposed on a model with twice the field content. Nonetheless, we argue that the GOOFy procedure has been shown to produce physical

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¹The acronym GOOFy consists of the first letters in the surnames of the authors that suggested this symmetry; Grzadkowski, OGREID, OSland, FERreira.

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consequences, such as mass degeneracies or parameter RGE stability, for a growing number of different models. And in this work, we will show that invariance under the GOOFy transformations is not only found at tree level but also for the one-loop effective potential. We therefore argue that indeed the epithet “symmetries” seems to apply to the field transformations herein discussed, no matter how bizarre they may seem.

II. THE TWO-HIGGS DOUBLET MODEL

The 2HDM is one of the simplest extensions of the SM, wherein one considers two $SU(2)$ doublets with hypercharge one instead of just one doublet. In the following we will briefly review the basic aspects of a useful formalism to understand the structure of the scalar sector of the model, and the global symmetries one can impose upon it. The most general scalar potential involving two hypercharge $Y = 1$ scalar doublets invariant under the electroweak gauge group $SU(2)_L \times U(1)_Y$ is given by

$$V_{\text{tree}} = m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - [m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{H.c.}] + \frac{1}{2} \lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \frac{1}{2} \lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1) (\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_1) + \left\{ \frac{1}{2} \lambda_5 (\Phi_1^\dagger \Phi_2)^2 + [\lambda_6 (\Phi_1^\dagger \Phi_1) + \lambda_7 (\Phi_2^\dagger \Phi_2)] \Phi_1^\dagger \Phi_2 + \text{H.c.} \right\}, \quad (1)$$

where, other than m_{12}^2 and $\lambda_{5,6,7}$, all parameters are real. An alternative notation uses four gauge-invariant bilinears constructed from the doublets [8–18],

$$\begin{aligned} r_0 &= \frac{1}{2} (\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2), \\ r_1 &= \frac{1}{2} (\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1) = \text{Re}(\Phi_1^\dagger \Phi_2), \\ r_2 &= -\frac{i}{2} (\Phi_1^\dagger \Phi_2 - \Phi_2^\dagger \Phi_1) = \text{Im}(\Phi_1^\dagger \Phi_2), \\ r_3 &= \frac{1}{2} (\Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_2). \end{aligned} \quad (2)$$

In terms of these quantities, then, the potential of Eq. (1) may be written as

$$V = M_\mu r^\mu + \Lambda_{\mu\nu} r^\mu r^\nu, \quad (3)$$

where we use a Minkowski-like formalism to define the four-vectors

$$\begin{aligned} r^\mu &= (r_0, r_1, r_2, r_3) = (r_0, \vec{r}), \\ M^\mu &= (m_{11}^2 + m_{22}^2, 2 \text{Re}(m_{12}^2), -2 \text{Im}(m_{12}^2), m_{22}^2 - m_{11}^2) \\ &= (M_0, \vec{M}), \end{aligned} \quad (4)$$

as well as the tensor

$$\Lambda^{\mu\nu} = \begin{pmatrix} \Lambda_{00} & \vec{\Lambda} \\ \vec{\Lambda}^T & \Lambda \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\lambda_1 + \lambda_2) + \lambda_3 & -\text{Re}(\lambda_6 + \lambda_7) & \text{Im}(\lambda_6 + \lambda_7) & \frac{1}{2}(\lambda_2 - \lambda_1) \\ -\text{Re}(\lambda_6 + \lambda_7) & \lambda_4 + \text{Re}(\lambda_5) & -\text{Im}(\lambda_5) & \text{Re}(\lambda_6 - \lambda_7) \\ \text{Im}(\lambda_6 + \lambda_7) & -\text{Im}(\lambda_5) & \lambda_4 - \text{Re}(\lambda_5) & -\text{Im}(\lambda_6 - \lambda_7) \\ \frac{1}{2}(\lambda_2 - \lambda_1) & \text{Re}(\lambda_6 - \lambda_7) & -\text{Im}(\lambda_6 - \lambda_7) & \frac{1}{2}(\lambda_1 + \lambda_2) - \lambda_3 \end{pmatrix}. \quad (5)$$

