Updated analysis of charmonium states in a relativized quark potential model

Xiu-Li Gao, Jun-Xi Cui, Yu-Hui Zhou, and Zhi-Yong Zhou*

School of Physics, Southeast University, Nanjing 211189, People's Republic of China

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Motivated by recent experimental observations of charmonium(like) states, we investigate their interpretation as conventional charmonium states within the framework of relativized quark potential model. We find a consistent description of the updated masses of charmonium, bottomonium and some selected charmed mesons, then systematically compute open-charm strong decay widths and electromagnetic transition widths for radially excited charmonium states up to $n = 5$. The numerical results show significantly improved agreement with experimental masses and widths. Notably, the newly observed $\chi_{c1}(4010)$ and $\chi_{c1}(4274)$ are tentatively associated with the $\chi_{c1}(2P)$ and $\chi_{c1}(3P)$, respectively, while the $\chi_{c0}(4500)$ and $\chi_{c0}(4700)$ are interpreted as the $\chi_{c0}(4P)$ and $\chi_{c0}(5P)$ states. For the controversial $\chi_{c1}(3872)$ state, we suggest its potential connection to $\chi_{c1}(4010)$, as a two-pole structure due to the $\chi_{c1}(2P)$ coupling to $D\bar{D}^*$ channels. Intriguingly, the predicted mass of $\chi_{c0}(2P)$ is about 3851 MeV, diverging from the original estimate of 3916 MeV in the Godfrey and Isgur's work but consistent with nonrelativistic potential model predictions. This calculation might provide more insight to the future experimental investigation of charmonium(like) states.

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I. INTRODUCTION

Since the discovery of the J/ψ meson in 1974, over 30 charmonium(like) states have been identified through experimental analyses. Because of a significant increase in the number of observed states, the charmonium spectrum appears denser than the predictions made by the conventional quark potential model [1]. To comprehend the richness of the meson spectrum, many theoretical efforts focus on elucidating the mechanisms behind the additional states observed experimentally. Many approaches are devoted to explore mechanisms beyond the $q\bar{q}$ paradigm, such as multiquark states, hadronic molecules, hybrid states, and other exotic degrees of freedom, to account for the surplus on charmonium(like) spectrum (see literature in Refs. [2,3]). There is also some other attempts aiming to refine the potential model to accommodate a more densely structured spectrum [4,5].

Puzzles surrounding the charmonium spectrum can be exemplified by the $J^{PC} = 1^{++}$ states. The $\chi_{c1}(3872)$ [6], identified as the first excited 1^{++} charmonium state, has sparked intense debate since its discovery. However, positioned near the $D_0\bar{D}_0^*$ threshold, the $\chi_{c1}(3872)$ cannot be easily categorized as the $\chi_{c1}(2P)$ state. First, its mass notably deviates from predictions in relativized Godfrey-Isgur (GI) models [7] and nonrelativistic potential (NR) models [8]. Furthermore, its decay width falls significantly below the anticipated value for $\chi_{c1}(2P)$ in different decay models. The higher state $\chi_{c1}(4140)$ [9] also exhibits a mass and a width far exceeding expectations of $\chi_{c1}(2P)$. More and more research suggests an exotic nature of the $\chi_{c1}(3872)$, while the whereabouts of the $\chi_{c1}(2P)$ state remains an open query. A recent breakthrough by the LHCb collaboration revealed the presence of a new state, $\chi_{c1}(4010)$, in the amplitude analysis of $B^+ \to D^{*\pm}D^{\mp}K^+$ decays [10]. This discovery seems to bridge the long-standing gap of the 1^{++} charmonium spectroscopy, but more evidence from other experiments is required to confirm this discovery. Similar challenges are encountered in categorizing 0^{++} states. There also exist long-time debates about $\chi_{c0}(2P)$ state and its potential correlation with the reported $\chi_{c0}(3860)$ [11] or $\chi_{c0}(3915)$ states [12,13]. The conclusion has not been made until now. For the higher states see, recent observations by LHCb of the 0^{++} $\chi_{c0}(4500)$ and $\chi_{c0}(4700)$ [14,15], which are also suggested to possess exotic origins, as reviewed in the PDG review [1]. These observed states have further enriched discussions about how to understand the charmoniumlike spectrum.

It is agreed that QCD permits the formation of exotic meson states beyond conventional $q\bar{q}$ configurations [16].

*Contact author: zhouzhy@seu.edu.cn

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However, at present it is still absent for a unified theoretical framework which could simultaneously describe conventional $q\bar{q}$ and exotic states. Thus, identifying conventional $q\bar{q}$ candidates provides a critical baseline for distinguishing non-$q\bar{q}$ configurations. The GI model has long served as a reference for such studies. Yet accumulating evidence demonstrates that its predictive power diminishes for states above the open-flavor threshold, where significant discrepancies between its predictions and observed masses emerge. Despite these shortcomings, the GI framework retains conceptual validity for understanding meson spectroscopy. In our opinion, one possibility of its weak prediction power above the open-flavor threshold is that it is too ambitious to describe the states containing quarks from the light to the heavy. If it is limited in studying a subset of the mesonic states, the predictive scope of this model might be extended after recalibrating its parameters.

Motivated by the advances in experimental investigation, we revisit charmonium spectroscopy within the framework of relativized quark potential model, and systematically evaluate open-charm decay properties and electromagnetic transitions. While this study focuses on charmonium states, we incorporate bottomonium states and ground-state $c\bar{u}/c\bar{s}$ systems to constrain model parameters through comparative analysis. With the spectrum and wave functions extracted from the potential model, open-charm strong decay widths for states with radial quantum numbers up to $n = 5$ are computed within the quark pair creation (QPC) model [17], and then compared with experimental measurements. Furthermore, we calculate electromagnetic transition rates, including E1 and M1 partial widths, which serve as references for future experimental explorations. It should be emphasized that the relativized quark potential model and the QPC model employed in this work are QCD-motivated phenomenological models rather than approaches rigorously derived from QCD. Although they have achieved considerable success in describing hadron spectra and decay patterns, the resulting predictions remain model dependent and should not be regarded as first-principles QCD results. Nevertheless, within the scope of these phenomenological models, our findings demonstrate that the updated implementation provides a more coherent spectroscopic interpretation of the current experimental data, including the strong-decay patterns of the 5^3S_1, 4^3P_0, and 5^3P_0 charmonium states.

The paper is organized as follows. Theoretical background is presented in Sec. II, in which we briefly review the relativistic quark model in Sec. II A and the QPC model in Sec. II B, and then illustrate how to compute the radiative transitions in Sec. II C. The numerical results and subsequent discussions are devoted in Sec. III. The last section is the summary and conclusions.

II. THEORETICAL BACKGROUND

A. Brief review about the relativized quark potential model

First, we briefly review the calculation scheme of the GI model [7]. The Hamiltonian of a system containing a quark and an antiquark can be expressed as the sum of the kinematic term and the interaction potential

$$H = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2} + V(\mathbf{p}, \mathbf{r}), \tag{1}$$

where m_1 and m_2 represent the masses of constituent quarks and $\mathbf{p}$ denote the c.m. momentum. In the nonrelativistic limit, the momentum-dependent potential, defined as $V(\mathbf{p}, \mathbf{r}) = H^{\text{conf}} + H^{\text{hyp}} + H^{\text{so}}$, includes the linear confinement and color-Coulomb interactions

$$H^{\text{conf}} = -\left(\frac{3}{4}c + \frac{3}{4}br - \frac{\alpha_s(r)}{r}\right), \tag{2}$$

the color-hyperfine interaction

$$H^{\text{hyp}} = -\frac{\alpha_s(r)}{m_1 m_2}\left[\frac{8\pi}{3}\mathbf{S_1}\cdot\mathbf{S_2}\delta^3(\mathbf{r}) + \frac{1}{r^3}\left(\frac{3\mathbf{S_1}\cdot\mathbf{r}\mathbf{S_2}\cdot\mathbf{r}}{r^2} - \mathbf{S_1}\cdot\mathbf{S_2}\right)\right], \tag{3}$$

and the spin-orbit interaction

$$H^{\text{so}} = -\frac{\alpha_s(r)}{r^3}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\left(\frac{\mathbf{S_1}}{m_1} + \frac{\mathbf{S_2}}{m_2}\right)\cdot\mathbf{L} - \frac{1}{2r}\frac{\partial H^{\text{conf}}}{\partial r}\left(\frac{\mathbf{S_1}}{m_1^2} + \frac{\mathbf{S_2}}{m_2^2}\right)\cdot\mathbf{L}. \tag{4}$$

The explicit treatment of corrections addresses the nonnegligible velocity-dependent effects, particularly manifest in their hyperfine splitting. The "relativization" procedure starts from parameterizing the running coupling constant with respect to the transition momentum Q as $\alpha(Q^2) = \sum_k \alpha_k e^{-Q^2/4\gamma_k^2} = 0.25e^{-Q^2} + 0.15e^{-Q^2/10} + 0.20e^{-Q^2/1000}$. To regularize the long-range interaction behavior, a smearing function $\rho(\mathbf{r} - \mathbf{r}') = \frac{\sigma^3}{\pi^{3/2}} e^{-\sigma^2(\mathbf{r}'-\mathbf{r})^2}$ is introduced to find the smearing potentials through

$$\tilde{V}(r) = \int \mathrm{d}^3\mathbf{r}'\rho(\mathbf{r} - \mathbf{r}')V(r') \tag{5}$$

where σ is a smearing parameter related to the quark flavors.

The smeared color Coulomb potential between quark i and quark j in the configuration space is defined by Fourier transformation as

$$\tilde{G}(r) = -\sum_k \frac{4\alpha_k}{3r}\left(\frac{2}{\sqrt{\pi}}\int_0^{\tau_{kij}r} e^{-x^2}\mathrm{d}x\right), \tag{6}$$

where $\tau_{kij}^2 = (\gamma_k^{-2} + \sigma_{ij}^{-2})^{-1}$ and $\sigma_{ij}^2 = \sigma_0^2[\frac{1}{2} + \frac{1}{2}(\frac{4m_i m_j}{(m_i+m_j)^2})^4] + s^2(\frac{2m_i m_j}{m_i+m_j})^2$. Similarly, the linear confinement potential in the configuration space could be transformed to a smeared form as

$$\tilde{S}(r) = br\left[\frac{1}{\sqrt{\pi}\sigma_{ij}r}e^{-\sigma_{ij}^2 r^2} + \left(1+\frac{1}{2\sigma_{ij}^2 r^2}\right)\frac{2}{\sqrt{\pi}}\int_0^{\sigma_{ij}r} e^{-x^2}\mathrm{d}x\right] + c. \tag{7}$$

With the help of these two important ingredients, the interaction potential is expressed as the sum of the six effective terms as

$$V(\mathbf{p},\mathbf{r}) = G_{\mathrm{eff}}^{\mathrm{Coul}}(r) + G_{\mathrm{eff}}^{\mathrm{so}(v)}(r) + G_{\mathrm{eff}}^{\mathrm{con}}(r) + G_{\mathrm{eff}}^{\mathrm{ten}}(r) + \tilde{S}(r) + S_{\mathrm{eff}}^{\mathrm{so}(s)}(r), \tag{8}$$

and the terms of the right side are, respectively, the effective color Coulomb, spin-orbit, contact, tensor, linear confinement, and scalar spin-orbit interaction potential. These terms are written down as

$$\begin{aligned}
G_{\mathrm{eff}}^{\mathrm{coul}}(r) &= \left(1+\frac{p^2}{E_1E_2}\right)^{1/2}\tilde{G}(r)\left(1+\frac{p^2}{E_1E_2}\right)^{1/2},\\
G_{\mathrm{eff}}^{\mathrm{so}(v)}(r) &= \left(\frac{\mathbf{S}_1\cdot\mathbf{L}}{2m_1^2}\frac{1}{r}\frac{\partial\tilde{G}_{11}^{\mathrm{so}(v)}}{\partial r} + \frac{\mathbf{S}_2\cdot\mathbf{L}}{2m_2^2}\frac{1}{r}\frac{\partial\tilde{G}_{22}^{\mathrm{so}(v)}}{\partial r} + \frac{(\mathbf{S}_1+\mathbf{S}_2)\cdot\mathbf{L}}{m_1m_2}\frac{1}{r}\frac{\partial\tilde{G}_{12}^{\mathrm{so}(v)}}{\partial r}\right),\\
G_{\mathrm{eff}}^{\mathrm{con}}(r) &= \frac{2\mathbf{S}_1\cdot\mathbf{S}_2}{3m_1m_2}\nabla^2\tilde{G}_{12}^{c},\\
G_{\mathrm{eff}}^{\mathrm{ten}}(r) &= -\left(\frac{\mathbf{S}_1\cdot\hat{r}\mathbf{S}_2\cdot\hat{r}-\frac{1}{3}\mathbf{S}_1\cdot\mathbf{S}_2}{m_1m_2}\right)\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)\tilde{G}_{12}^{t},\\
S_{\mathrm{eff}}^{\mathrm{so}(s)}(r) &= -\frac{\mathbf{S}_1\cdot\mathbf{L}}{2m_1^2}\frac{1}{r}\frac{\partial\tilde{S}_{11}^{\mathrm{so}(s)}}{\partial r} - \frac{\mathbf{S}_2\cdot\mathbf{L}}{2m_2^2}\frac{1}{r}\frac{\partial\tilde{S}_{22}^{\mathrm{so}(s)}}{\partial r}.
\end{aligned} \tag{9}$$

