

Dynamical response of twin stars to perturbations

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If a strong first-order phase transition takes place at sufficiently high rest-mass densities in the equation of state (EOS) modeling compact stars, a new branch will appear in the mass-radius sequence of stable equilibria. This branch will be populated by stars comprising a quark-matter core and a hadronic-matter envelope, i.e., hybrid stars, which represent “twin-star” solutions to equilibria having the same mass but a fully hadronic EOS. While both branches are stable to linear perturbations, it is unclear which of the twin solutions is the “favored” one, that is, which of the two configurations is expected to be found in nature. We assess this point by performing a large campaign of general-relativistic simulations aimed at assessing the response of compact stars on the two branches to perturbations of various strength. In this way, we find that, independently of whether the stars populate the hadronic or the twin branch, their response is characterized by a critical-perturbation strength such that the star will oscillate on the original branch for subcritical perturbations and migrate to the neighboring branch for supercritical perturbations while conserving rest mass. Because the critical values are different for stars with the same rest mass but sitting on either branch, it is possible to define as favored the part of the branch that has the largest critical perturbation, thus correcting the common wisdom that stellar models on the twin branch are the favored ones. Interestingly, we show that the binding energies on the two branches can be used to deduce without simulations which of the stellar configurations is more likely to be found in nature.

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I. INTRODUCTION

Neutron stars (NSs) gain pressing priority when it comes to understanding the behavior of matter at high density and low temperatures [1–3]. These astrophysical objects are harnessed by immensely strong gravity, which allows them to compress the matter at the core well beyond the nuclear saturation density, $\rho_{\text{sat}} \simeq 2.7 \times 10^{14} \text{ g/cm}^3$. Quantum chromodynamics (QCD) predicts a phase transition (PT) from hadronic matter to quark matter in such conditions, which could take place when a binary system of neutron stars collides (see, e.g., [4–12]) or a supernova explosion takes place (see, e.g., [13–16]). This is a decades-old hypothesis that is yet to meet its complete theoretical

understanding with complementing experimental (and observational) evidence.

The equation of state (EOS) of the cold neutron-rich matter is sufficiently comprehended and mainly constrained by chiral effective field theory (see, e.g., [17,18]). Relativistic heavy-ion collider experiments [19–22] and their theoretical description (see, e.g., [23–25]) efficiently probe the matter at high temperatures and low densities, and perturbative QCD calculations [26–31] give reliable results only at asymptotically high densities. Hence, the NS cores, either when isolated or in binary systems, have become the promising candidates to probe the hadron-to-quark PT.

The onset of hadron-to-quark PT and the nature of this phenomenon are not known *a priori* [32–34], which leaves a plethora of possible hybrid EOSs that dictates the expression of PT inside the NS core [35,36]. Under these assumptions, alternatives exist to the traditional NSs that are described by purely hadronic EOSs. In particular, it is in principle possible to consider the existence of “hybrid stars,” that is, compact stars comprising a quark-matter core and an hadronic-matter envelope.

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In literature, PT has mainly been modeled as a transition using density discontinuity at a constant pressure (Maxwell construction) or a mixed-phase where hadrons and quarks coexist (Gibbs construction) [37]. Importantly, the presence of a PT allows for the existence of a new branch in the mass-radius sequence of stable equilibria corresponding to hybrid stars. Two equilibria are then possible for the same mass on either of the stable branches. Hereafter, we will refer to as the “twin branch” (TB) the set of more compact hybrid stars, i.e., the “twin stars,” and as the “hadronic branch” (HB) the set of stellar models that are less compact for the same mass (see also, e.g., [38–40]).

The bulk of the work produced over the last decade on twin stars is vast, and it is difficult to summarize it properly (the vast number of directions pursued in the study of twin stars that has been recently reviewed by Ref. [41] and Refs. [90, 105–159] therein are particularly relevant for our results). Despite this rich literature, the rather fundamental issue of the “degeneracy of states” in twin stars has not been addressed yet. Stated differently, given the possible existence of two equilibria for the same gravitational mass, *which of these two equilibria is the favored one?* Since equilibrium models are unable to break this degeneracy, we here address, this simple and yet challenging question by studying for the first time the response of twin and hadronic stars to nonlinear perturbations. Using general-relativistic hydrodynamical simulations we establish under what conditions a star on one of the two branches will “migrate” to the neighboring branch to seek a new and favored equilibrium. Because these migrations are not symmetric but reflect the different binding energies of the equilibria on the two branches, we provide a simple and intuitive criterion to determine which of the two solutions is the favored one. More in general, this study improves the long-standing question on the nonlinear stability of equal-mass hydrostatic solutions of compact stars in general relativity.

II. METHODS

A. Equation of state

A strong PT in EOS is identified by a region with vanishing speed of sound, formed due to a jump in the energy density (or rest-mass density) keeping the transition pressure constant [37]. This region defines a Maxwell-type first-order PT separating lower densities and higher

densities given by hadronic-matter and quark-matter EOSs, respectively. When performing simulations in general-relativistic hydrodynamics, however, a vanishingly small speed of sound in the PT region needs special care to avoid an artifact in the numerical evolutions [42–46]. A simple solution to this problem consists of constructing the EOS such that the pressure is not exactly constant in the PT region, and hence the speed of sound is very small but nonzero.

As a generic representative of an EOS leading to a twin-star configuration, we here consider the piecewise polytropic prescription (see, e.g., [47]) adopted in Table 1 of [48] where additional information can be found. This study used the EOS for general-relativistic hydrodynamic simulations of isolated hybrid stars, guaranteeing that the initial value problem remains well posed when the PT region is included in the stellar profile. The six segments of the piecewise polytropic prescription employed here are listed in Table I, where the crust is described by the polytropes $i = 1, 2$, the hadronic part by the polytropes $i = 3$, the PT region by $i = 4$, and, finally, the quark region is described by the polytropes $i = 5, 6$ (note that because of our simplified treatment of the crust, we use here only six of the eight segments presented by [48]).

