

Positivity bounds from thermal field theory entropy

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We present an approach to deriving positivity bounds on effective field theories by analyzing the thermodynamic behavior of thermal quantum field systems. Focusing on scalar theories with higher-dimensional operators, we compute the finite-temperature entropy using thermal field theory techniques. We argue that consistency with fundamental thermodynamic principles—specifically, the expectation that entropy increases with the introduction of new degrees of freedom—imposes nontrivial constraints on Wilson coefficients. In particular, we show that the coefficient of the leading dimension-eight operator must be strictly positive. This thermodynamic perspective offers an alternative to traditional S -matrix-based derivations of positivity bounds and provides a complementary perspective on the interplay between entropy, unitarity, and causality in quantum field theory.

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I. INTRODUCTION

Effective field theory (EFT) provides a systematic framework for describing low-energy phenomena in systems with a hierarchy of scales. It is especially useful when the ultraviolet (UV) completion is unknown or strongly coupled, as often occurs in gravity and hadronic physics. In practice, EFTs are built in two complementary ways: a top-down approach, which integrates out heavy degrees of freedom from a known UV theory, and a bottom-up approach, which enumerates all operators consistent with low-energy symmetries and treats their coefficients as free parameters.

However, if we assume the existence of a consistent UV completion that respects causality, locality, and unitarity, then the low-energy Wilson coefficients are not entirely arbitrary [1]. Constraints arising from these fundamental principles—commonly referred to as positivity bounds—have traditionally been derived using analyticity properties of scattering amplitudes and dispersion relations [2–56].

In this work we pursue an alternative route. Motivated by studies relating black hole thermodynamics to quantum-gravity consistency conditions [57–67], we ask whether

analogous constraints can be obtained directly in finite-temperature quantum field theory. Our central proposal is that thermodynamic consistency—specifically, requiring that the thermal entropy does not decrease when additional microscopic degrees of freedom are integrated out—imposes positivity constraints on EFT Wilson coefficients.

To make this idea concrete, we analyze a shift-symmetric scalar EFT, compute its finite-temperature entropy, and show that the leading dimension-eight coefficient must be strictly positive. This identifies a direct link between thermal statistical mechanics and EFT consistency, providing a complement to standard amplitude-based positivity arguments.

II. BASIC IDEA

Consider a system in thermal equilibrium characterized by two widely separated mass scales: a heavy mass M and a light mass m . We focus on an intermediate temperature regime

$$m \ll T \ll M, \quad (1)$$

in which thermal excitations of the heavy degrees of freedom are exponentially suppressed, while the light modes remain thermally active. In this regime, the thermal entropy of the full theory can be well approximated by that of an EFT containing only the light degrees of freedom. The effects of heavy degrees of freedom are encoded in higher-dimensional interaction terms that arise from integrating out the heavy fields,

$$S_{m+M, \text{thermal}} \simeq S_{m\text{-EFT}, \text{thermal}}. \quad (2)$$

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A naive classical intuition suggests that introducing additional microscopic degrees of freedom should increase the number of accessible states and hence the thermal entropy. This motivates the inequality

$$S_{m+M,\text{thermal}} > S_{m\text{-alone},\text{thermal}}, \quad (3)$$

where $S_{m\text{-alone},\text{thermal}}$ denotes the thermal entropy of a free theory containing only the light sector.

However, this intuition does not directly carry over to quantum theory when entropy is defined as the von Neumann entropy of a reduced density matrix, due to the presence of quantum entanglement. For example, a bipartite system $A+B$ prepared in a pure state has vanishing total entropy

$$S_{A+B} = 0, \quad (4)$$

while the reduced state of subsystem A obtained by tracing out B is generally mixed and has nonzero entropy

$$S_A > 0. \quad (5)$$

In the present context, tracing out heavy degrees of freedom therefore introduces entanglement contributions that must be treated with care. To make this explicit, let S_m and S_M denote the von Neumann entropies of the reduced density matrices of the light and heavy sectors, respectively, obtained from the full theory. Schematically, one may write