In our previous publication [1] it was shown that there exist hitherto unknown relations between parameters of the scalar potential which are stable under RGE to all orders in the scalar and gauge sectors, and at least up to two loops when the Yukawa sector is included [19]. In other words, the corresponding beta functions have fixed points. In [1] we hypothesized the existence of a class of new symmetries, called r_0 symmetries for the 2HDM since they seem to arise from the transformation $r_0 \rightarrow -r_0$, which would explain the presence of those relations. It should be mentioned here that according to common knowledge, there exist only six possible symmetries of the 2HDM

scalar potential [10,11] that are consistent with gauge $SU(2) \times U(1)$ invariance. This work states that there exist six extra symmetries that adopt the r_0 transformation [1]. Phenomenology of the 2HDM with the r_0 symmetries has been presented in Sec. IV of our earlier work [1], where we showed that these symmetries could be used to explain mass degeneracies (exact or approximate) in the scalar sector, and were also responsible for specific tree-level relations between masses and couplings. Furthermore, it was also shown that for GOOFy 2HDMs the decoupling of heavy degrees of freedom is not possible, so that the masses of the extra scalars (other than the SM Higgs) are

necessarily below ~ 700 GeV. Further details of phenomenology of GOOFy 2HDMs, involving the fermionic sector as well, will be presented in [20], and an extension of GOOFy symmetries to three-Higgs doublet models is being investigated [21].

III. THE r_0 SYMMETRY AND ITS CONSEQUENCES

In the previous publication [1] we identified the following relations between potential parameters of the 2HDM that are fixed points of the renormalization group equations valid to all orders of perturbation theory in the scalar and gauge sectors:

$$\vec{\Lambda} = \vec{0} \quad \text{and} \quad M_0 = 0. \quad (6)$$

More explicitly, beta functions corresponding to $m_{11}^2 + m_{22}^2$, $\lambda_1 - \lambda_2$ and $\lambda_6 + \lambda_7$ vanish at the fixed point (6). In terms of parameters of the potential (1) the above relations read as $m_{11}^2 + m_{22}^2 = 0$, $\lambda_1 = \lambda_2$ and $\lambda_6 = -\lambda_7$.

Let us define the following basis in eight-dimensional space for the real components of the two doublets,

$$\phi_i \equiv \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8\}, \quad (7)$$

such that

$$\Phi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}, \quad \Phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_5 + i\phi_6 \\ \phi_7 + i\phi_8 \end{pmatrix}. \quad (8)$$

Given that r_0 of Eq. (2) is obviously non-negative, it is impossible to change the sign of r_0 without resorting to transforming real fields into purely imaginary fields. It is easy to verify that the field transformations of the eight real components of $\Phi_{1,2}$ that imply (6) are the following:

$$\begin{aligned} \phi_1 &\rightarrow i\phi_6, & \phi_2 &\rightarrow i\phi_5, & \phi_3 &\rightarrow i\phi_8, & \phi_4 &\rightarrow i\phi_7, \\ \phi_5 &\rightarrow -i\phi_2, & \phi_6 &\rightarrow -i\phi_1, & \phi_7 &\rightarrow -i\phi_4, & \phi_8 &\rightarrow -i\phi_3. \end{aligned} \quad (9)$$

The above transformations imply $r_0 \rightarrow -r_0$, as one could have anticipated. It has been also shown in [1] that in order to keep scalar and gauge kinetic terms invariant under the above transformation, the additional transformations of the coordinate x^μ and vector fields are needed:

$$\begin{aligned} x^\mu &\rightarrow ix^\mu, & B_\mu &\rightarrow iB_\mu, \\ W_{1\mu} &\rightarrow iW_{1\mu}, & W_{2\mu} &\rightarrow -iW_{2\mu}, & W_{3\mu} &\rightarrow iW_{3\mu}. \end{aligned} \quad (10)$$

Hereafter we will refer to (9) together with (10) as the r_0 transformations. Note that, by construction, the Lagrangian density $\mathcal{L}$ is invariant under the r_0 transformation, and it is easily shown that the classical action is invariant as well.