The sequences of stellar equilibrium models for the EOS considered are reported in Fig. 1 in terms of the gravitational mass M and radius R , with the gray-shaded area marking the “twin region,” that is, the region in the mass-radius diagrams where the HB and the TB coexist in terms of rest masses. Within the twin region, it is possible to use the turning-point criterion $dM/d\rho_c > 0$, where ρ_c is the central rest-mass density of the star to distinguish stable stellar models (solid lines) from the unstable ones (dashed lines); see [49] for a discussion of the validity of this criterion for rotating and nonrotating stars. We should note that a very similar picture would be obtained if the gravitational mass M in Fig. 1 were to be replaced by the rest mass M_b . This is because for these low-mass stars the two masses scale essentially linearly [see, e.g., Fig. 3 (left) of [50]].

In both branches, we mark two stellar models having the same rest mass $M_b = 1.4000M_\odot$ (yellow star for the TB and yellow box for the HB), which will be used to explain the migration process. The properties of these stellar models are listed in Table II, where they are referred to as TB.1.4 and HB.1.4, respectively. Note that despite

TABLE I. Properties of the polytropic segments for the EOS used in this work in terms of the polytropic constants k_i , adiabatic indices Γ_i , and of the junction rest-mass densities ρ_i in CGS units.

i	1	2	3	4	5	6
$\log k_i$	−4.245	−21.197	−35.544	10.108	−64.283	−101.225
Γ_i	1.139	2.354	3.346	0.258	5.188	7.610
$\log \rho_i$	13.943	14.471	14.783	15.089	15.250	...

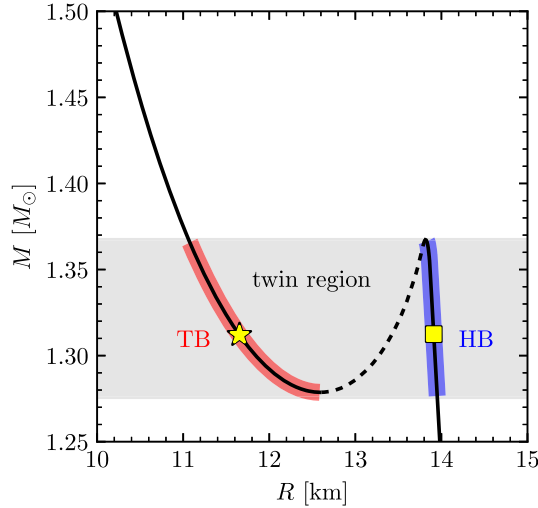


FIG. 1. Representative example of a mass-radius sequence for an EOS leading to the coexistence of a HB and of a TB. Solid and dashed lines are used to indicate branches that are stable or unstable to linear perturbations, respectively. Shown with a shaded area is the range in gravitational masses where the twin solutions exist, i.e., the twin region. The yellow star on TB and the yellow box on HB indicate the models TB . 1 . 4 and HB . 1 . 4 which are selected to discuss the migration process in Sec. III B. The properties of these configurations are listed in Table II.

having the same rest mass and essentially the same gravitational mass,¹ the two configurations have radii that differ by $\gtrsim 2$ km, making TB . 1 . 4 significantly more compact than HB . 1 . 4. As a result, the matter in the core of TB . 1 . 4 is compressed enough such that the central core is described by a quark-matter EOS, while HB . 1 . 4 is purely made of hadronic matter.

B. Numerical setup

To study the nonlinear response to perturbations of twin and hadronic stars we have employed the open-source code GRID [52], which solves the spherically symmetric general-relativistic hydrodynamic equations with a finite-volume scheme, piecewise-parabolic reconstruction and an HLLE Riemann solver (see, e.g., [47]). The initial data are obtained via the solution of the Tolman-Oppenheimer-Volkoff (TOV) equations. A uniform one-dimensional (1D) spatial grid is used with a resolution of 100 m, and the outer boundary is extended to 200 km (additional resolutions at 50 m and 25 m have been studied to explore the converge of the solutions). The temporal evolution uses the method-of-lines to a second-order Runge-Kutta scheme with a Courant-Friedrichs-Lewy condition set to $1/2$. An atmosphere with a constant density of 1 g/cm^3 is set outside the stellar profile. We finally note that because of the

¹We recall that a quasiuniversal relation can be used to relate M and M_b analytically [50,51].

TABLE II. Properties of the representative stellar models indicated in Fig. 1 with a yellow star and the yellow box. These models are selected to explore the migration process in Sec. III B.

Model	ρ_c/ρ_{sat}	$M[M_\odot]$	$M_b[M_\odot]$	R [km]
TB . 1 . 4	5.7342	1.3119	1.4000	11.6540
HB . 1 . 4	2.2227	1.3124	1.4000	13.9110

computational costs associated with our large exploration of the space of parameters, which counts more than 500 simulations, we have here resorted to the use of 1D simulations.

III. RESULTS

A. Initial perturbations

As first shown in Ref. [42] using three-dimensional general-relativistic simulations, a stellar model that is in an unstable equilibrium can “migrate” to an equilibrium configuration as a response to perturbations and while conserving rest mass. A similar process is expected to take place also in the case of stars on the TB and HB, with the important difference that, in this case, both of the stellar models are expected to be in a stable equilibrium. Despite stability with respect to *linear* perturbations, one of the two stable configuration may be “favored” over the other, so that, when subject to *nonlinear* perturbations, a star on the TB or HB will either oscillate in its original branch or migrate to the neighboring one. This migration can either take place via an expansion to larger radii, i.e., a TB $\rightarrow$ HB transition, or via a compression to more compact configurations, i.e., a HB $\rightarrow$ TB transition.

