

Yang-Baxter sigma model from twistor space

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We derive a novel two-field four-dimensional integrable field theory (IFT) from 6D holomorphic Chern-Simons theory on twistor space. The four-dimensional IFT depends on a skew-symmetric linear operator acting on a Lie algebra, and when this operator is specialized to a solution of the modified classical Yang-Baxter equation, the IFT develops a semilocal symmetry associated with this solution. The resulting 4D analog of the Yang-Baxter sigma model is related by symmetry reduction to the well-known 2D Yang-Baxter sigma model. An important implication that we find is the embedding of the equations of motion of the 2D Yang-Baxter sigma model in the anti-self-dual Yang-Mills equations. The 6D Chern-Simons theory on twistor space can alternatively be symmetry reduced to a 4D Chern-Simons theory configuration with disorder surface defects. The latter realizes the Yang-Baxter sigma model, implying a “diamond” for the Yang-Baxter sigma model obtained from twistor space. We also show that the homogeneous 2D Yang-Baxter sigma model can be derived from a limit of our setup.

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I. INTRODUCTION

Two-dimensional integrable field theories serve as valuable laboratories for exploring the structure of quantum field theory. Many such models, including the principal chiral model, are asymptotically free, and their constrained dynamics offer insight into the behavior of more realistic asymptotically free theories, such as four-dimensional Yang-Mills theory.

Four-dimensional Chern-Simons (CS) theory has emerged as a unifying framework for the study of two-dimensional integrable systems [1–4], encompassing integrable lattice models governed by Yangian symmetry, as well as integrable field theories (IFTs) [5–9], with further connections to integrable structures arising in supersymmetric field theories [10–12] and holographically dual theories [13,14]. In particular, Costello and Yamazaki [5] showed that a broad class of two-dimensional integrable field theories can be obtained from four-dimensional Chern-Simons theory through suitable choices of a meromorphic one-form, ω , and boundary conditions. The action

takes the form

$$S = \frac{1}{2\pi i} \int_{\Sigma \times C} \omega \wedge \text{CS}(A), \quad (1.1)$$

where $\text{CS}(A)$ is the Chern-Simons three-form, Σ is the world sheet of the integrable field theory, and C is a Riemann surface. This perspective on integrable field theories has since led to numerous generalizations and further developments; see Refs. [15–22] for recent progress.

Another well-known organizing principle for 2D integrable field theories is the anti-self-dual Yang-Mills (ASDYM) equations [23]. Various integrable systems such as the Korteweg–de Vries equation, nonlinear Schrödinger equation, and Toda field equation can be derived from the ASDYM equations via different symmetry reductions.

The relationship between the two frameworks of 4D CS and the ASDYM equations can be understood through the lens of 6D holomorphic Chern-Simons theory on twistor space. This was first elucidated in the works of Bittleston and Skinner [24] and Penna [25], where the embedding of the principal chiral model in 6D holomorphic CS was elucidated. Further work in this direction clarified the embedding of integrable deformations of the principal chiral model, as well as coset models and non-Abelian T-duals [26–29].

In this work, we shall present an embedding of the Yang-Baxter (YB) deformation of the principal chiral model (also known as the Yang-Baxter sigma model) in 6D holomorphic CS on twistor space. The Yang-Baxter sigma model is a well-known example of a deformation of the principal

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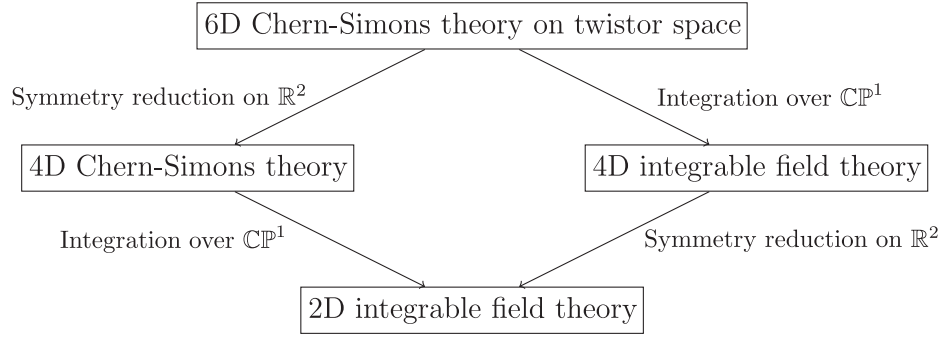


FIG. 1. Relationship between 4D Chern-Simons theory and 6D Chern-Simons theory on twistor space, as well as 4D and 2D integrable field theories, forming a “diamond”.

chiral model (PCM) that preserves integrability, that originates in the work of Klimcik [30,31]; see also Ref. [32]. It depends explicitly on a solution of the classical Yang-Baxter equation or its modified version. The deformation can be interpreted as deformation of the geometry of the Lie group target space of the principal chiral model, along with the inclusion of a nontrivial B -field.

The YB sigma model has proven to be a canonical example of an integrable deformation of the PCM, exhibiting rich dynamical structure. Notably, an analog of the YB deformation also exists for coset models, a prime example of which is the YB deformation of the $\text{AdS}_5 \times S^5$ superstring [33–36] (derived from 4D Chern-Simons theory in Ref. [37]). The existence of the latter shows that integrability in the AdS/CFT correspondence survives a deformation of the $\text{AdS}_5 \times S^5$ geometry. Moreover, integrability constrains the quantum dynamics of the YB model strongly enough such that we can derive its nonperturbative physics systematically [38,39].

We recall that the derivation of the Yang-Baxter sigma model from 4D Chern-Simons theory is well known from the work of Delduc *et al.* [40] (an alternative derivation is found in Ref. [41]). In this derivation, the datum of the solution of the classical Yang-Baxter equation enters via boundary conditions in the Chern-Simons theory. In this work, we shall identify this $4\text{D} \rightarrow 2\text{D}$ reduction with an edge in a “diamond” of reductions that start with 6D holomorphic CS on twistor space, depicted in Fig. 1. We shall derive a novel 4D IFT that is an analog of the Yang-Baxter sigma model from this 6D holomorphic CS theory. This 4D IFT can moreover be symmetry reduced to the 2D Yang-Baxter sigma model, completing the diamond. In the process, we shall show that the equations of motion of the Yang-Baxter sigma model can be embedded in the 4D ASDYM equations.

The remainder of this article is organized as follows: in Sec. II, we shall show that 6D holomorphic CS with boundary conditions depend on a linear, antisymmetric operator reduces to a two-field 4D integrable field theory, also referred to as an IFT_4 . We further describe how the

equations of motion of this theory are equivalent to the anti-self-dual Yang-Mills equations. In Sec. III, we describe the symmetry reduction of our 6D CS action and boundary conditions to a 4D CS action with disorder defects and compatible boundary conditions. In Sec. IV, we describe the features of the 4D IFT when the linear operator is specialized to a solution of the modified classical Yang-Baxter equation. We refer to the 4D IFT with such a specialization as IFT_4^{YB} , or the 4D Yang-Baxter sigma model. In Sec. V, we show that IFT_4 can be related via a symmetry reduction to a two-field 2D theory, that can be further identified with the (inhomogeneous) Yang-Baxter sigma model in the presence of a gauge symmetry. In Sec. VI, we explain how the aforementioned two-field 2D theory arises from a 4D CS setup, completing a diamond of reductions. In Secs. VII and VIII, we describe how to derive the Yang-Baxter sigma model associated with the homogeneous classical Yang-Baxter equation from a limit of our setup. Finally, in Sec. IX, we summarize our results and discuss possible future directions.

II. 4D IFT FROM 6D CS ON TWISTOR SPACE

We begin with the holomorphic 6D Chern-Simons theory on twistor space, whose action is given by

$$S_{\text{hCS}_6} = \frac{1}{2\pi i} \int_{\mathbb{PT}} \Omega \wedge \text{Tr} \left(\mathcal{A} \wedge \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right), \quad (2.1)$$

with the (3,0)-form given by

$$\Omega = \frac{1}{2} \Phi e^0 \wedge e^A \wedge e_A, \quad \Phi = \frac{K}{\langle \pi \alpha \rangle \langle \pi \tilde{\alpha} \rangle \langle \pi \beta \rangle^2}. \quad (2.2)$$

Our conventions for twistor space are collected in the Appendix A. The gauge field in the (0, 1)-form basis is given by

$$\mathcal{A} = \mathcal{A}_0 \bar{e}^0 + \mathcal{A}_A \bar{e}^A, \quad (2.3)$$

and the variation of the holomorphic Chern-Simons action is

$$\delta S = \frac{1}{2\pi i} \int_{\mathbb{P}^1} [2\Omega \wedge \text{Tr}(\delta \mathcal{A} \wedge \mathcal{F}^{0,2}) + \bar{\partial}\Omega \wedge \text{Tr}(\mathcal{A} \wedge \delta \mathcal{A})]. \quad (2.4)$$

For the action to be stationary for arbitrary $\delta \mathcal{A}$, one requires that the bulk equation of motion,

$$\mathcal{F}^{0,2} = \bar{\partial}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0, \quad (2.5)$$

and the boundary equation of motion,

$$0 = \int_{\mathbb{P}^1} \bar{\partial}\Omega \wedge \text{Tr}(\mathcal{A} \wedge \delta \mathcal{A}), \quad (2.6)$$

hold. Since all singularities of Ω are encoded in the scalar prefactor Φ , the $\bar{\partial}$ -operator localises the integral to the poles on the $\mathbb{CP}^1$ fiber. From Eq. (2.2), Ω has poles on $\mathbb{CP}^1$ at $\pi = \alpha$ (simple pole), $\pi = \tilde{\alpha}$ (simple pole), and $\pi = \beta$ (double pole). We consider the Dirichlet boundary conditions $\mathcal{A}_A|_{\pi=\beta} = 0$ for the second-order pole. The resulting residues yield pole-localized terms proportional to $\int_{E^4} \text{vol}_4 e^{AB} \text{Tr}(\mathcal{A}_A \delta \mathcal{A}_B)$ evaluated at the respective poles. Using the antisymmetry of the spinor bracket, $\langle \tilde{\alpha}\alpha \rangle = -\langle \alpha\tilde{\alpha} \rangle$, the boundary equation of motion can be stated as

$$\begin{aligned} & \frac{1}{\langle \alpha\tilde{\alpha} \rangle \langle \alpha\beta \rangle^2} \int_{E^4} \text{vol}_4 e^{AB} \text{Tr}(\mathcal{A}_A \delta \mathcal{A}_B) \Big|_{\pi=\alpha} \\ &= \frac{1}{\langle \alpha\tilde{\alpha} \rangle \langle \tilde{\alpha}\beta \rangle^2} \int_{E^4} \text{vol}_4 e^{AB} \text{Tr}(\mathcal{A}_A \delta \mathcal{A}_B) \Big|_{\pi=\tilde{\alpha}}. \end{aligned} \quad (2.7)$$

$$\begin{aligned} & \frac{1}{\langle \alpha\beta \rangle^2} \text{Tr} \left(\sigma \frac{\langle \alpha\beta \rangle}{\langle \tilde{\alpha}\beta \rangle} \frac{(\mathcal{O} + c)}{(\mathcal{O} - c)} [\mathcal{A}\mu] \Big|_{\pi=\tilde{\alpha}} \cdot \sigma^{-1} \frac{\langle \alpha\beta \rangle}{\langle \tilde{\alpha}\beta \rangle} \frac{(\mathcal{O} + c)}{(\mathcal{O} - c)} [\delta \mathcal{A}\hat{\mu}] \Big|_{\pi=\tilde{\alpha}} - \sigma^{-1} \frac{\langle \alpha\beta \rangle}{\langle \tilde{\alpha}\beta \rangle} \frac{(\mathcal{O} + c)}{(\mathcal{O} - c)} [\mathcal{A}\hat{\mu}] \Big|_{\pi=\tilde{\alpha}} \cdot \sigma \frac{\langle \alpha\beta \rangle}{\langle \tilde{\alpha}\beta \rangle} \frac{(\mathcal{O} + c)}{(\mathcal{O} - c)} [\delta \mathcal{A}\mu] \Big|_{\pi=\tilde{\alpha}} \right) \\ &= \frac{1}{\langle \tilde{\alpha}\beta \rangle^2} \text{Tr}(\mathcal{A}\mu)[\delta \mathcal{A}\hat{\mu}] - [\mathcal{A}\hat{\mu}][\delta \mathcal{A}\mu] \Big|_{\pi=\tilde{\alpha}}. \end{aligned}$$

Although the dependence on $\mathcal{O}$ in these boundary conditions may seem arbitrary, in later sections, we shall investigate the specialization of the operator $\mathcal{O}$ to a skew-symmetric solution of the modified classical Yang-Baxter equation associated with $c = 1$ or i , whereby the boundary conditions shall resemble those used to derive the 2D Yang-Baxter sigma model from 4D Chern-Simons theory in Ref. [40]. As shown in Ref. [40], although the derivations of the 2D Yang-Baxter sigma model in the two cases of $c = 1$ and $c = i$ are similar, they differ in a subtle way in that the field space of the models (prior to gauge fixing) are different, and correspond to $G \times G$ for $c = 1$ and $G_{\mathbb{C}}$ for $c = i$.

Let μ^A and $\hat{\mu}^A$ be a basis of the unprimed spinor space, normalized such that $[\mu\hat{\mu}] = \epsilon_{AB}\mu^A\hat{\mu}^B = 1$. Since the space is two-dimensional, any spinor X^A admits the expansion $X^A = p\mu^A + q\hat{\mu}^A$. Contracting with $\hat{\mu}_A$ and μ_A and using antisymmetry of the spinor inner product $[XY] = \epsilon_{AB}X^AY^B$ yields $p = [X\hat{\mu}]$ and $q = -[X\mu]$, so that

$$X^A = [X\hat{\mu}]\mu^A - [X\mu]\hat{\mu}^A. \quad (2.8)$$

Using Eq. (2.8) in the gauge-field component spinors $\mathcal{A}_A$ and $\delta \mathcal{A}_A$ (at fixed π) gives $\epsilon^{AB}\text{Tr}(\mathcal{A}_A \delta \mathcal{A}_B) = \text{Tr}([\mathcal{A}\mu][\delta \mathcal{A}\hat{\mu}] - [\mathcal{A}\hat{\mu}][\delta \mathcal{A}\mu])$, which leads directly to

$$\begin{aligned} & \frac{1}{\langle \alpha\beta \rangle^2} \text{Tr}([\mathcal{A}\mu][\delta \mathcal{A}\hat{\mu}] - [\mathcal{A}\hat{\mu}][\delta \mathcal{A}\mu]) \Big|_{\pi=\alpha} \\ &= \frac{1}{\langle \tilde{\alpha}\beta \rangle^2} \text{Tr}([\mathcal{A}\mu][\delta \mathcal{A}\hat{\mu}] - [\mathcal{A}\hat{\mu}][\delta \mathcal{A}\mu]) \Big|_{\pi=\tilde{\alpha}}. \end{aligned} \quad (2.9)$$

Now, the boundary condition we impose at the two simple poles of Ω is

$$\begin{aligned} (\mathcal{O} - c)[\mathcal{A}\mu] \Big|_{\pi=\alpha} &= \sigma \frac{\langle \alpha\beta \rangle}{\langle \tilde{\alpha}\beta \rangle} (\mathcal{O} + c)[\mathcal{A}\mu] \Big|_{\pi=\tilde{\alpha}}, \\ (\mathcal{O} - c)[\mathcal{A}\hat{\mu}] \Big|_{\pi=\alpha} &= \sigma^{-1} \frac{\langle \alpha\beta \rangle}{\langle \tilde{\alpha}\beta \rangle} (\mathcal{O} + c)[\mathcal{A}\hat{\mu}] \Big|_{\pi=\tilde{\alpha}}, \end{aligned} \quad (2.10)$$

where we require the operator $\mathcal{O} \in \text{End } \mathfrak{g}$ to be skew-symmetric, i.e., $\mathcal{O}^t = -\mathcal{O}$, and where c is a complex constant. Consequently, the boundary equation of motion is satisfied using the boundary conditions (2.10), since

In fact, we could be more general, and consider boundary conditions of the form

$$\begin{aligned} P[\mathcal{A}\mu] \Big|_{\pi=\alpha} &= \sigma \frac{\langle \alpha\beta \rangle}{\langle \tilde{\alpha}\beta \rangle} [\mathcal{A}\mu] \Big|_{\pi=\tilde{\alpha}}, \\ P[\mathcal{A}\hat{\mu}] \Big|_{\pi=\alpha} &= \sigma^{-1} \frac{\langle \alpha\beta \rangle}{\langle \tilde{\alpha}\beta \rangle} [\mathcal{A}\hat{\mu}] \Big|_{\pi=\tilde{\alpha}}, \end{aligned} \quad (2.11)$$

where $P \in \text{End } \mathfrak{g}$ satisfies $P^t = P^{-1}$, with the choice $P = \frac{(\mathcal{O}-c)}{(\mathcal{O}+c)}$ corresponding to the boundary conditions in Eq. (2.10). In what follows, although our focus will be on the boundary conditions (2.10), we shall express many of

our results such that they depend on the operator $P = \left(\frac{\mathcal{O}-c}{\mathcal{O}+c}\right)$.

A. Derivation of 4D field theory

We shall now proceed to derive a four-dimensional field theory from the 6D Chern-Simons theory on twistor space with boundary conditions (2.10). To perform the localization analysis, it shall be convenient to rewrite the (0,1)-connection $\mathcal{A}$ in terms of new variables $\mathcal{A}'$ and a group-valued field $\hat{h}$ via

$$\mathcal{A} = \hat{h}^{-1} \mathcal{A}' \hat{h} + \hat{h}^{-1} \bar{\partial} \hat{h}. \quad (2.12)$$

This relation has the form of a gauge transformation, but is in fact a change of variables that separates the part of the connection which is pure gauge, particularly along the $\mathbb{CP}^1$ fiber. This part of the connection is pure gauge since any complex bundle on a $\mathbb{CP}^1$ fiber that is topologically trivial is also generically holomorphically trivial. Notably, the fiber component of $\mathcal{A}'$ is equal to zero, so that the remaining field may be interpreted as a four-dimensional anti-self-dual Yang-Mills connection.

As a result, $\mathcal{A}'$ may be regarded as a (0,1)-connection with support entirely along the $\mathbb{E}^4$ directions, which is precisely the structure required for its interpretation, via the

Penrose-Ward correspondence, as an anti-self-dual Yang-Mills connection on $\mathbb{E}^4$. After fixing $\mathcal{A}'_0 = 0$, there remains a residual internal gauge freedom consisting of gauge transformations involving smooth functions that are independent of the $\mathbb{CP}^1$ fiber coordinate. We can fix this symmetry by fixing the value of the group-valued field $\hat{h}$ at one of the poles, which we choose to be $\pi = \beta$, by imposing $\hat{h}|_\beta = \text{id}$.

However, because gauge transformations are restricted at the remaining poles by boundary conditions associated with the meromorphic structure of the theory, the values of $\hat{h}$ at $\pi = \alpha$ and $\pi = \bar{\alpha}$, as follows:

$$h = \hat{h}|_\alpha, \quad \tilde{h} = \hat{h}|_{\bar{\alpha}}, \quad (2.13)$$

cannot be gauged away. These fields therefore become genuine dynamical degrees of freedom, representing edge modes localized at the poles. As a consequence, the holomorphic Chern-Simons action localises to an effective four-dimensional theory on $\mathbb{E}^4$ whose dynamics depend only on these edge-mode fields. Also, after imposing $\mathcal{A}'_0 = 0$, $\mathcal{A}'$ only has legs in a two-dimensional (0,1) subspace, so any triple wedge product is forced to vanish. In terms of the redefined gauge field $\mathcal{A}'$, the transformed action takes the form

$$\begin{aligned} S_{\text{hcs}_6} = & \frac{1}{2\pi i} \int_{\mathbb{PT}} \Omega \wedge \text{Tr}(\mathcal{A}' \wedge \bar{\partial} \mathcal{A}') + \frac{1}{2\pi i} \int_{\mathbb{PT}} \bar{\partial} \Omega \wedge \text{Tr}(\mathcal{A}' \wedge \bar{\partial} \hat{h} \hat{h}^{-1}) \\ & - \frac{1}{6\pi i} \int_{\mathbb{PT}} \Omega \wedge \text{Tr}(\hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h}). \end{aligned} \quad (2.14)$$

The latter two terms in the action (2.14) localize to the poles of the meromorphic form Ω , and the remaining obstruction to deriving the four-dimensional theory is the presence of a genuine bulk term and the residual dependence on the field $\mathcal{A}'$. Both issues are resolved by invoking the bulk equations of motion for $\mathcal{A}'$. Varying the action, which is the only bulk contribution, yields the equation of motion

$$\bar{\partial} \mathcal{A}' + \mathcal{A}' \wedge \mathcal{A}' = 0. \quad (2.15)$$

Imposing the gauge condition $\mathcal{A}'_0 = 0$ and decomposing $\mathcal{A}' = \mathcal{A}'_A \bar{e}^A$, this equation implies

$$\bar{\partial}_0 \mathcal{A}'_A = 0, \quad (2.16)$$

so that the components $\mathcal{A}'_A$ are holomorphic along the $\mathbb{CP}^1$ fiber, and thus there are no nontrivial bulk contributions to the action (2.14).