The superscripts and subscripts of the $\tilde{G}$ and $\tilde{S}$ terms in Eq. (9) stand for explicit treatment of relativistic corrections, where the modifications could be generally represented as

$$\tilde{G}_{ij}^{\alpha} = \left(\frac{m_i m_j}{E_iE_j}\right)^{1/2+\epsilon_\alpha}\tilde{G}(r)\left(\frac{m_i m_j}{E_iE_j}\right)^{1/2+\epsilon_\alpha}, \tag{10}$$

and $\alpha = c, t, \mathrm{so}(v), \mathrm{so}(s)$ represent contact, tensor, vector spin-orbit, and scalar spin-orbit interactions, respectively.

The wave function of the "mock meson" states is factorized into flavor (ϕ), color (ω), spin (χ) and spatial (ψ) components, and can be explicitly written as

$$\begin{aligned}
|^{2S+1}L_{JM_J}\rangle \equiv \sqrt{2E}\sum_{M_L,M_S}&\langle LM_LSM_S|JM_J\rangle\\
&\times\int \mathrm{d}^3\mathbf{p}\psi_{LM_L}(\mathbf{p})\chi_{SM_S}^{12}\phi^{12}\omega^{12}|\\
&\times q_1(\mu_1\mathbf{P}+\mathbf{p})\bar{q}_2(\mu_2\mathbf{P}-\mathbf{p})\rangle,
\end{aligned} \tag{11}$$

where $\mu_i = \frac{m_i}{m_1+m_2}$. Here, S, L, and J denote the total spin, relative orbital angular momentum and total angular momentum of the $q\bar{q}$ pair, respectively, with M_S, M_L, M_J representing their corresponding magnetic projections. The relative momentum is denoted by $\mathbf{p}$ and the center of mass momentum by $\mathbf{P}$. Usually, the wave function is solved in their c.m. frame, which means $\mathbf{P} = \mathbf{0}$. The eigenenergy and radial part of the wave function is obtained by solving through diagonalization within an extensive basis of simple harmonic oscillator (SHO) wave functions. This computational approach leverages the unique analytic structure of SHO wave functions, whose isomorphic function forms in momentum space, $\psi_{LM_L}(\mathbf{p})$, and coordinate space, $\phi_{LM_L}(\mathbf{r})$, are given by similar Hermite polynomials multiplied by Gaussian factors. Such duality allows the separation of momentum-dependent and coordinate-dependent components in the interaction matrix elements, enabling computational efficient evaluation through factorized integrals in a simple way. That means the wave function of a certain meson in the momentum space and the configuration space are expanded in series of the SHO wave functions as

$$\begin{aligned}
\psi_{LM_L}(\mathbf{p}) &= \sum_n c_n R_{nL}(p)Y_{LM_L}(\Omega_p),\\
\phi_{LM_L}(\mathbf{r}) &= \sum_n c_n \tilde{R}_{nL}(r)Y_{LM_L}(\Omega_r),
\end{aligned} \tag{12}$$

where c_n is the coefficient of the SHO wave function basis, and the radial parts in momentum space R_{nL} and that in configuration space $\tilde{R}_{nL}$ can be, respectively, expressed as

$$R_{nL}(p) = \frac{(-1)^n(-i)^L}{\beta^{\frac{3}{2}}}\sqrt{\frac{2^{L+2-n}(2L+2n+1)!!}{\sqrt{\pi}n![(2L+1)!!]^2}}\left(\frac{p}{\beta}\right)^L \times e^{-\frac{1}{2}\frac{p^2}{\beta^2}}{}_1F_1\left(-n, L+\frac{3}{2}, \frac{p^2}{\beta^2}\right), \tag{13}$$

$$\tilde{R}_{nL}(r) = \beta^{\frac{3}{2}}\sqrt{\frac{2^{L+2-n}(2L+2n+1)!!}{\sqrt{\pi}n![(2L+1)!!]^2}}(\beta r)^L \times e^{-\frac{1}{2}\beta^2r^2}{}_1F_1\left(-n, L+\frac{3}{2}, \beta^2r^2\right), \tag{14}$$

where β is a harmonic oscillator parameter in wave function and ${}_1F_1$ the confluent hypergeometric function.

When we calculate the Hamiltonian matrix element of the operator with both the momentum and the spacial coordinate as $f_1(p)f_2(r)$, a complete set of the wave function is inserted so that the element can be represented as the matrix product of two terms as

$$\langle i|f_1(p)f_2(r)|j\rangle = \sum_n \langle i|f_1(p)|n\rangle\langle n|f_2(r)|j\rangle. \quad (15)$$

The matrix elements $\langle i|f_1(p)|n\rangle$ could be calculated in momentum space while the matrix elements $\langle n|f_2(r)|j\rangle$ are calculated in configuration space. The similarity of the SHO wave function in the momentum and configuration space makes numerical calculation very simple. Finally, we could find the wave functions of the eigenstates by diagonalizing the Hamiltonian matrix in a large SHO basis. In the practical procedure, we find that numerical convergence is achieved by using 30 SHO basis states with the harmonic oscillator parameter $\beta = 0.4$ GeV.

B. The quark pair creation model

The QPC model, alternatively known as the ${}^3P_0$ model due to its creation of $J^{PC} = 0^{++}$ quark-antiquark pairs from the vacuum [17–20], is employed to calculate the open-charm decay widths of charmonium states. This model has been demonstrated to provide reliable predictions for Okubo-Zweig-Iizuka (OZI) allowed strong decays into two-body hadronic final states, which dominate the decay patterns of conventional quarkonium states above the open-flavor threshold [21–24]. The interaction could be derived from the quantum field theory Hamiltonian

$$H_I = \gamma \int \mathrm{d}^3x\bar{\psi}(x)\psi(x), \quad t = 0, \quad (16)$$

where $\psi(x)$ is a Dirac field operator associated with the space-time point x. One could find, for the process of $A \to BC$, the S matrix is expressed as

$$\langle BC|S|A\rangle = I - 2\pi i\delta(E_f - E_i)\langle BC|T|A\rangle, \quad (17)$$

where the transition operator T can be expressed as

$$T = -3\gamma\sum_m \langle 1m;1-m|00\rangle \int \mathrm{d}^3\mathbf{p}_3\mathrm{d}^3\mathbf{p}_4\delta^3(\mathbf{p}_3+\mathbf{p}_4) \times \mathcal{Y}_{1m}\left(\frac{\mathbf{p}_3-\mathbf{p}_4}{2}\right)\chi^{34}_{1,-m}\phi^{34}_0\omega^{34}_0 b^\dagger_3(p_3)d^\dagger_4(p_4). \quad (18)$$

The γ parameter represents the production strength of a quark-antiquark pair from the vacuum. The superscripts and subscripts "3" and "4" represent the quark and antiquark. $\chi^{34}_{1,-m}$, ϕ^{34}_0, ω^{34}_0 are spin, flavor and color wave functions of the quark pair generated from the vacuum, and the solid harmonic $\mathcal{Y}^m_1(\mathbf{p}) = pY^m_1(\Omega_p)$ represents its relative space wave function. The mock state of the meson is defined consistently with the wave function defined in the relativized potential model in the previous Sec. [7]. Therefore, the matrix element of the T operator in the c.m. frame is

$$\begin{aligned}\langle BC|T|A\rangle &= \delta^3(\mathbf{P}_B+\mathbf{P}_C-\mathbf{P}_A)\mathcal{M}^{M_{J_A}M_{J_B}M_{J_C}} \\ &= \gamma\sqrt{8E_AE_BE_C}\sum_{M_{L_A},M_{S_A},M_{L_B},M_{S_B},M_{L_C},M_{S_C},m}\langle L_AM_{L_A}S_AM_{S_A}|J_AM_{J_A}\rangle \\ &\quad\times\langle L_BM_{L_B}S_BM_{S_B}|J_BM_{J_B}\rangle\langle L_CM_{L_C}S_CM_{S_C}|J_CM_{J_C}\rangle\langle 1m1-m|00\rangle\langle\chi^{14}_{S_BM_{S_B}}\chi^{32}_{S_CM_{S_C}}|\chi^{12}_{S_AM_{S_A}}\chi^{34}_{1,-m}\rangle \\ &\quad\times\langle\phi^{14}_B\phi^{32}_C|\phi^{12}_A\phi^{34}_0\rangle I(\mathbf{P},m_1,m_2,m_3), \end{aligned} \quad (19)$$

where $\langle\chi^{14}_{S_BM_{S_B}}\chi^{32}_{S_CM_{S_C}}|\chi^{12}_{S_AM_{S_A}}\chi^{34}_{1,-m}\rangle$ is the spin matrix element, the color matrix element $\langle\omega^{14}_B\omega^{32}_C|\omega^{12}_A\omega^{34}_0\rangle = 1/3$, and the flavor matrix element $\langle\phi^{14}_B\phi^{32}_C|\phi^{12}_A\phi^{34}_0\rangle = 1/\sqrt{3}$ in this study. The space integral factor $I(\mathbf{P},m_1,m_2,m_3)$ is given by

$$\begin{aligned} I(\mathbf{P},m_1,m_2,m_3) &= \int \mathrm{d}^3\mathbf{p}\psi^*_{n_BL_BM_{L_B}}\left(\frac{m_3}{m_1+m_3}\mathbf{P}+\mathbf{p}\right)\psi^*_{n_CL_CM_{L_C}}\left(\frac{m_3}{m_2+m_3}\mathbf{P}+\mathbf{p}\right) \\ &\quad\times\psi_{n_AL_AM_{L_A}}(\mathbf{P}+\mathbf{p})\mathcal{Y}_{1m}(\mathbf{p}), \end{aligned} \quad (20)$$

where $\mathbf{P} = \mathbf{P}_B = -\mathbf{P}_C$ represents the three momentum of B in the c.m. frame of the BC system.

By choosing the momentum $\mathbf{P}$ along the z axis and applying the Jacob-Wick formula [25], the partial wave amplitude is expressed by the helicity amplitude $\mathcal{M}^{M_{J_A}M_{J_B}M_{J_C}}$ as

$$\mathcal{M}^{JL}(P) = \frac{\sqrt{4\pi(2L+1)}}{2J_A+1}\times\sum_{M_{J_B},M_{J_C}}\langle L0JM_{J_A}|J_AM_{J_A}\rangle\langle J_BM_{J_B}J_CM_{J_C}|J_AM_{J_A}\rangle\times\mathcal{M}^{M_{J_A}M_{J_B}M_{J_C}}(P\hat{z}). \quad (21)$$

Thus, the partial decay width of meson A could be calculated through the following formula

$$\Gamma = \frac{\pi}{4}\frac{P}{M_A^2}\sum_{J,L}|M^{JL}(P)|^2. \tag{22}$$

C. Radiative transition model

The quark-photon interaction in the tree-level QED framework is governed by the Hamiltonian density

$$H_{em} = -\sum_j e_j \overline{\psi_j}\gamma_\mu^j A^\mu(\mathbf{k},\mathbf{r})\psi_j, \tag{23}$$

where ψ_j denotes the Dirac field operator for the jth constituent quark with electric charge e_j and $\mathbf{k}$ the three momentum of the photon. $A^\mu(\mathbf{k},\mathbf{r}) = \epsilon^\mu e^{i\mathbf{k}\cdot\mathbf{r}_j}$ represents the photon's quantized electromaganetic field in the Coulomb gauge with polarization vector ϵ^μ.