To trigger such a transition, we introduce a perturbation that could reflect a plausible astrophysical scenario, such as a stellar spin-down, or the accretion of matter on the stellar surface, we introduce a constant inward radial velocity of different strength on the initial static stellar models, i.e.,

$$v^r = 0 \rightarrow v^r = -\lambda_{H,T}, \quad (1)$$

where $\lambda_{H,T} > 0$ measures the strength of the perturbation, and the index H, T indicates whether the star is initially on the HB or on the TB. Because extremely large perturbations would be incompatible with the assumption of adiabatic evolutions, we limit the strength of the initial perturbation to $\lambda_{H,T} < 0.1 c$; as we discuss below, such strength is sufficient to obtain all the relevant results.

Ideally, it would be preferable to introduce perturbations with suitable perturbative eigenfunctions in all hydrodynamical quantities. In practice, and as done customarily in this type of study (see, e.g., [43,53–55]), the solution of the full set of hydrodynamical equations is such that over a couple of dynamical timescales the star recovers the correct behavior with the eigenfunctions in the hydrodynamical variables matching well the perturbative ones.

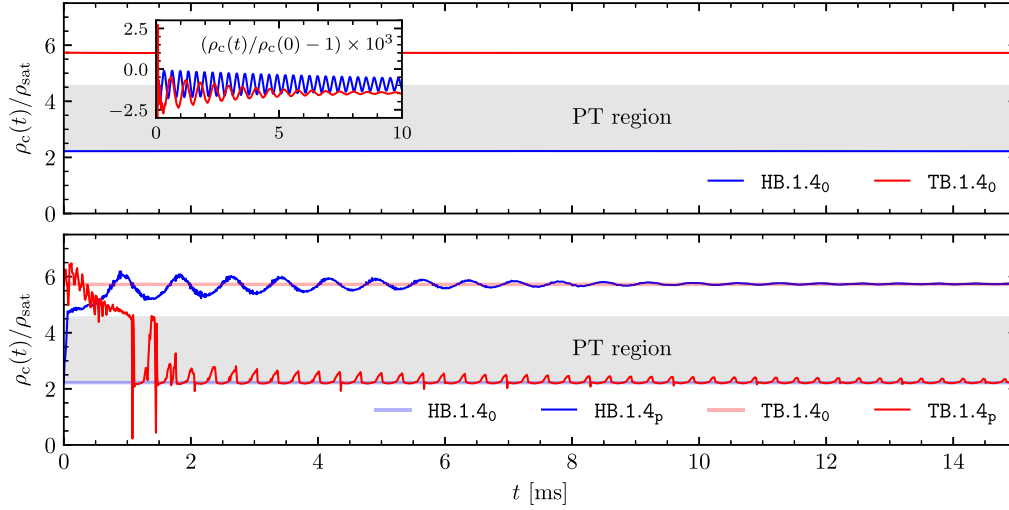


FIG. 2. Top panel: evolution of the central rest-mass density ρ_c normalized to the nuclear-saturation density for the representative stellar configurations TB.1.4 and HB.1.4 that are shown with yellow symbols in Fig. 1 (see also Table II). Because no perturbations are introduced in this case, the inset is needed to show the minute relative oscillations and their decay with time. The shaded area reports the range of rest-mass densities across which the PT takes place. Clearly, both TB.1.4 and HB.1.4 are stable to linear perturbations. Bottom panel: the same as in the top but when the stars experience critical perturbations. In both cases, the stars migrate to the neighboring branch of static solutions.

As mentioned above, the EOS employed here is meant to describe zero-temperature nuclear matter in beta equilibrium. To reflect a purely adiabatic treatment of the perturbations and concentrate on the simpler interplay between kinetic energy and binding energy—and hence not to contaminate the dynamics with the inclusion of dissipative heating—no additional thermal contribution is added to the pressure during the evolution, which is therefore isentropic. This is a reasonable approximation given the modest velocity perturbations introduced.

B. Migration dynamics

The top panel of Fig. 2 reports the evolutions of our two reference stellar configurations without external perturbations, which we will refer to as HB.1.4₀ (black solid line) and TB.1.4₀ (red solid line), respectively. More specifically, shown is the central rest-mass density ρ_c when normalized to ρ_{sat} . The two lines appear as perfectly horizontal here, but this is just the consequence of the scale used and the minuteness of the perturbations that are triggered uniquely by the truncation error in the discretization of the equilibrium models. Shown however in the inset is the central rest-mass density normalized to its initial value $\rho_c(t=0)$; the scale is sufficiently magnified so as to appreciate that there indeed are oscillations and that these are of $\approx 0.1\%$. It is then clear that in both cases ρ_c oscillates around an equilibrium value² and that the oscillations

²We note that the equilibrium is toward slightly larger central rest-mass density values because the numerical import naturally introduces a difference in the equilibrium model (see Ref. [56] for a first discussion of this issue).