Since A'_A has homogeneous weight one under rescalings of the fiber coordinate, it follows that its $\mathbb{CP}^1$ -dependence is necessarily linear, and may be written as

$$\mathcal{A}'_A = \pi^{A'} A_{AA'}, \quad (2.17)$$

where $A_{AA'}$ is independent of π . We now recall the boundary condition at the double pole $\pi = \beta$, which is $\mathcal{A}_A|_\beta = 0$. Recalling that $\hat{h}|_\beta = \text{id}$, this condition translates directly into $\mathcal{A}'_A|_\beta = 0$, which implies $\beta^{A'} A_{AA'} = 0$. Hence $A_{AA'}$ must be proportional to $\langle \pi \beta \rangle$, and we may write

$$\mathcal{A}'_A = \langle \pi \beta \rangle B_A, \quad (2.18)$$

for some π -independent field B_A . Substituting this result into the field redefinition

$$\mathcal{A} = \hat{h}^{-1} \mathcal{A}' \hat{h} + \hat{h}^{-1} \bar{\partial} \hat{h}, \quad (2.19)$$

and extracting the spacetime component yields

$$\mathcal{A}_A = \langle \pi \beta \rangle \text{Ad}_{\hat{h}}^{-1}(B_A) + \pi^{A'} \hat{h}^{-1} \partial_{AA'} \hat{h}. \quad (2.20)$$

From Eqs. (2.10) and (2.20), we find

$$(\mathcal{O} - c) \left[\langle \alpha \beta \rangle \text{Ad}_h^{-1}(B_A) + \alpha^{A'} h^{-1} \partial_{AA'} h \right] \mu^A \quad (2.21)$$

$$= \sigma \frac{\langle \alpha \beta \rangle}{\langle \tilde{\alpha} \beta \rangle} (\mathcal{O} + c) \left[\langle \tilde{\alpha} \beta \rangle \text{Ad}_h^{-1}(B_A) + \tilde{\alpha}^{A'} \tilde{h}^{-1} \partial_{AA'} \tilde{h} \right] \mu^A, \quad (2.22)$$

$$(\mathcal{O} - c) \left[\langle \alpha \beta \rangle \text{Ad}_h^{-1}(B_A) + \alpha^{A'} h^{-1} \partial_{AA'} h \right] \hat{\mu}^A \quad (2.23)$$

It follows that the solutions to these constraints can be expressed as

$$\begin{aligned} \text{Ad}_{h^{-1}}(B_A) &= \hat{b} \mu_A - b \hat{\mu}_A, \\ \text{Ad}_{\tilde{h}^{-1}}(B_A) &= \tilde{b} \mu_A - \tilde{b} \hat{\mu}_A. \end{aligned} \quad (2.24)$$

We define $b = \text{Ad}_h^{-1}[B\mu]$ and $\hat{b} = \text{Ad}_h^{-1}[B\hat{\mu}]$, together with their tilded counterparts. By using Eq. (2.10), we find that the resulting fields are related by the following relations:

$$\begin{aligned} \tilde{b} - \tilde{j} &= \sigma^{-1} \left(\frac{\mathcal{O} - c}{\mathcal{O} + c} \right) (b - j), \\ \tilde{\tilde{b}} - \tilde{\tilde{j}} &= \sigma \left(\frac{\mathcal{O} - c}{\mathcal{O} + c} \right) (\hat{b} - \hat{j}), \end{aligned} \quad (2.25)$$

and moreover from Eq. (2.24), they satisfy $\tilde{b} = \Lambda b$ and $\tilde{\tilde{b}} = \Lambda \hat{b}$, where $\Lambda = \text{Ad}_h^{-1} \text{Ad}_h$, and we obtain

$$b = U_+(Pj - \sigma \tilde{j}), \quad \hat{b} = U_-(P\hat{j} - \sigma^{-1} \tilde{\hat{j}}), \quad (2.26)$$

where we define

$$U_{\pm} = (P - \sigma^{\pm 1} \Lambda)^{-1}, \quad P = \left(\frac{\mathcal{O} - c}{\mathcal{O} + c} \right), \quad (2.27)$$

as well as the currents $j, \tilde{j}, \hat{j}$, and $\tilde{\hat{j}}$ as

$$\begin{aligned} j &= \langle \alpha \beta \rangle^{-1} \mu^A \alpha^{A'} h^{-1} \partial_{AA'} h, & \hat{j} &= \langle \alpha \beta \rangle^{-1} \hat{\mu}^A \alpha^{A'} h^{-1} \partial_{AA'} h, \\ \tilde{j} &= \langle \tilde{\alpha} \beta \rangle^{-1} \mu^A \tilde{\alpha}^{A'} \tilde{h}^{-1} \partial_{AA'} \tilde{h}, & \tilde{\hat{j}} &= \langle \tilde{\alpha} \beta \rangle^{-1} \hat{\mu}^A \tilde{\alpha}^{A'} \tilde{h}^{-1} \partial_{AA'} \tilde{h}. \end{aligned} \quad (2.28)$$

The operators $U_{\pm}$ are subject to the following identities, which play an important role in the analysis:

$$\begin{aligned} \Lambda^T U_{\pm}^T &= -\sigma^{\mp 1} U_{\mp} P, & U_{\pm}^T \Lambda^T &= -\sigma^{\mp 1} P U_{\mp}, \\ P U_{\pm} + U_{\mp}^T P^T &= 1, & U_{\pm} P + P^T U_{\mp}^T &= 1. \end{aligned} \quad (2.29)$$

Coming back to Eq. (2.14), since the first term vanishes upon imposing the bulk equations of motion, we are left with

$$\begin{aligned} S_{\text{IFT}_4} &= \frac{1}{2\pi i} \int_{\mathbb{P}^1} \bar{\partial} \Omega \wedge \text{Tr}(\mathcal{A}' \wedge \bar{\partial} \hat{h} \hat{h}^{-1}) - \frac{1}{6\pi i} \int_{\mathbb{P}^1} \Omega \wedge \text{Tr}(\hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h}), \\ &= \frac{K}{2\pi i} \int \bar{e}_0 \bar{\partial}_0 \left(\frac{1}{\langle \pi \alpha \rangle} \right) \frac{1}{\langle \pi \tilde{\alpha} \rangle \langle \pi \beta \rangle^2} \wedge e^0 \wedge e^A \wedge e_A \wedge \bar{e}^C \wedge \bar{e}^D \text{Tr}(A'_C \bar{\partial}_D \hat{h} \hat{h}^{-1}) \\ &\quad + \frac{K}{2\pi i} \int \bar{e}_0 \bar{\partial}_0 \left(\frac{1}{\langle \pi \tilde{\alpha} \rangle} \right) \frac{1}{\langle \pi \alpha \rangle \langle \pi \beta \rangle^2} \wedge e^0 \wedge e^A \wedge e_A \wedge \bar{e}^C \wedge \bar{e}^D \text{Tr}(A'_C \bar{\partial}_D \tilde{h} \tilde{h}^{-1}) \\ &\quad - \frac{1}{6\pi i} \int_{\mathbb{P}^1} \Omega \wedge \text{Tr}(\hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h}). \end{aligned} \quad (2.30)$$

In evaluating the boundary variation of the action, we make use of the identities $e^C \wedge e_C \wedge \bar{e}^A \wedge \bar{e}^B = -2 \text{vol}_4 \epsilon^{AB}$, where $\text{vol}_4 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$, together with $\frac{1}{2\pi i} \int_{\mathbb{C}P^1} e^0 \wedge \bar{e}^0 \bar{\partial}_0 \left(\frac{1}{\langle \pi \alpha \rangle} \right) f(\pi) = f(\alpha)$ and Eq. (2.18). The effective four-dimensional action takes the form

$$\begin{aligned} S_{\text{IFT}_4} &= \frac{K}{\langle \alpha \tilde{\alpha} \rangle} \int \frac{1}{\langle \alpha \beta \rangle^2} \text{vol}_4 \epsilon^{CD} \text{Tr}(B_C \bar{\partial}_D h h^{-1}) \Big|_{\pi=\alpha} - \frac{K}{\langle \alpha \tilde{\alpha} \rangle} \int \frac{1}{\langle \tilde{\alpha} \beta \rangle^2} \text{vol}_4 \epsilon^{CD} \text{Tr}(B_C \bar{\partial}_D \tilde{h} \tilde{h}^{-1}) \Big|_{\pi=\tilde{\alpha}} \\ &\quad - \frac{1}{6\pi i} \int_{\mathbb{P}^1} \Omega \wedge \text{Tr}(\hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h}), \\ &= \frac{K}{\langle \alpha \tilde{\alpha} \rangle} \int \frac{1}{\langle \alpha \beta \rangle} \text{vol}_4 \epsilon^{AB} \text{Tr}(\text{Ad}_{h^{-1}}(B_A) h^{-1} \bar{\partial}_B h) \Big|_{\pi=\alpha} - \frac{K}{\langle \alpha \tilde{\alpha} \rangle} \int \frac{1}{\langle \tilde{\alpha} \beta \rangle} \text{vol}_4 \epsilon^{AB} \text{Tr}(\text{Ad}_{\tilde{h}^{-1}}(B_A) \tilde{h}^{-1} \bar{\partial}_B \tilde{h}) \Big|_{\pi=\tilde{\alpha}} \\ &\quad - \frac{1}{6\pi i} \int_{\mathbb{P}^1} \Omega \wedge \text{Tr}(\hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h}). \end{aligned} \quad (2.31)$$

Now, using $\bar{\partial}_A = \pi^{A'} \partial_{AA'}$ and (2.24), we finally get

$$\begin{aligned}
S_{\text{IFT}_4} &= \frac{K}{\langle \alpha \tilde{\alpha} \rangle} \int \frac{1}{\langle \alpha \beta \rangle} \text{vol}_4 \epsilon^{AB} \text{Tr}[(\hat{b}\mu_A - b\hat{\mu}_A) \alpha^{B'} h^{-1} \bar{\partial}_{BB'} h] - \frac{K}{\langle \alpha \tilde{\alpha} \rangle} \int \frac{1}{\langle \tilde{\alpha} \beta \rangle} \text{vol}_4 \epsilon^{AB} \text{Tr}[(\tilde{b}\mu_A - \tilde{b}\hat{\mu}_A) \tilde{\alpha}^{B'} \tilde{h}^{-1} \bar{\partial}_{BB'} \tilde{h}] \\
&\quad - \frac{1}{6\pi i} \int_{\mathbb{P}^1} \Omega \wedge \text{Tr}(\hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h}), \\
&= \frac{K}{\langle \alpha \tilde{\alpha} \rangle} \int_{\mathbb{E}^4} \text{vol}_4 \text{Tr}(b(\hat{j} - \Lambda^T \tilde{j}) - \hat{b}(j - \Lambda^T \tilde{j})) + S_{\text{WZ}_4},
\end{aligned} \tag{2.32}$$

where

$$S_{\text{WZ}_4} = \frac{K}{\langle \alpha \tilde{\alpha} \rangle} \int_{\mathbb{E}^4 \times [0,1]} \text{vol}_4 \wedge d\rho \text{Tr}(h^{-1} \partial_\rho h \cdot [j, \hat{j}] - \tilde{h}^{-1} \partial_\rho \tilde{h} \cdot [\tilde{j}, \hat{\tilde{j}}]). \tag{2.33}$$

Here, the final term $-\frac{1}{6\pi i} \int_{\mathbb{P}^1} \Omega \wedge \text{Tr}(\hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h} \wedge \hat{h}^{-1} \bar{\partial} \hat{h})$ in Eq. (2.31) has been identified with a four-dimensional Wess-Zumino term, denoted by S_{WZ_4} . A detailed derivation of this term can be found in Ref. [28]. Importantly, this term is unaffected by the deformation and depends only on the fields h and $\tilde{h}$. Finally, by using Eqs. (2.26), (2.27), and (2.29), we find a 4D field theory given by

$$\begin{aligned}
S_{\text{IFT}_4} &= \frac{K}{\langle \alpha \tilde{\alpha} \rangle} \int_{\mathbb{E}^4} \text{vol}_4 \text{Tr}(U_+(Pj - \sigma \tilde{j})(\hat{j} - \Lambda^T \hat{\tilde{j}}) - U_-(P\hat{j} - \sigma^{-1} \tilde{j})(j - \Lambda^T \tilde{j})) + S_{\text{WZ}_4}, \\
&= \frac{K}{\langle \alpha \tilde{\alpha} \rangle} \int_{\mathbb{E}^4} \text{vol}_4 \text{Tr}(j(P^T U_+^T - U_- P)\hat{j} + \tilde{j}(U_+^T P^T - P U_-)\hat{\tilde{j}} - 2\sigma \tilde{j} U_+^T \hat{j} + 2\sigma^{-1} j U_- \tilde{j}) + S_{\text{WZ}_4}.
\end{aligned} \tag{2.34}$$

B. Equations of motion

Let us now describe the equations of motion of the 4D field theory we have derived with the aim of relating them to the anti-self-dual Yang-Mills equations. Consider an infinitesimal variation of the group element introduced through $g \rightarrow g e^\epsilon \approx g(1 + \epsilon)$, with ϵ belonging to the Lie algebra $\mathfrak{g}$. Hence, $g^{-1} \delta g = \epsilon$ and varying $j_\mu = g^{-1} \partial_\mu g$ gives $\delta j_\mu = \partial_\mu \epsilon + [j_\mu, \epsilon]$. Together with the identities listed in Eq. (2.29), the variation of the operator $U_\pm$ plays a crucial role. This leads to the following relations, which are used in the derivation of the equations of motion, as follows:

$$\delta(U_\pm X) = U_\pm(\delta X) + U_\pm[\tilde{h}^{-1} \delta \tilde{h}, U_\mp^T P^T(X)] - P^T U_\mp^T[h^{-1} \delta h, U_\pm(X)]. \tag{2.35}$$

The variation of the 4D field theory action is then

$$\begin{aligned}
\delta_h S_4 &= \frac{K}{\langle \alpha \tilde{\alpha} \rangle} \int_{\mathbb{E}^4} \text{vol}_4 \text{Tr}(\delta j \hat{j} - 2\delta j(U_- P \hat{j} - \sigma^{-1} U_- \tilde{j}) - j \delta \hat{j} + 2(U_+ P j - U_+ \tilde{j}) \delta \hat{j} \\
&\quad + 2\epsilon[U_- P \hat{j}, U_+ P j] + 2\epsilon[U_- \tilde{j}, U_+ \tilde{j}] - 2\epsilon\sigma[U_- P \hat{j}, U_+ \tilde{j}] - 2\epsilon\sigma^{-1}[U_- \tilde{j}, U_+ P j] + \epsilon[j, \hat{j}]) \\
&= \frac{K}{\langle \alpha \tilde{\alpha} \rangle} \int_{\mathbb{E}^4} \text{vol}_4 \text{Tr}(\delta j \hat{j} - 2\delta j(\hat{b}) - j \delta \hat{j} + 2(b) \delta \hat{j} + \epsilon[\hat{b}, b] + \epsilon[j, \hat{j}]).
\end{aligned} \tag{2.36}$$

For the variation with respect to h , the corresponding equation of motion takes the form

$$-\frac{\mu^A \alpha^{A'}}{\langle \alpha \beta \rangle} \partial_{AA'} \hat{b} + \frac{\hat{\mu}^A \alpha^{A'}}{\langle \alpha \beta \rangle} \partial_{AA'} b + [\hat{j}, b] - [j, \hat{b}] - [\hat{b}, b] = 0. \tag{2.37}$$

A similar variation with respect to $\tilde{h}$ yields the equation

$$-\frac{\mu^A \tilde{\alpha}^{A'}}{\langle \tilde{\alpha} \beta \rangle} \partial_{AA'} \hat{\tilde{b}} + \frac{\hat{\mu}^A \tilde{\alpha}^{A'}}{\langle \tilde{\alpha} \beta \rangle} \partial_{AA'} \tilde{b} + [\hat{\tilde{j}}, \tilde{b}] - [\tilde{j}, \hat{\tilde{b}}] - [\hat{\tilde{b}}, \tilde{b}] = 0. \tag{2.38}$$

C. 4D ASDYM and Lax formulation

We now turn to the details of the integrability of the 4D field theory. This can be established by showing that the resulting equations of motion (2.37) and (2.38) are equivalent to the four-dimensional anti-self-dual Yang-Mills equations. To this end, it suffices to rewrite the equations of motion in terms of the solution for B^A .