Let us now consider a more general system that consists of a number of real scalar fields in order to find out what is the scaling for the momenta of the system that is implied by a given scaling of fields and coordinates. Relativistic quantum mechanics suggests that $x^\mu \rightarrow ix^\mu$ implies $p_\mu \equiv -i\partial_\mu \rightarrow -ip_\mu$. In fact, in field theory the 4-momentum is given in terms of the energy-momentum tensor $\Theta_{\alpha\beta}$ as

$$p_\mu \equiv \int d^3x \Theta_{0\mu}, \quad (11)$$

with

$$\Theta_{\alpha\beta} = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial^\alpha \phi_i)} \partial_\beta \phi_i - \eta_{\alpha\beta} \mathcal{L}. \quad (12)$$

The Lagrangian $\mathcal{L}$ is invariant under the r_0 transformation (9) and (10); therefore $\Theta_{\alpha\beta}$ is invariant as well. On the other hand $d^3x \rightarrow -id^3x$; therefore, taking into account integration limits, we confirm that indeed $p_\mu \rightarrow -ip_\mu$, in agreement with our initial intuition.

IV. INVARIANCE OF THE 2HDM EFFECTIVE POTENTIAL UNDER THE r_0 TRANSFORMATION

In order to test the invariance of the effective potential under the r_0 transformation, it is convenient to invoke its functional definition [22–24], since it allows the formulation of the potential in terms of coordinates and fields only. Adopting, e.g., [23] one finds that the one-loop contribution to the effective potential in a generic real scalar model reads as

$$\begin{aligned} V_{\text{eff}}(\phi_{cl}) &\propto \int \prod_k \mathcal{D}(\phi_k) \exp \left\{ -i \int d^4x \phi_i(x) [\square_x \delta_{ij} \right. \\ &\quad \left. + (M_S^2(\phi_{cl}))_{ij}] \phi_j(x) \right\}, \end{aligned} \quad (13)$$

where $\square_x \equiv \partial_\mu \partial^\mu$ and $M_S^2(\phi_{cl})$ is the scalar mass squared matrix,

$$(M_S^2)_{ij} \equiv \frac{\partial^2 V_{\text{tree}}}{\partial \phi_i \partial \phi_j}, \quad (14)$$

calculated for a given tree-level potential for constant classical field configuration ϕ_{cl} . The d'Alembertian is odd under r_0 , i.e., $\square_x \rightarrow -\square_x$. Since $\phi_i \rightarrow \pm i\phi_i$, it becomes clear that, for both models considered in this paper, the measure $\prod_k \mathcal{D}(\phi_k)$ is invariant under the r_0 transformation, so we conclude that the one-loop effective potential is indeed r_0 invariant.

Recently, within the 2HDM, the one-loop effective potential has been calculated adopting the bilinear notation

[7,25,26]. We will concentrate hereafter only on the scalar ($V_{\text{eff}}^{(S)}$) contribution to the potential that can be, after dropping field-independent contributions, written in the following form [27]:

$$V_{\text{eff}}^{(S)} = \frac{1}{2} \int \frac{d^4 p_E}{(2\pi)^4} \text{Tr}[\ln(p_E^2 + M_S^2)] \\ = -\frac{1}{2} \int \frac{d^4 p_E}{(2\pi)^4} \left[\text{Tr} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{M_S^2}{p_E^2} \right)^n \right], \quad (15)$$

where p_E is the Euclidian momentum obtained by a standard Wick rotation from the 4-momentum of virtual quanta propagating in a loop. At the lowest order, $n = 1$, specializing to the 2HDM fixed point (6), and calculating the trace in the i, j indices, we obtain the r_0 -odd result

$$\text{Tr}(M_S^2) = 4[5\Lambda_{00} + \text{tr}(\Lambda)]r_0. \quad (16)$$

In order to verify the transformation properties of higher order terms let us denote by $\lambda_i(r_0)$ an eigenvalue of $M_S^2(r_0)$. For the 2HDM, the characteristic equation for M_S^2 could be written as follows:

$$\lambda^8 + c_7(r_0)\lambda^7 + c_6(r_0)\lambda^6 + \cdots + c_0(r_0) = 0, \quad (17)$$

where $c_i(r_0)$ are coefficients determined in terms of entries of the matrix M_S^2 , in particular $c_0(r_0) = \det(M_S^2)$ and $c_7(r_0) = \text{Tr}(M_S^2)$. It turns out that the coefficients satisfy the following symmetry relations:

$$c_{2n}(r_0) = c_{2n}(-r_0) \quad \text{and} \quad c_{2n+1}(r_0) = -c_{2n+1}(-r_0). \quad (18)$$