eventually damp out at later time steps as a result of the small but nonzero to numerical bulk viscosity [57,58]; we have verified that both the amplitude and the decay rate of the oscillations decrease as the resolution is increased, thus validating the computational approach as numerically consistent (see [47] and the discussion in the Appendix). Also clear is that the oscillations have different frequencies as expected for stars of different compactnesses, with HB.1.4₀ having an f -mode frequency that is ~ 1.74 times larger than that of TB.1.4₀. Overall, the results of the top panel of Fig. 2 indicate that both HB.1.4 and TB.1.4 are stable with respect to linear perturbations. For a discussion of oscillation modes and stability of twin stars, see Refs. [45,59–61] and, for comparison with perturbative predictions in our case, see the discussion in the Appendix. By performing a very large number of simulations in which the strength of the initial perturbation, i.e., $\lambda_{\text{H,T}}$, has revealed that a threshold critical perturbation determines whether a perturbed star (either on the HB or the TB) will migrate to the neighboring branch. In other words, referring to Eq. (1) there appears a critical value $\lambda_{\text{H,crit}}$ such that a perturbed star initially on the HB remains on the HB for perturbations $\lambda_{\text{H}} < \lambda_{\text{H,crit}}$ while it will migrate to the TB for $\lambda_{\text{H}} \geq \lambda_{\text{H,crit}}$. Similar considerations apply to stars initially on the TB, such that the star will remain on the TB for perturbations $\lambda_{\text{T}} < \lambda_{\text{T,crit}}$ while it will migrate to the HB for $\lambda_{\text{T}} \geq \lambda_{\text{T,crit}}$.

To illustrate the dynamics of the migration, the bottom panel of Fig. 2 reports the evolution of the same models discussed in the top panel but when a critical perturbation is used for the migration on either side, i.e., $\lambda_{\text{H}} = \lambda_{\text{H,crit}} = 0.021c$ and $\lambda_{\text{T}} = \lambda_{\text{T,crit}} = 0.040c$. We refer to these models

as HB.1.4_p and TB.1.4_p, respectively and report as a reference also in the bottom panel the evolution of the models HB.1.4₀ and TB.1.4₀.

For the stellar model HB.1.4_p, the initial infall velocity perturbation induces a minicollapse, with a rapid contraction of the star and an increase in the central rest-mass density ρ_c over a timescale of ~ 0.1 ms. Since ρ_c of HB.1.4_p (black solid line) is located close to the PT region, the perturbation is strong enough to trigger a PT inside the core and hence generate quark matter in the star. As a result, ρ_c continues to grow until the matter becomes sufficiently stiff enough to exert an outward balancing pressure to halt the collapse around $t \sim 1$ ms. This results in ρ_c starting to oscillate around an equilibrium value. As the evolution proceeds (we have carried out the simulations up to $t \simeq 50$ ms), the oscillations tend to decay, and ρ_c settles around the value corresponding to model TB.1.4₀ with a precision of 0.36%. After about ~ 20 ms, the system has reached its new equilibrium, and a successful HB $\rightarrow$ TB migration has taken place.

A complementary dynamics is shown by the stellar model TB.1.4_p, where the initial radial kick similarly induces an increase in ρ_c , which however excites the high-frequency oscillation modes. In this case, however, the collapse is immediately halted by the stiff quark-matter EOS, thus resulting in a rapid bounce and outward expansion of the whole star. The resulting decompression allows for the central rest-mass density to enter at $t \sim 1$ ms the high-density edge of the PT region in the EOS (the EOS at the PT region can be thought to be extremely soft) and hence undergo a very rapid expansion. As a result of the reverse PT, the star is purely hadronic, and the central density oscillates around the low-density edge of the PT region while attaining a new equilibrium, which corresponds to that of HB.1.4₀ with a precision of 0.18%. Also in this case, the oscillations are gradually damped, and after about ~ 20 ms, the system has reached its new equilibrium, and a successful TB $\rightarrow$ HB migration has taken place.

C. Critical perturbations and favored twin solutions

The simulations described above for the HB.1.4 and TB.1.4 models have been repeated for a large number of stellar models with different baryonic (and gravitational) masses where λ_{crit} was adaptively computed using a bisection process up to an accuracy of $0.001c$. The dynamics is then essentially the same but with the important difference that the critical perturbation velocity is a function of the mass, i.e., $\lambda_{\text{T,crit}} = \lambda_{\text{T,crit}}(M)$ and $\lambda_{\text{H,crit}} = \lambda_{\text{H,crit}}(M)$. Figure 3 highlights this mass dependence by reporting the critical values for both branches as a function of the gravitational mass. Interestingly, the behavior of the two critical perturbations with M is very different, so that, in general, $\lambda_{\text{T,crit}} \neq \lambda_{\text{H,crit}}$.

These results are very revealing as they show that although migrations in both directions are possible, i.e.,

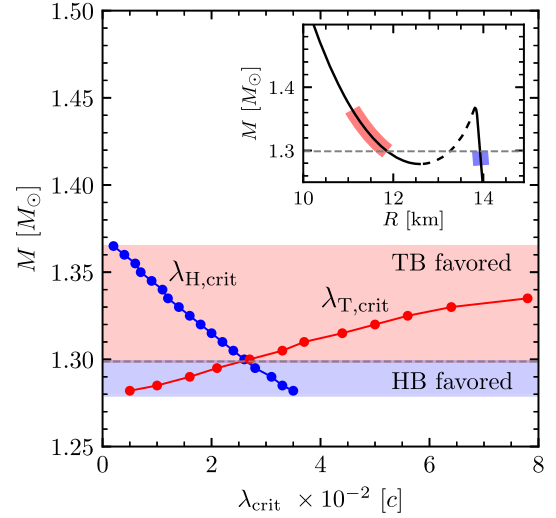


FIG. 3. Numerical values of the critical perturbation velocities $\lambda_{\text{T,crit}}$ for models on the TB (filled red circles) and the corresponding values $\lambda_{\text{H,crit}}$ for models on the HB (filled blue circles) for the range of masses in the twin region. The different colors in the two shaded regions show which of the two branches is favored when comparing $\lambda_{\text{T,crit}}$ and $\lambda_{\text{H,crit}}$; shown with a horizontal dashed line is the value of the neutral-favoritism mass $M_{\text{neut}} = 1.2988M_{\odot}$, where the two branches are equally favored. Shown in the inset is a representation of the favorite branches in an (M, R) diagram.