Applying the Foldy-Wouthuysen transformation to derive the nonrelativistic reduction, the interaction Hamiltonian H_{em} reduces to the leading-order multipole expansion as [26–28]

$$h_e \simeq \sum_j \left[e_j \mathbf{r}_j\cdot\boldsymbol{\epsilon} - \frac{e_j}{2m_j}\boldsymbol{\sigma}_j(\boldsymbol{\epsilon}\times\hat{\mathbf{k}})\right]e^{-i\mathbf{k}\cdot\mathbf{r}_j}, \tag{24}$$

where m_j, $\mathbf{r}_j$ and $\boldsymbol{\sigma}_j$ are the mass, coordinate and Pauli matrix of the jth quark, respectively. The $\boldsymbol{\epsilon}$ denotes the transverse polarization vector with $\epsilon\cdot\mathbf{k} = 0$. The first and second terms generate electric dipole (E1) and magnetic dipole (M1) transitions, respectively.

The helicity amplitude of the electric and magnetic transitions between the initial state $J\lambda$ and final state $J'\lambda'$ can be expressed as

$$\mathcal{A}_\lambda = -i\sqrt{\frac{\omega_\gamma}{2}}\langle J'\lambda'|h_e|J\lambda\rangle, \tag{25}$$

where

$$\mathcal{A}_\lambda^E = -i\sqrt{\frac{\omega_\gamma}{2}}\langle J'\lambda'|\sum_j e_j\mathbf{r}_j\cdot\boldsymbol{\epsilon}e^{-i\mathbf{k}\cdot\mathbf{r}_j}|J\lambda\rangle, \tag{26}$$

$$\mathcal{A}_\lambda^M = +i\sqrt{\frac{\omega_\gamma}{2}}\langle J'\lambda'|\sum_j \frac{e_j}{2m_j}\boldsymbol{\sigma}_j(\boldsymbol{\epsilon}\times\hat{\mathbf{k}})e^{-i\mathbf{k}\cdot\mathbf{r}_j}|J\lambda\rangle, \tag{27}$$

and ω_γ is the energy of photon.

In the analysis of radiative transitions, we adopt the canonical reference frame where the initial hadron is at rest, $\mathbf{P_i} = 0$, and the final hadron carries momentum $\mathbf{P_f} = -\mathbf{k}$. The photon momentum is selected along the z axis ($\mathbf{k} = k\hat{\mathbf{z}}$), and its polarization vector is conventionally represented in a right-handed form, specifically as $\epsilon = \frac{1}{\sqrt{2}}(1, i, 0)$.

To systematically evaluate the transition matrix elements, we employ the method of multipole expansion for the plane wave

$$e^{-i\mathbf{k}\cdot\mathbf{r}_j} = e^{-ikz_j} = \sum_l \sqrt{4\pi(2l+1)}(-i)^l j_l(kr_j)Y_{l0}(\Omega), \tag{28}$$

where $j_l(x)$ denotes the spherical Bessel function of the first kind and $Y_{lm}(\Omega)$ denotes the spherical harmonics. The matrix elements corresponding to the electric multipole transition amplitude and the magnetic transition amplitude are derived through angular momentum recoupling techniques [5,29,30]

$$\mathcal{A}_\lambda^{El} = \sqrt{\frac{\omega_\gamma}{2}}\langle J'\lambda'|\sum_j(-i)^l B_l e_j j_{l+1}(kr_j)r_j Y_{l1}|J\lambda\rangle + \sqrt{\frac{\omega_\gamma}{2}}\langle J'\lambda'|\sum_j(-i)^l B_l e_j j_{l-1}(kr_j)r_j Y_{l1}|J\lambda\rangle, \tag{29}$$

$$\mathcal{A}_\lambda^{Ml} = \sqrt{\frac{\omega_\gamma}{2}}\langle J'\lambda'|\sum_j(-i)^l C_l\frac{e_j}{2m_j}j_{l-1}(kr_j)\sigma_j^+ Y_{l-10}|J\lambda\rangle, \tag{30}$$

with the coefficients $B_l = \sqrt{2\pi l(l+1)/(2l+1)}$, $C_l = i\sqrt{8\pi(2l-1)}$, where $\sigma^+ = (\sigma_x + i\sigma_y)/2$ represents the spin raising operator in the spherical tensor basis.

The electric and magnetic transition widths are given by

$$\Gamma^{El} = \frac{|\mathbf{k}|^2}{\pi}\frac{2}{2J_i+1}\frac{M_f}{M_i}\sum_{J_{zf},J_{zi}}|\mathcal{A}^{El}_{J_{zf},J_{zi}}|^2, \tag{31}$$

$$\Gamma^{Ml} = \frac{|\mathbf{k}|^2}{\pi}\frac{2}{2J_i+1}\frac{M_f}{M_i}\sum_{J_{zf},J_{zi}}|\mathcal{A}^{Ml}_{J_{zf},J_{zi}}|^2, \tag{32}$$

where M_i and M_f are the masses of the initial and final charmonium states, respectively. Here, J_i denotes the total angular momentum of the initial hadron, while J_{zi} and J_{zf} represent the z-axis projections of the total angular momentum of the initial and final hadrons.

Contributions from the higher angular momentum contributions are negligible compared with the lowest-order approximation ($l = 1$). Therefore, the electric and magnetic transition widths reduce to

$$\Gamma^{E1} = \frac{4}{3}\alpha e_c^2 \omega_\gamma^3 \delta_{SS'}\delta_{LL'\pm1}\max(L,L')(2J'+1) \times \begin{Bmatrix} L' & J' & S \\ J & L & 1 \end{Bmatrix}^2 \left\langle \sum_{n'} c'_{n'}\tilde{R}_{n'L'}(r)|r|\sum_n c_n \tilde{R}_{nL}(r)\right\rangle^2, \tag{33}$$

$$\Gamma^{M1} = \frac{4}{3}\frac{\alpha}{m_c^2} e_c^2 \omega_\gamma^3 \delta_{LL'}\delta_{SS'\pm1}\frac{2J'+1}{2L+1} \times \left\langle \sum_{n'} c'_{n'}\tilde{R}_{n'L'}(r)\left| j_0\left(\omega_\gamma \frac{r}{2}\right)\right|\sum_n c_n \tilde{R}_{nL}(r)\right\rangle^2, \tag{34}$$

where $\sum_n c_n \tilde{R}_{nL}(r)$ and $\sum_{n'} c'_{n'}\tilde{R}_{n'L'}(r)$ represent the radial wave functions of the initial and final states in coordinate space, respectively, as defined in Eq. (12). The charm quark charge $e_c = 2/3$, the fine-structure constant $\alpha = 1/137$, and $\{\ldots\}$ represents the $6j$ symbol.

III. NUMERICAL RESULTS AND DISCUSSION

In this calculation, we first redetermined the model parameters based on the newly observed meson states and calculated their wave functions in the GI' model. For the strong-decay calculations, the numerical eigenfunctions obtained from the relativized potential model are directly employed in the QPC spatial-overlap integral in Eq. (20). In practice, each initial- and final-state momentum-space wave function appearing in this integral is represented by the 30-SHO-basis expansion of Eq. (12), with all the corresponding expansion coefficients c_n retained in the numerical evaluation. Thus, the SHO basis is used only as a numerical representation of the eigenfunctions, rather than any other simplification. The same numerical eigenfunctions are also used in the electromagnetic-transition calculations.

The 50 selected candidates include 22 charmonium states, 17 bottomonium states, ten charmed and charmed-strange states and the pion, as listed in the "Candidates" columns of Tables I and II. Their masses and widths are average values quoted in the PDG table [1]. The mass spectra other than the charmonium states are used as additional constraints to refine model parameters. The agreement between the predicted and the observed masses is quantified using a χ^2 function defined as

$$\chi^2 = \sum_n \left(\frac{y_{\mathrm{exp},n} - y_{th,n}}{\sigma_n}\right)^2$$

where the sum runs over the selected candidate states. Given that the theoretical mass spectrum does not precisely match the experiment values and the discrepancies are significantly larger than the quoted experiment uncertainties, we assign a fixed value of $\sigma_n = 30$ MeV for all the states, rather than using the PDG uncertainties. This choice reflects a realistic tolerance for the model's accuracy. Using the original GI's values, we find $\frac{\chi^2}{dnf} = 2.3$ which is reduced to approximately 1.1 with our new parameter set. This demonstrates a significant improvement in the description of the mass spectrum. However, the resultant χ^2 value loses its formal statistics meaning but serves as a useful qualitative measure of the overall consistency between the theoretical predictions and the PDG values. The central values of model parameters determined here are listed as follows:

$$\begin{aligned} &m_{u/d} = 0.346 \pm 0.016\ \mathrm{GeV}, \quad m_s = 0.548 \pm 0.024\ \mathrm{GeV}, \quad m_c = 1.748 \pm 0.002\ \mathrm{GeV}, \quad m_b = 5.077 \pm 0.002\ \mathrm{GeV}, \\ &b = 0.163 \pm 0.001\ \mathrm{GeV}^2, \quad c = -0.409 \pm 0.003\ \mathrm{GeV}, \quad \sigma_0 = 2.020 \pm 0.087, \quad s_0 = 2.017 \pm 0.062, \\ &\epsilon_c = -0.217 \pm 0.012, \quad \epsilon_t = -0.716 \pm 0.028, \quad \epsilon_{\mathrm{sov}} = -0.212 \pm 0.016, \quad \epsilon_{\mathrm{sos}} = -0.031 \pm 0.175. \end{aligned} \tag{35}$$

This set of parameters are typically different from the original GI's values. With these set of parameters, the computed mass spectrum and candidate assignments are tabulated in Tables I and II. It is found that the updated results exhibit a denser charmonium mass spectrum, with the $n = 5$ states approximately 100 MeV lower in mass. In contrast, the bottomonium mass spectrum remains changes slightly. This resulting charmonium spectrum and the corresponding decay widths addressed later, which are not calculated in the GI model, are more consistent with the observed values. For example, the $\psi(4660)$ is difficult to accommodate in the original GI's prediction, but it is well described by our new results. The same holds for the $\chi_{c0}(4500)$ and $\chi_{c0}(4700)$. When we calculate the open-charm strong decay widths in the QPC model, the production strength from the vacuum is empirically chosen to be $\gamma = 6.3$ for $u\bar{u}$ or $d\bar{d}$ and $\gamma_s = 6.3/\sqrt{3}$ for $s\bar{s}$ [31]. The masses of the final states are chosen according to averages of the physical masses in the PDG table to ensure the correct threshold energies and phase space factors. For the initial charmonium states, the average BW masses quoted in PDG table are used when the related candidates have been reported. For those states without candidates observed, we use theoretically predicted masses to compute the decay widths. The total width is the sum over partial widths across all kinematically allowed decay channels, which are also

TABLE I. Mass spectrum and obtained strong decay widths of charmonium are compared with those of candidates in PDG. The unit is MeV. The candidates denoted with a question mark need further confirmation [1].