Writing the characteristic equation for $M_S^2(-r_0)$

$$\lambda^8 + c_7(-r_0)\lambda^7 + c_6(-r_0)\lambda^6 + \cdots + c_0(-r_0) = 0, \quad (19)$$

and adopting (18) one can conclude that if $\lambda_i(r_0)$ is an eigenvalue of $M_S^2(r_0)$ then $-\lambda_i(r_0)$ is an eigenvalue of $M_S^2(-r_0)$, i.e., there exists an eigenvalue $\lambda_j(r_0)$ such that

$$\lambda_i(r_0) \xrightarrow{r_0} \lambda_i(-r_0) = -\lambda_j(r_0), \quad \text{and} \\ \lambda_j(r_0) \xrightarrow{r_0} \lambda_j(-r_0) = -\lambda_i(r_0). \quad (20)$$

So there are four pairs of eigenvalues transforming within each pair, so that the set of the eigenvalues $\{\lambda_i(r_0)\}$ transforms as follows:

$$\{\lambda_i(r_0)\} \xrightarrow{r_0} -\{\lambda_i(r_0)\}. \quad (21)$$

An important conclusion from the above is that

$$\text{Tr}[(M_S^2)^{2n}](r_0) = \text{Tr}[(M_S^2)^{2n}](-r_0), \quad \text{and} \\ \text{Tr}[(M_S^2)^{2n+1}](r_0) = -\text{Tr}[(M_S^2)^{2n+1}](-r_0). \quad (22)$$

For any finite n the trace of higher powers of M_S^2 could be calculated explicitly. Indeed, considering the definition of M_S^2 , Eq. (14), it is simple to verify Eq. (22) for all values of n . Therefore we see that terms with even/odd powers of (M_S^2/p_E^2) appearing in (15) are even/odd functions of r_0 . But since under the r_0 transformation we must have $p_E^2 \rightarrow -p_E^2$ (a consequence of $p_\mu \rightarrow -ip_\mu$) this results in the r_0 invariance of the effective potential, equivalent to the r_0 invariance of (13).

In order to complete the calculation of the effective potential (15) one needs to perform the integration over $d^4 p_E$, where some regularization strategy must be adopted. For instance, the cutoff regularization implies that

$$V_{\text{eff}}^{(S)} = -\frac{1}{32\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n} \int_0^{\Lambda_{\text{UV}}^2} d\rho \rho \text{Tr} \left(\frac{M_S^2}{\rho} \right)^n \right], \quad (23)$$

where $\rho \equiv p_E^2$. Note that if we replace Λ_{UV}^2 by $-\Lambda_{\text{UV}}^2$ this is equivalent to replacing ρ by $-\rho$ under the trace. Consequently, for odd n a minus sign appears. Since for odd n $\text{Tr}(M_S^2(r_0))^n$ is an odd function of r_0 , but under the r_0 transformation we have $\Lambda_{\text{UV}}^2 \rightarrow -\Lambda_{\text{UV}}^2$, we hence arrive at an invariant one-loop effective potential [28]. This is indeed consistent with our earlier conclusion that under r_0 , $p_\mu \rightarrow -ip_\mu$.

The r_0 invariance could be conveniently checked in the basis in which M_S^2 is diagonal: $(M_S^2)_{ij} = \delta_{ij}\lambda_j = \delta_{ij}M_j^2$, where there is no summation over j . Adopting the cutoff regularization, neglecting field independent terms and terms which vanish in the limit $\Lambda_{\text{UV}} \rightarrow \infty$, one finds the effective potential

$$V_{\text{eff}}^{1\text{-loop}}(r_0) = \frac{\Lambda_{\text{UV}}^2}{32\pi^2} \sum_{i=1}^8 M_i^2(r_0) \\ + \frac{1}{64\pi^2} \sum_{i=1}^8 M_i^4(r_0) \left[\log \frac{M_i^2(r_0)}{\Lambda_{\text{UV}}^2} - \frac{1}{2} \right]. \quad (24)$$

Using the result (21) it is clear that the first term above is invariant since $\Lambda_{\text{UV}}^2 \rightarrow -\Lambda_{\text{UV}}^2$. On the other hand (21) implies that the same transformation of the cutoff guarantees invariance of the second, thus proving the invariance of the one-loop effective potential under the r_0 transformation.