We first evaluate $\text{Ad}_{h^{-1}}[B_A, B^A]$ in the h -frame. Using the fact that the adjoint action is an automorphism, together with Eqs. (2.24) and (B1), we obtain

$$\begin{aligned} \text{Ad}_{h^{-1}}[B_A, B^A] &= [\text{Ad}_{h^{-1}}(B_A), \text{Ad}_{h^{-1}}(B^A)] \\ &= [\hat{b}\mu_A - b\hat{\mu}_A, \hat{b}\mu^A - b\hat{\mu}^A] \\ &= \mu_A\mu^A[\hat{b}, \hat{b}] - \mu_A\hat{\mu}^A[\hat{b}, b] \\ &\quad - \hat{\mu}_A\mu^A[b, \hat{b}] + \hat{\mu}_A\hat{\mu}^A[b, b] \\ &= 2[\hat{b}, b]. \end{aligned} \quad (2.39)$$

Here, we use the fact that the spinors μ_A and $\hat{\mu}_A$ satisfy $\mu_A\mu^A = 0, \hat{\mu}_A\hat{\mu}^A = 0, \mu_A\hat{\mu}^A = -1$ and $\hat{\mu}_A\mu^A = 1$. Next, we compute $\text{Ad}_{h^{-1}}(\alpha^{A'}\partial_{AA'}B^A)$. This calculation is carried out in detail in the Appendix B, and yields

$$\begin{aligned} \text{Ad}_{h^{-1}}(\alpha^{A'}\partial_{AA'}B^A) &= \mu^A\alpha^{A'}\partial_{AA'}\hat{b} - \hat{\mu}^A\alpha^{A'}\partial_{AA'}b \\ &\quad + \langle\alpha\beta\rangle[j, \hat{b}] - \langle\alpha\beta\rangle[\hat{j}, b]. \end{aligned} \quad (2.40)$$

Combining the contributions (2.39) and (2.40), we obtain an equation of motion equivalent to Eq. (2.37), expressed in terms of B^A , as follows:

$$\alpha^{A'}\partial_{AA'}B^A + \frac{1}{2}\langle\alpha\beta\rangle[B_A, B^A] = 0. \quad (2.41)$$

Similarly, for $\tilde{h}$, we obtain

$$\tilde{\alpha}^{A'}\partial_{AA'}B^A + \frac{1}{2}\langle\tilde{\alpha}\tilde{\beta}\rangle[B_A, B^A] = 0. \quad (2.42)$$

These equations of motions can be recast as ASDYM equations. From Eq. (2.15), the “0A” component of the curvature, $\mathcal{F}'_{0A} = \bar{\partial}_0\mathcal{A}'_A - \bar{\partial}_A\mathcal{A}'_0 + [\mathcal{A}'_0, \mathcal{A}'_A]$, simplifies upon imposing the gauge condition $\mathcal{A}'_0 = 0$, leading to the constraint $\bar{\partial}_0\mathcal{A}'_A = 0$. The remaining $\mathcal{F}'_{AB}$ component of Eq. (2.15) then yields the equation $\bar{\partial}_A\mathcal{A}'_B - \bar{\partial}_B\mathcal{A}'_A + [\mathcal{A}'_A, \mathcal{A}'_B] = 0$, which expresses the integrability condition for the gauge field in the (0,1) directions. Recall that, by definition of the frame, $\bar{\partial}_A = \pi^{A'}\partial_{AA'}$ and using Eq. (2.17), we obtain $\bar{D}'_A = \bar{\partial}_A + \mathcal{A}'_A = \pi^{A'}\partial_{AA'} + \pi^{A'}A_{AA'} = \pi^{A'}(\partial_{AA'} + A_{AA'})$. We define the spacetime covariant derivative $D_{AA'} = \partial_{AA'} + A_{AA'}$. Then we can write

$$\bar{D}'_A = \pi^{A'}D_{AA'}. \quad (2.43)$$

This is the key relation linking the twistor connection to a connection on $\mathbb{E}^4$. Now we turn to the second hCS equation $\mathcal{F}'_{AB} = 0$. By definition, $\mathcal{F}'_{AB} = [\bar{D}'_A, \bar{D}'_B]$, and using the equation of motion, we get

$$\begin{aligned} \mathcal{F}'_{AB} &= [\pi^{A'}D_{AA'}, \pi^{B'}D_{BB'}] = \pi^{A'}\pi^{B'}[D_{AA'}, D_{BB'}] \\ &= \pi^{A'}\pi^{B'}\mathcal{F}_{AA'BB'} = 0, \end{aligned} \quad (2.44)$$

where

$$\begin{aligned} [D_{AA'}, D_{BB'}] &= \mathcal{F}_{AA'BB'} \\ &= \partial_{AA'}A_{BB'} - \partial_{BB'}A_{AA'} + [A_{AA'}, A_{BB'}]. \end{aligned}$$

Next, we show that Eq. (2.44) is precisely the ASDYM condition. For that, we use the spinor decomposition of the field strength. Any real two-form $\mathcal{F}_{\mu\nu}$ in four dimensions can be written in spinor notation as $\mathcal{F}_{AA'BB'} = \mathcal{F}_{\mu\nu}\sigma_{AA'}^\mu\sigma_{BB'}^\nu$. Representation theory of $\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$ tells us that any such $\mathcal{F}_{AA'BB'}$ satisfies

$$\mathcal{F}_{AA'BB'} = \epsilon_{AB}\Phi_{A'B'} + \epsilon_{A'B'}\Psi_{AB}, \quad (2.45)$$

with

$$\Phi_{A'B'} = \Phi_{(A'B')}, \quad \Psi_{AB} = \Psi_{(AB)}. \quad (2.46)$$

The symmetric spinor $\Phi_{A'B'}$ encodes the self-dual part $\mathcal{F}^{(+)}$, and Ψ_{AB} encodes the anti-self-dual part $\mathcal{F}^{(-)}$. Now compute

$$\begin{aligned} \pi^{A'}\pi^{B'}\mathcal{F}_{AA'BB'} &= \pi^{A'}\pi^{B'}(\epsilon_{AB}\Phi_{A'B'} + \epsilon_{A'B'}\Psi_{AB}) \\ &= \epsilon_{AB}\pi^{A'}\pi^{B'}\Phi_{A'B'} + \pi^{A'}\pi^{B'}\epsilon_{A'B'}\Psi_{AB}. \end{aligned} \quad (2.47)$$

Since $\pi^{A'}\pi^{B'}$ is symmetric and $\epsilon_{A'B'}$ is antisymmetric, we have $\pi^{A'}\pi^{B'}\epsilon_{A'B'} = 0$. So, we get $\pi^{A'}\pi^{B'}\mathcal{F}_{AA'BB'} = \epsilon_{AB}\pi^{A'}\pi^{B'}\Phi_{A'B'}$. Hence the twistor equation (2.44) becomes

$$\epsilon_{AB}\pi^{A'}\pi^{B'}\Phi_{A'B'} = 0 \quad \forall \pi^{A'}. \quad (2.48)$$

Since the factor ϵ_{AB} is nonzero, we must have $\pi^{A'}\pi^{B'}\Phi_{A'B'} = 0, \forall \pi^{A'}$. But $\Phi_{A'B'}$ is symmetric, and we may regard it as a 2×2 symmetric matrix. We define $P(\pi) := \pi^{A'}\pi^{B'}\Phi_{A'B'}$, a homogeneous quadratic polynomial in π . If $P(\pi) = 0$ for all $\pi^{A'}$, every coefficient must vanish, so

$$\Phi_{A'B'} = 0. \quad (2.49)$$

As recalled earlier, $\Phi_{A'B'}$ is exactly the self-dual part of $\mathcal{F}$. Thus

$$\Phi_{A'B'} = 0 \Leftrightarrow \mathcal{F}^{(+)} = 0. \quad (2.50)$$

This is precisely the ASDYM equation:

$$\mathcal{F} = \mathcal{F}^{(-)} \quad (\text{purely anti-self-dual curvature}). \quad (2.51)$$

Thus, we have that

$$\mathcal{F}'_{AB} = \pi^{A'} \pi^{B'} \mathcal{F}_{AA'BB'} = 0 \Leftrightarrow \text{ASD Yang-Mills on } \mathbb{E}^4. \quad (2.52)$$

From the twistor discussion, the ASDYM is equivalent to Eq. (2.52). Also, from Eq. (2.17), we have $A_{AA'} = \beta_{A'} B_A$. We assume, $\beta_{A'}$ is a fixed spinor (constant in spacetime), $B_A(x)$ is Lie-algebra valued. So $A_{BB'} = \beta_{B'} B_B$, and $A_{AA'} = \beta_{A'} B_A$. Using Eqs. (2.17) and (2.52), the ASYDM simplifies to

$$\langle \pi \beta \rangle \pi^{A'} \partial_{AA'} B_B - \langle \pi \beta \rangle \pi^{B'} \partial_{BB'} B_A + \langle \pi \beta \rangle^2 [B_A, B_B] = 0, \quad (2.53)$$

where $\pi^{A'}$ is the homogeneous coordinate on the twistor $\mathbb{CP}^1$. This condition must hold for every $\pi^{A'} \in \mathbb{CP}^1$. Pick a fixed spinor basis $\alpha^{A'}$, $\tilde{\alpha}^{A'}$ with $\langle \alpha \tilde{\alpha} \rangle \neq 0$. Any $\pi^{A'}$ can be written uniquely as

$$\pi^{A'} = c_1 \alpha^{A'} + c_2 \tilde{\alpha}^{A'}, \quad (2.54)$$

for some complex coefficients c_1, c_2 . To express c_1, c_2 in invariant form, contract with $\alpha_{A'}$ and $\tilde{\alpha}_{A'}$, as follows:

$$\begin{aligned} \langle \pi \alpha \rangle &= \pi^{A'} \alpha_{A'} = c_1 \langle \alpha \alpha \rangle + c_2 \langle \tilde{\alpha} \alpha \rangle = c_2 \langle \tilde{\alpha} \alpha \rangle = -c_2 \langle \alpha \tilde{\alpha} \rangle, \\ \langle \pi \tilde{\alpha} \rangle &= \pi^{A'} \tilde{\alpha}_{A'} = c_1 \langle \alpha \tilde{\alpha} \rangle + c_2 \langle \tilde{\alpha} \tilde{\alpha} \rangle = c_1 \langle \alpha \tilde{\alpha} \rangle. \end{aligned} \quad (2.55)$$

Hence,

$$c_1 = \frac{\langle \pi \tilde{\alpha} \rangle}{\langle \alpha \tilde{\alpha} \rangle}, \quad c_2 = -\frac{\langle \pi \alpha \rangle}{\langle \alpha \tilde{\alpha} \rangle}. \quad (2.56)$$

So,

$$\pi^{A'} = \frac{1}{\langle \alpha \tilde{\alpha} \rangle} (\langle \pi \tilde{\alpha} \rangle \alpha^{A'} - \langle \pi \alpha \rangle \tilde{\alpha}^{A'}). \quad (2.57)$$

Contract Eq. (2.57) with $\beta_{A'}$, as follows:

$$\langle \pi \beta \rangle = \pi^{A'} \beta_{A'} = \frac{\langle \pi \tilde{\alpha} \rangle \langle \alpha \beta \rangle - \langle \pi \alpha \rangle \langle \tilde{\alpha} \beta \rangle}{\langle \alpha \tilde{\alpha} \rangle}. \quad (2.58)$$

It is convenient to define the antisymmetric derivative combination $K_{ABA'} = \partial_{AA'} B_B - \partial_{BA'} B_A$. Using Eq. (2.57), we get

$$\pi^{A'} K_{ABA'} = \frac{1}{\langle \alpha \tilde{\alpha} \rangle} (\langle \pi \tilde{\alpha} \rangle \alpha^{A'} K_{ABA'} - \langle \pi \alpha \rangle \tilde{\alpha}^{A'} K_{ABA'}). \quad (2.59)$$

Using Eqs. (2.57)–(2.59), we get

$$\begin{aligned} 0 &= \langle \pi \beta \rangle (\pi^{A'} K_{ABA'} + \langle \pi \beta \rangle [B_A, B_B]) \\ &= \frac{\langle \pi \tilde{\alpha} \rangle \langle \alpha \beta \rangle - \langle \pi \alpha \rangle \langle \tilde{\alpha} \beta \rangle}{\langle \alpha \tilde{\alpha} \rangle} \left\{ \frac{1}{\langle \alpha \tilde{\alpha} \rangle} (\langle \pi \tilde{\alpha} \rangle \alpha^{A'} K_{ABA'} - \langle \pi \alpha \rangle \tilde{\alpha}^{A'} K_{ABA'}) + \frac{\langle \pi \tilde{\alpha} \rangle \langle \alpha \beta \rangle - \langle \pi \alpha \rangle \langle \tilde{\alpha} \beta \rangle}{\langle \alpha \tilde{\alpha} \rangle} [B_A, B_B] \right\}. \end{aligned} \quad (2.60)$$

We can pull out the common constant factor $1/\langle \alpha \tilde{\alpha} \rangle^2$. The remaining expression is a polynomial in $\langle \pi \tilde{\alpha} \rangle$ and $\langle \pi \alpha \rangle$. So the coefficient of $\langle \pi \beta \rangle \langle \pi \tilde{\alpha} \rangle$ is proportional to

$$\alpha^{A'} K_{ABA'} + \langle \alpha \beta \rangle [B_A, B_B] = 0, \quad (2.61)$$

and the coefficient of $\langle \pi \beta \rangle \langle \pi \alpha \rangle$ gives

$$\tilde{\alpha}^{A'} K_{ABA'} + \langle \tilde{\alpha} \beta \rangle [B_A, B_B] = 0. \quad (2.62)$$

These are the same as the spacetime equations of motion of the 4D field theory in Eqs. (2.41) and (2.42).

The ASDYM equations are integrable, with a Lax pair given by

$$\begin{aligned} L^{(B)} &= \langle \pi \hat{\gamma} \rangle^{-1} \hat{\mu}^A \pi^{A'} D_{AA'}, \\ M^{(B)} &= \langle \pi \gamma \rangle^{-1} \mu^A \pi^{A'} D_{AA'}, \end{aligned} \quad (2.63)$$

following Ref. [27]. Here, convenient normalizations have been chosen that involve constant spinors γ and $\hat{\gamma}$ that will

appear when we consider symmetry reductions in subsequent sections. These operators satisfy the zero-curvature (flatness) condition

$$[L^{(B)}, M^{(B)}] = 0 \quad (2.64)$$

for all $\pi^{A'} \in \mathbb{CP}^1$, which is equivalent to the anti-self-dual Yang-Mills equations. We shall thus refer to the 4D field theory we have derived as IFT₄ in what follows.

III. SYMMETRY REDUCTION OF 6D CS TO 4D CS

In this section, we shall describe how the 6D holomorphic Chern-Simons theory on twistor space that we considered in Sec. II can be symmetry reduced to 4D Chern-Simons theory with disorder surface defects. The symmetry reduction of the action essentially follows from Refs. [24,27]; what is new here is the novel set of boundary conditions for the 6D twistor Chern-Simons theory.

The symmetry reduction relies on our freedom to add any (1,0)-form to the 6D gauge field; i.e., we may perform

the shift

$$\mathcal{A} \rightarrow \rho_0 e^0 + \rho_A e^A + \mathcal{A} \quad (3.1)$$

without changing the action (2.1). A judicious choice of ρ_A results in a four-dimensional gauge field that can be identified with the gauge field of 4D Chern-Simons theory.

To perform the reduction, we impose invariance of the six-dimensional gauge field with respect to certain vector fields. To this end, we shall pick a unit norm spinor $\gamma_{A'}$, that selects a complex structure on $\mathbb{E}^4 \subset \mathbb{PT}$ corresponding to the point $\pi_{A'} = \gamma_{A'}$. The spinor μ^A then allows us to define the following basis of one-forms in this complex structure:

$$\begin{aligned} dz &= \mu_A \gamma_{A'} dx^{AA'}, & d\bar{z} &= \hat{\mu}_A \hat{\gamma}_{A'} dx^{AA'}, \\ dw &= -\hat{\mu}_A \gamma_{A'} dx^{AA'}, & d\bar{w} &= \mu_A \hat{\gamma}_{A'} dx^{AA'}. \end{aligned} \quad (3.2)$$

Given a coframe $\{dz, d\bar{z}, dw, d\bar{w}\}$, its dual frame $\{\partial_z, \partial_{\bar{z}}, \partial_w, \partial_{\bar{w}}\}$ is defined by

$$\langle \partial_j, \theta^i \rangle = \delta_j^i, \quad (3.3)$$

i.e.,

$$\begin{aligned} \langle \partial_w, dw \rangle &= 1, & \langle \partial_w, dz \rangle &= 0, \\ \langle \partial_w, d\bar{z} \rangle &= 0, & \langle \partial_w, d\bar{w} \rangle &= 0, \end{aligned} \quad (3.4)$$

and similarly for the other directions.

The vector fields dual to dz and $d\bar{z}$ are then

$$\chi = \hat{\mu}^A \hat{\gamma}^{A'} \partial_{AA'} = \partial_z, \quad \bar{\chi} = \mu^A \gamma^{A'} \partial_{AA'} = \partial_{\bar{z}}, \quad (3.5)$$

and symmetry reduction shall be performed with respect to these vector fields. The remaining dual vector fields are

$$\kappa = \mu^A \hat{\gamma}^{A'} \partial_{AA'} = -\partial_w, \quad \bar{\kappa} = \hat{\mu}^A \gamma^{A'} \partial_{AA'} = \partial_{\bar{w}}. \quad (3.6)$$

The reduction is performed by demanding

$$\mathcal{L}_\chi \mathcal{A} = \mathcal{L}_{\bar{\chi}} \mathcal{A} = 0. \quad (3.7)$$

The shifted gauge field also satisfies $\iota_\chi \mathcal{A} = \iota_{\bar{\chi}} \mathcal{A} = 0$, meaning it has no dz or $d\bar{z}$ legs, and the explicit form of the 4D gauge field is

$$\begin{aligned} A &= \bar{e}^0 \mathcal{A}_0 + \left(\iota_\kappa \mathcal{A} - \frac{\langle \pi \hat{\gamma} \rangle}{\langle \pi \gamma \rangle} \iota_{\bar{\chi}} \mathcal{A} \right) dw \\ &+ \left(\iota_{\bar{\kappa}} \mathcal{A} + \frac{\langle \pi \gamma \rangle}{\langle \pi \hat{\gamma} \rangle} \iota_\chi \mathcal{A} \right) d\bar{w}. \end{aligned} \quad (3.8)$$

The 6D to 4D reduction is then performed by contracting the bivector $\chi \wedge \bar{\chi}$ with the Lagrangian density of the twistor space Chern-Simons theory. Using

$$\iota_{\chi \wedge \bar{\chi}} \Omega = -K \frac{\langle \pi \gamma \rangle \langle \pi \hat{\gamma} \rangle}{\langle \pi \alpha \rangle \langle \pi \tilde{\alpha} \rangle \langle \pi \beta \rangle^2} e^0 = \omega, \quad (3.9)$$

it can be shown that

$$\iota_{\chi \wedge \bar{\chi}} (\Omega \wedge CS(\mathcal{A})) = \omega \wedge CS(A), \quad (3.10)$$

and the symmetry reduction results in the 4D Chern-Simons theory action

$$\frac{1}{2\pi i} \int_{\Sigma \times \mathbb{CP}^1} \omega \wedge \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (3.11)$$

where, crucially,

$$\begin{aligned} A_w &= -\frac{[\mathcal{A}\mu]}{\langle \pi \gamma \rangle} \\ A_{\bar{w}} &= -\frac{[\mathcal{A}\hat{\mu}]}{\langle \pi \hat{\gamma} \rangle} \end{aligned} \quad (3.12)$$

have appropriate singularities at γ and $\hat{\gamma}$. In other words, the symmetry reduction has generated disorder surface defects at the locations of zeroes of ω , despite the fact that such defects are not present in the 6D CS theory.

Now, let us turn to the reduction of 6D Chern-Simons boundary conditions to 4D Chern-Simons boundary conditions. Recall that the 6D Chern-Simons boundary conditions are

$$\begin{aligned} (\mathcal{O} - c)[\mathcal{A}\mu] \Big|_{\pi=\alpha} &= \sigma \frac{\langle \alpha \beta \rangle}{\langle \tilde{\alpha} \beta \rangle} (\mathcal{O} + c)[\mathcal{A}\mu] \Big|_{\pi=\tilde{\alpha}} \\ (\mathcal{O} - c)[\mathcal{A}\hat{\mu}] \Big|_{\pi=\alpha} &= \sigma^{-1} \frac{\langle \alpha \beta \rangle}{\langle \tilde{\alpha} \beta \rangle} (\mathcal{O} + c)[\mathcal{A}\hat{\mu}] \Big|_{\pi=\tilde{\alpha}}, \end{aligned} \quad (3.13)$$

and $\mathcal{A}|_{\pi=\beta} = 0$. Using Eq. (3.12), we immediately find the 4D Chern-Simons boundary conditions

$$\begin{aligned} (\mathcal{O} - c)A_w|_{\pi=\alpha} &= (\mathcal{O} + c)t s A_w|_{\pi=\tilde{\alpha}}, \\ (\mathcal{O} - c)A_{\bar{w}}|_{\pi=\alpha} &= (\mathcal{O} + c)t^{-1} s A_{\bar{w}}|_{\pi=\tilde{\alpha}}, \end{aligned} \quad (3.14)$$

and $A|_{\pi=\beta} = 0$, where we have defined (following the conventions of Ref. [27])

$$r_+ = K \frac{\langle \alpha \gamma \rangle \langle \alpha \hat{\gamma} \rangle}{\langle \alpha \tilde{\alpha} \rangle \langle \alpha \beta \rangle^2}, \quad r_- = -K \frac{\langle \tilde{\alpha} \gamma \rangle \langle \tilde{\alpha} \hat{\gamma} \rangle}{\langle \tilde{\alpha} \tilde{\alpha} \rangle \langle \tilde{\alpha} \beta \rangle^2}, \quad (3.15)$$

$$s = \sqrt{-\frac{r_-}{r_+}} = \frac{\langle \alpha \beta \rangle}{\langle \tilde{\alpha} \beta \rangle} \sqrt{\frac{\langle \tilde{\alpha} \gamma \rangle \langle \tilde{\alpha} \hat{\gamma} \rangle}{\langle \alpha \gamma \rangle \langle \alpha \hat{\gamma} \rangle}}, \quad (3.16)$$

$$t = \sigma s \frac{\langle \tilde{\alpha} \beta \rangle \langle \alpha \hat{\gamma} \rangle}{\langle \alpha \beta \rangle \langle \tilde{\alpha} \hat{\gamma} \rangle}. \quad (3.17)$$

It shall be convenient to specify an inhomogeneous coordinate on $\mathbb{CP}^1$, that is, $\zeta = \pi_{2'}/\pi_{1'}$ on the patch where $\pi_{1'} \neq 0$. Moreover, we shall specify the remaining spinors in terms of the inhomogeneous coordinates, that is, we pick

$$\alpha_{A'} = (1, \alpha_+), \quad \tilde{\alpha}_{A'} = (1, \alpha_-), \quad \beta_{A'} = (0, 1), \quad (3.18)$$

implying

$$\langle \tilde{\alpha}\beta \rangle = \langle \alpha\beta \rangle = 1, \quad \langle \tilde{\alpha}\alpha \rangle = \alpha_+ - \alpha_-. \quad (3.19)$$

Now, we shall further specify σ in order to make a connection a more familiar integrable field theory. That is, we shall pick

$$\sigma = \frac{\langle \alpha\gamma \rangle}{\langle \tilde{\alpha}\gamma \rangle} = \frac{\langle \tilde{\alpha}\hat{\gamma} \rangle}{\langle \alpha\hat{\gamma} \rangle}, \quad (3.20)$$

whereby $t = 1$ and $s = 1$. Moreover, we shall pick $\mathcal{O}$ to be a solution, $R \in \text{End}\mathfrak{g}$, of the modified classical Yang-Baxter equation given by

$$[Rx, Ry] - R([Rx, y] + [x, Ry]) = -c^2[x, y], \quad (3.21)$$

for $c = i$ or $c = 1$, and $x, y \in \mathfrak{g}$. The 4D Chern-Simons boundary conditions then become

$$\begin{aligned} (R - c)A_w|_{\zeta=\alpha_+} &= (R + c)A_w|_{\zeta=\alpha_-}, \\ (R - c)A_{\bar{w}}|_{\zeta=\alpha_+} &= (R + c)A_{\bar{w}}|_{\zeta=\alpha_-}, \end{aligned} \quad (3.22)$$

and $A|_{\pi=\beta} = 0$ that are the familiar boundary conditions that realize the Yang-Baxter sigma model, as shown in Ref. [40], as long as we identify α_+ with $-c\eta$ and α_- with $c\eta$ in Ref. [40]. The choice (3.20) is a sensible one, since, if we pick γ to correspond to $\zeta = 1$ and $\hat{\gamma}$ to correspond to $\zeta = -1$, the second equality in Eq. (3.20) just corresponds to

$$\frac{-c\eta - 1}{c\eta - 1} = \frac{c\eta + 1}{-c\eta + 1}. \quad (3.23)$$

In the following, we shall explore the consequences of the restrictions (3.20) and $\mathcal{O} = R$.