System	State	Our result (mass)	Our result (width)	Candidate	PDG mass	PDG width	GI [7]
$c\bar{c}$	1^1S_0	2997		η_c	2984		2974
	2^1S_0	3628		η_c'	3637	20 ± 4	3630
	3^1S_0	4039	93				4068
	4^1S_0	4372	70				4424
	5^1S_0	4663	117				5018
	1^3S_1	3122		J/ψ	3096		3092
	2^3S_1	3682		$\psi(3686)$	3686		3680
	3^3S_1	4076	65	$\psi(4040)$	4040	84 ± 12	4100
	4^3S_1	4401	86	$\psi(4415)$	4415 ± 5	110 ± 22	4450
	5^3S_1	4687	71	$\psi(4660)$	4641 ± 10	73^{+13}_{-11}	4757
	1^3D_1	3808	26	$\psi(3770)$	3773.7 ± 0.7	27.2 ± 1.0	3818
	2^3D_1	4155	74	$\psi(4160)$	4191 ± 5	69 ± 10	4194
	3^3D_1	4457	107				4519
	4^3D_1	4728	120				4812
	5^3D_1	4977	82				5080
	1^3P_0	3402		χ_{c0}	3414		3443
	2^3P_0	3851	73	$\chi_{c0}(3860)$?	3862^{+50}_{-35}	200^{+180}_{-110}	3917
	3^3P_0	4204	85				4291
	4^3P_0	4508	93	$\chi_{c0}(4500)$	4474 ± 4	77^{+12}_{-10}	4614
	5^3P_0	4781	79	$\chi_{c0}(4700)$	4694^{+16}_{-5}	87^{+18}_{-10}	4903
	1^3P_1	3529		χ_{c1}	3510.67 ± 0.05		3508
	2^3P_1	3948	48	$\chi_{c1}(4010)$	4012.5	62.7	3953
	3^3P_1	4286	19	$\chi_{c1}(4274)$	4286^{+8}_{-9}	51 ± 7	4316
	4^3P_1	4581	242	$\chi_{c1}(4685)$	4684^{+15}_{-17}	130 ± 40	4633
	5^3P_1	4840	111				4920
	1^3P_2	3567		χ_{c2}	3556.17 ± 0.07		3548
	2^3P_2	3971	20	$\chi_{c2}(3930)$	3922.5 ± 1.0	35.2 ± 2.2	3979
	3^3P_2	4304	46	$X(4350)$?	4351 ± 5	13^{+18}_{-10}	4336
	4^3P_2	4595	91				4650
	5^3P_2	4858	91				4934
	1^1P_1	3534		h_c	3525.37 ± 0.14		3515
	2^1P_1	3946	63	$X(3940)$?	3942 ± 9	37^{+27}_{-17}	3956
	3^1P_1	4283	31				4318
	4^1P_1	4576	113				4634
	5^1P_1	4841	111				4920

presented in Table I. In this calculation, the physical charmed mesons $D_1(2420)$ and $D_1(2430)$, as well as their charmed-strange counterparts $D_{s1}(2536)$ and $D_{s1}(2460)$, are interpreted as mixtures of the 1^1P_1 and 1^3P_1 configurations. The mixed states are defined as

$$|P_1'\rangle = \cos\theta|1^1P_1\rangle + \sin\theta|1^3P_1\rangle, \tag{36}$$

$$|P_1\rangle = -\sin\theta|1^1P_1\rangle + \cos\theta|1^3P_1\rangle, \tag{37}$$

where the mixing angle is empirically fixed at $\theta = -54.7°$ extracted in the heavy quark limit [32,33]. To determine the mixing angle require the accurate partial width data, which is still absent. A more significant challenge, however, is that traditional potential models consistently struggle to accurately describe the properties of $D_1(2420)$ and $D_1(2430)$ pair, and particularly the $D_{s1}(2460)$ and $D_{s1}(2536)$ states. The failure of potential models to reproduce the physical mixing and mass orderings of these states suggests that they may be better described within a molecular picture or that coupled-channel effects play a dominant role in their structure. Nevertheless, we think the heavy quark limit is a reasonable approximation for the charmed states, and we also found the total decay widths are not significantly influenced with different choice of the mixing angle. In practice, the predicted masses of these states are not

TABLE II. Mass spectra of bottomonium, selected charmed and charmed-strange states, and the pion, which is used to constrain the constituent u/d quark mass, are compared with the corresponding PDG values. The unit is MeV [1].

System	State	Our result (mass)	Candidate	PDG mass	GI [7]
$b\bar{b}$	1^1S_0	9449	$\eta_b(1S)$	9398.7 ± 2.0	9431
	2^1S_0	9998	$\eta_b(2S)$	9999 ± 4	9998
	3^1S_0	10335			10354
	4^1S_0	10601			10636
	5^1S_0	10831			10881
	1^3S_1	9490	$\Upsilon(1S)$	9460.4 ± 0.10	9472
	2^3S_1	10014	$\Upsilon(2S)$	10023.4 ± 0.5	10015
	3^3S_1	10345	$\Upsilon(3S)$	10355.1 ± 0.5	10365
	4^3S_1	10609	$\Upsilon(4S)$	10579.4 ± 1.2	10645
	5^3S_1	10837	$\Upsilon(10860)$	$10885.2^{+2.6}_{-1.6}$	10887
	1^3D_1	10134			10138
	2^3D_1	10420			10442
	3^3D_1	10662			10700
	4^3D_1	10877			10930
	5^3D_1	11074			11140
	1^3P_0	9856	$\chi_{b0}(1P)$	9859.44 ± 0.73	9850
	2^3P_0	10218	$\chi_{b0}(2P)$	10232.5 ± 0.9	10232
	3^3P_0	10498			10529
	4^3P_0	10736			10783
	5^3P_0	10948			11001
	1^3P_1	9886	$\chi_{b1}(1P)$	9892.78 ± 0.57	9876
	2^3P_1	10239	$\chi_{b1}(2P)$	10255.46 ± 0.72	10249
	3^3P_1	10514	$\chi_{b1}(3P)$	10513.4 ± 0.7	10542
	4^3P_1	10749			10792
	5^3P_1	10959			11017
	1^3P_2	9905	$\chi_{b1}(1P)$	9912.21 ± 0.57	9896
	2^3P_2	10252	$\chi_{b2}(2P)$	10268.65 ± 0.72	10262
	3^3P_2	10524	$\chi_{b2}(3P)$	10524.0 ± 0.8	10551
	4^3P_2	10757			10800
	5^3P_2	10967			11024
	1^1P_1	9891	$h_b(1P)$	9899.3 ± 0.8	9882
	2^1P_1	10242	$h_b(2P)$	10259.8 ± 1.2	10252
	3^1P_1	10516			10544
	4^1P_1	10751			10794
	5^1P_1	10961			11019
$c\bar{u}$	1^1S_0	1819	D^0	1864.84 ± 0.05	1871
	1^3S_1	2003	$D^{*0}(2007)$	2006.85 ± 0.05	2038
	1^3P_2	2466	$D_2^*(2460)$	2461.1 ± 0.8	2501
	1^3P_1	2441	$D_1(2430)$	2412.1 ± 9	2490
	1^1P_1	2405	$D_1(2420)$	2422.1 ± 0.6	2440
$c\bar{s}$	1^1S_0	1947	D_s^+	1968.35 ± 0.07	1975
	1^3S_1	2119	D_s^{*+}	2112.2 ± 0.4	2124
	1^3P_2	2579	$D_{s2}^{*+}(2573)$	2569.1 ± 0.8	2591
	1^3P_1	2546	$D_{s1}^+(2536)$	2535.11 ± 0.06	2570
	1^1P_1	2528	$D_{s1}^+(2460)$	2459.5 ± 0.6	2530
$(u\bar{u} - d\bar{d})/\sqrt{2}$	1^1S_0	144	π	137 ± 0.07	150

particularly sensitive to the precise value of the mixing angles, as the mass difference between the unperturbed states is only about 20 MeV difference.

While parameters of relativized potential model are determined by fitting the mass spectrum, two significant uncertainties must be acknowledged: First, states with quantum numbers matching predictions of the quenched quark model do not necessarily correspond exclusively to conventional $q\bar{q}$ configurations. This ambiguity arises due to potential contamination from non-$q\bar{q}$ states or near-threshold structures that may be interpreted as resonances in PDG listings. A second uncertainty stems from the extraction of mass parameters via Breit-Wigner (BW) parametrization. The BW masses precisely correspond neither to the poles mass from the scattering amplitudes nor to bare masses from quark model, particularly when the narrow-resonance approximation breaks down for broad or overlapping states. The number of states reported in experiments now significantly exceeds the capacity of the conventional quark potential model. Consequently, our fitting procedure is necessarily restricted to those states whose mass splittings are most consistent with the model's predictions. The systematic uncertainty associated with this specific selection, and the sensitivity to alternative assignments of the observed states to the theoretical spectrum, cannot be easily assessed.

A. $^3S_1$ and $^3D_1$ states

A series of $J^{PC} = 1^{--}$ charmonium states such as $\psi(3686)$, $\psi(3770)$, $\psi(4040)$, $\psi(4160)$, and $\psi(4415)$ have been observed and identified as the radial and orbital excitations $\psi(2S)$, $\psi(1D)$, $\psi(3S)$, $\psi(2D)$, and $\psi(4S)$, respectively, in different potential models [5,7,34,35]. In recent years, high-precision data from e^+e^- collider experiments at Belle, *BABAR*, BES, and CLEO-c have revealed additional resonance structures in the charmoniumlike spectrum. The BES collaboration reported the hyperfine structure of the broad $Y(4260)$ state, identifying the lower $\psi(4230)$ and higher $\psi(4360)$ states. Very recently, BESIII reported the observation of another 1^{--} state, $G(3900)$, in the $e^+e^- \to D\bar{D}$ process [36]. Furthermore, the Belle have discovered another heavy 1^{--} state, $\psi(4660)$ [37], which was confirmed by *BABAR* [38]. This $\psi(4660)$ state has been the subject of extensive theoretical debate with many non-$q\bar{q}$ interpretations including molecular states composed of $D_s^{(*)}\bar{D}_{s1}(2536)$, $D_s^{(*)}\bar{D}_{s2}(2573)$ [39–41], $\psi(2S)f_0(980)$ [42], and $\Lambda_c^+\bar{\Lambda}_c^-$ [43] or a $[cs][\bar{c}\,\bar{s}]$ tetraquark state [44].

In this calculation, we can accommodate J/ψ, $\psi(3686)$, $\psi(4040)$, and $\psi(4415)$ as a $\psi(nS)$ state with $n = 1, \ldots, 4$, while $\psi(3770)$ and $\psi(4160)$ are consistent with the $\psi(1D)$ and $\psi(2D)$. This classification matches the assignments in the GI model [7]. It is found that the calculated widths of n^3S_1 and n^3D_1 states are consistent with the extracted BW widths of the observed states. This implies the empirical choice of $\gamma = 6.3$ is reasonable for most open-flavor decays.

The discrepancy emerges for the $\psi(5S)$ state. The predicted mass of $\psi(5S)$ state is about 4757 MeV, which is about 110 MeV higher than the extracted BW mass of $\psi(4660)$. In the PDG review [1], it is claimed that the $\psi(4660)$ challenges the quark model since its mass and decay properties are in conflicted with expectations. However, in this calculation, the mass of $\psi(5S)$ is $M_{\psi(5S)} = 4687$ MeV, which is in agreement with the BW mass of $\psi(4660)$, which is $M_{\psi(4660)} = 4641 \pm 10$ MeV. This suggests that the $\psi(4660)$ could be identified as the $\psi(5S)$ state. Further evidence could be found from the strong decay widths. The open-flavor strong width obtained here is about 71 MeV, which is comparable with the PDG-averaged value $\Gamma_{\psi(4660)} = 73^{+13}_{-11}$ MeV. Nevertheless, it is worth noting that the PDG-quoted widths are averaged over different experimental analyses, which deviate from each other. The recent observations by BESIII prefer a width larger than 100 MeV [45,46], while the older measurements by Belle and *BABAR* reported much narrower results [47,48]. Further experimental investigation are expected to determine the properties of this charmonium state more accurately.