Note that we have proven invariance without needing to diagonalize the 8×8 mass matrix M_S^2 . It was sufficient to find out how the eigenvalues of M_S^2 transform under r_0 .

V. THE TOY MODEL—2RSM

Even though we have managed, for the 2HDM, to verify transformation properties of the effective potential without calculating the M_S^2 eigenvalues, hereafter we are going to present a simple, minimal toy model [2] which shares many properties with the 2HDM and for which the diagonalization could be done explicitly. This is a two real scalar model described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2) - V(\phi_1, \phi_2), \quad (25)$$

with the potential

$$\begin{aligned} V^0(\phi_1, \phi_2) = & \frac{1}{2}m_1^2(\phi_1^2 - \phi_2^2) + m_{12}^2\phi_1\phi_2 \\ & + \frac{1}{2}\lambda_1(\phi_1^4 + \phi_2^4) + \lambda_3(\phi_1\phi_2)^2 \\ & + \lambda_6(\phi_1^2 - \phi_2^2)\phi_1\phi_2. \end{aligned} \quad (26)$$

The model is invariant under the following r_0 -like transformation:

$$x^\mu \rightarrow ix^\mu, \quad \phi_1 \rightarrow i\phi_2, \quad \phi_2 \rightarrow -i\phi_1. \quad (27)$$

It turns out that for this symmetric potential it is possible to choose a (ϕ_1, ϕ_2) basis such that $\lambda_6 = 0$. This is an analog of the statement proven for the 2HDM in [29,30] where the authors showed that if $\lambda_1 = \lambda_2$ and $\lambda_6 = -\lambda_7$ ($CP2$ invariant quartic sector) then there exists a basis such that all λ_i are real and $\lambda_6 = \lambda_7 = 0$. Therefore λ_6 will be dropped from now on. The squared mass matrix then reads as

$$(M_S^2)_{ij} = \begin{pmatrix} m_1^2 + 6\lambda_1\phi_1^2 + 2\lambda_3\phi_2^2 & m_{12}^2 + 4\lambda_3\phi_1\phi_2 \\ m_{12}^2 + 4\lambda_3\phi_1\phi_2 & -m_1^2 + 6\lambda_1\phi_2^2 + 2\lambda_3\phi_1^2 \end{pmatrix}.$$

One can notice that near the origin, $\phi_1 = \phi_2 = 0$, we have

$$(M_S^2)_{ij} = \begin{pmatrix} m_1^2 & m_{12}^2 \\ m_{12}^2 & -m_1^2 \end{pmatrix} + \mathcal{O}(\phi^2). \quad (28)$$

At the origin the M_S^2 eigenvalues are $\pm\sqrt{m_{12}^4 + m_1^4}$, so there is a saddle point at $\phi_1 = \phi_2 = 0$ with two perpendicular directions of opposite curvature (a similar situation appears in the r_0 -symmetric 2HDM where, at the origin, the Hessian has four positive and four negative eigenvalues). This implies that there is a global minimum away from the origin, and therefore the r_0 symmetry is always spontaneously broken. It turns out that there are two local minima in the toy model determined by the minimization conditions: $(v_1^2 - v_2^2) = -m_1^2/\lambda_1$ and $v_1 v_2 = -m_{12}^2/(\lambda_1 + \lambda_3)$, where $\langle \phi_{1,2} \rangle \equiv v_{1,2}/\sqrt{2}$.