IV. MODIFIED CLASSICAL YANG-BAXTER EQUATION, SEMILOCAL SYMMETRY AND A 4D YANG-BAXTER SIGMA MODEL

Let us now investigate the implications of restricting σ as in Eq. (3.20) and picking $\mathcal{O} = R$, i.e., a solution of the modified classical Yang-Baxter equation, for the IFT₄.

Recall that the boundary conditions (3.22) used to derive the two-dimensional YB sigma model exhibit gauge invariance, and fixing this gauge invariance leads us to

an IFT₂ with a single field [40].¹ The gauge transformation of the boundary condition is

$$\begin{aligned} (R - c)(u^{-1}\partial_i u|_{\zeta=\alpha_+} + u^{-1}A_i u|_{\zeta=\alpha_+}) \\ = (R + c)(u^{-1}\partial_i u|_{\zeta=\alpha_-} + u^{-1}A_i u|_{\zeta=\alpha_-}), \end{aligned} \quad (4.1)$$

where $i = w, \bar{w}$. We can show that the boundary condition is gauge invariant when $(u|_{\alpha_+}, u|_{\alpha_-}) = (r, \tilde{r})$, where $(r, \tilde{r}) \in G_R$, the Lie group generated by the Lie algebra

$$\mathfrak{g}_R := \{((R - c)x, (R + c)x) | x \in \mathfrak{g}\}. \quad (4.2)$$

Gauge invariance follows, since

$$(R - c)r^{-1}\partial_i r = (R + c)\tilde{r}^{-1}\partial_i \tilde{r}, \quad (4.3)$$

and since

$$(R - c)(r^{-1}A_i|_{\zeta=\alpha_+}r) = (R + c)(\tilde{r}^{-1}A_i|_{\zeta=\alpha_-}\tilde{r}), \quad (4.4)$$

is equivalent to the original boundary condition. To show the latter, we need to express the boundary restrictions of the gauge fields as $(A_i|_{\alpha_+}, A_i|_{\alpha_-}) = ((R + c)\tilde{x}_i, (R - c)\tilde{x}_i)$ for $\tilde{x}_i \in \mathfrak{g}$, and express $(r, \tilde{r})$ as exponentials of the Lie algebra elements $((R + c)y, (R - c)y)$ for $y \in \mathfrak{g}$. The original boundary condition can then be retrieved using the identity $\text{Ad}_{\exp X} = \exp(\text{ad}X)$ for $X \in \mathfrak{g}$ and the modified classical Yang-Baxter equation, stated as

$$\begin{aligned} [(R \pm c)X, (R \pm c)Y] \\ = (R \pm c)([X, RY] + [RX, Y]), \quad X, Y \in \mathfrak{g}, \end{aligned} \quad (4.5)$$

which implies that $\mathfrak{h}_{\pm} := \text{im}(R \pm c)$ are Lie algebras.

In the case of 6D CS on twistor space, the restrictions (3.20) and $\mathcal{O} = R$ lead us not to a gauge symmetry of the boundary conditions and the IFT₄, but to a *semilocal symmetry*. We shall refer to the IFT₄ with these restrictions as IFT₄^{YB}, or the 4D Yang-Baxter sigma model. Indeed, using Eq. (3.19), the boundary conditions with these restrictions are

$$\begin{aligned} (R - c)[\mathcal{A}\mu] \Big|_{\pi=\alpha} &= \frac{\langle \alpha\gamma \rangle}{\langle \tilde{\alpha}\gamma \rangle} (R + c)[\mathcal{A}\mu] \Big|_{\pi=\tilde{\alpha}} \\ (R - c)[\mathcal{A}\hat{\mu}] \Big|_{\pi=\alpha} &= \frac{\langle \alpha\hat{\gamma} \rangle}{\langle \tilde{\alpha}\hat{\gamma} \rangle} (R + c)[\mathcal{A}\hat{\mu}] \Big|_{\pi=\tilde{\alpha}}. \end{aligned} \quad (4.6)$$

These boundary conditions can also be stated as

¹If we did not fix this gauge symmetry in the boundary conditions, the two-dimensional IFT would take the form of a two-field model that admits a nontrivial gauge symmetry.

$$\begin{aligned}
 [\mathcal{A}\mu]|_{\pi=\alpha} &= \langle\alpha\gamma\rangle(R+c)x \\
 [\mathcal{A}\mu]|_{\pi=\tilde{\alpha}} &= \langle\tilde{\alpha}\gamma\rangle(R-c)x \\
 [\mathcal{A}\hat{\mu}]|_{\pi=\alpha} &= \langle\alpha\hat{\gamma}\rangle(R+c)y \\
 [\mathcal{A}\hat{\mu}]|_{\pi=\tilde{\alpha}} &= \langle\tilde{\alpha}\hat{\gamma}\rangle(R-c)y,
 \end{aligned} \tag{4.7}$$

where $x, y \in \mathfrak{g}$.

If we perform a gauge transformation with respect to a gauge parameter u which satisfies $(u|_{\alpha_+}, u|_{\alpha_-}) \in G_R$, then we find that the first boundary condition in Eq. (4.6) is transformed to (using $[A\mu] = -[\mu A]$)

$$\begin{aligned}
 &(R-c)(-\mu^A \langle\alpha\gamma\rangle \hat{\gamma}^{A'} u^{-1} \partial_{AA'} u + \mu^A \langle\alpha\hat{\gamma}\rangle \gamma^{A'} u^{-1} \partial_{AA'} u)|_{\pi=\alpha} \\
 &= (R+c) \frac{\langle\alpha\gamma\rangle}{\langle\tilde{\alpha}\gamma\rangle} (-\mu^A \langle\tilde{\alpha}\gamma\rangle \hat{\gamma}^{A'} u^{-1} \partial_{AA'} u + \mu^A \langle\tilde{\alpha}\hat{\gamma}\rangle \gamma^{A'} u^{-1} \partial_{AA'} u)|_{\pi=\tilde{\alpha}},
 \end{aligned} \tag{4.10}$$

while the remaining terms are equivalent to the original boundary condition using Eq. (4.7). Invariance of the boundary condition then follows if

$$\mu^A \gamma^{A'} \partial_{AA'} u = 0, \tag{4.11}$$

which implies that the gauge invariance is only semilocal. Analogously, the remaining boundary condition is invariant if

$$\hat{\mu}^A \hat{\gamma}^{A'} \partial_{AA'} u = 0. \tag{4.12}$$

These semilocal symmetries of the boundary conditions should be manifest in the IFT_4^{YB} . Semilocal symmetries are common in the study of 4D IFTs, and occur even in the 4DWess–Zumino–Witten (WZW) model [42,43]. As explained in Ref. [27], there is also a semilocal symmetry that arises from transformations that preserve the conditions $\mathcal{A}|_\beta = 0$. The conserved current associated with this symmetry determines the B -Lax operator that we constructed in Sec. II C. It is expected that the semilocal symmetry described in this section determines another Lax operator, called the C -Lax operator in Ref. [27].

In the preceding discussion, we have shown that $\mathcal{O} = R$ is a sufficient condition for a semilocal symmetry in the IFT_4 . It is in fact also a necessary condition, as we explain in the Appendix D.

V. SYMMETRY REDUCTION OF IFT_4 TO IFT_2

In this section, we shall perform the symmetry reduction of the IFT_4 defined by Eq. (2.34) to a two-dimensional field theory, that corresponds to the 2D Yang-Baxter sigma model when we specialize to IFT_4^{YB} . The symmetry reduction is performed along the vector fields dual to dz and $d\bar{z}$. Under this reduction, w and $\bar{w}$ become the

$$\begin{aligned}
 &(R-c)\mu^A(\alpha^{A'}u^{-1}\partial_{AA'}u+u^{-1}\mathcal{A}_Au)\Big|_{\pi=\alpha} \\
 &= \frac{\langle\alpha\gamma\rangle}{\langle\tilde{\alpha}\gamma\rangle}(R+c)\mu^A(\tilde{\alpha}^{A'}u^{-1}\partial_{AA'}u+u^{-1}\mathcal{A}_Au)\Big|_{\pi=\tilde{\alpha}}.
 \end{aligned} \tag{4.8}$$

Using

$$X_{A'} = \langle X\hat{\gamma}\rangle\gamma_{A'} - \langle X\gamma\rangle\hat{\gamma}_{A'} \tag{4.9}$$

we then find that the derivative terms in this expression can be rewritten as

coordinates of the two-dimensional world sheet Σ . The symmetry reduction imposes the condition

$$\partial_z h = \partial_{\bar{z}} h = 0, \quad \partial_z \tilde{h} = \partial_{\bar{z}} \tilde{h} = 0. \tag{5.1}$$

Let us start by writing the IFT_4 action in terms of the currents (2.28). In two dimensions, we set

$$\partial_+ \equiv \partial_w, \quad \partial_- \equiv \partial_{\bar{w}}, \tag{5.2}$$

Starting from the definition $j = \langle\alpha\beta\rangle^{-1}\mu^A\alpha^{A'}h^{-1}\partial_{AA'}h$, we use the spinor expansion (4.9) with $X = \alpha$, as follows:

$$\alpha^{A'} = \langle\alpha\hat{\gamma}\rangle\gamma^{A'} - \langle\alpha\gamma\rangle\hat{\gamma}^{A'}. \tag{5.3}$$

Therefore, the derivative term $h^{-1}\partial_{AA'}h$ appearing in j becomes

$$\mu^A\alpha^{A'}\partial_{AA'} = \langle\alpha\hat{\gamma}\rangle\mu^A\gamma^{A'}\partial_{AA'} - \langle\alpha\gamma\rangle\mu^A\hat{\gamma}^{A'}\partial_{AA'}. \tag{5.4}$$

Now we identify the first term using Eq. (3.2), which is $\mu^A\gamma^{A'}\partial_{AA'} = \partial_{\bar{z}}$. However, symmetry reduction imposes $\partial_z h = \partial_{\bar{z}} h = 0$, so the first term drops out on reduced fields. The second term is precisely the derivative along w (i.e., the surviving world sheet direction), since w and $\bar{w}$ are the coordinates that remain after reducing along z and $\bar{z}$. Thus, on reduced fields, $\mu^A\alpha^{A'}\partial_{AA'} \longrightarrow \langle\alpha\gamma\rangle\partial_w \equiv \langle\alpha\gamma\rangle\partial_+$. Substituting this back into j , we find

$$j \longrightarrow \langle\alpha\beta\rangle^{-1}\langle\alpha\gamma\rangle h^{-1}\partial_+ h = \frac{\langle\alpha\gamma\rangle}{\langle\alpha\beta\rangle} J_+. \tag{5.5}$$

Similarly, replace $\hat{j} \longrightarrow \frac{\langle\alpha\hat{\gamma}\rangle}{\langle\alpha\beta\rangle} J_-$, $\tilde{j} \longrightarrow \frac{\langle\tilde{\alpha}\gamma\rangle}{\langle\tilde{\alpha}\beta\rangle} \tilde{J}_+$ and $\tilde{\tilde{j}} \longrightarrow \frac{\langle\tilde{\alpha}\hat{\gamma}\rangle}{\langle\tilde{\alpha}\beta\rangle} \tilde{\tilde{J}}_-$, where,

$$J_+ = h^{-1} \partial_+ h, \quad J_- = h^{-1} \partial_- h, \quad \tilde{J}_+ = \tilde{h}^{-1} \partial_+ \tilde{h}, \quad \tilde{J}_- = \tilde{h}^{-1} \partial_- \tilde{h}.$$

Then, the resulting two-dimensional action is given by

$$S_{\text{IFT}_4} \rightsquigarrow S_{\text{IFT}_2} = \int_{\Sigma} \text{vol}_2 \text{Tr}(r_+ J_+ (P^T U_+^T - U_- P) J_- - r_- \tilde{J}_+ (U_+^T P^T - P U_-) \tilde{J}_- + r_+ \mathcal{L}_{\text{WZ}}(h) + r_- \mathcal{L}_{\text{WZ}}(\tilde{h}) - 2t \sqrt{-r_+ r_-} \tilde{J}_+ U_+^T J_- + 2t^{-1} \sqrt{-r_+ r_-} J_+ U_- \tilde{J}_-). \quad (5.6)$$

The symmetry reduction along ∂_z and $\partial_{\bar{z}}$ maps the four-form $\text{vol}_4 = \frac{1}{12} \varepsilon_{AB} \varepsilon_{CD} \varepsilon_{A'C'} \varepsilon_{B'D'} dx^{AA'} \wedge dx^{BB'} \wedge dx^{CC'} \wedge dx^{DD'}$ to the two-form vol_2 , defined by $\text{vol}_2 = \iota_{\partial_z} \iota_{\partial_{\bar{z}}} \text{vol}_4 = d\bar{w} \wedge dw = d\sigma^- \wedge d\sigma^+$.

A. Reduction to Yang-Baxter sigma model

We shall now specialize the IFT₂ in Eq. (5.6) to the case where

$$\mathcal{O} = R, \quad (5.7)$$

$$\sigma = \frac{1 + \eta c}{1 - \eta c}. \quad (5.8)$$

This would correspond to the symmetry reduction of IFT₄^{YB} that admits a semilocal symmetry associated with R . In this case,

$$\begin{aligned} r_+ &= -r_- \\ s &= 1 \\ t &= 1. \end{aligned} \quad (5.9)$$

The resulting 2D field theory is expected to enjoy a gauge symmetry corresponding to the group G_R , due to the gauge invariance of the 4D CS boundary condition (4.1). As explained in the Appendix D, $\mathcal{O} = R$ is a necessary and sufficient condition for gauge symmetry of the 2D field

theory defined by Eq. (5.6). This gauge symmetry can be fixed using $h = \tilde{h}$, whereby this two-field model ought to reduce to the Yang-Baxter sigma model. The reason that $h = \tilde{h}$ is the correct gauge fixing is because the field space is $D = G \times G$ or $G_{\mathbb{C}}$ when c is equal to 1 or i , respectively. Since there is a G_R gauge symmetry, modding out the corresponding redundancies in field space gives $G_R \backslash D$ or $G_R \backslash G_{\mathbb{C}}$. As explained, e.g., in Ref. [40], $G_R \backslash D$ can be parametrized by the *diagonal* subgroup G^{δ} , which corresponds to subgroup of D determined by $h = \tilde{h}$. Analogously, $G_R \backslash G_{\mathbb{C}}$ can be parametrized by G with elements determined by $h = \tilde{h}$.

Let us show that this two-field model reduces to the Yang-Baxter sigma model when $h = \tilde{h}$, whereby $\Lambda|_{h=\tilde{h}} = 1$. Since $\Lambda = 1$, we have

$$U_{\pm} = (P - \sigma^{\pm 1})^{-1}. \quad (5.10)$$

Also, because $R^T = -R$, we have

$$P^T = P^{-1}. \quad (5.11)$$

Therefore, we have

$$U_{\pm}^T = (P^T - \sigma^{\pm 1})^{-1} = (P^{-1} - \sigma^{\pm 1})^{-1}. \quad (5.12)$$

Making the identification $\tilde{J}_{\pm} = J_{\pm}$, the action becomes

$$S_{\text{IFT}_2} = \int_{\Sigma} \text{vol}_2 \text{Tr}(r_+ J_+ (P^T U_+^T - U_- P) J_- - r_- J_+ (U_+^T P^T - P U_-) J_- - 2t \sqrt{-r_+ r_-} J_+ U_+^T J_- + 2t^{-1} \sqrt{-r_+ r_-} J_+ U_- J_- + (r_+ + r_-) \mathcal{L}_{\text{WZ}}(h)). \quad (5.13)$$

Using the identities in Eq. (2.29) and setting $\Lambda^T = 1$ (since $\Lambda = 1$), we obtain

$$U_+^T = -\sigma^{-1} U_- P, \quad U_-^T = -\sigma U_+ P. \quad (5.14)$$

Define $\mathcal{O}_1 = P^T U_+^T - U_- P$ and $\mathcal{O}_2 = U_+^T P^T - P U_-$ in Eq. (5.13). Simplifying $\mathcal{O}_1$ using $U_+^T = -\sigma^{-1} U_- P$, we obtain

$$\begin{aligned} \mathcal{O}_1 &= P^T (-\sigma^{-1} U_- P) - U_- P \\ &= -(\sigma^{-1} P^T + 1) U_- P. \end{aligned} \quad (5.15)$$

Now using $P^T = P^{-1}$ and P, U_+ and U_- commute with each other, which simplifies $\mathcal{O}_1$ as

$$\mathcal{O}_1 = -(\sigma^{-1} + P) U_-. \quad (5.16)$$

Similarly, to simplify $\mathcal{O}_2$, we use $U_+^T = -\sigma^{-1} U_- P$ to obtain

$$\mathcal{O}_2 = -(\sigma^{-1} + P) U_-. \quad (5.17)$$

Therefore,

$$\mathcal{O}_1 = \mathcal{O}_2 = -(\sigma^{-1} + P)U_- \quad \text{when } \Lambda = 1. \quad (5.18)$$

Rewriting the action using Eq. (5.18), it becomes

$$\begin{aligned} S_{\text{IFT}_2} = & \int_{\Sigma} \text{vol}_2 \text{Tr} \left(-(r_+ - r_-)J_+(\sigma^{-1} + P)U_-J_- \right. \\ & - 2t\sqrt{-r_+r_-}J_+U_+^TJ_- + 2t^{-1}\sqrt{-r_+r_-}J_+U_-J_- \\ & \left. + (r_+ + r_-)\mathcal{L}_{\text{WZ}}(h) \right). \end{aligned} \quad (5.19)$$