In this classification of conventional charmonium, there is no room for the $G(3900)$, $\psi(4230)$ and $\psi(4360)$ states, strongly suggesting their non-$q\bar{q}$ nature. Notably, the $G(3900)$ has been proposed as a $D\bar{D}$ molecular state in Ref. [49]. A recent K-matrix analysis of $e^+e^- \to D\bar{D}$ data also argued that the $G(3900)$ does not originate from a conventional $c\bar{c}$ resonance but instead arises as a threshold enhancement linked to the $D\bar{D}^*/\bar{D}D^*$ threshold [50]. It is also worth mentioning that in the classical Cornell model [51], a bump at about 3900 MeV in the $D\bar{D}$ invariant mass was predicted without introducing any new resonance other than the conventional charmonium states. Similarly, the $\psi(4230)$ and $\psi(4360)$ states are usually understood in molecular interpretations due to the proximity of strongly coupled hadronic threshold near their observed masses [52]. These observations also implies that the traditional BW parametrization method might introduce more and more observed resonance shapes in the era of high-precision experimental investigation. Other parametrization methods beyond the BW form, such as coupled-channel analyses or effective range expansions, are essential for distinguishing true resonant states from threshold-induced enhancements. Such advances will clarify the fundamental nature of these exotic candidates.

B. $^3P_0$ states

Long-standing challenges persist in the classification of $J^{PC} = 0^{++}$ charmonium states. The $\chi_{c0}(3915)$, first observed by Belle [12] and later assigned quantum numbers 0^{++} by *BABAR* [53], initially appeared consistent with

the $\chi_{c0}(2P)$ prediction of the GI model. However, its absence in the $\gamma\gamma \to D\bar{D}$ process, where a narrow peak near 3930 MeV is identified as the $\chi_{c2}(2P)$ state, raised doubts about this assignment and inspired a suggestion of a broad structure at 3860 MeV linked to $\chi_{c0}(2P)$ [54]. It was also pointed out that, if the helicity-2 dominance is abandoned, the peak at about 3930 MeV in $\gamma\gamma \to J/\psi\omega$ could be a $J^{PC} = 2^{++}$ $\chi_{c2}(2P)$ state as in the $\gamma\gamma \to D\bar{D}$ process [55], which also implies the large helicity-0 component in this peak. Later, Belle reported a broad $\chi_{c0}(3860)$ state in the $e^+e^- \to J/\psi X$ process [11], which seems to be related to the board structure in the $\gamma\gamma \to D\bar{D}$ process. Different from the earlier analysis, LHCb made an amplitude analysis of $B^+ \to D^+D^-K^+$ by modeling the data points about 3930 MeV in D^+D^- invariant mass as contributed by nearly degenerate $\chi_{c0}(3915)$ and $\chi_{c2}(3930)$ in the BW parametrization form [56]. Their results do not accommodate the existence of a broad $\chi_{c0}(3860)$. These analyses add controversies at low mass scalar charmonium. In the higher energy region, the $\chi_{c0}(4500)$ and $\chi_{c0}(4700)$ states are observed by LHCb in $B^+ \to J/\psi\phi K^+$ [15], both of which are more than 100 MeV lower than the predicted masses of conventional $c\bar{c}$ states, so they are also considered to be of exotic or non-$q\bar{q}$ origins in the PDG review.

In our calculation, the mass spectrum become denser than the predicted values in the original GI's work. The predicted mass of $\chi_{c0}(2P)$ state is located at about 3851 MeV. This mass is similar to the results of other nonrelativistic quark potential models [8,57], but much lower than the original predicted mass of 3916 MeV by GI's work. The strong decay width of $\chi_{c0}(2P)$ obtained in the QPC model is quite large to about ~73 MeV, so these results disfavor assigning the $\chi_{c0}(3915)$ as the $\chi_{c0}(2P)$ state. To further investigate the possibility of $\chi_{c0}(3915)$ as the $\chi_{c0}(2P)$ state, we performed a constrained calculation using the same $\chi_{c0}(2P)$ wave function while fixing the mass at 3915 MeV by hand, and we found that the open-charm decay width results in a much smaller value of ~20 MeV, which seems to be consistent with the measured width of $\chi_{c0}(3915)$. This comparison might imply that the calculated width is sensitive to nodal structures in the wave function, so the information of the decay width is not sufficient for determining the property of $\chi_{c0}(3915)$. At first sight, this smaller width may seem to support the assignment of the $\chi_{c0}(3915)$ as the $\chi_{c0}(2P)$ state. However, this result should be interpreted with caution. In the constrained calculation, the same $\chi_{c0}(2P)$ wave function is used, and only the input mass of the initial state is changed. When the input mass is increased from about 3.86 GeV to 3.93 GeV, the explicit phase-space factor $2P/M_A$ increases. Therefore, the reduction of the calculated width from 73 MeV to about 20 MeV indicates that the QPC amplitude $|\mathcal{M}^{SL}(P)|^2$ decreases significantly in this momentum region, which implies the nodal structure of the wave function in the spatial-overlap integral. Therefore, the decay width alone is not sufficient for determining the nature of the $\chi_{c0}(3915)$.

For the excited states, the mass and width of the $\chi_{c0}(4P)$ state align closely with the observed $\chi_{c0}(4500)$ resonance reported by LHCb [15]. The calculated mass of the $\chi_{c0}(5P)$ state is 4781 MeV, approximately 87 MeV above the BW mass of $\chi_{c0}(4700)$ quoted in the PDG table. However, it is important to note that in the $B^+ \to J/\psi\phi K^+$ process, the upper limit of $J/\psi\phi$ energy is constrained by $m_{B^+} - m_{K^+} = 4785.7$ MeV. This phase space restriction could distort the resonance line shape, potentially resulting in a lower extracted BW mass. It is also noted that in an analysis of $B_s^0 \to J/\psi\pi^+\pi^-K^+K^-$ by LHCb [58], which has a larger phase space for $J/\psi\phi$, a similar structure, denoted as $X(4740)$, is observed and its BW mass and width are determined to be $m_X = 4741 \pm 6 \pm 6$ MeV and $\Gamma_X = 53 \pm 15 \pm 11$ MeV. The measured values are more consistent with the results of the $\chi_{c0}(5P)$ state obtained here, where $M_{\chi_{c0}(5P)} = 4781$ MeV and $\Gamma_{\chi_{c0}(5P)} = 79$ MeV.

Although the experimental evidence for these structures comes from the hidden-charm $J/\psi\phi$ channel, this does not necessarily imply that $J/\psi\phi$ is their dominant decay mode. For a conventional charmonium assignment, the $J/\psi\phi$ channel is OZI suppressed, while OZI-allowed open-charm channels are expected to provide the dominant contributions to the total width. A discovery channel is not always the dominant decay channel, since the observed event yield also depends on the production rate, trigger and reconstruction efficiencies, and background level. In particular, the fully reconstructed $J/\psi\phi$ final state, with $J/\psi \to \mu^+\mu^-$ and $\phi \to K^+K^-$, provides a comparatively clean experimental signature, which is favorable for observing resonance structures in a B^+ decay amplitude analysis. By contrast, open-charm final states usually involve hadronic decays of charmed mesons and may suffer from larger combinatorial backgrounds and more complicated amplitude-analysis issues, especially for broad or overlapping structures near thresholds. Moreover, the Breit-Wigner parametrization in the $J/\psi\phi$ channel determines the resonance mass and total width from the line shape, but does not determine the absolute branching fraction into $J/\psi\phi$. Therefore, the observation of the $\chi_{c0}(4500)$ and $\chi_{c0}(4700)$ in the $J/\psi\phi$ channel alone is insufficient to definitively determine whether these states are dominated by $c\bar{c}$ components, tetraquark configurations, or molecular degrees of freedom. Within a conventional charmonium interpretation, comparing the sum of our calculated OZI-allowed open-charm partial widths with the total width extracted from experiment can be regarded as a reasonable first approximation. Future measurements of their open-charm decay modes would provide critical information for clarifying their internal structures.

At present, no confirmed candidate for the $\chi_{c0}(3P)$ charmonium state has been observed. In this work,

the $\chi_{c0}(3P)$ is predicted to be located at approximately 4204 MeV with a width of 85 MeV. It is noticed that in the $B^+ \to J/\psi\phi K^+$ process analyzed by LHCb [15], where the higher-mass $\chi_{c0}(4500)$ and $\chi_{c0}(4700)$ states were identified, an unexplained structure (e.g., a kink or localized excess) appears near 4200 MeV in the $J/\psi\phi$ invariant mass spectrum. This feature is not accounted for in the model used in the experimental analysis in Ref. [15]. The presence of this weak signal suggests the need for further experimental studies in this mass region to clarify its origin and explore potential resonance contributions.

C. $^3P_1$ states

The quantum numbers of $^3P_1$ state are $J^{PC} = 1^{++}$. The first excited 1^{++} charmoniumlike state, the prominent $\chi_{c1}(3872)$, was the first heavy meson that could not be explained within the conventional $q\bar{q}$ quark model. The $\chi_{c1}(3872)$ lies near the $D^0\bar{D}^{*0}$ threshold and has an exceptionally narrow BW width of $\Gamma = 1.19 \pm 0.21$ MeV, strongly suggesting a non-$c\bar{c}$ origin. Additional higher 1^{++} charmoniumlike states include the $\chi_{c1}(4140)$, $\chi_{c1}(4274)$ and $\chi_{c1}(4685)$. Recently, the LHCb collaboration reported the observation of the $\chi_{c1}(4010)$ in the $B^+ \to D^{*+}D^-K^+$ process [10], further enriching the 1^{++} spectrum and highlighting its growing complexity.

In this analysis, the $\chi_{c1}(2P)$ state is tentatively associated with the recently observed $\chi_{c1}(4010)$ by LHCb [10]. Although the predicted mass of $\chi_{c1}(2P)$ state is approximately 3948 MeV, which is about 60 MeV smaller than the BW mass of $\chi_{c1}(4010)$, a plausible explanation for this discrepancy is the inherent difference between the quenched model predictions and the BW mass determinations. Furthermore, interactions between the $\chi_{c1}(2P)$ and the $D\bar{D}^*$ scattering channels may play a role. In our prior work using the Lee-Friedrichs model [59], coupling the bare $\chi_{c1}(2P)$ state to $D\bar{D}^*$ channels was shown to generate two poles in the complex energy plane: one dynamically generated bound-state pole at the $D\bar{D}^*$ threshold and another shifted from the bare state into the complex energy plane. This framework aligns with recent findings in Ref. [60], where the interplay with $\chi_{c1}(2P)$ produces a $D\bar{D}^*$ lineshape characterized by a sharp threshold enhancement and a broad resonance near 3990 MeV. These features are attributed to the $\chi_{c1}(3872)$ and $\chi_{c1}(4010)$, respectively.

The $\chi_{c1}(3P)$ state in our calculation corresponds to the $\chi_{c1}(4274)$, while the $\chi_{c1}(4P)$ is predicted at 4581 MeV with a broad width of ∼242 MeV within the QPC model. This $\chi_{c1}(4P)$ state might relate to the $\chi_{c1}(4685)$, but the mass difference of about 100 MeV is the largest one in the list. Since this $\chi_{c1}(4685)$ is only observed by one experiment, more experimental investigations are expected to verify or determine the physical parameters more accurately. However, the $\chi_{c1}(4140)$ does not naturally align with this theoretical framework. Current experimental determinations of its properties show significant discrepancies: For instance, analyses of the $B^+ \to J/\psi\phi K^+$ decay by the D0 and CMS collaborations report a narrow width of ∼20 MeV [61,62], whereas LHCb measures a substantially larger width of $\Gamma = 162 \pm 21^{+24}_{-49}$ MeV in the same channel [63]. These inconsistencies highlight the need for further experimental exploration to clarify the $\chi_{c1}(4140)$'s parameters and resolve its nature.

D. Radiative transitions

We calculate the radiative E1 and M1 transitions of all relevant charmonium states. Results are presented in Tables III–VII (E1 transitions) and VIII–IX (M1 transitions), with experimentally measured radiative partial widths included for comparison to theoretical predictions.

For the ground states, the computed E1 transitions of $\psi(1D), \chi_{c0}(1P), \chi_{c1}(1P), \chi_{c2}(1P)$, and $h_c(1P)$ agree with the experimental values. Among excited states, only a few S-wave states have been measured. For instance, we find $\psi(2S) \to \gamma\chi_{cJ}$ yield $\Gamma(\psi(2S) \to \gamma\chi_{c0}/\chi_{c1}/\chi_{c2}) = 23/37/23$ keV, respectively, consistent with the experimental results in Table III. For higher excitations, only upper limits on transition rates exists for $\psi(3S)$ and $\psi(4S)$.