One can also express the potential in terms of bilinear variables,

$$r_0 \equiv \frac{1}{2}(\phi_1^2 + \phi_2^2), \quad r_1 \equiv \phi_1\phi_2, \quad r_2 \equiv \frac{1}{2}(\phi_1^2 - \phi_2^2). \quad (29)$$

Note that r_0 , r_1 , and r_2 are not independent, since one has $r_0^2 - r_1^2 - r_2^2 = 0$. Upon the r_0 transformation,

$$(r_0, r_1, r_2) \xrightarrow{r_0} (-r_0, r_1, r_2). \quad (30)$$

The r_0 symmetry thus requires the coefficient of r_0 in the potential to vanish, and the potential may be written as

$$V(r^\mu) = -M_\mu r^\mu + \Lambda_{\mu\nu} r^\mu r^\nu,$$

for $\mu, \nu = 0, 1, 2$ with $r^\mu \equiv (r_0, r_1, r_2)$; $M^\mu \equiv (0, m_{12}^2, m_1^2)$, the “Minkowski metric tensor”; $\eta_{\mu\nu} = \text{diag}(1, -1, -1)$; and

$$\Lambda^{\mu\nu} \equiv \begin{pmatrix} \Lambda_{00} & 0 & 0 \\ 0 & \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{21} & \Lambda_{22} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

Note that $M_0 = 0$ and $\vec{\Lambda} = 0$ are consequences of the r_0 symmetry, similar to the 2HDM case, Eq. (6). Indeed, for the toy model $M_0 = 0$ and $\vec{\Lambda} = 0$ mean that $m_1^2 + m_2^2 = 0$ and $\lambda_1 = \lambda_2$.

Then one finds that

$$\text{Tr}(M_S^2) = 4(3\lambda_1 + \lambda_3)r^0, \quad (31)$$

and therefore under the r_0 transformation this trace is odd, $\text{Tr}(M_S^2) \xrightarrow{r_0} -\text{Tr}(M_S^2)$, so that we reproduce the analogous property observed in the 2HDM one-loop effective potential.

Since in this model there are only 2 real degrees of freedom one can easily diagonalize the mass squared matrix and calculate the one-loop effective potential. It turns out that the eigenvalues of M_S^2 can be expressed through bilinears as follows:

$$M_{1,2}^2(r^\mu) = 2(3\lambda_1 + \lambda_3)r_0 \pm \sqrt{\Delta}, \quad (32)$$

where

$$\begin{aligned} \Delta = & (m_1^2)^2 + (m_{12}^2)^2 + 4m_1^2(3\lambda_1 - \lambda_3)r_2 + 8m_{12}^2\lambda_3r_1 \\ & + 16\lambda_3^2r_0^2 + 12(3\lambda_1 + \lambda_3)(\lambda_1 - \lambda_3)r_2^2. \end{aligned} \quad (33)$$

Note that the eigenvalues of M_S^2 are linearly dependent on r_0 . It is easy to see that they transform under the r_0 symmetry as

$$M_1^2 \xrightarrow{r_0} -M_2^2 \quad \text{and} \quad M_2^2 \xrightarrow{r_0} -M_1^2. \quad (34)$$

In order to illustrate explicitly the r_0 invariance at the one-loop level we present the one-loop effective potential

adopting cutoff regularization and dropping irrelevant terms:

$$V_{\text{eff}}^{1\text{-loop}}(r^\mu) = \frac{\Lambda_{\text{UV}}^2}{32\pi^2} \sum_{i=1,2} M_i^2(r^\mu) + \frac{1}{64\pi^2} \sum_{i=1,2} M_i^4(r^\mu) \left[\log \frac{M_i^2(r^\mu)}{\Lambda_{\text{UV}}^2} - \frac{1}{2} \right]. \quad (35)$$

Note that

$$M_1^4 + M_2^4 = 2\{[2(3\lambda_1 + \lambda_3)r_0]^2 + \Delta\}, \quad (36)$$

so this quantity is r_0 invariant. Here one can explicitly see that the one-loop effective potential is indeed r_0 invariant for $\Lambda_{\text{UV}}^2 \xrightarrow{r_0} -\Lambda_{\text{UV}}^2$.