We can use $U_+^T = -\sigma^{-1}U_-P$ again to find

$$-2\sqrt{-r_+r_-}J_+U_+^TJ_- = 2\sigma^{-1}\sqrt{-r_+r_-}J_+U_-PJ_-. \quad (5.20)$$

Thus,

$$S_{\text{IFT}_2} = \int_{\Sigma} \text{vol}_2 \text{Tr}(J_+U_- \mathcal{M}(P)J_-) + S_{\text{WZ}}, \quad (5.21)$$

where the remaining operator coefficient is

$$\begin{aligned} \mathcal{M}(P) = & -(r_+ - r_-)(\sigma^{-1} + P) + 2t\sigma^{-1}\sqrt{-r_+r_-}P \\ & + 2t^{-1}\sqrt{-r_+r_-}. \end{aligned} \quad (5.22)$$

Imposing Eq. (5.9), which eliminates the Wess-Zumino part of the action, and $\mathcal{M}(P)$ simplifies to

$$\mathcal{M}(P) = r_+(-2(\sigma^{-1} + P) + 2\sigma^{-1}P + 2). \quad (5.23)$$

The operator sandwiched between J_+ and J_- is $U_-(\mathcal{M}(P))$ is given by $U_- = (P - \sigma^{-1})^{-1} = -\sigma(1 - \sigma P)^{-1}$. Therefore,

$$S_{\text{IFT}_2} = 2(\sigma - 1)r_+ \int_{\Sigma} \text{vol}_2 \text{Tr}(J_+((1 - \sigma P)^{-1})(1 - P)J_-). \quad (5.24)$$

An identity can be derived for the YB operator that can relate R and P , that is,

$$\begin{aligned} & (1 - \sigma P)^{-1}(1 - P) \\ & = \frac{2}{1 + \sigma}(1 - \eta R)^{-1} \quad \eta = \frac{\sigma - 1}{(\sigma + 1)c}, \\ & = (1 - \eta c)(1 - \eta R)^{-1}, \quad \sigma = \frac{1 + \eta c}{1 - \eta c}. \end{aligned} \quad (5.25)$$

This identity is derived in the Appendix C. From Eq. (5.25), the standard Yang-Baxter operator $(1 - \eta R)^{-1}$ is expressed in terms of the P -operator, that is, it is proportional to $(1 - \sigma P)^{-1}(1 - P)$. Also, from Eq. (5.24), we see that the action is also proportional to $(1 - \sigma P)^{-1}(1 - P)$. As a result, the action can be written as

$$S_{\text{IFT}_2} = C_1 \int_{\Sigma} \text{vol}_2 \text{Tr} \left(J_+ \frac{1}{1 - \eta R} J_- \right), \quad (5.26)$$

where

$$C_1 = 4r_+ \frac{(\sigma - 1)}{(\sigma + 1)} = 4r_+ c\eta.$$

This is precisely the action for the Yang-Baxter sigma model. Given that we have obtained the Yang-Baxter sigma model as a symmetry reduction of IFT_4^{YB} , and the equations of motion of the latter can be identified with the ASDYM equations, this means that we have obtained an embedding of the equations of motion of the Yang-Baxter sigma model in the ASDYM equations!

VI. 2D IFT FROM 4D CS

In this section, we shall derive the two-dimensional field theory given in Eq. (2.34) from the 4D Chern-Simons theory setup that was obtained via symmetry reduction in Sec. III. We begin by parametrizing the gauge field $\hat{A}$ by

$$\hat{A}_{\bar{\zeta}} = \hat{h}^{-1} \partial_{\bar{\zeta}} \hat{h}, \quad \hat{A}_I = \hat{h}^{-1} \mathcal{L}_I \hat{h} + \hat{h}^{-1} \partial_I \hat{h}, \quad I = w, \bar{w}, \quad (6.1)$$

which corresponds to a formal gauge choice in which the $\bar{\zeta}$ -component of the connection is pure gauge. In this parametrization, the fields $\mathcal{L}_I$ turn out to define a meromorphic Lax connection on Σ , while residual gauge transformations act as ordinary two-dimensional gauge transformations. With this setup, the 4D CS action becomes

$$S_{\text{CS4}} = -\frac{1}{2\pi i} \int_{\Sigma \times \mathbb{CP}^1} d\omega \wedge \text{Tr}(\hat{J} \wedge \hat{h}^{-1} \mathcal{L} \hat{h}) + \text{WZ terms}. \quad (6.2)$$

Since ω is a meromorphic one-form on $\mathbb{CP}^1$, given explicitly by $\omega = -K \frac{\langle \pi \gamma \rangle \langle \pi \hat{\gamma} \rangle}{\langle \pi \alpha \rangle \langle \pi \hat{\alpha} \rangle \langle \pi \beta \rangle^2} e^0$, it follows, in the distributional sense, that $d\omega = 2\pi i \sum_{p \in \text{poles}(\omega)} \times \text{res}_p(\omega) \delta_p^{(2)}$. Using this identity, the action reduces to

$$S_{\text{CS4}} = -\frac{1}{2\pi i} \int_{\Sigma} \int_{\mathbb{CP}^1} \left(2\pi i \sum_p \text{Res}_p(\omega) \delta^{(2)}(p) \right) \wedge \text{Tr}(\hat{J} \wedge \hat{h}^{-1} \mathcal{L} \hat{h}) \quad (6.3)$$

$$= -\sum_p \text{Res}_p(\omega) \int_{\Sigma} \text{Tr}(\hat{J} \wedge \hat{h}^{-1} \mathcal{L} \hat{h}) \Big|_p. \quad (6.4)$$

The resulting localized 2D action is given by

$$S_{2d} = r_+ \int_{\Sigma} \text{Tr}(\hat{J} \wedge \hat{h}^{-1} \mathcal{L} \hat{h}) \Big|_{\alpha} + r_- \int_{\Sigma} \text{Tr}(\hat{J} \wedge \hat{h}^{-1} \mathcal{L} \hat{h}) \Big|_{\bar{\alpha}} + \text{WZ terms}, \quad (6.5)$$

where

$$r_+ = K \frac{\langle \alpha \gamma \rangle \langle \alpha \hat{\gamma} \rangle}{\langle \alpha \hat{\alpha} \rangle \langle \alpha \beta \rangle^2}, \quad \text{and} \quad r_- = -K \frac{\langle \tilde{\alpha} \gamma \rangle \langle \tilde{\alpha} \hat{\gamma} \rangle}{\langle \alpha \hat{\alpha} \rangle \langle \tilde{\alpha} \beta \rangle^2}. \quad (6.6)$$

Note that we set

$$\hat{h} = \text{id} \quad (6.7)$$

in order to fix the residual gauge symmetry arising from transformations involving smooth functions on Σ . In what follows, we shall also define

$$\hat{h}|_{\alpha} = h, \quad \hat{h}|_{\bar{\alpha}} = \tilde{h}, \quad \hat{J}|_{\alpha} = J, \quad \text{and} \quad \hat{J}|_{\bar{\alpha}} = \tilde{J}.$$

To complete the construction, one must fix the meromorphic dependence of the Lax connection $\mathcal{L}$ on the

twistor coordinate $\pi \in \mathbb{CP}^1$ in a manner consistent with the meromorphic one-form ω and the boundary conditions at its poles. Since ω has zeros at $\pi \sim \gamma$ and $\pi \sim \hat{\gamma}$, regularity of the four-dimensional Chern-Simons action requires that $\mathcal{L}$ may have at most simple poles at these points. The most general well-defined ansatz with this property is therefore

$$\mathcal{L}_w = \frac{\langle \pi \beta \rangle}{\langle \pi \gamma \rangle} M_w + N_w, \quad \mathcal{L}_{\bar{w}} = \frac{\langle \pi \beta \rangle}{\langle \pi \hat{\gamma} \rangle} M_{\bar{w}} + N_{\bar{w}}, \quad (6.8)$$

where M_I and N_I are $\mathfrak{g}$ -valued fields on Σ independent of π . The boundary conditions $A|_{\beta} = 0$ and

$$\begin{aligned} (\mathcal{O} - c)A_w|_{\pi=\alpha} &= (\mathcal{O} + c)tsA_w|_{\pi=\bar{\alpha}}, \\ (\mathcal{O} - c)A_{\bar{w}}|_{\pi=\alpha} &= (\mathcal{O} + c)t^{-1}sA_{\bar{w}}|_{\pi=\bar{\alpha}}, \end{aligned} \quad (6.9)$$

then uniquely determine the allowed Lax connection. Solving for M_w , $M_{\bar{w}}$, N_w , and $N_{\bar{w}}$, we obtain

$$\begin{aligned} \frac{\langle \alpha \beta \rangle}{\langle \alpha \gamma \rangle} \text{Ad}_h^{-1}(M_w) &= [P - \sigma \text{Ad}_h^{-1} \text{Ad}_h]^{-1} (ts\tilde{J}_w - PJ_w), \quad \mathcal{N}_w = 0, \\ \frac{\langle \alpha \beta \rangle}{\langle \alpha \hat{\gamma} \rangle} \text{Ad}_h^{-1}(M_{\bar{w}}) &= [P - \sigma^{-1} \text{Ad}_h^{-1} \text{Ad}_h]^{-1} (t^{-1}s\tilde{J}_{\bar{w}} - PJ_{\bar{w}}), \quad \mathcal{N}_{\bar{w}} = 0. \end{aligned} \quad (6.10)$$

We also find it convenient to write these expressions in the alternative forms

$$\begin{aligned} \frac{\langle \tilde{\alpha} \beta \rangle}{\langle \tilde{\alpha} \gamma \rangle} \text{Ad}_{\tilde{h}}^{-1}(M_w) &= [P^T - \sigma^{-1} \text{Ad}_{\tilde{h}}^{-1} \text{Ad}_{\tilde{h}}]^{-1} (t^{-1}s^{-1}J_w - P^T\tilde{J}_w), \quad \mathcal{N}_w = 0, \\ \frac{\langle \tilde{\alpha} \beta \rangle}{\langle \tilde{\alpha} \hat{\gamma} \rangle} \text{Ad}_{\tilde{h}}^{-1}(M_{\bar{w}}) &= [P^T - \sigma \text{Ad}_{\tilde{h}}^{-1} \text{Ad}_{\tilde{h}}]^{-1} (ts^{-1}J_{\bar{w}} - P^T\tilde{J}_{\bar{w}}), \quad \mathcal{N}_{\bar{w}} = 0. \end{aligned} \quad (6.11)$$

Finally, using Eqs. (6.10) and (6.11), we obtain the final form of the two-dimensional action, given by

$$\begin{aligned} S_{2d} &= r_+ \int_{\Sigma} \text{vol}_2 \text{Tr} \left(J_w \frac{\langle \alpha \beta \rangle}{\langle \alpha \hat{\gamma} \rangle} \text{Ad}_h^{-1}(M_{\bar{w}}) - J_{\bar{w}} \frac{\langle \alpha \beta \rangle}{\langle \alpha \gamma \rangle} \text{Ad}_h^{-1}(M_w) \right) \\ &\quad + r_- \int_{\Sigma} \text{vol}_2 \text{Tr} \left(\tilde{J}_w \frac{\langle \tilde{\alpha} \beta \rangle}{\langle \tilde{\alpha} \hat{\gamma} \rangle} \text{Ad}_{\tilde{h}}^{-1}(M_{\bar{w}}) - \tilde{J}_{\bar{w}} \frac{\langle \tilde{\alpha} \beta \rangle}{\langle \tilde{\alpha} \gamma \rangle} \text{Ad}_{\tilde{h}}^{-1}(M_w) \right) + \text{WZ terms} \\ &= \int_{\Sigma} \text{vol}_2 \text{Tr} (r_+ J_w (P^T U_+^T - U_- P) J_{\bar{w}} - r_- \tilde{J}_w (U_+^T P^T - P U_-) \tilde{J}_{\bar{w}} \\ &\quad - 2t\sqrt{-r_+ r_-} \tilde{J}_w U_+^T J_{\bar{w}} + 2t^{-1}\sqrt{-r_+ r_-} J_w U_- \tilde{J}_{\bar{w}}) + \text{WZ terms}, \end{aligned} \quad (6.12)$$

which is exactly same as the two-dimensional field theory derived in Eq. (5.6).

We have described how this 2D field theory with two fields includes the Yang-Baxter sigma model as a special case in Eq. VA. We would like to address the question of when this 2D two-field theory is integrable in general. This requires additional investigation despite the fact that we

have derived a flat Lax connection as part of the derivation from 4D CS, because such field theories with two fields may require more than one Lax connection to realize all nonlocal integrals of motion. Indeed, two-field models such as the one we derive here are expected to arise from coupled sigma models in a certain limit, as explained in Ref. [44]. In this limit, two possible Lax operators can arise depending

on how one takes the limit of the Lax operator of the original theory. Each of the resulting Lax operators loses complementary information about the parameters of the original theory (corresponding to poles of the twist function) in the limit. Hence, both Lax operators are expected to be necessary for integrability of the resulting theory. We expect that this is also the case for the two-field model we have derived.

As explained in Sec. IV, when we specialize $\mathcal{O}$ to R and σ to $\frac{1+\eta c}{1-\eta c}$, the IFT_4 admits a semilocal symmetry, which in turn is expected to give rise to an additional 4D Lax operator called a C -Lax. Both the B -Lax and C -Lax can be symmetry reduced to two-dimensional Lax operators, as explained in Ref. [27]. In general, we expect that whenever the IFT_4 derived in Eq. (2.34) admits a semilocal symmetry, there ought to be an additional C -Lax that further symmetry reduces to a second Lax pair in 2D, ensuring the integrability of the 2D model. It would be interesting to find explicit examples of such semilocal symmetries that do not correspond to the IFT_4^{YB} case of $\mathcal{O} = R$ and $\sigma = \frac{1+\eta c}{1-\eta c}$ that we considered in this work.

VII. HOMOGENEOUS YANG-BAXTER MODEL FROM THE CHERN-SIMONS ROUTE

In this section, we derive the homogeneous Yang-Baxter sigma model along the Chern-Simons side of the diamond,

$$\text{hCS}_6 \rightarrow \text{CS}_4 \rightarrow \text{IFT}_2. \quad (7.1)$$

We start from the Yang-Baxter boundary condition obtained in Sec. III after reducing the six-dimensional twistor theory to four-dimensional Chern-Simons theory. The homogeneous limit is obtained by bringing the pair of simple-pole locations of the inhomogeneous Yang-Baxter construction to a common point on the auxiliary $\mathbb{CP}^1$ fiber. In this limit, the boundary condition is no longer expressed in terms of the values of the Chern-Simons gauge field at two separated simple poles. Instead, it relates the value of the gauge field at the coalesced double pole to its first ζ -coefficient there.

We work in the affine patch

$$\zeta = \frac{\pi_{2'}}{\pi_{1'}} \quad (7.2)$$

and choose

$$\begin{aligned} \alpha_{A'} &= (1, -a), & \tilde{\alpha}_{A'} &= (1, a), \\ \beta_{A'} &= (0, 1), & a &= c\eta. \end{aligned} \quad (7.3)$$

Thus, $\pi = \alpha$ and $\pi = \tilde{\alpha}$ are located at $\zeta = -a$ and $\zeta = a$, respectively. The condition at the double pole $\pi = \beta$ is kept fixed, namely

$$\mathcal{A}_A|_{\pi=\beta} = 0. \quad (7.4)$$

We also keep the zeroes of the meromorphic one-form ω fixed, as in the inhomogeneous Yang-Baxter construction before taking the coalescence limit,

$$\gamma: \zeta = 1, \quad \hat{\gamma}: \zeta = -1, \quad (7.5)$$

with representatives

$$\gamma_{A'} = (1, 1), \quad \hat{\gamma}_{A'} = (1, -1). \quad (7.6)$$

For these spinors, one finds

$$\frac{\langle \alpha \gamma \rangle}{\langle \tilde{\alpha} \gamma \rangle} = \frac{1+a}{1-a}, \quad \frac{\langle \alpha \hat{\gamma} \rangle}{\langle \tilde{\alpha} \hat{\gamma} \rangle} = \frac{1-a}{1+a}. \quad (7.7)$$

Therefore the specialization (3.20) gives

$$\sigma = \frac{\langle \alpha \gamma \rangle}{\langle \tilde{\alpha} \gamma \rangle} = \frac{\langle \tilde{\alpha} \hat{\gamma} \rangle}{\langle \alpha \hat{\gamma} \rangle} = \frac{1+a}{1-a}. \quad (7.8)$$

With this choice, the parameters defined in Eqs. (3.16) and (3.17) reduce to

$$s = 1, \quad t = 1. \quad (7.9)$$

We now reduce the six-dimensional boundary condition to the Chern-Simons boundary condition. The matching relation between the twistor connection and the reduced Chern-Simons gauge field is

$$A_w = -\frac{[\mathcal{A}\mu]}{\langle \pi \gamma \rangle}, \quad A_{\bar{w}} = -\frac{[\mathcal{A}\hat{\mu}]}{\langle \pi \hat{\gamma} \rangle}. \quad (7.10)$$

At $\pi = \alpha$ and $\pi = \tilde{\alpha}$, this gives

$$\begin{aligned} [\mathcal{A}\mu]|_{\pi=\alpha} &= -\langle \alpha \gamma \rangle A_w(-a), \\ [\mathcal{A}\mu]|_{\pi=\tilde{\alpha}} &= -\langle \tilde{\alpha} \gamma \rangle A_w(a). \end{aligned} \quad (7.11)$$

Substituting these expressions into the first equation in Eq. (2.10), and using Eq. (7.8), the spinor factors cancel. Hence,

$$(R - c)A_w(-a) = (R + c)A_w(a). \quad (7.12)$$

The hatted component gives the same relation for $A_{\bar{w}}$ because

$$\sigma^{-1} = \frac{\langle \alpha \hat{\gamma} \rangle}{\langle \tilde{\alpha} \hat{\gamma} \rangle}. \quad (7.13)$$

Therefore, both components obey

$$(R - c)A_i(-a) = (R + c)A_i(a), \quad i = w, \bar{w}. \quad (7.14)$$

This is the Yang-Baxter boundary condition (3.22), written in the present affine parametrization with $\alpha_+ = -a$ and $\alpha_- = a$.

We now take the homogeneous scaling limit. The combination kept fixed is

$$R_h = \eta R = \frac{a}{c} R, \quad (7.15)$$

while $a \rightarrow 0$. Equivalently,

$$R = \frac{c}{a} R_h. \quad (7.16)$$

Thus,

$$R - c = \frac{c}{a} (R_h - a), \quad R + c = \frac{c}{a} (R_h + a). \quad (7.17)$$

The boundary relation before the homogeneous limit, Eq. (7.14), becomes

$$(R_h - a)A_i(-a) = (R_h + a)A_i(a), \quad i = w, \bar{w}. \quad (7.18)$$

Expanding around the coalesced point $\zeta = 0$, we obtain

$$A_i(-a) = A_i(0) - a\partial_\zeta A_i(0) + O(a^2), \quad (7.19)$$

$$A_i(a) = A_i(0) + a\partial_\zeta A_i(0) + O(a^2), \quad (7.20)$$

and substituting into Eq. (7.18), the leading terms cancel. The first nontrivial order gives

$$A_i(0) + R_h \partial_\zeta A_i(0) = 0, \quad i = w, \bar{w}. \quad (7.21)$$

This is the homogeneous boundary condition at the double pole. Here, $\partial_\zeta A_i(0)$ is the first ζ -coefficient in the expansion of the Chern-Simons gauge field around the coalesced pole.