For states lacking experimental data, we compare our results with those from established models (e.g., LP and SP in Ref. [5], NR and GI in Ref. [8] and SNR in Ref. [57]). Due to the ambiguity of $\chi_{c0}(2P)$, we provide two sets of predictions for transitions involving the $\chi_{c0}(2P)$ state, corresponding to its tentative assignments as either the $\chi_{c0}(3860)$ or $\chi_{c0}(3915)$.

Radiative transitions, such as E1 and M1 decays, provide a powerful tool for discriminating between the underlying natures of controversial states and facilitating the classification of the observed charmoniumlike spectrum. The identification of the $\chi_{c0}(2P)$ state serves as a clear case in point. (1) The numerical results indicate that the E1 transition rations are highly sensitive to the spatial wave functions of the candidates. For instance, the ratio of the E1 transition widths $\mathcal{B}[\psi(3S) \to \gamma\chi_{c0}(2P)]/\mathcal{B}[\psi(3S) \to \gamma\chi_{c0}(1P)]$ is predicted to be approximately 15/4 if the $\chi_{c0}(2P)$ is identified as the $X(3860)$, whereas the ratio is nearly 1∶1 if the $\chi_{c0}(3915)$ is assigned to be the $2P$ state. Such disparities suggest that experimental measurement of these ratios could effectively discriminate between these two candidates. (2) The $\psi(2D) \to \gamma\chi_{c0}(2P)$ transition may provide a potentially useful but experimentally challenging probe of the $\chi_{c0}(2P)$ state. The identification of the $\chi_{c0}(2P)$ state has historically been complicated by the proximity of the $\chi_{c2}(3930)$. It is worth recalling that early observations, such as the Belle identification of the $\chi_{c2}(2P)$ in $\gamma\gamma \to D\bar{D}$ processes [64], initially assumed a single resonance at 3930 MeV. This observation also inspires the argument about where the $\chi_{c0}(2P)$ is located since its decay to $D\bar{D}$ is OZI allowed [54]. Subsequent debates regarding the helicity of this peak also rested on the single-resonance assumption [55]. The LHCb amplitude analysis

TABLE III. E1 radiative transitions of S-wave states. The unit is keV. In the "our result" column, values separated by a slash denote the results for $\chi_{c0}(3860)$ (left) and $\chi_{c0}(3915)$ (right) assignments of the $\chi_{c0}(2P)$ state.

Initial state	Final state	Γ_i (our result)	NR [8]	GI [8]	LP [5]	SP [5]	Γ_{exp}
$\eta_c(2S)$	$\gamma h_c(1P)$	33	49	36	49	52	
$\eta_c(3S)$	$\gamma h_c(1P)$	43	105	64	104	128	
	$\gamma h_c(2P)$	45	9.1	28	0.045	1.4	
$\eta_c(4S)$	$\gamma h_c(1P)$	18	5.2	9.6	0.29	0.63	
	$\gamma h_c(2P)$	43	10.1	31.3	20	7.9	
	$\gamma h_c(3P)$	69	159	101	102	70	
$\psi(2S)$	$\gamma\chi_{c0}(1P)$	21	63	26	27	26	29 ± 2
	$\gamma\chi_{c1}(1P)$	35	54	29	45	48	29 ± 2
	$\gamma\chi_{c2}(1P)$	22	38	24	36	44	27 ± 2
$\psi(3S)$	$\gamma\chi_{c0}(1P)$	4	0.27	0.63	9.1	10.6	
	$\gamma\chi_{c0}(2P)$	15/4	54	22	21	31	
	$\gamma\chi_{c1}(1P)$	1	0.53	0.85	4.8	8.0	<326
	$\gamma\chi_{c1}(2P)$	3×10^{-1}	39	43	58	71	
	$\gamma\chi_{c2}(1P)$	14	0.7	13	0.21	2.1	<480
	$\gamma\chi_{c2}(2P)$	51	14	48	65	79	
$\psi(4S)$	$\gamma\chi_{c0}(1P)$	3	0.18	1.13	7.5	6.2	
	$\gamma\chi_{c0}(2P)$	6/4	0.24	0.39	17	12	
	$\gamma\chi_{c0}(3P)$	45	0.003	25	40	23	
	$\gamma\chi_{c1}(1P)$	1	0.41	0.53	3.8	3.9	<106
	$\gamma\chi_{c1}(2P)$	1	0.49	0.92	24	12	
	$\gamma\chi_{c1}(3P)$	60	126	54	74	38	
	$\gamma\chi_{c2}(1P)$	9	0.61	5.2	0.13	0.66	<528
	$\gamma\chi_{c2}(2P)$	31	0.62	15	11	4.7	
	$\gamma\chi_{c2}(3P)$	72	68	66	80	39	
$\psi(5S)$	$\gamma\chi_{c0}(1P)$	2					
	$\gamma\chi_{c0}(2P)$	3/2					
	$\gamma\chi_{c0}(3P)$	4					
	$\gamma\chi_{c0}(4P)$	32					
	$\gamma\chi_{c1}(1P)$	1					
	$\gamma\chi_{c1}(2P)$	5×10^{-1}					
	$\gamma\chi_{c1}(3P)$	2					
	$\gamma\chi_{c2}(1P)$	5					
	$\gamma\chi_{c2}(2P)$	9					
	$\gamma\chi_{c2}(3P)$	14					
	$\gamma\chi_{c2}(4P)$	7					

of $B^+ \to D^+D^-K^+$ suggested the presence of two nearly degenerate states: the $\chi_{c0}(3930)$ and $\chi_{c2}(3930)$. The historical context demonstrates that the presence of the $\chi_{c2}(2P)$ significantly hinders the clean identification of its scalar partner, the $\chi_{c0}(2P)$.

In this regard, the $\psi(2D) \to \gamma\chi_{c0}(2P)$ transition may still be useful for reducing the possible contamination from the nearby $\chi_{c2}(2P)$ state. However, because of the small estimated branching fraction and the possible broad width of the $\chi_{c0}(2P)$, its experimental reconstruction is expected to be challenging. Our calculations show that the transition width for $\psi(2D) \to \gamma\chi_{c2}(2P)$ is approximately 16 keV, while the width for $\psi(2D) \to \gamma\chi_{c0}(2P)$ is significantly large, which is estimated at 288 keV or 154 keV depending on the candidate assigned. Consequently, the signal from this channel would be an order of magnitude stronger than that of the $\chi_{c2}(2P)$, greatly reducing the difficulty of experimental disentanglement. Furthermore, while the E1 transitions of the $\chi(3D)$ state would also suppress the $\chi_{c2}(2P)$ contribution relative to the $\chi_{c0}(2P)$, the $\psi(3D)$ remains unidentified as it likely possesses a smaller production branching ratio, making the $\psi(2D)$ a more viable probe in this problem.

From the summary Tables III–VII and VIII–IX, we identify several common principles governing dominant radiative transition channels:

(1) S-wave states: The dominant E1 transitions of n^1S_0 and n^3S_1 occur between states with adjacent

TABLE IV. E1 radiative transitions of some $^3D_1$ charmonium states. The unit is keV. In the "our result" column, values separated by a slash denote the results for $\chi_{c0}(3860)$ (left) and $\chi_{c0}(3915)$ (right) assignments of the $\chi_{c0}(2P)$ state.

Initial state	Final state	Γ_i (our result)	NR [8]	GI [8]	LP [5]	SP [5]	$\Gamma_{\rm exp}$
$\psi(1D)$	$\gamma\chi_{c0}(1P)$	229	403	213	243	232	188 ± 23
	$\gamma\chi_{c1}(1P)$	102	125	77	115	111	68 ± 9
	$\gamma\chi_{c2}(1P)$	4	4.9	3.3	5.4	5.7	<18
$\psi(2D)$	$\gamma\chi_{c0}(1P)$	117					
	$\gamma\chi_{c0}(2P)$	288/154					
	$\gamma\chi_{c1}(1P)$	1					
	$\gamma\chi_{c1}(2P)$	59					
	$\gamma\chi_{c2}(1P)$	2×10^{-1}					
	$\gamma\chi_{c2}(2P)$	16					

TABLE V. E1 radiative transitions of some $^3P_0$ and $^3P_1$ charmonium states. The unit is keV. In the "our result" column, values separated by a slash denote the results for $\chi_{c0}(3860)$ (left) and $\chi_{c0}(3915)$ (right) assignments of the $\chi_{c0}(2P)$ state.

Initial state	Final state	Γ_i (our result)	Γ_i (LP/SP) [5]	Γ_i (NR/GI) [8]	Γ_i ($\mathrm{SNR}_{0/1}$) [57]	$\Gamma_{\rm exp}$
$\chi_{c0}(1P)$	$\gamma J/\psi(1S)$	145	141/146	152/114	167/117	151 ± 18
$\chi_{c0}(2P)$	$\gamma J/\psi(1S)$	43/56	4.0/1.5	56/1.3	74/9.3	
	$\gamma\psi(2S)$	80/197	108/89	64/135	61/44	
	$\gamma\psi_1(1D)$	15/71	21/12	13/51	⋯	
$\chi_{c0}(3P)$	$\gamma J/\psi(1S)$	51	0.14/0.08	27/1.5	⋯	
	$\gamma\psi(2S)$	27	13/6.9	32/0.045	⋯	
	$\gamma\psi(3S)$	125	214/56	109/145	⋯	
	$\gamma\psi_1(1D)$	40	0.31/0.44	0.037/9.7	⋯	
	$\gamma\psi_1(2D)$	1×10^{-1}	⋯	⋯	⋯	
$\chi_{c0}(4P)$	$\gamma J/\psi(1S)$	46				
	$\gamma\psi(2S)$	23				
	$\gamma\psi(3S)$	29				
	$\gamma\psi(4S)$	9				
	$\gamma\psi_1(1D)$	10				
	$\gamma\psi_1(2D)$	23				
	$\gamma\psi_1(3D)$	4×10^{-1}				
$\chi_{c0}(5P)$	$\gamma J/\psi(1S)$	39				
	$\gamma\psi(2S)$	19				
	$\gamma\psi(3S)$	17				
	$\gamma\psi(4S)$	11				
	$\gamma\psi(5S)$	9				
	$\gamma\psi_1(1D)$	4				
	$\gamma\psi_1(2D)$	7				
	$\gamma\psi_1(3D)$	20				
$\chi_{c1}(1P)$	$\gamma J/\psi(1S)$	339	269/278	314/239	354/244	288 ± 24
$\chi_{c1}(2P)$	$\gamma J/\psi(1S)$	54	64/68	71/14	83/45	
	$\gamma\psi(2S)$	416	129/145	183/183	103/60	
	$\gamma\psi_1(1D)$	51	8.6/10.8	22/21	⋯	
	$\gamma\psi_2(1D)$	75	2.8/3.4	35/18	⋯	
$\chi_{c1}(3P)$	$\gamma J/\psi(1S)$	15	36/33	31/2.2	⋯	
	$\gamma\psi(2S)$	21	78/63	45/8.9	⋯	
	$\gamma\psi(3S)$	107	305/111	303/181	⋯	
	$\gamma\psi_1(1D)$	2×10^{-2}	4.4/2.7	0.0014/0.39	⋯	
	$\gamma\psi_1(2D)$	7	8.6/0	19/15	⋯	
	$\gamma\psi_2(1D)$	4	0.13/0.09	0.0035/4.6	⋯	
	$\gamma\psi_2(2D)$	33	22/11	58/35	⋯	

TABLE VI. E1 radiative transitions of some $^3P_2$ charmonium states. The unit is keV.