HB $\rightleftharpoons$ TB, they are not equally likely. On the contrary, a star on a given branch will prefer to remain on that branch unless it is subject to a very large (and possibly unrealistic) perturbation. In turn, this allows us to introduce criterion discriminating which of the two twins is the “favored” one. More specifically, we conclude that

$$\text{TB is favored if } \lambda_{\text{T,crit}}(M) < \lambda_{\text{H,crit}}(M), \quad (2)$$

$$\text{HB is favored if } \lambda_{\text{T,crit}}(M) > \lambda_{\text{H,crit}}(M), \quad (3)$$

with a “neutral favoritism” condition (analogous to a “neutral stability” condition) when

$$\lambda_{\text{T,crit}}(M = M_{\text{neut}}) = \lambda_{\text{H,crit}}(M = M_{\text{neut}}). \quad (4)$$

For the specific EOS considered here, this latter condition is met for a neutral mass $M_{\text{neut}} = 1.2988M_{\odot}$ and is marked by a horizontal dashed line in Fig. 3. It may be useful to remark that while the precise values of $\lambda_{\text{T,crit}}$ and $\lambda_{\text{H,crit}}$ will depend (weakly) on the type of perturbations chosen (see the Appendix), the definitions (2) and (3) will continue to hold. More importantly, the neutral mass M_{neut} will be independent of the perturbation chosen and thus always the same for a given EOS.

Overall, the response of twin and hadronic stars to perturbations nicely summarized in Fig. 3 points out what is possibly the most important result of this paper, namely,

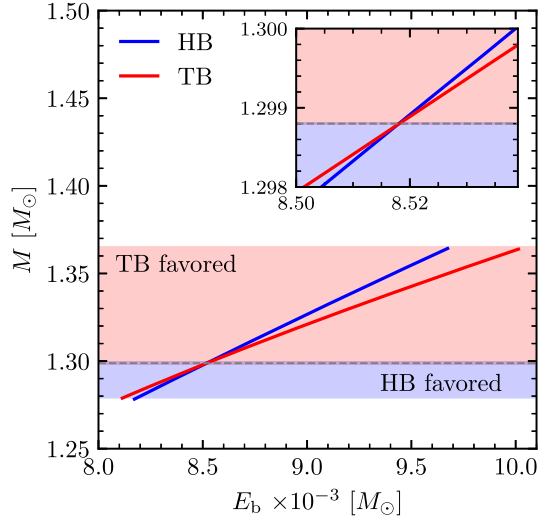


FIG. 4. The same as in Fig. 3 but in terms of the binding energy $E_b := M_b - M$; the inset shows a magnified view of the binding energies near the neutral mass $M_{\text{neut}} = 1.2988 M_\odot$. Clearly, the representations in terms of E_b and λ_{crit} are very similar, so that the binding energies can be used to deduce the favored twin without resorting to numerical simulations.

that the prevailing notion that stellar models on the TB are those more likely to be found in nature is incomplete and rather conditional on the position of M_{crit} in the twin region. The response to perturbations analyzed here clearly shows that low-mass hadronic stars are favored over the corresponding twin solution. Furthermore, as we will discuss in Sec. III D, the actual existence of stars on the TB may be highly unlikely if the PT is very large.

The existence of favored and neutral stellar configurations can be intuitively understood if one bears in mind that stable equilibrium stellar models can be associated in a classical sense to local ground states of the Einstein equations, therefore as sitting at the bottom of the local minimum of an effective potential that depends on the perturbation strength. When subject to small (linear) perturbations, their energy will be varied by the introduction of the energy in the perturbation, which will induce oscillations around the ground states. So long as the energy perturbation is smaller than the potential barrier, the configuration will remain in the same local (and false) minimum or, equivalently, a star in a favored branch will remain in such a branch. However, if the energy in the perturbation is sufficiently large to overcome the potential barrier, a second local minimum will become the true minimum, and the star will move to it. This is what happens when a star on a given branch migrates to the neighboring one for $\lambda_{\text{H,T}} \geq (\lambda_{\text{H,T}})_{\text{crit}}$. Clearly, if the two minima are identical, both branches are equally favored, hence the existence of a neutral mass where this happens. The concept of neutral mass and the distinction between the favored branches in the (M, R) plane are summarized in the inset of Fig. 3.

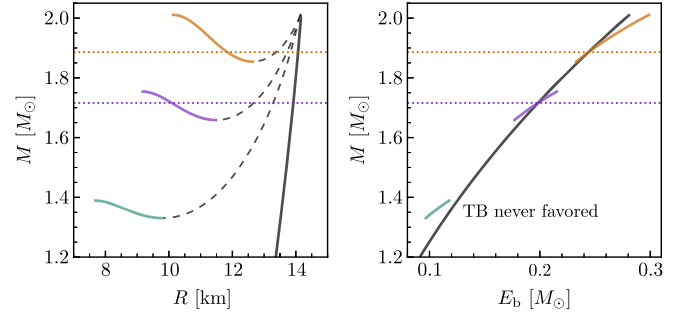


FIG. 5. Left panel: the same as in Fig. 1 but for three EOSs that lead to twin stars of category II. The stable HB is marked with a black solid line, while the stable TBs are marked with solid lines of different colors; dashed and dotted lines mark the (linearly) unstable branches and the neutral masses M_{neut} , respectively. Right panel: the same as in Fig. 4 but for the three EOSs on the left panel; also here, the colored dotted lines mark the neutral masses. Note that no neutral mass exists for the EOS with the smallest maximum mass on the TB; hence, no stellar model on the TB is favored for this EOS. Prescriptions of these EOSs are discussed in the Appendix.