The same relation follows directly in six-dimensional variables. Let

$$B(\zeta) = [\mathcal{A}\mu](\zeta). \quad (7.22)$$

The homogeneous limit of the first equation in Eq. (2.10) gives

$$B(0) + R_h \partial_\zeta B(0) + R_h B(0) = 0. \quad (7.23)$$

Using

$$B(\zeta) = -\langle \pi\gamma \rangle A_w(\zeta), \quad \langle \pi\gamma \rangle = \zeta - 1, \quad (7.24)$$

we have

$$B(\zeta) = (1 - \zeta)A_w(\zeta). \quad (7.25)$$

Therefore,

$$B(0) = A_w(0), \quad \partial_\zeta B(0) = -A_w(0) + \partial_\zeta A_w(0). \quad (7.26)$$

Substituting these expressions into Eq. (7.23) gives

$$A_w(0) + R_h \partial_\zeta A_w(0) = 0. \quad (7.27)$$

The hatted component works in the same way. Since

$$[\mathcal{A}\hat{\mu}](\zeta) = -\langle \pi\hat{\gamma} \rangle A_{\bar{w}}(\zeta), \quad \langle \pi\hat{\gamma} \rangle = \zeta + 1, \quad (7.28)$$

the homogeneous limit of the second equation in Eq. (2.10) gives

$$A_{\bar{w}}(0) + R_h \partial_\zeta A_{\bar{w}}(0) = 0. \quad (7.29)$$

Thus the twistor boundary condition and the reduced Chern-Simons boundary condition give the same double-pole relation.

The meromorphic one-form of the reduced four-dimensional Chern-Simons theory is inherited from the contraction formula (3.10). With the normalization used there,

$$\omega_a = -K \frac{\langle \pi\gamma \rangle \langle \pi\hat{\gamma} \rangle}{\langle \pi\alpha \rangle \langle \pi\tilde{\alpha} \rangle \langle \pi\beta \rangle^2} d\zeta. \quad (7.30)$$

In the affine patch,

$$\langle \pi\beta \rangle = 1, \quad \langle \pi\gamma \rangle = \zeta - 1, \quad \langle \pi\hat{\gamma} \rangle = \zeta + 1, \quad (7.31)$$

and

$$\langle \pi\alpha \rangle = \zeta + a, \quad \langle \pi\tilde{\alpha} \rangle = \zeta - a. \quad (7.32)$$

Therefore,

$$\omega_a = -K \frac{(\zeta - 1)(\zeta + 1)}{(\zeta + a)(\zeta - a)} d\zeta = K \frac{1 - \zeta^2}{\zeta^2 - a^2} d\zeta. \quad (7.33)$$

Taking $a \rightarrow 0$ gives

$$\omega_h = K \left(\frac{1}{\zeta^2} - 1 \right) d\zeta. \quad (7.34)$$

Thus, the simple poles at $\zeta = \pm a$ combine into a double pole at $\zeta = 0$, while the zeroes at $\zeta = \pm 1$ remain fixed.

The algebraic Yang-Baxter condition degenerates in parallel. Substituting $R = (c/a)R_h$ into the modified classical Yang-Baxter equation gives

$$[R_h x, R_h y] - R_h([R_h x, y] + [x, R_h y]) = -a^2[x, y]. \quad (7.35)$$

Taking $a \rightarrow 0$, one obtains

$$[R_h x, R_h y] - R_h([R_h x, y] + [x, R_h y]) = 0. \quad (7.36)$$

Thus, R_h satisfies the homogeneous classical Yang-Baxter equation. Equivalently, the double-pole boundary condition (7.21) selects the graph of $-R_h$ in the space of boundary data $(A_i(0), \partial_\zeta A_i(0))$. Closure of this boundary algebra is equivalent to Eq. (7.36), while $R_h^t = -R_h$ gives isotropy of the corresponding boundary pairing.

We now localize the four-dimensional Chern-Simons theory to a two-dimensional integrable field theory. The derivation is well known from Ref. [40], and we shall only quote the result:

$$S_{\text{IFT}_2}^{\text{hom}} = 2K \int_{\Sigma} \text{vol}_2 \text{Tr}[J_w(1 - R_h)^{-1} J_{\bar{w}}]. \quad (7.37)$$

Using

$$J_+ = J_w, \quad J_- = J_{\bar{w}}, \quad (7.38)$$

this becomes

$$S_{\text{IFT}_2}^{\text{hom}} = 2K \int_{\Sigma} \text{vol}_2 \text{Tr}[J_+(1 - R_h)^{-1} J_-]. \quad (7.39)$$

Equivalently, after defining $\kappa = 2K$, we obtain

$$S_{\text{YB}}^{\text{hom}} = \kappa \int_{\Sigma} \text{vol}_2 \text{Tr}[J_+(1 - R_h)^{-1} J_-]. \quad (7.40)$$

The same result is obtained as a consistency check by taking the homogeneous limit of the inhomogeneous Yang-Baxter action derived in Sec. VA. The inhomogeneous Yang-Baxter action is

$$S_{\text{IFT}_2} = C_1 \int_{\Sigma} \text{vol}_2 \text{Tr} \left[J_+ \frac{1}{1 - \eta R} J_- \right], \quad (7.41)$$

$$C_1 = 4r_+ c\eta. \quad (7.41)$$

In the homogeneous scaling (7.15),

$$1 - \eta R = 1 - R_h. \quad (7.42)$$

Furthermore,

$$r_+ = K \frac{\langle \alpha \gamma \rangle \langle \alpha \hat{\gamma} \rangle}{\langle \alpha \tilde{\alpha} \rangle \langle \alpha \beta \rangle^2}. \quad (7.43)$$

For the affine representatives used above,

$$\begin{aligned} \langle \alpha \gamma \rangle &= -(1 + c\eta), & \langle \alpha \hat{\gamma} \rangle &= 1 - c\eta, \\ \langle \alpha \tilde{\alpha} \rangle &= -2c\eta, & \langle \alpha \beta \rangle &= 1. \end{aligned} \quad (7.44)$$

Therefore,

$$r_+ = K \frac{1 - c^2 \eta^2}{2c\eta}. \quad (7.45)$$

It follows that

$$C_1 = 4r_+ c\eta = 2K(1 - c^2 \eta^2) \rightarrow 2K \quad \text{as } \eta \rightarrow 0. \quad (7.46)$$

Thus, Eq. (7.41) reduces precisely to Eq. (7.39).

The homogeneous Yang-Baxter model is therefore obtained from the same twistor-space construction by the coalescence

$$\alpha: \zeta = -c\eta, \quad \tilde{\alpha}: \zeta = c\eta \quad \rightarrow \quad \zeta = 0. \quad (7.47)$$

The combination

$$R_h = \eta R \quad (7.48)$$

survives in the first ζ -coefficient of the reduced Chern-Simons gauge field at the double pole. At the same time, the modified classical Yang-Baxter equation degenerates to the homogeneous classical Yang-Baxter equation, and the localized two-dimensional theory becomes the homogeneous Yang-Baxter sigma model. In the next section, we reproduce the same result by following the other side of the diamond, namely by taking the homogeneous limit of IFT_4 and then reducing to two dimensions.

VIII. HOMOGENEOUS LIMIT AND SYMMETRY REDUCTION

In Sec. VII, we obtained the homogeneous Yang-Baxter sigma model by first reducing the twistor theory to four-dimensional Chern-Simons theory and then localizing the resulting Chern-Simons theory to two dimensions. We now follow the other side of the diamond. We take the homogeneous limit directly in the four-dimensional integrable field theory (2.34), and then reduce the resulting homogeneous IFT_4 to two dimensions. The final answer agrees with Eq. (7.39).

A. Homogeneous limit of the four-dimensional field theory

We start from the four-dimensional action before the homogeneous limit, Eq. (2.34),

$$\begin{aligned} S_{\text{IFT}_4} &= \frac{K}{\langle \alpha \tilde{\alpha} \rangle} \int_{\mathbb{E}^4} \text{vol}_4 \text{Tr} [U_+(Pj - \sigma \tilde{j})(\hat{j} - \Lambda^T \tilde{j}) \\ &\quad - U_-(P\hat{j} - \sigma^{-1} \tilde{j})(j - \Lambda^T \tilde{j})] + S_{\text{WZ}_4}. \end{aligned} \quad (8.1)$$

Here, P , $U_{\pm}$ and Λ are defined in Eq. (2.27). In the Yang-Baxter specialization,

$$P = \frac{R - c}{R + c}, \quad U_{\pm} = (P - \sigma^{\pm 1} \Lambda)^{-1}, \quad \Lambda = \text{Ad}_h^{-1} \text{Ad}_h. \quad (8.2)$$

The currents $j, \hat{j}, \tilde{j}, \hat{\tilde{j}}$ are those in Eq. (2.28).

We now use the same coalescence data as in Sec. VII. Thus, the two simple-pole locations are $\zeta = -a$ and $\zeta = a$, with $a = c\eta$, and the coalesced point is denoted by α_0 . In spinor notation, this is

$$\alpha_{A'} = \alpha_{0A'} - a\beta_{A'}, \quad \tilde{\alpha}_{A'} = \alpha_{0A'} + a\beta_{A'}. \quad (8.3)$$

After raising the primed index,

$$\alpha^{A'} = \alpha_0^{A'} - a\beta^{A'}, \quad \tilde{\alpha}^{A'} = \alpha_0^{A'} + a\beta^{A'}. \quad (8.4)$$

The parameter σ is given by Eq. (7.8), and therefore

$$\sigma = 1 + 2a + O(a^2), \quad \sigma^{-1} = 1 - 2a + O(a^2). \quad (8.5)$$

The homogeneous Yang-Baxter operator is the fixed combination (7.15),

$$R_h = \eta R = \frac{a}{c} R. \quad (8.6)$$

Equivalently,

$$R = \frac{c}{a} R_h, \quad P = (R_h - a)(R_h + a)^{-1}. \quad (8.7)$$

The operator R_h is skew symmetric and satisfies the homogeneous classical Yang-Baxter equation (7.36).

The standard homogeneous Yang-Baxter model arises from the diagonal coalescence branch,

$$h = \tilde{h}. \quad (8.8)$$

On this branch,

$$\Lambda = 1, \quad \Lambda^T = 1, \quad (8.9)$$

so that

$$U_+ = (P - \sigma)^{-1}, \quad U_- = (P - \sigma^{-1})^{-1}. \quad (8.10)$$

This branch is the four-dimensional counterpart of the one-field homogeneous Yang-Baxter model. More general double-pole limits may keep extra edge data, but they are not needed for the standard homogeneous model.

Using Eq. (8.4), the currents have the following small- a expansion:

$$\begin{aligned} j &= j_0 - aj_{\beta}, & \tilde{j} &= j_0 + aj_{\beta}, \\ \hat{j} &= \hat{j}_0 - a\hat{j}_{\beta}, & \hat{\tilde{j}} &= \hat{j}_0 + a\hat{j}_{\beta}, \end{aligned} \quad (8.11)$$

where

$$\begin{aligned} j_0 &= \mu^A \alpha_0^{A'} h^{-1} \partial_{AA'} h, & j_{\beta} &= \mu^A \beta^{A'} h^{-1} \partial_{AA'} h, \\ \hat{j}_0 &= \hat{\mu}^A \alpha_0^{A'} h^{-1} \partial_{AA'} h, & \hat{j}_{\beta} &= \hat{\mu}^A \beta^{A'} h^{-1} \partial_{AA'} h. \end{aligned} \quad (8.12)$$

Therefore,

$$j - \tilde{j} = -2aj_{\beta}, \quad \hat{j} - \hat{\tilde{j}} = -2a\hat{j}_{\beta}. \quad (8.13)$$

At the same time,

$$\langle \alpha \tilde{\alpha} \rangle = 2a. \quad (8.14)$$

Thus, the factor $1/\langle \alpha \tilde{\alpha} \rangle$ in Eq. (8.1) is compensated by the difference of the two currents. This cancellation gives a well-defined homogeneous four-dimensional action.

We now evaluate the two combinations appearing in Eq. (8.1). First, define

$$B_+(a) = U_+(Pj - \sigma \tilde{j}). \quad (8.15)$$

Using Eq. (8.7), this equation is equivalent to

$$[(R_h - a) - \sigma(R_h + a)]B_+(a) = (R_h - a)j - \sigma(R_h + a)\tilde{j}. \quad (8.16)$$

Substituting Eqs. (8.5) and (8.11), and keeping the first nontrivial order in a , gives

$$B_+(a) \rightarrow b_h = j_0 + R_h(1 + R_h)^{-1}j_{\beta}. \quad (8.17)$$

Similarly, define

$$B_-(a) = U_-(P\hat{j} - \sigma^{-1}\hat{\tilde{j}}). \quad (8.18)$$

Using Eq. (8.7), this becomes

$$\begin{aligned} &[(R_h - a) - \sigma^{-1}(R_h + a)]B_-(a) \\ &= (R_h - a)\hat{j} - \sigma^{-1}(R_h + a)\hat{\tilde{j}}. \end{aligned} \quad (8.19)$$

Substituting Eqs. (8.5) and (8.11) gives

$$B_-(a) \rightarrow \hat{b}_h = \hat{j}_0 + R_h(1 - R_h)^{-1}\hat{j}_{\beta}. \quad (8.20)$$

Using Eqs. (8.13), (8.14), (8.17), and (8.20), the kinetic part of the action becomes

$$S_{\text{kin}}^{\text{hom}} = K \int_{\mathbb{E}^4} \text{vol}_4 \text{Tr} [\hat{b}_h j_{\beta} - b_h \hat{j}_{\beta}]. \quad (8.21)$$

The Wess-Zumino term also gives a well-defined homogeneous limit. On the diagonal branch,

$$[j, \hat{j}] - [\tilde{j}, \hat{\tilde{j}}] = -2a([j_\beta, \hat{j}_0] + [j_0, \hat{j}_\beta]). \quad (8.22)$$

Therefore,

$$S_{\text{WZ}}^{\text{hom}} = -K \int_{\mathbb{E}^4 \times [0,1]} \text{vol}_4 \wedge d\rho \text{Tr}[h^{-1} \partial_\rho h([j_\beta, \hat{j}_0] + [j_0, \hat{j}_\beta])]. \quad (8.23)$$

The homogeneous four-dimensional theory obtained from the IFT₄ side of the diamond is therefore

$$S_{\text{IFT}_4}^{\text{hom}} = S_{\text{kin}}^{\text{hom}} + S_{\text{WZ}}^{\text{hom}}. \quad (8.24)$$

B. Reduction to two dimensions

We now reduce Eq. (8.24) to two dimensions. The reduction is the same as in Sec. VA: the field is independent of the two directions removed by the reduction,

$$\partial_z h = 0, \quad \partial_{\bar{z}} h = 0. \quad (8.25)$$

The remaining coordinates are

$$\sigma^+ = w, \quad \sigma^- = \bar{w}, \quad (8.26)$$

and we define

$$J_+ = h^{-1} \partial_+ h, \quad J_- = h^{-1} \partial_- h, \\ \partial_+ = \partial_w, \quad \partial_- = \partial_{\bar{w}}. \quad (8.27)$$

For any fixed primed spinor $X^{A'}$, we use the spinor decomposition (4.9). Together with the reduction constraints, it gives

$$\mu^A X^{A'} h^{-1} \partial_{AA'} h \rightarrow \langle X \gamma \rangle J_+, \\ \hat{\mu}^A X^{A'} h^{-1} \partial_{AA'} h \rightarrow \langle X \hat{\gamma} \rangle J_-. \quad (8.28)$$

It is convenient to introduce

$$A = \langle \alpha_0 \gamma \rangle, \quad B = \langle \beta \gamma \rangle, \\ C = \langle \alpha_0 \hat{\gamma} \rangle, \quad D = \langle \beta \hat{\gamma} \rangle. \quad (8.29)$$

Then

$$j_0 \longrightarrow A J_+, \quad j_\beta \longrightarrow B J_+, \\ \hat{j}_0 \longrightarrow C J_-, \quad \hat{j}_\beta \longrightarrow D J_-. \quad (8.30)$$

Thus,

$$b_h \rightarrow [A + B R_h (1 + R_h)^{-1}] J_+, \\ \hat{b}_h \rightarrow [C + D R_h (1 - R_h)^{-1}] J_-. \quad (8.31)$$

The kinetic term (8.21) reduces to

$$S_{\text{kin}}^{\text{hom}} \rightarrow K \int_{\Sigma} \text{vol}_2 \text{Tr}[J_+ \mathcal{M}_h J_-], \quad (8.32)$$

where the operator $\mathcal{M}_h$ is obtained by using $R_h' = -R_h$, as follows:

$$\mathcal{M}_h = (BC - AD) + 2BD R_h (1 - R_h)^{-1}. \quad (8.33)$$

The Wess-Zumino term reduces as

$$[j_\beta, \hat{j}_0] + [j_0, \hat{j}_\beta] \rightarrow (BC + AD)[J_+, J_-]. \quad (8.34)$$

Hence,

$$S_{\text{WZ}}^{\text{hom}} \rightarrow -K(BC + AD) \int_{\Sigma \times [0,1]} \text{vol}_2 \\ \wedge d\rho \text{Tr}[h^{-1} \partial_\rho h[J_+, J_-]]. \quad (8.35)$$

Equivalently, defining

$$\mathcal{L}_{\text{WZ}}(h) = - \int_0^1 d\rho \text{Tr}[h^{-1} \partial_\rho h[J_+, J_-]], \quad (8.36)$$

the reduced homogeneous action is

$$S_{\text{IFT}_2}^{\text{hom}} = K \int_{\Sigma} \text{vol}_2 \text{Tr}[J_+ \mathcal{M}_h J_-] \\ + K(BC + AD) \int_{\Sigma} \text{vol}_2 \mathcal{L}_{\text{WZ}}(h). \quad (8.37)$$

We finally impose the same zeroes as in Eq. (7.5). With the affine convention of Sec. VII, the required spinor products are

$$BC = 1, \quad AD = -1, \\ BD = 1, \quad BC + AD = 0. \quad (8.38)$$

The Wess-Zumino term therefore vanishes. The kinetic operator becomes

$$\mathcal{M}_h = (BC - AD) + 2BD R_h (1 - R_h)^{-1} \\ = 2 + 2R_h (1 - R_h)^{-1} \\ = 2(1 - R_h)^{-1}. \quad (8.39)$$

Substituting this into Eq. (8.37), we obtain

$$S_{\text{IFT}_2}^{\text{hom}} = 2K \int_{\Sigma} \text{vol}_2 \text{Tr}[J_+ (1 - R_h)^{-1} J_-]. \quad (8.40)$$

This is exactly the result obtained from the Chern-Simons route in Eq. (7.39). After the coupling redefinition $\kappa = 2K$, we get

$$S_{\text{YB}}^{\text{hom}} = \kappa \int_{\Sigma} \text{vol}_2 \text{Tr}[J_+(1 - R_h)^{-1} J_-]. \quad (8.41)$$

Thus, the homogeneous limit is compatible with both routes of the diamond. On the Chern-Simons side, it appears as the double-pole boundary condition (7.21). On the four-dimensional field-theory side, it appears as the one-field homogeneous action (8.24). After symmetry reduction, both routes give the same homogeneous Yang-Baxter sigma model.

IX. CONCLUSION

We have derived a new two-dimensional field theory with action (5.6) from 4D Chern-Simons theory and its associated twistor space Chern-Simons theory, in particular, by starting with the action (2.1) and boundary conditions (2.10). Its origin from 4D Chern-Simons theory suggests that is integrable, given that the 4D equations of motion include the flatness of a gauge field that can be interpreted as a Lax connection. It would be interesting to derive the flatness of the Lax connection predicted by 4D Chern-Simons theory explicitly using the equations of motion of the two-dimensional theory.

The two-dimensional field theory was shown to be related via symmetry reduction to a novel 4D field theory depending on a skew-symmetric operator, $\mathcal{O}$. We showed that by choosing the operator $\mathcal{O}$ to be a solution R of the modified classical Yang-Baxter equation, the 4D theory gains a semilocal symmetry. Symmetry reducing this 4D Yang-Baxter sigma model results in the familiar 2D Yang-Baxter sigma model associated with the modified classical Yang-Baxter sigma model. Finally, we showed how the homogenous Yang-Baxter sigma model associated with ordinary classical Yang-Baxter sigma model arises in the limit where simple poles in the meromorphic three-form of the twistor space Chern-Simons theory coalesce to become a double pole.