The one-loop, renormalized, effective potential can be written (see, e.g., [31]) as

$$V_{\text{eff}}^{1\text{-loop}}(r^\mu) = \delta m_0^2 r_0 + (m_1^2 + \delta m_1^2) r_2 + (m_{12}^2 + \delta m_{12}^2) r_1 + (\lambda_1 + \delta \lambda_1 + \lambda_3 + \delta \lambda_3) r_1^2 + 2(\lambda_1 + \delta \lambda_1) r_2^2 \quad (37)$$

$$+ \frac{\Lambda_{\text{UV}}^2}{8\pi^2} (3\lambda_1 + \lambda_3) r_0 + \quad (38)$$

$$- \frac{\log \Lambda_{\text{UV}}^2}{32\pi^2} [4m_1^2(3\lambda_1 - \lambda_3)r_2 + 8m_{12}^2\lambda_3 r_1 + 4(9\lambda_1^2 + 6\lambda_1\lambda_3 + 5\lambda_3^2)r_1^2 + 8(9\lambda_1^2 + \lambda_3^2)r_2^2] + \quad (39)$$

$$+ \text{finite contributions}, \quad (40)$$

where $\delta \dots$ denotes a corresponding counterterm. Since during renormalization all dependence on the cutoff shall disappear, it is interesting to see how the r_0 symmetry is realized in the renormalized potential.

Field-independent one-loop generated terms, even if divergent, have been dropped in (38) and (39). Note that, since the cutoff Λ_{UV}^2 is odd under r_0 , the counterterm $\propto \Lambda_{\text{UV}}^2 \cdot r_0$ is invariant, so that it can cancel the quadratic divergence in (38) if

$$\delta m_0^2 = -\frac{(3\lambda_1 + \lambda_3)}{8\pi^2} \Lambda_{\text{UV}}^2. \quad (41)$$

The remaining counterterms in (37) could be also adjusted to cancel divergent terms present in (39),

$$\frac{\partial V_{\text{eff}}^{1\text{-loop}}}{\partial r_2} \Big|_{r^\mu=M} = m_1^2 \Rightarrow \delta m_1^2 = + \frac{m_1^2(3\lambda_1 - \lambda_3)}{8\pi^2} \log(\Lambda_{\text{UV}}^2), \quad (42)$$

$$\frac{\partial V_{\text{eff}}^{1\text{-loop}}}{\partial r_1} \Big|_{r^\mu=M} = m_{12}^2 \Rightarrow \delta m_{12}^2 = + \frac{m_{12}^2 \lambda_3}{4\pi^2} \log(\Lambda_{\text{UV}}^2), \quad (43)$$

for the quadratic terms and for the quartic ones,

$$\frac{\partial^2 V_{\text{eff}}^{1\text{-loop}}}{\partial r_2^2} \Big|_{r^\mu=M} = 4\lambda_1 \Rightarrow \delta \lambda_1 = + \frac{i}{128\pi} \frac{\sqrt{\Delta_0}}{M^2} (3\lambda_1 + \lambda_3) + \frac{(9\lambda_1^2 + \lambda_3^2)}{8\pi^2} \log(\Lambda_{\text{UV}}^2), \quad (44)$$

$$\frac{\partial^2 V_{\text{eff}}^{1\text{-loop}}}{\partial r_1^2} \Big|_{r^\mu=M} = 2(\lambda_1 + \lambda_3) \Rightarrow \delta \lambda_3 = + \frac{i}{128\pi} \frac{\sqrt{\Delta_0}}{M^2} (3\lambda_1 + \lambda_3) + \frac{\lambda_3(3\lambda_1 + 2\lambda_3)}{4\pi^2} \log(\Lambda_{\text{UV}}^2), \quad (45)$$

where $\Delta_0 \equiv \Delta|_{r^\mu=0} = (m_1^2)^2 + (m_{12}^2)^2$. In order to regularize infrared (IR) singularities present at $r^\mu = 0$, the above renormalization conditions have been specified at $r^\mu = M$ with M being a IR-regularization scale such that $r_1 = r_2 = M^2/\sqrt{2}$. The counterterms have been chosen to contain only IR divergent ($M \rightarrow 0$) and/or the UV divergent ($\Lambda_{\text{UV}} \rightarrow \infty$) terms. After dropping irrelevant field-independent terms (even if they are divergent), the renormalized one-loop effective potential reads as

$$V_{\text{eff}}^{1\text{-loop}}(r^\mu) = V^0(r_1, r_2) + \frac{i}{64\pi} \frac{\sqrt{\Delta_0}}{M^2} (3\lambda_1 + \lambda_3) (r_1^2 + r_2^2) + \sum_{i=1,2} \frac{M_i^4(r^\mu)}{64\pi} \left[\log M_i^2 - \frac{1}{2} \right] + \dots, \quad (46)$$