Initial state	Final state	Γ_i (our result)	Γ_i (LP/SP) [5]	Γ_i (NR/GI) [8]	Γ_i (SNR$_{0/1}$) [57]	$\Gamma_{\rm exp}$
$\chi_{c2}(1P)$	$\gamma J/\psi(1S)$	493	327/338	424/313	473/309	386 ± 33
$\chi_{c2}(2P)$	$\gamma J/\psi(1S)$	103	118/119	81/53	101/109	
	$\gamma\psi(2S)$	131	146/163	304/207	225/100	
	$\gamma\psi_1(1D)$	4×10^{-1}	0.47/0.62	1.9/1.0	⋯	
	$\gamma\psi_2(1D)$	1	3.3/4.1	17/5.6	⋯	
	$\gamma\psi_3(1D)$	6	20/24	88/29	⋯	
$\chi_{c2}(3P)$	$\gamma J/\psi(1S)$	70	83/69	34/19	⋯	
	$\gamma\psi(2S)$	52	116/90	55/30	⋯	
	$\gamma\psi(3S)$	325	306/121	509/199	⋯	
	$\gamma\psi_1(1D)$	1×10^{-2}	1.9/1.0	∼0/0.001	⋯	
	$\gamma\psi_1(2D)$	4×10^{-1}	0.55/0.004	2.1/0.77	⋯	
	$\gamma\psi_2(1D)$	9×10^{-2}	4.6/2.5	0.0091/0.13	⋯	
	$\gamma\psi_2(2D)$	9	18/10	31/9.9	⋯	
	$\gamma\psi_3(1D)$	4	15/10	0.049/6.8	⋯	
	$\gamma\psi_3(2D)$	47	116/64	148/51	⋯	

TABLE VII. E1 radiative transitions of some $^1P_1$ charmonium states. The unit is keV.

Initial state	Final state	Γ_i (our result)	Γ_i (LP/SP) [5]	Γ_i (NR/GI) [8]	Γ_i (SNR$_{0/1}$) [57]	$\Gamma_{\rm exp}$
$h_c(1P)$	$\gamma\eta_c(1S)$	559	361/373	498/352	765/323	468^{+210}_{-188}
$h_c(2P)$	$\gamma\eta_c(1S)$	204	135/134	140/85	134/250	
	$\gamma\eta_c(2S)$	251	160/176	280/218	309/108	
	$\gamma\eta_{c2}(1D)$	24	25/25	60/27	⋯	
$h_c(3P)$	$\gamma\eta_c(1S)$	124	90/77	72/38		
	$\gamma\eta_c(2S)$	68	124/96	75/43		
	$\gamma\eta_c(3S)$	224	237/146	276/208		
	$\gamma\eta_{c2}(1D)$	6	15/8.7	0.16/5.7		
	$\gamma\eta_{c2}(2D)$	43	93/47	99/48		

radial quantum number n. This pattern is systematically observed for S-wave charmonium (Table III).

(2) D-wave states: The dominant E1 transitions of n^3D_1 charmonium states follow $n^3D_1 \to n^3P_0 + \gamma$, where the final state retains the same radial quantum number and total spin S, but with the orbital angular momentum L and total angular momentum J reduced by one unit (Table IV).

(3) P-wave states: The dominant E1 transitions of $\chi_{cJ}(nP)$ ($J = 0, 1, 2$) usually proceed via $\gamma\psi(nS)$, preserving the radial quantum number n (Tables V and VI). However, exceptions arise for the processes $\chi_{c0}(4P) \to \gamma\psi(4S)$ and $\chi_{c0}(5P) \to \gamma\psi(5S)$, where the radiative partial widths are significantly suppressed. This suppression stems from the nodal structures of the initial- and final-state wave functions. At the same time, dominant channels of $h_c(nP)$ involve $\gamma\eta_c(nS)$ similarly conserving the radial quantum number n (VII).We note that, while S–D mixing is known to play a crucial role in explaining the large E1 widths of observed D-wave states and the suppressed widths of S-wave states, our calculated E1 transition widths for the P-wave charmonium states remain systematically larger than the available experimental measurements. To further investigate whether configuration mixing could resolve the overestimated E1 widths in the P-wave sector, we focused on the $J^{PC} = 2^{++}$ channel—the only conventional charmonium channel where $^3P_2 - {}^3F_2$ mixing is allowed. The evaluated E1 transition widths for the unmixed $^3P_2$ and $^3F_2$ configurations are approximately 493 keV and 568 keV, respectively. This result suggests that $^3P_2 - {}^3F_2$ mixing is insufficient to explain the suppression of the E1 widths of excited P-wave charmonium states. The discrepancy likely arises because E1 transitions between excited states are highly sensitive to the nodal structures of the initial- and final-state wave functions. Consequently,

TABLE VIII. M1 radiative transitions of $^3S_1$ and $^3D_1$ charmonium states. The unit is keV.

Initial state	Final state	Γ_i (our result)	$\Gamma_{\rm exp}$
$\eta_c(2S)$	$\gamma J/\psi(1S)$	9	<188
$\eta_c(3S)$	$\gamma J/\psi(1S)$	12	
	$\gamma\psi(2S)$	1	
$\eta_c(4S)$	$\gamma J/\psi(1S)$	14	
	$\gamma\psi(2S)$	3	
	$\gamma\psi(3S)$	2×10^{-1}	
$\eta_c(5S)$	$\gamma J/\psi(1S)$	13	
	$\gamma\psi(2S)$	4	
	$\gamma\psi(3S)$	1	
	$\gamma\psi(4S)$	1×10^{-1}	
	$\gamma\psi(5S)$	5×10^{-2}	
$\psi(1S)$	$\gamma\eta_c(1S)$	2	
$\psi(2S)$	$\gamma\eta_c(1S)$	23	1
	$\gamma\eta_c(2S)$	2×10^{-1}	$(2\pm1)\times10^{-1}$
$\psi(3S)$	$\gamma\eta_c(1S)$	29	
	$\gamma\eta_c(2S)$	3	
	$\gamma\eta_c(3S)$	1×10^{-7}	
$\psi(4S)$	$\gamma\eta_c(1S)$	51	
	$\gamma\eta_c(2S)$	7	
	$\gamma\eta_c(3S)$	3	
	$\gamma\eta_c(4S)$	1×10^{-2}	
$\psi(5S)$	$\gamma\eta_c(1S)$	54	
	$\gamma\eta_c(2S)$	8	
	$\gamma\eta_c(3S)$	3	
	$\gamma\eta_c(4S)$	1	
$\psi(2D)$	$\gamma\eta_{c2}(1D)$	8×10^{-2}	
	$\gamma\eta_{c2}(2D)$	2×10^{-3}	
$\psi(3D)$	$\gamma\eta_{c2}(1D)$	3×10^{-1}	
	$\gamma\eta_{c2}(2D)$	8×10^{-2}	
$\psi(4D)$	$\gamma\eta_{c2}(1D)$	5×10^{-1}	
	$\gamma\eta_{c2}(2D)$	3×10^{-1}	
	$\gamma\eta_{c2}(3D)$	6×10^{-2}	
$\psi(5D)$	$\gamma\eta_{c2}(1D)$	1	
	$\gamma\eta_{c2}(2D)$	1	
	$\gamma\eta_{c2}(3D)$	3×10^{-1}	
	$\gamma\eta_{c2}(4D)$	4×10^{-2}	

TABLE IX. M1 radiative transitions of $^3P_0$, $^3P_1$ and $^3P_2$ charmonium states. The unit is keV.

Initial state	Final state	Γ_i (our result)	Γ_i (NR/GI) [8]
$\chi_{c0}(2P)$	$\gamma h_c(1P)$	3/5	0.19/0.50
$\chi_{c0}(3P)$	$\gamma h_c(1P)$	6	
	$\gamma h_c(2P)$	2	
$\chi_{c0}(4P)$	$\gamma h_c(1P)$	7	
	$\gamma h_c(2P)$	3	
	$\gamma h_c(3P)$	1	
$\chi_{c0}(5P)$	$\gamma h_c(1P)$	7	
	$\gamma h_c(2P)$	4	
	$\gamma h_c(3P)$	2	
	$\gamma h_c(4P)$	2×10^{-1}	
$\chi_{c1}(2P)$	$\gamma h_c(1P)$	1	0.51/0.045
	$\gamma h_c(2P)$	5×10^{-1}	
$\chi_{c1}(3P)$	$\gamma h_c(1P)$	2×10^{-1}	
	$\gamma h_c(2P)$	2×10^{-1}	
	$\gamma h_c(3P)$	4×10^{-5}	
$\chi_{c1}(4P)$	$\gamma h_c(1P)$	1	
	$\gamma h_c(2P)$	1	
	$\gamma h_c(3P)$	2	
	$\gamma h_c(4P)$	2	
$\chi_{c1}(5P)$	$\gamma h_c(1P)$	5×10^{-1}	
	$\gamma h_c(2P)$	4×10^{-1}	
	$\gamma h_c(3P)$	4×10^{-1}	
	$\gamma h_c(4P)$	3×10^{-1}	
	$\gamma h_c(5P)$	3×10^{-4}	
$\chi_{c2}(1P)$	$\gamma h_c(1P)$	1×10^{-1}	
$\chi_{c2}(2P)$	$\gamma h_c(1P)$	1	0.90/1.3
$\chi_{c2}(3P)$	$\gamma h_c(1P)$	3	
	$\gamma h_c(2P)$	1	
	$\gamma h_c(3P)$	2×10^{-2}	
$\chi_{c2}(4P)$	$\gamma h_c(1P)$	4	
	$\gamma h_c(2P)$	2	
	$\gamma h_c(3P)$	1	
	$\gamma h_c(4P)$	1×10^{-2}	
$\chi_{c2}(5P)$	$\gamma h_c(1P)$	4	
	$\gamma h_c(2P)$	3	
	$\gamma h_c(3P)$	2	
	$\gamma h_c(4P)$	1	
	$\gamma h_c(5P)$	8×10^{-3}	

accurately reproducing the experimental data may require a more sophisticated framework [5] that accounts for node-induced cancellations, relativistic corrections, coupled-channel effects, and higher-order multipole contributions. Furthermore, since the radiative branching fractions of highly excited states are expected to be intrinsically small, significantly higher experimental statistics than currently available will be essential to measure these values and discriminate between various theoretical models.

(4) M1 transitions: The dominant channels predominantly feature transitions to ground-state mesons, as listed in VIII–IX.

These principles could be understood by viewing the Eqs. (33) and (34), which illustrate that the E1 transition mainly depends on the overlapping of the wave functions between the initial and final states, while the spherical Bessel function plays an important role in the M1 transition.

1. Futher remarks about theoretical uncertainties

We have reanalyze the mass spectrum of charmonium in terms of the GI' relativized potential model, computed their open-charm strong decay widths based on a nonrelativistic QPC model, and calculated the corresponding radiative transitions by multipole expansion method by using the wave functions obtained in the relativized potential model. In this section, we would like to analyze possible theoretical uncertainties in this calculation.

The unquenched effect is not included in the quark potential model. Such a restriction of the simplest Fock space might limit the accuracy of the theoretical predictions. It is expected that the mass shifts arising from their decay channels (both open and virtual ones) is of the order of magnitude of about 30 MeV [7], thus we use a uniform tolerance of $\sigma_n = 30$ MeV empirically. For the high excited states, coupled channel models or potential models including the vacuum fluctuation effects might be helpful to improve our understanding of these states [4].

The GI's model is a relativized potential model, while the present study employ a nonrelativistic formulation of the QPC model. Several considerations justify this approach for the charmonium system. First, from the perspective of heavy quark dynamics, the charm quark is sufficiently massive ($m_c \gg \Lambda_{\mathrm{QCD}}$) such that the velocity of the constituents ($v^2/c^2 \approx 0.2$–0.3) allows nonrelativistic approximations to capture the essential physics of the leading-order transitions. Second, the use of nonrelativistic operators ensures internal consistency with wave functions derived from the potential model, which are solved in the center of mass frame. Furthermore, it is important to note that the QCD model includes an effective pair-creation strength parameter (γ). In many phenomenological applications, this parameter can partially absorb the systematic uncertainties introduced by neglecting higher order relativistic corrections, such as Lorentz boosts and Wigner rotations. Although a fully relativistic treatment, which incorporates the complex entanglement of spin and meomentum integrations as discussed in Ref. [65], would theoretically enhance the description of high-velocity final states, such methods often introduce additional model dependencies regarding the specific form of the relativistic boost. Given the successful track record of the nonrelativistic QPC model in describing heavy quarkonium phenomenology, we believe the current approach provides reliable estimate of the decay widths. A more rigorous treatment of relativistic effects remains a significant challenge, and perhaps the method based on Bethe-Salpeter equation could provide more insight into our understanding of the relativistic bound state or resonance state [66].