There are several future directions that may be pursued based on this work. Firstly, a natural generalization of our results would be the derivation of the Yang-Baxter sigma model with Wess-Zumino term from twistor space. Secondly, given that the twistor space origins of both the Yang-Baxter sigma model and λ -model [27] are known, it would be interesting to investigate the twistor space origin of the Poisson-Lie T-duality between these theories. Going beyond classical aspects, it may also be interesting to derive the quantization of both the 4D and 2D Yang-Baxter sigma models from twistor space Chern-Simons theory, leveraging known results on quantizing 6D holomorphic field theories by Costello [45].

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: TWISTOR CONVENTIONS

The twistor space, $\mathbb{PT}$, of complexified Minkowski space, $\mathbb{CM}^4$, is the total space of the holomorphic vector bundle

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \rightarrow \mathbb{CP}^1, \quad (\text{A1})$$

which can be endowed with homogeneous coordinates $Z^\alpha = (\omega^A, \pi_{A'})$ defined with respect to the equivalence relation $Z^\alpha \sim tZ^\alpha$ for $t \in \mathbb{C}^*$. Here, the unprimed (primed) indices $A, B, C, \dots$ ($A', B', C', \dots$) label elements of the right-handed (left-handed) spin bundle on $\mathbb{CM}^4$. In other words, we have $\omega^A = (\omega^0, \omega^1)$ and $\pi_{A'} = (\pi_{0'}, \pi_{1'})$, that are, respectively, coordinates on the base and fiber of Eq. (A1). The points $x^{AA'} \in \mathbb{CM}^4$ are in bijection with holomorphic lines, that is,

$$\iota_x: \mathbb{CP}_x^1 \hookrightarrow \mathbb{PT}, \quad \pi_{A'} \mapsto (\omega^A, \pi_{A'}) = (x^{AB'} \pi_{B'}, \pi_{A'}). \quad (\text{A2})$$

We shall impose the reality structure associated with 4D Euclidean space $\mathbb{E}^4$. In Euclidean signature, primed (unprimed) spinors are mapped to primed (unprimed) spinors; that is, for $\omega^A = (\omega^0, \omega^1)$ and $\pi_{A'} = (\pi_{0'}, \pi_{1'})$, we have

$$\omega^A \rightarrow \hat{\omega}^A = \left(-\overline{\omega^1}, \overline{\omega^0} \right) \quad \text{and} \quad \pi_{A'} \rightarrow \hat{\pi}_{A'} = (-\overline{\pi_{1'}}, \overline{\pi_{0'}}). \quad (\text{A3})$$

Having imposed the Euclidean reality structure, primed indices label elements of the left-handed spin bundle over $\mathbb{E}^4$, while unprimed indices label elements of the right-handed spin bundle over $\mathbb{E}^4$.

Inner products for left-handed and right-handed spinors shall be defined respectively as

$$\begin{aligned} \langle \pi \hat{\pi} \rangle &= \pi^{A'} \hat{\pi}_{A'} = \varepsilon_{A'B'} \pi^{A'} \hat{\pi}^{B'} = \varepsilon^{A'B'} \pi_{B'} \hat{\pi}_{A'}, \\ [\omega \hat{\omega}] &= \omega^A \hat{\omega}_A = \varepsilon_{AB} \omega^A \hat{\omega}^B = \varepsilon^{AB} \omega_B \hat{\omega}_A. \end{aligned} \quad (\text{A4})$$

We utilize a frame of holomorphic (0,1)-forms adapted to the nonholomorphic coordinates $(x^{AA'}, \pi_{A'})$ on $\mathbb{PT}$, as follows:

$$\begin{aligned}\bar{e}^0 &= \frac{\langle d\hat{\pi} \hat{\pi} \rangle}{\|\pi\|^4} \in \Omega^{0,1}(\mathbb{P}\mathbb{T}, \mathcal{O}(-2)), \\ \hat{e}^A &= \frac{dx^{AA'} \hat{\pi}_{A'}}{\|\pi\|^2} \in \Omega^{0,1}(\mathbb{P}\mathbb{T}, \mathcal{O}(-1)),\end{aligned}\quad (\text{A5})$$

where $\|\pi\|^2 = \pi_{A'} \hat{\pi}^{A'}$. The dual frame of (0,1)-vectors is given by

$$\bar{\partial}_0 = \|\pi\|^2 \pi^{A'} \frac{\partial}{\partial \hat{\pi}^{A'}}, \quad \hat{\partial}_A = \pi^{A'} \frac{\partial}{\partial x^{AA'}} = \pi^{A'} \partial_{AA'}.\quad (\text{A6})$$

We shall also use

$$e^0 = \langle d\pi \pi \rangle.\quad (\text{A7})$$

APPENDIX B: SOME USEFUL IDENTITIES

We would like to first derive the identity

$$\text{Ad}_{h^{-1}}[B_A, B^A] = [\text{Ad}_{h^{-1}} B_A, \text{Ad}_{h^{-1}} B^A].\quad (\text{B1})$$

This is simply the statement that the adjoint action defines a Lie-algebra automorphism, i.e.,

$$\text{Ad}_h([X, Y]) = [\text{Ad}_h X, \text{Ad}_h Y].\quad (\text{B2})$$

We begin by recalling the definition of the commutator,

$$[X, Y] := XY - YX.\quad (\text{B3})$$

Then

$$\begin{aligned}\text{Ad}_h[X, Y] &= h[X, Y]h^{-1} \\ &= h(XY - YX)h^{-1} \\ &= hXYh^{-1} - hYXh^{-1}.\end{aligned}\quad (\text{B4})$$

We now insert the identity $1 = hh^{-1}$ at appropriate places in order to factor each term, as follows:

$$hXYh^{-1} = (hXh^{-1})(hYh^{-1}),\quad (\text{B5})$$

$$hYXh^{-1} = (hYh^{-1})(hXh^{-1}).\quad (\text{B6})$$

Therefore, Eq. (B2) follows from

$$\begin{aligned}\text{Ad}_h[X, Y] &= (hXh^{-1})(hYh^{-1}) - (hYh^{-1})(hXh^{-1}) \\ &= [hXh^{-1}, hYh^{-1}] \\ &= [\text{Ad}_h X, \text{Ad}_h Y].\end{aligned}\quad (\text{B7})$$

Next, we would like to compute $\partial_{AA'}(\text{Ad}_h X)$. Now, taking a derivative of $\text{Ad}_h X$, we obtain

$$\begin{aligned}\partial_{AA'}(hXh^{-1}) &= (\partial_{AA'} h)Xh^{-1} + h(\partial_{AA'} X)h^{-1} \\ &\quad + hX(\partial_{AA'} h^{-1}).\end{aligned}\quad (\text{B8})$$

Using the basic identity $\partial_{AA'} h^{-1} = -h^{-1}(\partial_{AA'} h)h^{-1}$, this expression can be simplified to

$$\begin{aligned}\partial_{AA'}(hXh^{-1}) &= (\partial_{AA'} h)Xh^{-1} + h(\partial_{AA'} X)h^{-1} - hXh^{-1}(\partial_{AA'} h)h^{-1} \\ &= h(h^{-1}(\partial_{AA'} h)X + \partial_{AA'} X - Xh^{-1}(\partial_{AA'} h)h^{-1}) \\ &= h(\partial_{AA'} X + [h^{-1}\partial_{AA'} h, X])h^{-1} \\ &= h(D_{AA'} X)h^{-1}.\end{aligned}\quad (\text{B9})$$

Here, we defined $D_{AA'} X = \partial_{AA'} X + [h^{-1}\partial_{AA'} h, X]$. Thus, we obtain

$$\partial_{AA'}(\text{Ad}_h X) = \text{Ad}_h(D_{AA'} X).\quad (\text{B10})$$

Applying Eq. (B10) to B^A , we obtain

$$\begin{aligned}\partial_{AA'} B^A &= \partial_{AA'}(\text{Ad}_h(\hat{b}\mu^A - b\hat{\mu}^A)) \\ &= \text{Ad}_h(D_{AA'}(\hat{b}\mu^A - b\hat{\mu}^A)).\end{aligned}\quad (\text{B11})$$

Contracting with $\alpha^{A'}$, we obtain

$$\alpha^{A'} \partial_{AA'} B^A = \text{Ad}_h(\alpha^{A'} D_{AA'}(\hat{b}\mu^A - b\hat{\mu}^A)).\quad (\text{B12})$$

Therefore,

$$\text{Ad}_{h^{-1}}(\alpha^{A'} \partial_{AA'} B^A) = \alpha^{A'} D_{AA'}(\hat{b}\mu^A - b\hat{\mu}^A).\quad (\text{B13})$$

We now expand the rhs using the definition of $D_{AA'}$, as follows:

$$\begin{aligned}\alpha^{A'} D_{AA'}(\hat{b}\mu^A - b\hat{\mu}^A) &= \alpha^{A'} \partial_{AA'}(\hat{b}\mu^A - b\hat{\mu}^A) + \alpha^{A'} [h^{-1}\partial_{AA'} h, \hat{b}\mu^A - b\hat{\mu}^A].\end{aligned}\quad (\text{B14})$$

Since the spinors μ^A and $\hat{\mu}^A$ are constant, the ordinary derivative acts only on b and $\hat{b}$, as follows:

$$\alpha^{A'} \partial_{AA'}(\hat{b}\mu^A - b\hat{\mu}^A) = \mu^A \alpha^{A'} \partial_{AA'} \hat{b} - \hat{\mu}^A \alpha^{A'} \partial_{AA'} b.\quad (\text{B15})$$

For the commutator term, we similarly obtain

$$\begin{aligned}\alpha^{A'} [h^{-1}\partial_{AA'} h, \hat{b}\mu^A - b\hat{\mu}^A] &= \mu^A \alpha^{A'} [h^{-1}\partial_{AA'} h, \hat{b}] - \hat{\mu}^A \alpha^{A'} [h^{-1}\partial_{AA'} h, b].\end{aligned}\quad (\text{B16})$$

Altogether, Eq. (B13) becomes

$$\text{Ad}_{h^{-1}}(\alpha^{A'} \partial_{AA'} B^A) = \mu^A \alpha^{A'} \partial_{AA'} \hat{b} - \hat{\mu}^A \alpha^{A'} \partial_{AA'} b\quad (\text{B17})$$

$$+\mu^A\alpha^{A'}[h^{-1}\partial_{AA'}h, \hat{b}] - \hat{\mu}^A\alpha^{A'}[h^{-1}\partial_{AA'}h, b]. \quad (\text{B18})$$

We now rewrite the final two terms using the currents j and $\hat{j}$. By the definition given in Eq. (2.28) for j and $\hat{j}$, we seek an explicit decomposition of $\alpha^{A'}h^{-1}\partial_{AA'}h$ in the $\mu, \hat{\mu}$ basis. Accordingly, we write

$$\frac{1}{\langle\alpha\beta\rangle}(\alpha^{A'}h^{-1}\partial_{AA'}h) = X\mu_A + Y\hat{\mu}_A, \quad (\text{B19})$$

for some Lie-algebra-valued coefficients X and Y (with no spinor indices). Contracting with μ^A and $\hat{\mu}^A$ gives

$$\frac{1}{\langle\alpha\beta\rangle}(\alpha^{A'}h^{-1}\partial_{AA'}h) = j\hat{\mu}_A - \hat{j}\mu_A. \quad (\text{B20})$$

Then, we find

$$\begin{aligned} \text{Ad}_{h^{-1}}(\alpha^{A'}\partial_{AA'}B^A) \\ = \mu^A\alpha^{A'}\partial_{AA'}\hat{b} - \hat{\mu}^A\alpha^{A'}\partial_{AA'}b + \langle\alpha\beta\rangle\mu^A[j\hat{\mu}_A - \hat{j}\mu_A, \hat{b}] \end{aligned} \quad (\text{B21})$$

$$\begin{aligned} - \langle\alpha\beta\rangle\hat{\mu}^A[j\hat{\mu}_A - \hat{j}\mu_A, b] \\ = \mu^A\alpha^{A'}\partial_{AA'}\hat{b} - \hat{\mu}^A\alpha^{A'}\partial_{AA'}b + \langle\alpha\beta\rangle[j, \hat{b}] - \langle\alpha\beta\rangle[\hat{j}, b]. \end{aligned} \quad (\text{B22})$$

APPENDIX C: PROOF OF EQ. (5.25)

We would like to rewrite the operator $\mathcal{K} = (1 - \sigma P)^{-1}(1 - P)$ in terms of R . We proceed in two steps: first compute $1 - P$ in terms of R , and then compute $1 - \sigma P$ in terms of R . Starting from $P = (R - c)(R + c)^{-1}$, we compute

$$1 - P = 1 - \frac{R - c}{R + c} = \frac{2c}{R + c}, \quad (\text{C1})$$

$$1 - \sigma P = \frac{((1 - \sigma)R + (1 + \sigma)c)}{(R + c)}. \quad (\text{C2})$$

Inverting $1 - \sigma P$, we get

$$\mathcal{K} = \frac{2c}{((1 - \sigma)R + (1 + \sigma)c)}. \quad (\text{C3})$$

We intend to write $\mathcal{N} = (1 - \sigma)R + (1 + \sigma)c$ as a scalar multiple of $(1 - \eta R)$. So $\mathcal{N} = (1 + \sigma)c(1 + \frac{1 - \sigma}{(1 + \sigma)c}R)$. We want to identify the expression in the parenthesis with $1 - \eta R$, where η is the deformation parameter of the Yang-Baxter sigma model. We therefore define η by

$$1 - \eta R = 1 + \frac{1 - \sigma}{(1 + \sigma)c}R \Rightarrow \eta = \frac{\sigma - 1}{(\sigma + 1)c}. \quad (\text{C4})$$

Equivalently, we solve for σ in terms of η , as follows:

$$\begin{aligned} \eta c = \frac{\sigma - 1}{\sigma + 1} &\Rightarrow \sigma(1 - \eta c) = 1 + \eta c \\ &\Rightarrow \sigma = \frac{1 + \eta c}{1 - \eta c}. \end{aligned} \quad (\text{C5})$$

We find

$$\mathcal{N}^{-1} = \frac{1}{(1 + \sigma)c}(1 - \eta R)^{-1}. \quad (\text{C6})$$

Plugging into Eq. (C3), we obtain

$$\begin{aligned} (1 - \sigma P)^{-1}(1 - P) &= 2c \cdot \frac{1}{(1 + \sigma)c}(1 - \eta R)^{-1} \\ &= \frac{2}{1 + \sigma}(1 - \eta R)^{-1}. \end{aligned} \quad (\text{C7})$$

Thus, the identity we require is

$$(1 - \sigma P)^{-1}(1 - P) = \frac{2}{1 + \sigma}(1 - \eta R)^{-1}, \quad \eta = \frac{\sigma - 1}{(\sigma + 1)c}. \quad (\text{C8})$$

Or equivalently (solving for σ), we have

$$(1 - \sigma P)^{-1}(1 - P) = (1 - \eta c)(1 - \eta R)^{-1}, \quad \sigma = \frac{1 + \eta c}{1 - \eta c}. \quad (\text{C9})$$

APPENDIX D: GAUGE TRANSFORMATIONS FOR A GENERAL OPERATOR $\mathcal{O}$

We first discuss the gauge symmetry of the two-pole boundary condition for a general linear operator,

$$\mathcal{O}: \mathfrak{g} \rightarrow \mathfrak{g}. \quad (\text{D1})$$

The Yang-Baxter case is obtained only after the specialization $\mathcal{O} = R$. Keeping $\mathcal{O}$ general is useful because it makes clear which algebraic properties are required for the boundary condition to be preserved by gauge transformations.

The reduced four-dimensional Chern-Simons boundary condition has the form

$$(\mathcal{O} - c)A_i|_{\zeta=\alpha_+} = (\mathcal{O} + c)A_i|_{\zeta=\alpha_-}, \quad i = w, \bar{w}. \quad (\text{D2})$$

Here, α_+ and α_- denote the two simple poles of the four-dimensional Chern-Simons one-form. The finite gauge transformation of the Chern-Simons connection is

$$A_i \mapsto A_i^u = u^{-1}\partial_i u + u^{-1}A_i u. \quad (\text{D3})$$

This is the same convention used later in the parametrization

$$A_I = \hat{h}^{-1} \mathcal{L}_I \hat{h} + \hat{h}^{-1} \partial_I \hat{h}, \quad I = w, \bar{w}. \quad (\text{D4})$$

Indeed, this parametrization is obtained by applying a finite gauge transformation to the meromorphic connection $\mathcal{L}_I$ with group element $\hat{h}^{-1}$.

Let

$$u_+ = u|_{\zeta=\alpha_+}, \quad u_- = u|_{\zeta=\alpha_-}. \quad (\text{D5})$$

After the gauge transformation, the boundary condition becomes

$$\begin{aligned} & (\mathcal{O} - c)(u_+^{-1} \partial_i u_+ + u_+^{-1} A_i|_{\zeta=\alpha_+} u_+) \\ &= (\mathcal{O} + c)(u_-^{-1} \partial_i u_- + u_-^{-1} A_i|_{\zeta=\alpha_-} u_-). \end{aligned} \quad (\text{D6})$$

Thus, there are two parts to check. The first is the Maurer-Cartan part $u_{\pm}^{-1} \partial_i u_{\pm}$. The second is the conjugated gauge-field part $u_{\pm}^{-1} A_i u_{\pm}$.

The original boundary condition (D2) can be solved by writing the pair of boundary values as

$$A_i|_{\zeta=\alpha_+} = (\mathcal{O} + c)x_i, \quad A_i|_{\zeta=\alpha_-} = (\mathcal{O} - c)x_i, \quad (\text{D7})$$

for some $x_i \in \mathfrak{g}$. This parametrization solves the boundary condition because

$$(\mathcal{O} - c)A_i|_{\zeta=\alpha_+} = (\mathcal{O} - c)(\mathcal{O} + c)x_i, \quad (\text{D8})$$

$$(\mathcal{O} + c)A_i|_{\zeta=\alpha_-} = (\mathcal{O} + c)(\mathcal{O} - c)x_i, \quad (\text{D9})$$

and the two operators commute since both are polynomials in $\mathcal{O}$. Thus the boundary condition selects the subspace

$$\mathfrak{g}_{\mathcal{O}} = \{((\mathcal{O} + c)x, (\mathcal{O} - c)x) | x \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{g}. \quad (\text{D10})$$

For the transformed boundary condition to have the same form, the boundary values of the gauge transformation must lie in some corresponding group $G_{\mathcal{O}}$. Infinitesimally, this means that the Maurer-Cartan pair is tangent to $\mathfrak{g}_{\mathcal{O}}$, as follows:

$$u_+^{-1} \partial_i u_+ = (\mathcal{O} + c)y_i, \quad u_-^{-1} \partial_i u_- = (\mathcal{O} - c)y_i, \quad (\text{D11})$$

for some $y_i \in \mathfrak{g}$. Then

$$(\mathcal{O} - c)u_+^{-1} \partial_i u_+ = (\mathcal{O} - c)(\mathcal{O} + c)y_i, \quad (\text{D12})$$

$$(\mathcal{O} + c)u_-^{-1} \partial_i u_- = (\mathcal{O} + c)(\mathcal{O} - c)y_i, \quad (\text{D13})$$

and these two expressions are equal. Therefore, the Maurer-Cartan part of Eq. (D6) preserves the boundary condition.

It remains to check the conjugated gauge-field part. If the original boundary pair lies in $\mathfrak{g}_{\mathcal{O}}$, then the conjugated pair will remain in $\mathfrak{g}_{\mathcal{O}}$ provided $\mathfrak{g}_{\mathcal{O}}$ is stable under the adjoint action of $G_{\mathcal{O}}$. This is guaranteed if $\mathfrak{g}_{\mathcal{O}}$ is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$. We therefore compute the closure condition.