where the ellipsis denotes the terms hidden in (40). Since the curvature of the tree-level potential might be negative (e.g., in the vicinity of the origin), the one-loop effective potential $V_{\text{eff}}^{1\text{-loop}}$ might be a complex function [32]. Note also that in the scale-invariant limit $\Delta_0 \rightarrow 0$ the imaginary part of the potential vanishes in accordance with our intuition gained from a single massless real field model [33]. It is seen that the first two terms in (46) are trivially r_0 invariant, and since $M_1^2 \xrightarrow{r_0} -M_2^2$, the third term is invariant as well.

The ultimate verification of the quality of the toy model is to check if β functions for squared mass terms which are forbidden by the r_0 symmetry do vanish as they did in 2HDM. Using generic results from [34] it is easy to show that for the potential without the r_0 symmetry (27) imposed, the beta functions read as

$$\beta_{m_1^2+m_2^2} = 12(\lambda_1 m_1^2 + \lambda_2 m_2^2) + 2\lambda_3(m_1^2 + m_2^2) + 6(\lambda_6 + \lambda_7)m_{12}^2 \quad (47)$$

$$\beta_{m_{12}^2} = 3(\lambda_6 m_1^2 + \lambda_7 m_2^2) + 4\lambda_3 m_{12}^2, \quad (48)$$

where, for pedagogical reasons, here we have kept $m_2^2 \neq -m_1^2$ and $\lambda_{6,7} \neq 0$. It is then clear that when the r_0 symmetry (27) is imposed $\beta_{m_1^2+m_2^2}$ vanishes. This was also verified at two loops in [3]. Therefore we conclude that the toy model constitutes the minimal model possessing the r_0 symmetry.

VI. SUMMARY AND CONCLUSIONS

In this paper we have discussed implications of the r_0 symmetry for the one-loop effective potential within the 2HDM and the minimal (toy) model defined by (26). The novel aspect of the symmetry lies in the necessity to transform, via an imaginary scaling, coordinates together with fields. In this paper we showed that an imaginary scaling of scalar fields is accompanied by spacetime coordinates also being scaled by imaginary factors, so that invariance of the theory at tree level is achieved. We then showed that that procedure is consistent at one loop, and the effective potential was shown to be invariant under the r_0 transformation. The one-loop effective potential has been calculated and renormalized. For the invariance to hold the key property is that the imaginary scaling of spacetime coordinates implies an imaginary scaling of 4-momenta, which in turn implies a nontrivial transformation of the cutoff, Λ_{UV} , or renormalization, μ , scales—namely $\Lambda_{UV}^2 \xrightarrow{r_0} -\Lambda_{UV}^2$ or $\mu^2 \xrightarrow{r_0} -\mu^2$.

It is worth noticing that the invariance of beta functions, which contain coefficients of singular contributions to Green's functions, does not require imaginary scaling of coordinates. However, finite parts of loop corrections, like those that emerge in the effective potential, do need coordinate scaling.

Relations between parameters of a theory invariant under the r_0 transformation are stable under RGE evolution to all orders. This is why we name the r_0 transformation a symmetry. However, this is not an ordinary symmetry since

the invariance requires complexification of real fields and coordinates. Nevertheless it is unquestionable that the parameter relations implied by the r_0 transformation are stable under RGE running. This is the power and essence of the symmetry.

Symmetries in field theories imply the presence of relations between, *a priori*, independent parameters of a given model. Sometimes they explain various fundamental facts, like, e.g., the masslessness of the photon, as a consequence of $U(1)$ gauge symmetry. Symmetries may also protect decays of dark matter candidates or, by forbidding certain mass terms, be relevant for the SM hierarchy problem. It is worth mentioning that field theories are applicable also in soft matter physics, statistical physics, and other fields; therefore the novel symmetries described in this work may prove to be useful in other contexts. Hence, proving the existence of hitherto unknown symmetries will broaden our understanding of bosonic field theories, and it might be relevant while looking for solutions of SM difficulties. It should be emphasized that within the 2HDM adopting the new transformations we obtain seven extra symmetries on top of the classical six ones.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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