IV. SUMMARY AND CONCLUSIONS

We have made a reinvestigation of charmoniumlike states in the GI's relativized potential model and calculated their strong decays and radiative transitions by using the wave functions obtained in the GI's model. The mass spectrum of bottomonium state and some heavy-light ground states are also included in the fit to constrain the fit parameters. It is demonstrated that the new parameters could present a consistent description of the updated charmonium spectroscopy as well as their decay widths. We give a summary of the candidates of different sets of quantum numbers:

(1) For the $J^{PC} = 1^{--}$ state, the J/ψ, $\psi(3686)$, $\psi(4040)$, $\psi(4415)$ and $\psi(4660)$ could be associated with $c\bar{c}(n^3S_1)$ states of $n = 1, \ldots, 5$ respectively, while $\psi(3770)$ and $\psi(4160)$ are consistent with the $c\bar{c}(1^3D_1)$ and $c\bar{c}(2^3D_1)$. Now the mass of $c\bar{c}(5^3S_1)$ is consistent with the observed value of $\psi(4660)$.

(2) For the $J^{PC} = 0^{++}$ state, the χ_{c0}, $\chi_{c0}(4500)$ and $\chi_{c0}(4700)$ are consistent with the $c\bar{c}(n^3P_0)$ states with $n = 1$, 4 and 5, respectively. The predicted mass of $c\bar{c}(2^3P_0)$ is located at about 3851 MeV, which is different from the observed $\chi_{c0}(3915)$. The candidate of $c\bar{c}(3^3P_0)$ is still absent. The kink structure of the $J/\psi\phi$ invariant mass at about 4200 MeV is suggested to further explore and check whether there exist the $c\bar{c}(3^3P_0)$.

(3) For the $J^{PC} = 1^{++}$ state, $\chi_{c1}(3510)$, $\chi_{c1}(4010)$, $\chi_{c1}(4274)$ and $\chi_{c1}(4685)$ are consistent with the $c\bar{c}(n^3P_1)$ states with $n = 1, \ldots, 4$, respectively. $\chi_{c1}(3872)$ could be regarded as the dynamically generated state due to the coupling of $c\bar{c}(2^3P_1)$ to the $D\bar{D}$ channel, at the same time the mass of the $c\bar{c}(2^3P_1)$ pole is also shifted to a higher mass due to a coupled channel effect.

(4) For the $J^{PC} = 2^{++}$ states, the $\chi_{c2}(3556)$ and $\chi_{c2}(3930)$ are associated with $c\bar{c}(1^3P_2)$ and $c\bar{c}(2^3P_2)$. At present, there are no other 2^{++} candidates reported in experiments.

In this calculation, most charmonium states could find their candidates observed in experiments. The remaining puzzle is only the $c\bar{c}(2^3P_0)$, whose predicted mass and width are about 3851 MeV and 73 MeV, respectively. This mass value is about 65 MeV lower than that of the GI's work (about 3916 MeV), but it is close to those predicted values of the nonrelativistic potential model [8,35].

Other states, such as $G(3900)$, $\chi_{c1}(4140)$, $\psi(4230)$ and $\psi(4360)$, are usually regarded as non-$q\bar{q}$ states so they could not be accommodated in this calculation.

Our analysis suggests that the GI model's original predictions for high charmonium states above the open-flavor threshold exhibit systematically larger mass splittings than experimental observations when using PDG-listed states as benchmarks. This discrepancy could be amended by recalibrating the model parameters to align theoretical predictions with data. Despite the challenges of describing the non-$q\bar{q}$ states, the quenched quark model remains to be a valuable tool for rapidly categorizing charmonium states and distinguishing conventional configurations in the experiments.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

[1] S. Navas *et al.* (Particle Data Group Collaboration), Phys. Rev. D **110**, 030001 (2024).
[2] F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, and B.-S. Zou, Rev. Mod. Phys. **90**, 015004 (2018); **94**, 029901(E) (2022).
[3] H.-X. Chen, W. Chen, X. Liu, and S.-L. Zhu, Phys. Rep. **639**, 1 (2016).
[4] B.-Q. Li, C. Meng, and K.-T. Chao, Phys. Rev. D **80**, 014012 (2009).
[5] W.-J. Deng, H. Liu, L.-C. Gui, and X.-H. Zhong, Phys. Rev. D **95**, 034026 (2017).
[6] S. K. Choi *et al.* (Belle Collaboration), Phys. Rev. Lett. **91**, 262001 (2003).
[7] S. Godfrey and N. Isgur, Phys. Rev. D **32**, 189 (1985).
[8] T. Barnes, S. Godfrey, and E. S. Swanson, Phys. Rev. D **72**, 054026 (2005).
[9] T. Aaltonen *et al.* (CDF Collaboration), Phys. Rev. Lett. **102**, 242002 (2009).
[10] R. Aaij *et al.* (LHCb Collaboration), Phys. Rev. Lett. **133**, 131902 (2024).
[11] K. Chilikin *et al.* (Belle Collaboration), Phys. Rev. D **95**, 112003 (2017).
[12] K. Abe *et al.* (Belle Collaboration), Phys. Rev. Lett. **94**, 182002 (2005).
[13] B. Aubert *et al.* (*BABAR* Collaboration), Phys. Rev. Lett. **101**, 082001 (2008).
[14] R. Aaij *et al.* (LHCb Collaboration), Phys. Rev. D **95**, 012002 (2017).
[15] R. Aaij *et al.* (LHCb Collaboration), Phys. Rev. Lett. **118**, 022003 (2017).
[16] S. Weinberg, Phys. Rev. Lett. **110**, 261601 (2013).
[17] H. G. Blundell and S. Godfrey, Phys. Rev. D **53**, 3700 (1996).
[18] L. Micu, Nucl. Phys. **B10**, 521 (1969).
[19] A. Le Yaouanc, L. Oliver, O. Pène, and J. C. Raynal, Phys. Rev. D **8**, 2223 (1973).
[20] A. Le Yaouanc, L. Oliver, O. Pène, and J.-C. Raynal, Phys. Rev. D **9**, 1415 (1974).
[21] X. Liu, Z.-G. Luo, and Z.-F. Sun, Phys. Rev. Lett. **104**, 122001 (2010).
[22] T. Barnes and S. Godfrey, Phys. Rev. D **69**, 054008 (2004).
[23] Y.-c. Yang, Z. Xia, and J. Ping, Phys. Rev. D **81**, 094003 (2010).
[24] J. Segovia, D. Entem, and F. Fernández, Phys. Lett. B **715**, 322 (2012).
[25] M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) **7**, 404 (1959).
[26] F. Close and L. Copley, Nucl. Phys. **B19**, 477 (1970).
[27] Z. Li, Phys. Rev. D **50**, 5639 (1994).
[28] Z. Li, H. Ye, and M. Lu, Phys. Rev. C **56**, 1099 (1997).
[29] S.-F. Chen, J. Liu, H.-Q. Zhou, and D.-Y. Chen, Eur. Phys. J. C **80**, 290 (2020).
[30] J.-L. Li and D.-Y. Chen, Chin. Phys. C **46**, 073106 (2022).
[31] D.-Y. Chen, C.-Q. Pang, J. He, and Z.-Y. Zhou, Phys. Rev. D **100**, 074016 (2019).
[32] T. Barnes, N. Black, and P. R. Page, Phys. Rev. D **68**, 054014 (2003).
[33] T. Matsuki, T. Morii, and K. Seo, Prog. Theor. Phys. **124**, 285 (2010).
[34] E. Eichten, S. Godfrey, H. Mahlke, and J. L. Rosner, Rev. Mod. Phys. **80**, 1161 (2008).
[35] D. Ebert, R. N. Faustov, and V. O. Galkin, Phys. Rev. D **67**, 014027 (2003).
[36] M. Ablikim *et al.* (BESIII Collaboration), Phys. Rev. Lett. **133**, 081901 (2024).
[37] X. L. Wang *et al.* (Belle Collaboration), Phys. Rev. Lett. **99**, 142002 (2007).
[38] J. P. Lees *et al.* (*BABAR* Collaboration), Phys. Rev. D **89**, 111103 (2014).
[39] J. He, Y. Liu, J.-T. Zhu, and D.-Y. Chen, Eur. Phys. J. C **80**, 246 (2020).
[40] F.-L. Wang and X. Liu, Phys. Rev. D **102**, 094006 (2020).
[41] H.-W. Ke, X.-H. Liu, and X.-Q. Li, Chin. Phys. C **44**, 093104 (2020).
[42] F.-K. Guo, C. Hanhart, and U.-G. Meissner, Phys. Rev. Lett. **102**, 242004 (2009).
[43] G. Cotugno, R. Faccini, A. D. Polosa, and C. Sabelli, Phys. Rev. Lett. **104**, 132005 (2010).
[44] J.-R. Zhang, Phys. Rev. D **102**, 054006 (2020).
[45] M. Ablikim *et al.* (BESIII Collaboration), Phys. Rev. Lett. **131**, 211902 (2023).
[46] M. Ablikim *et al.* (BESIII Collaboration), Phys. Rev. Lett. **130**, 121901 (2023).
[47] X. L. Wang *et al.* (Belle Collaboration), Phys. Rev. D **91**, 112007 (2015).
[48] G. Pakhlova *et al.* (Belle Collaboration), Phys. Rev. Lett. **101**, 172001 (2008).
[49] Z.-Y. Lin, J.-Z. Wang, J.-B. Cheng, L. Meng, and S.-L. Zhu, Phys. Rev. Lett. **133**, 241903 (2024).
[50] N. Hüsken, R. F. Lebed, R. E. Mitchell, E. S. Swanson, Y.-Q. Wang, and C.-Z. Yuan, Phys. Rev. D **109**, 114010 (2024).
[51] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T.-M. Yan, Phys. Rev. D **21**, 203 (1980).
[52] X.-H. Liu and G. Li, Phys. Rev. D **88**, 014013 (2013).
[53] J. P. Lees *et al.* (*BABAR* Collaboration), Phys. Rev. D **86**, 072002 (2012).
[54] F.-K. Guo and U.-G. Meissner, Phys. Rev. D **86**, 091501 (2012).

[55] Z.-Y. Zhou, Z. Xiao, and H.-Q. Zhou, Phys. Rev. Lett. **115**, 022001 (2015).
[56] R. Aaij *et al.* (LHCb Collaboration), Phys. Rev. D **102**, 112003 (2020).
[57] B.-Q. Li and K.-T. Chao, Phys. Rev. D **79**, 094004 (2009).
[58] R. Aaij *et al.* (LHCb Collaboration), J. High Energy Phys. 02 (2021) 024; 04 (2021) 170(E).
[59] Z.-Y. Zhou and Z. Xiao, Phys. Rev. D **96**, 054031 (2017); **96**, 099905(E) (2017).
[60] Q. Deng, R.-H. Ni, Q. Li, and X.-H. Zhong, Phys. Rev. D **110**, 056034 (2024).
[61] V. M. Abazov *et al.* (D0 Collaboration), Phys. Rev. Lett. **115**, 232001 (2015).
[62] T. Aaltonen *et al.* (CDF Collaboration), Mod. Phys. Lett. A **32**, 1750139 (2017).
[63] R. Aaij *et al.* (LHCb Collaboration), Phys. Rev. Lett. **127**, 082001 (2021).
[64] S. Uehara *et al.* (Belle Collaboration), Phys. Rev. Lett. **96**, 082003 (2006).
[65] M. G. Fuda, Phys. Rev. C **86**, 055205 (2012).
[66] Y.-Y. Rui, T. Wang, Z.-H. Wang, T.-F. Feng, and G.-L. Wang, arXiv:2512.12531.