Take two elements of $\mathfrak{g}_{\mathcal{O}}$, as follows:

$$\begin{aligned} X_{\mathcal{O}} &= ((\mathcal{O} + c)X, (\mathcal{O} - c)X), \\ Y_{\mathcal{O}} &= ((\mathcal{O} + c)Y, (\mathcal{O} - c)Y). \end{aligned} \quad (\text{D14})$$

Their commutator in $\mathfrak{g} \oplus \mathfrak{g}$ is

$$[X_{\mathcal{O}}, Y_{\mathcal{O}}] = \left([(\mathcal{O} + c)X, (\mathcal{O} + c)Y], [(\mathcal{O} - c)X, (\mathcal{O} - c)Y] \right). \quad (\text{D15})$$

For closure, this must again be of the form

$$((\mathcal{O} + c)Z, (\mathcal{O} - c)Z) \quad (\text{D16})$$

for some $Z \in \mathfrak{g}$. Hence, we require

$$[(\mathcal{O} + c)X, (\mathcal{O} + c)Y] = (\mathcal{O} + c)Z, \quad (\text{D17})$$

$$[(\mathcal{O} - c)X, (\mathcal{O} - c)Y] = (\mathcal{O} - c)Z. \quad (\text{D18})$$

We now expand both components explicitly. The first component gives

$$\begin{aligned} [(\mathcal{O} + c)X, (\mathcal{O} + c)Y] &= [\mathcal{O}X + cX, \mathcal{O}Y + cY] \\ &= [\mathcal{O}X, \mathcal{O}Y] + c[\mathcal{O}X, Y] \\ &\quad + c[X, \mathcal{O}Y] + c^2[X, Y]. \end{aligned} \quad (\text{D19})$$

The second component gives

$$\begin{aligned} [(\mathcal{O} - c)X, (\mathcal{O} - c)Y] &= [\mathcal{O}X - cX, \mathcal{O}Y - cY] \\ &= [\mathcal{O}X, \mathcal{O}Y] - c[\mathcal{O}X, Y] \\ &\quad - c[X, \mathcal{O}Y] + c^2[X, Y]. \end{aligned} \quad (\text{D20})$$

The terms proportional to c fix Z uniquely as

$$Z = [\mathcal{O}X, Y] + [X, \mathcal{O}Y]. \quad (\text{D21})$$

Substituting this into $(\mathcal{O} \pm c)Z$, we get

$$\begin{aligned} (\mathcal{O} + c)Z &= \mathcal{O}([\mathcal{O}X, Y] + [X, \mathcal{O}Y]) \\ &\quad + c([\mathcal{O}X, Y] + [X, \mathcal{O}Y]), \end{aligned} \quad (\text{D22})$$

$$\begin{aligned} (\mathcal{O} - c)Z &= \mathcal{O}([\mathcal{O}X, Y] + [X, \mathcal{O}Y]) \\ &\quad - c([\mathcal{O}X, Y] + [X, \mathcal{O}Y]). \end{aligned} \quad (\text{D23})$$

Comparing with Eqs. (D19) and (D20), the closure condition is

$$[\mathcal{O}X, \mathcal{O}Y] + c^2[X, Y] = \mathcal{O}([\mathcal{O}X, Y] + [X, \mathcal{O}Y]). \quad (\text{D24})$$

Equivalently,

$$[\mathcal{O}X, \mathcal{O}Y] - \mathcal{O}([\mathcal{O}X, Y] + [X, \mathcal{O}Y]) = -c^2[X, Y]. \quad (\text{D25})$$

Thus $\mathfrak{g}_{\mathcal{O}}$ is a Lie subalgebra precisely when $\mathcal{O}$ satisfies the modified classical Yang-Baxter equation.

There is also an isotropy requirement from the boundary term in the Chern-Simons variation. The relevant pairing on $\mathfrak{g} \oplus \mathfrak{g}$ is the difference of the two trace pairings. For two elements of $\mathfrak{g}_{\mathcal{O}}$, this gives

$$\begin{aligned} & \text{Tr}[(\mathcal{O} + c)X(\mathcal{O} + c)Y] - \text{Tr}[(\mathcal{O} - c)X(\mathcal{O} - c)Y] \\ &= 2c \text{Tr}[\mathcal{O}XY + X\mathcal{O}Y]. \end{aligned} \quad (\text{D26})$$

This vanishes if

$$\mathcal{O}^t = -\mathcal{O}, \quad (\text{D27})$$

where the transpose is defined with respect to the invariant trace pairing. Thus the boundary condition is compatible with the Chern-Simons variational principle and finite gauge transformations when

$$\begin{aligned} & \mathcal{O}^t = -\mathcal{O}, \\ & [\mathcal{O}X, \mathcal{O}Y] - \mathcal{O}([\mathcal{O}X, Y] + [X, \mathcal{O}Y]) = -c^2[X, Y]. \end{aligned} \quad (\text{D28})$$

If a general operator $\mathcal{O}$ does not satisfy these conditions, the boundary values do not define a stable boundary algebra. In that case, the boundary condition is not gauge invariant.

The Yang-Baxter case used in the main construction is obtained by setting

$$\mathcal{O} = \mathcal{R}. \quad (\text{D29})$$

Then Eq. (D25) becomes the modified classical Yang-Baxter equation for R , and the closure identity may be written as

$$\begin{aligned} & [(R \pm c)X, (R \pm c)Y] \\ &= (R \pm c)([RX, Y] + [X, RY]). \end{aligned} \quad (\text{D30})$$

This is the form used in the finite gauge-transformation argument. It shows that the images of $R + c$ and $R - c$ are Lie subalgebras, and therefore the finite boundary values of the allowed gauge transformations form the corresponding Yang-Baxter subgroup.

1. The same general operator in six dimensions

We now explain how the same algebraic condition appears before symmetry reduction, directly in the six-dimensional twistor theory. The six-dimensional boundary conditions for a general operator $\mathcal{O}$ are

$$(\mathcal{O} - c)[\mathcal{A}\mu]_{\pi=\alpha} = \sigma \frac{\langle \alpha\beta \rangle}{\langle \tilde{\alpha}\beta \rangle} (\mathcal{O} + c)[\mathcal{A}\mu]_{\pi=\tilde{\alpha}}, \quad (\text{D31})$$

$$(\mathcal{O} - c)[\mathcal{A}\hat{\mu}]_{\pi=\alpha} = \sigma^{-1} \frac{\langle \alpha\beta \rangle}{\langle \tilde{\alpha}\beta \rangle} (\mathcal{O} + c)[\mathcal{A}\hat{\mu}]_{\pi=\tilde{\alpha}}. \quad (\text{D32})$$

We take the same choice of σ as in the Yang-Baxter construction, so that after symmetry reduction the spinor factors convert Eqs. (D31) and (D32) into the four-dimensional boundary condition discussed above.

The algebraic part of the six-dimensional condition has the same structure as in CS_4 . In the μ component, the allowed boundary values may be written as

$$[\mathcal{A}\mu]_{\pi=\alpha} = \langle \alpha\gamma \rangle (\mathcal{O} + c)x, \quad [\mathcal{A}\mu]_{\pi=\tilde{\alpha}} = \langle \tilde{\alpha}\gamma \rangle (\mathcal{O} - c)x. \quad (\text{D33})$$

Similarly, the hatted component may be written as

$$[\mathcal{A}\hat{\mu}]_{\pi=\alpha} = \langle \alpha\hat{\gamma} \rangle (\mathcal{O} + c)y, \quad [\mathcal{A}\hat{\mu}]_{\pi=\tilde{\alpha}} = \langle \tilde{\alpha}\hat{\gamma} \rangle (\mathcal{O} - c)y. \quad (\text{D34})$$

The spinor factors only keep track of which component is being reduced to A_w or $A_{\bar{w}}$. The Lie-algebraic content is still the pair

$$((\mathcal{O} + c)x, (\mathcal{O} - c)x). \quad (\text{D35})$$

Therefore, the conjugation part of a six-dimensional gauge transformation preserves the algebraic form of the boundary values if and only if $\mathfrak{g}_{\mathcal{O}}$ is a Lie subalgebra. As shown above, this is equivalent to the modified classical Yang-Baxter equation (D25).

The new feature in six dimensions is the derivative part of the gauge transformation. The six-dimensional connection transforms as

$$A \mapsto A^u = u^{-1} A u + u^{-1} \bar{\partial} u. \quad (\text{D36})$$

For the components along $\mathbb{E}^4$, this gives

$$\mathcal{A}_A \mapsto \mathcal{A}_A^u = u^{-1} \mathcal{A}_A u + u^{-1} \pi^{A'} \partial_{AA'} u. \quad (\text{D37})$$

Therefore,

$$[\mathcal{A}^u \mu]_{\pi=\alpha} = u_{\alpha}^{-1} [\mathcal{A}\mu]_{\pi=\alpha} u_{\alpha} + \mu^A \alpha^{A'} u_{\alpha}^{-1} \partial_{AA'} u_{\alpha}, \quad (\text{D38})$$

and similarly at $\pi = \tilde{\alpha}$. The first term is the conjugation part, controlled by the boundary algebra above. The second term contains derivatives along the four spacetime directions before symmetry reduction.

To see the effect of the derivative term, decompose a primed spinor in the $\gamma, \hat{\gamma}$ basis:

$$X^{A'} = \langle X\hat{\gamma} \rangle \gamma^{A'} - \langle X\gamma \rangle \hat{\gamma}^{A'}. \quad (\text{D39})$$

For the μ component, this gives

$$\begin{aligned} \mu^A \alpha^{A'} u_{\alpha}^{-1} \partial_{AA'} u_{\alpha} &= \langle \alpha \hat{\gamma} \rangle \mu^A \gamma^{A'} u_{\alpha}^{-1} \partial_{AA'} u_{\alpha} \\ &\quad - \langle \alpha \gamma \rangle \mu^A \hat{\gamma}^{A'} u_{\alpha}^{-1} \partial_{AA'} u_{\alpha}. \end{aligned} \quad (\text{D40})$$

The term with $\hat{\gamma}^{A'}$ is the piece that becomes the w derivative after symmetry reduction. The term with $\gamma^{A'}$ points along the direction removed by the reduction. In reduction coordinates,

$$\mu^A \gamma^{A'} \partial_{AA'} = \partial_{\bar{z}}, \quad \mu^A \hat{\gamma}^{A'} \partial_{AA'} = -\partial_w, \quad (\text{D41})$$

up to the chosen orientation convention for w . The ∂_w part is the derivative term seen by the reduced CS_4 connection. The $\partial_{\bar{z}}$ part has no counterpart in the CS_4 boundary condition. Thus, preservation of the six-dimensional μ boundary condition requires

$$\mu^A \gamma^{A'} \partial_{AA'} u|_{\pi=\alpha, \bar{\alpha}} = 0. \quad (\text{D42})$$

Repeating the same analysis for the hatted component gives the complementary condition

$$\hat{\mu}^A \hat{\gamma}^{A'} \partial_{AA'} u|_{\pi=\alpha, \bar{\alpha}} = 0. \quad (\text{D43})$$

In the same reduction coordinates,

$$\hat{\mu}^A \hat{\gamma}^{A'} \partial_{AA'} = \partial_z, \quad \hat{\mu}^A \gamma^{A'} \partial_{AA'} = \partial_{\bar{w}}. \quad (\text{D44})$$

Hence, the allowed six-dimensional gauge transformations must obey

$$\partial_{\bar{z}} u|_{\pi=\alpha, \bar{\alpha}} = 0, \quad \partial_z u|_{\pi=\alpha, \bar{\alpha}} = 0. \quad (\text{D45})$$

This is the origin of semilocality. The algebraic part of the symmetry is the same as in CS_4 : it is governed by the subgroup generated by $\mathfrak{g}_0$. However, before symmetry reduction, the gauge parameter also appears through derivatives along z and $\bar{z}$. These directions are removed in the reduction, and their derivative contributions must vanish for the six-dimensional boundary condition to be preserved. Thus the symmetry is not a fully local gauge symmetry of the four-dimensional IFT. It is local only along the directions which survive the reduction, while being constrained along the removed directions.

For a general operator $\mathcal{O}$, the six-dimensional boundary condition therefore has a semilocal symmetry only if the following conditions hold:

$$\mathcal{O}^\dagger = -\mathcal{O}, \quad (\text{D46})$$

$$[\mathcal{O}X, \mathcal{O}Y] - \mathcal{O}([\mathcal{O}X, Y] + [X, \mathcal{O}Y]) = -c^2[X, Y], \quad (\text{D47})$$

and

$$\mu^A \gamma^{A'} \partial_{AA'} u = 0, \quad \hat{\mu}^A \hat{\gamma}^{A'} \partial_{AA'} u = 0 \quad (\text{D48})$$

at the relevant pole values. The first two conditions are algebraic conditions on $\mathcal{O}$. The last two conditions are differential conditions on the six-dimensional gauge parameter. If $\mathcal{O}$ is not of Yang-Baxter type, the algebraic boundary values do not close; if the differential conditions are not imposed, the six-dimensional derivative terms spoil the boundary condition.

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- [1] K. Costello, Supersymmetric gauge theory and the Yangian, [arXiv:1303.2632](#).
 - [2] K. Costello, E. Witten, and M. Yamazaki, Gauge theory and integrability, I, *ICCM Not.* **06**, 46 (2018).
 - [3] K. Costello, E. Witten, and M. Yamazaki, Gauge theory and integrability, II, *ICCM Not.* **06**, 120 (2018).
 - [4] K. Costello, D. Gaiotto, and J. Yagi, Q-operators are 't Hooft lines, *J. High Energy Phys.* **11** (2024) 003.
 - [5] K. Costello and M. Yamazaki, Gauge theory and integrability, III, [arXiv:1908.02289](#).
 - [6] M. Ashwinkumar, J.-i. Sakamoto, and M. Yamazaki, Dualities and discretizations of integrable quantum field theories from 4D Chern-Simons theory, *Adv. Theor. Math. Phys.* **29**, 1509 (2025).
 - [7] N. Levine, Equivalence of 1-loop RG flows in 4D Chern-Simons and integrable 2D sigma-models, *Phys. Lett. B* **862**, 139325 (2025).
 - [8] D. M. Schmidtt, A generalized 4d Chern-Simons theory, *J. High Energy Phys.* **11** (2023) 144.
 - [9] D. M. Schmidtt, Contact 4d Chern-Simons theory: Generalities, *J. High Energy Phys.* **12** (2025) 078.
 - [10] M. Ashwinkumar, M.-C. Tan, and Q. Zhao, Branes and categorifying integrable lattice models, *Adv. Theor. Math. Phys.* **24**, 1 (2020).
 - [11] K. Costello and J. Yagi, Unification of integrability in supersymmetric gauge theories, *Adv. Theor. Math. Phys.* **24**, 1931 (2020).
 - [12] M. Ashwinkumar and M.-C. Tan, Unifying lattice models, links and quantum geometric langlands via branes in string theory, *Adv. Theor. Math. Phys.* **24**, 1681 (2020).
 - [13] N. Ishtiaque, S. Farooq Moosavian, and Y. Zhou, Topological holography: The example of the D2-D4 brane system, *SciPost Phys.* **9**, 017 (2020).

- [14] M. Ashwinkumar, Integrable lattice models and holography, *J. High Energy Phys.* **02** (2021) 227.
- [15] J.-i. Sakamoto, R. Tateo, and M. Yamazaki, $T\bar{T}$ and root- $T\bar{T}$ deformations in four-dimensional Chern-Simons theory, *J. High Energy Phys.* **01** (2026) 084.
- [16] O. Fukushima, T. Matsumoto, and K. Yoshida, The Courant-Hilbert construction in 4D Chern-Simons theory, *J. High Energy Phys.* **01** (2026) 122.
- [17] O. Fukushima, T. Matsumoto, and K. Yoshida, Courant-Hilbert deformations of Yang-Baxter sigma models, *J. High Energy Phys.* **06** (2026) 084.
- [18] M. Benini, A. Schenkel, and B. Vicedo, The homological algebra of 2d integrable field theories.
- [19] J. Stedman, Hamiltonian analysis of doubled 4d Chern-Simons, [arXiv:2601.18647](#).
- [20] R. Bittleston, L. Mason, and S.F. Moosavian, The self-duality equations on a Riemann surface and four-dimensional Chern-Simons theory, [arXiv:2601.05309](#).
- [21] L.T. Cole and B. Hoare, Integrable models from 4d holomorphic BF theory, [arXiv:2512.15566](#).
- [22] R. Hamidi and B. Hoare, Twists of trigonometric sigma models, *J. High Energy Phys.* **08** (2025) 090.
- [23] L.J. Mason and N.M.J. Woodhouse, *Integrability, Self-Duality, and Twistor Theory* (Oxford University Press, New York, 1996).
- [24] R. Bittleston and D. Skinner, Twistors, the ASD Yang-Mills equations and 4d Chern-Simons theory, *J. High Energy Phys.* **02** (2023) 227.
- [25] R. F. Penna, Twistor actions for integrable systems, *J. High Energy Phys.* **09** (2021) 140.
- [26] Y.-J. He, J. Tian, and B. Chen, Deformed integrable models from holomorphic Chern-Simons theory, *Sci. China Phys. Mech. Astron.* **65**, 100413 (2022).
- [27] L. T. Cole, R. A. Cullinan, B. Hoare, J. Liniado, and D. C. Thompson, Integrable deformations from twistor space, *SciPost Phys.* **17**, 008 (2024).
- [28] L. T. Cole, R. A. Cullinan, B. Hoare, J. Liniado, and D. C. Thompson, Gauging the diamond: Integrable coset models from twistor space, *J. High Energy Phys.* **12** (2024) 202.
- [29] D. Chatzis, J.M. Marley, and D.C. Thompson, From diamond gaugings to dualisations, [arXiv:2512.17751](#).
- [30] C. Klimcik, Yang-Baxter sigma models and dS/AdS T duality, *J. High Energy Phys.* **12** (2002) 051.
- [31] C. Klimcik, On integrability of the Yang-Baxter sigma-model, *J. Math. Phys. (N.Y.)* **50**, 043508 (2009).
- [32] T. Matsumoto and K. Yoshida, Yang-Baxter sigma models based on the CYBE, *Nucl. Phys.* **B893**, 287 (2015).
- [33] F. Delduc, M. Magro, and B. Vicedo, An integrable deformation of the $\text{AdS}_5 \times S^5$ superstring action, *Phys. Rev. Lett.* **112**, 051601 (2014).
- [34] I. Kawaguchi, T. Matsumoto, and K. Yoshida, Jordanian deformations of the $\text{AdS}_5 \times S^5$ superstring, *J. High Energy Phys.* **04** (2014) 153.
- [35] F. Delduc, M. Magro, and B. Vicedo, Derivation of the action and symmetries of the q -deformed $\text{AdS}_5 \times S^5$ superstring, *J. High Energy Phys.* **10** (2014) 132.
- [36] S. J. van Tongeren, On classical Yang-Baxter based deformations of the $\text{AdS}_5 \times S^5$ superstring, *J. High Energy Phys.* **06** (2015) 048.
- [37] O. Fukushima, J.-i. Sakamoto, and K. Yoshida, Yang-Baxter deformations of the $\text{AdS}_5 \times S^5$ supercoset sigma model from 4D Chern-Simons theory, *J. High Energy Phys.* **09** (2020) 100.
- [38] L. Schepers and D.C. Thompson, Resurgence in the bi-Yang-Baxter model, *Nucl. Phys.* **B964**, 115308 (2021).
- [39] M. Ashwinkumar, D. Orlando, S. Reffert, and G. Sberveglieri, Asymptotics in the bi-Yang-Baxter sigma model, *Phys. Rev. D* **112**, 125023 (2025).
- [40] F. Delduc, S. Lacroix, M. Magro, and B. Vicedo, A unifying 2D action for integrable σ -models from 4D Chern-Simons theory, *Lett. Math. Phys.* **110**, 1645 (2020).
- [41] O. Fukushima, J.-i. Sakamoto, and K. Yoshida, Comments on η -deformed principal chiral model from 4D Chern-Simons theory, *Nucl. Phys.* **B957**, 115080 (2020).
- [42] V. P. Nair and J. Schiff, A Kahler-Chern-Simons theory and quantization of instanton moduli spaces, *Phys. Lett. B* **246**, 423 (1990).
- [43] V. P. Nair and J. Schiff, Kahler Chern-Simons theory and symmetries of antiselfdual gauge fields, *Nucl. Phys.* **B371**, 329 (1992).
- [44] C. Bassi and S. Lacroix, Integrable deformations of coupled σ -models, *J. High Energy Phys.* **05** (2020) 059.
- [45] K. J. Costello, Quantizing local holomorphic field theories on twistor space, [arXiv:2111.08879](#).