If the existence of a neutral mass M_{neut} is rather natural, a final point to clarify is what determines it. This can be easily addressed by comparing the gravitational binding energies $E_b := M_b - M > 0$ of the stellar models on the two branches. This is shown in Fig. 4, which reports the binding energy for stars on the HB (blue solid line) and on the TB (red solid line). Although these energies are very similar, they are not identical, and, interestingly, they coincide at $M = M_{\text{neut}}$. The mapping between λ_{crit} and E_b is true with a precision of 0.1%. This result clearly illustrates that, for any gravitational mass, the favored stars are those with the largest binding energy as they are sitting in the true equilibrium state. Furthermore, it shows that the potential-barrier differences—which are effectively mapped to the difference $|\lambda_{\text{H,crit}} - \lambda_{\text{T,crit}}|$ —are smaller for $M < M_{\text{neut}}$, which also explains why in Fig. 3 the difference in critical migration values, i.e., $|\lambda_{\text{H,crit}} - \lambda_{\text{T,crit}}|$, is smaller for masses below the neutral one.

D. Other classes of twin stars

The picture summarized above refers to twin stars that are classified to belong to category III [39,40], that is, stars for which the maximum mass of the HB is smaller than the maximum mass of the TB, i.e., $M_{\text{TOV,HB}} < M_{\text{TOV,TB}}$; for obvious reasons, a very similar response is expected also for categories I and IV of the same classification. However, there is a fourth category, for which the opposite is true, that is, $M_{\text{TOV,TB}} \leq M_{\text{TOV,HB}}$, and this is referred to as category II [39,40]. These configurations are shown in the left panel of Fig. 5, which reports three different EOSs with $M_{\text{TOV,TB}} = M_{\text{TOV,HB}} = 2.010 M_\odot$, but also $M_{\text{TOV,TB}} = 1.754 M_\odot$ and $1.389 M_\odot$.

The response of these compact-star configurations to perturbations can be easily deduced without performing numerical simulations (which we have however performed), by simply looking at the binding energies of the different stellar models. These are reported in the right panel of Fig. 5, and it then appears clear that while the first two EOSs from the top lead to binding-energy curves that cross at some specific value of the neutral mass (i.e., $M_{\text{neut}} = 1.886M_{\odot}$ and $1.716M_{\odot}$) the third EOS leads to sequences of stellar models whose binding energies never intersect. As a result, for this EOS, no neutral mass exists, and only one branch will be favored. Indeed, this is exactly what we find via numerical simulations, which reveal that $\lambda_{\text{T,crit}} < \lambda_{\text{H,crit}}$ for all stars on the TB with $M \lesssim 1.355M_{\odot}$. At the same time, the response to perturbations of stars on the TB having mass $M \gtrsim 1.355M_{\odot}$ is such that they will not migrate toward the HB, but rather collapse to black holes as a result of the radial-velocity perturbation. Unsurprisingly, smaller and smaller perturbations are needed as the mass of the perturbed star approaches $M_{\text{TOV,TB}}$. Overall, this phenomenology reveals a very intriguing new behavior of very compact twin-star configurations, where a new criticality in the behavior—i.e., the criticality to collapse to black holes—appears in addition to criticality to migration. We will explore this in more detail in a forthcoming paper.

IV. CONCLUSION

Twin stars are the manifestation of the occurrence of a first-order PT and have been the subject of numerous investigations over the last decade. While it is simple to distinguish which of the compact-star solutions resides on a branch that is stable to linear perturbations, it is much more difficult to predict which of the twin solutions is the “favored” one, that is, which of the two possible stable configurations is expected to be found in nature. We have addressed this simple but fundamental question by studying via fully general-relativistic simulations the response of stars on the TB and HB to nonlinear perturbations triggered by an initial inward radial velocity of various strength.

In this way, we have found that the response of all stellar models, independently of whether they sit on the hadronic or twin branch, is characterized by a critical-perturbation strength. For subcritical perturbations the star will oscillate on the original branch with damped oscillations, while for supercritical perturbations it will migrate to the neighboring branch, where it will eventually settle to the corresponding equilibrium having the same rest mass. More importantly, the critical values on the hadronic and twin branches depend on the mass of the star but are different for the same mass, so that it is possible to define as “favored” the part of the branch which has the largest critical perturbation, as one expects

that this represents the true ground system of the space of configurations.

Interestingly, we have also found that the differences in critical perturbations can be mapped into differences in the binding energies of the stellar models on the two branches. As a result, the favored star is simply that with the largest binding energy and a (neutral) mass appears where the favoritism is neutral and both configurations are equally favored. Hence, a simple investigation of the binding energies on the two branches will reveal which of the stellar configurations is more likely to be found in nature. Finally, while our results have been applied to an EOS leading to a specific class of twin stars, i.e., categories I, III, and IV of the classification in [39,40], we have shown, both with simulations and with the binding energies, that the conclusions drawn above hold true also in other classes of twin stars.

Overall, these results demonstrate that the common wisdom regarding stellar models on the twin branch being the ones expected to be revealed by observations is, at least in part, incorrect. Indeed, it is reasonable to expect that stars that should populate the low-mass part of the TB are not found in nature, and that the corresponding models should exist as stable hadronic stars or black holes.

Finally, while this study significantly improves our understanding of the long-standing issue of the existence of twin-star solutions, it can be extended in a number of ways: first, by a more extensive validation of the location of the critical perturbations via 3D simulations and the inclusion of nonspherical perturbations; second, by the inclusion of a broader set of EOSs that may reveal quasiuniversal properties in the neutral-favoritism mass M_{neut} , which, in turn, could be used to set tighter constraints on the existence of twin stars; and third, by extending the set of initial configurations considered to include configurations that are rotating and magnetized. We plan to explore these aspects in future work.

The following packages have been used in this work—GR1D [52], Numpy [62], Matplotlib [63], Jupyter [64].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX: SUPPLEMENTARY INFORMATION

1. Resolution dependence

We first discuss the robustness of our conclusion and present evidence that the resolution employed is adequate to measure the critical perturbation velocity inducing the migration. Taking as reference models TB.1.4 and HB.1.4 (see Table II), we recall that they have binding energies $E_b = 0.0881M_\odot$ and $E_b = 0.0876M_\odot$, respectively. Since the difference in these two binding energies is $\sim 10^{-4}M_\odot$, the evolutions must have a precision that is smaller (or much smaller) than this difference in order to capture robustly the migration process. In Fig. 6, we report the evolution of the relative differences in the rest mass M_b and of the gravitational mass M when compared to the initial values, i.e., $|1 - M_b(t)/M_b(0)|$ (left panel) and $|1 - M(t)/M(0)|$ (right panel),³ respectively. Note that the data refer to the perturbed models, i.e., HB.1.4_p (blue lines) and TB.1.4_p (red lines), since the unperturbed ones (HB.1.4₀, TB.1.4₀) have even smaller variations, which are about 2 orders of magnitude smaller.

Different lines show the variations for the three different resolutions considered here, namely $\Delta x = 150$ m (low), $\Delta x = 100$ m (medium and default resolution), $\Delta x = 50$ m (high), and $\Delta x = 25$ m (very high). Clearly, the figure highlights that even when employing the low resolution, the relative variations in the rest mass with and in the gravitational mass are of the order of 10^{-5} and that these reduce to values $\sim 10^{-6}$ when the resolution is increased. Similarly, Table IV reports the values of the critical velocity perturbation λ_{crit} and highlights that the values computed converge to a value that is constant with a precision of 0.1%. Overall, the data in Fig. 6 and Table IV demonstrate that the resolutions employed are more than sufficient to capture the nonlinear developments in the evolution.

2. Variations on the perturbation profiles

As mentioned in the main text, while the value of the neutral mass M_{neut} is independent of the perturbation chosen and thus always the same for a given EOS, the precise values of $\lambda_{\text{T,crit}}$ and $\lambda_{\text{H,crit}}$ will depend on the type of perturbations chosen. To validate this dependence, we have

³We recall that the gravitational mass is numerically extracted at the outer boundary after a volume integral across the whole domain [see Eq. (4) in [65]].

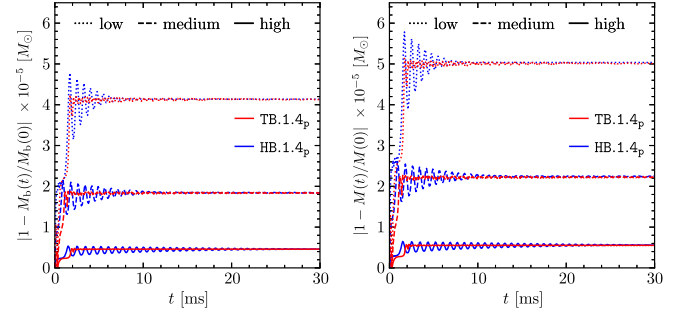


FIG. 6. Evolution of the relative differences in the rest-mass M_b (left panel) and of the gravitational mass M (right panel) when compared to the initial values, i.e., $|1 - M_b(t)/M_b(0)|$ (left panel) and $|1 - M(t)/M(0)|$ (right panel), respectively. Different lines show the variations for the three different resolutions considered here, namely, $\Delta x = 150$ m (low; dotted line), $\Delta x = 100$ m (medium; dashed line), and $\Delta x = 50$ m (high; solid line); the additional “very-high” resolution with $\Delta x = 25$ m was computed, but it is not reported as it would not be visible on these scales. Different colors refer to either the perturbed model TB.1.4_p (red lines) or to perturbed model HB.1.4_p (blue lines).

considered the migration dynamics using velocity perturbation profiles with different profiles. More specifically, we adopt a two-parameter perturbation velocity function proposed by [48],

$$v^r = -\tilde{\lambda} \left(\frac{r}{R} \right)^\alpha, \quad (\text{A1})$$

where $\tilde{\lambda}$ decides the perturbation strength, and the exponent α controls the radial profile of the perturbation. For $\alpha = 0$, the function reduces to the form of Eq. (1), i.e., the perturbation profile used for our study. For $\alpha > 0$, the perturbation smoothly goes to zero at the center.

In Table V, we summarize the values of $\tilde{\lambda}_{\text{crit}}$ for a variety of perturbation profiles. Note how the values of the critical perturbation strength depend only weakly on the profile chosen, but also that larger perturbations are needed for larger values of the exponent α . This has a simple justification: an increase in the exponent α decreases the perturbation strength inside the star, so that the linearly stable stars will require stronger perturbation strengths to migrate.

3. Comparison of stable evolution with perturbative results

As anticipated in the main text, the stars on TB and HB are linearly stable and hence will undergo (damped) oscillations when exposed to linear perturbations. The eigenfrequencies and eigenfunctions of such oscillations can be computed via linear perturbation theory or via nonlinear numerical simulations so long as the perturbations are very small, and the response can be considered to be in a linear regime. Obviously, the two approaches can be compared and must yield, for instance, the same

TABLE III. Properties of the polytropic segments for the EOSs used in Sec. III D in terms of the polytropic constants k_i , adiabatic indices Γ_i , and of the junction rest-mass densities ρ_i in CGS units. The M - R sequences are plotted in the left panel of Fig. 5. The top, middle, and bottom sets of values form EOSs that construct orange, purple, and green hybrid branches, respectively, with a common hadronic branch.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\log k_i$	-8.167	-5.974	1.725	-7.399	-24.415	-61.005	11.099	-64.768	-53.435	-45.867	-39.803	-35.246	-31.443	-28.394	-25.796
Γ_i	1.584	1.287	0.622	1.356	2.570	5.100	0.200	5.250	4.500	4.000	3.600	3.300	3.050	2.850	2.680
$\log \rho_i$	7.385	11.577	12.419	14.027	14.462	14.715	15.023	15.110	15.136	15.159	15.187	15.214	15.245	15.278	...
$\log k_i$	-8.167	-5.974	1.725	-7.399	-24.415	-61.005	11.099	-65.226	-53.810	-46.187	-40.079	-35.490	-31.658	-28.586	-25.970
Γ_i	1.584	1.287	0.622	1.356	2.570	5.100	0.200	5.250	4.500	4.000	3.600	3.300	3.050	2.850	2.680
$\log \rho_i$	7.385	11.577	12.419	14.027	14.462	14.715	15.113	15.220	15.245	15.270	15.298	15.326	15.357	15.389	...
$\log k_i$	-8.167	-5.974	1.725	-7.399	-24.415	-61.005	11.099	-66.000	-54.446	-46.730	-40.548	-35.903	-32.025	-28.917	-26.269
Γ_i	1.584	1.287	0.622	1.356	2.570	5.100	0.200	5.250	4.500	4.000	3.600	3.300	3.050	2.850	2.680
$\log \rho_i$	7.385	11.577	12.419	14.027	14.462	14.715	15.267	15.405	15.431	15.455	15.482	15.511	15.541	15.573	...

eigenfrequencies (see, e.g., Ref. [42] for a first systematic comparison in 3D simulations).

Here, we compare the numerical f -mode eigenfrequency as extracted from the numerical oscillations of the central rest-mass density $\rho_c(t)$ for the models HB.1.4₀ and TB.1.4₀ after a perturbation triggered by the truncation error only, with the radial f mode computed from the perturbative approach [66,67]. Figure 7 reports the power spectral density of the numerical evolution of $\rho_c(t)$ for models HB.1.4₀ (blue solid line) and TB.1.4₀ (red solid line) after an evolution of 50 ms. Also reported with vertical dotted lines of the same color are the corresponding eigenfrequencies of the f mode as computed from perturbation theory. Clearly, the match is very good, with only a very small difference ($\lesssim 4\%$), which is however inevitable as the numerical solution is always nonlinear even for very

small perturbations. Overall, this result demonstrates the ability of our numerical simulations to explore accurately both the linear and the nonlinear regimes of our stellar models.

4. Variations on the equation of state

As discussed in Sec. III D, we have also considered the evolution of other classes of twin stars. The fifteen segments of the piecewise polytropic prescription employed here are listed in Table III. A common crust part, implemented from [68], is described by the polytropes $i = 1-4$, and a common hadronic part is described by the polytropes $i = 5, 6$, making the EOSs identical up to ρ_6 . This results in a common hadronic branch, as indicated by a black curve in

TABLE IV. Variation of λ_{crit} for three different resolutions from low to high and for the representative stellar models given in Table II. Note how the values converge to a constant as the resolution is increased.

Model	$\lambda_{\text{crit}}[c]$			
	Low (150 m)	Medium (100 m)	High (50 m)	Very high (25 m)
TB.1.4	0.04790	0.03963	0.03547	0.03415
HB.1.4	0.02040	0.02028	0.02022	0.02019

TABLE V. Comparison of $\tilde{\lambda}_{\text{crit}}$ for different perturbation profiles given by Eq. (A1) for the representative stellar models given in Table II.

Model	$\tilde{\lambda}_{\text{crit}}[c]$			
	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 2.0$
TB.1.4	0.040	0.049	0.060	0.089
HB.1.4	0.021	0.030	0.040	0.066

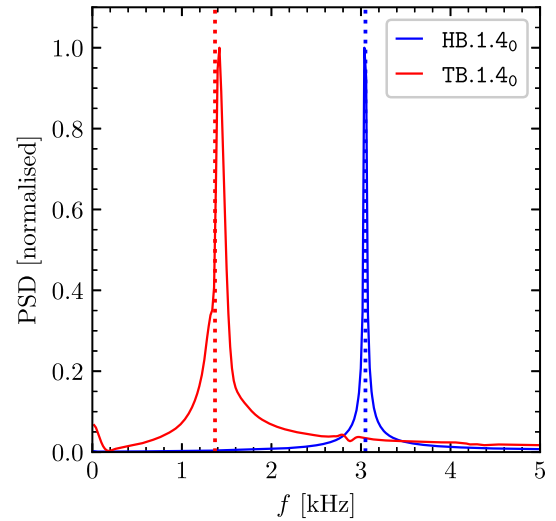


FIG. 7. Power spectral density (PSD) of the numerical evolution of central rest-mass density $\rho_c(t)$ for models HB.1.4₀ (blue solid line) and TB.1.4₀ (red solid line) after an evolution of 50 ms. Reported with vertical dotted lines of the same color are the corresponding eigenfrequencies of the f mode as computed from perturbation theory.

the left panel of Fig. 5. The PT region is described by the polytropic piece with $i = 7$, in which $\Delta\rho = \rho_7 - \rho_6$ dictates the strength of PT. We set $\Delta\rho_{\text{orange}} < \Delta\rho_{\text{purple}} < \Delta\rho_{\text{green}}$,

which leads to a more compact green hybrid branch with a smaller value for $M_{\text{TOV,TB}}$. Finally, the quark region is described by the polytropes $i = 8\text{--}15$.

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