$$S_m \simeq S_{m,\text{thermal}} + S_{m \rightarrow M,\text{ent}}, \quad (6)$$

$$S_M \simeq S_{M,\text{thermal}} + S_{M \rightarrow m,\text{ent}}, \quad (7)$$

where $S_{m,\text{thermal}}$ and $S_{M,\text{thermal}}$ denote the thermal-free (entropy-from-occupation) contributions, while $S_{m \rightarrow M,\text{ent}}$ and $S_{M \rightarrow m,\text{ent}}$ encode entanglement between the two sectors induced by interactions between the two fields. We stress that Eqs. (6) and (7) are not exact additive identities for von Neumann entropy; rather, they represent an approximate separation that is valid in the weakly coupled regime $m \ll T \ll M$, where thermal and entanglement contributions can be parametrically distinguished.

In quantum field theory, entanglement entropy is generally ultraviolet divergent and requires a regulator, reflecting short-distance correlations across the entangling surface. A standard regularization, familiar, for example, from conformal field theory, introduces a short-distance cutoff ϵ , which gives rise to local area-law divergences that are independent of temperature. The entanglement entropy in our setup behaves in the same way. However, since our analysis only involves differences of entropies, these divergent contributions cancel in the final result.

More precisely, consider the low-temperature regime $T \ll M$. In this limit, the thermal entropy associated with

the heavy sector is exponentially suppressed,

$$S_{M,\text{thermal}} \propto T^{3/2} e^{-M/T}, \quad (8)$$

and can therefore be neglected. Consequently, the entropy S_M is dominated by entanglement contributions and can be approximated as

$$S_M \simeq S_{M \rightarrow m,\text{ent}}. \quad (9)$$

Taking the difference between S_m and S_M isolates the thermal contribution of the light degrees of freedom,

$$S_{m,\text{thermal}} \simeq |S_m - S_M|_{T \ll M}. \quad (10)$$

We now invoke the Araki-Lieb inequality [68,69], which holds for the von Neumann entropies of any bipartite quantum system,

$$S_{m+M} \geq |S_m - S_M|. \quad (11)$$

Applying this inequality in the regime $m \ll T \ll M$, and combining it with Eq. (2), we obtain

$$S_{m,\text{EFT}} > S_{m,\text{thermal}}. \quad (12)$$

This result sharpens the heuristic expectation expressed in Eq. (3) by accounting for entanglement effects and regulator-dependent contributions in a controlled manner. It provides the thermodynamic inequality that underlies our subsequent entropy-based positivity bound.¹

III. CALCULATION OF THERMAL ENTROPY

We begin by considering a simple scalar field theory that possesses a shift symmetry, which physically corresponds to taking the massless limit ($m \rightarrow 0$). Such a field often emerges as a Goldstone boson in effective field theories, representing the low-energy fluctuations around spontaneously broken continuous symmetries. Throughout this section, we adopt natural units with $k_B = \hbar = 1$.

The low-energy effective Lagrangian for such a scalar field ϕ , truncated at dimension eight, is

$$\mathcal{L}_M = -\frac{1}{2}(\partial\phi)^2 + \frac{c}{M^4}(\partial\phi)^4 + \dots, \quad (13)$$

¹Although one might initially expect $S_{m,\text{EFT}}$ and $S_{m,\text{thermal}}$ to coincide, there is a crucial distinction between tracing out heavy degrees of freedom and integrating them out in the effective field theory (EFT). In constructing S_m , we perform a partial trace over the heavy sector M , retaining only the light degrees of freedom with mass m . In contrast, when evaluating $S_{m,\text{EFT}}$, we integrate out not only the heavy fields with mass M but also the high-energy modes of the light field itself. Consequently, S_m and $S_{m,\text{EFT}}$ are not identical: the latter includes additional contributions from integrating over the UV modes of the light field.

where c is a dimensionless coupling constant and M denotes a high-energy cutoff scale in the effective theory. The first term corresponds to the standard kinetic energy of a free scalar field, while the second term represents the leading-order self-interactions of the field.

To incorporate finite-temperature effects, we consider the system in thermal equilibrium with a heat bath at temperature T , which allows us to compute the statistical properties of the scalar field within a consistent quantum field theoretical framework.

From a theoretical perspective, to prepare a thermal equilibrium state at temperature $T = \frac{1}{\beta}$, we can employ the Euclidean path integral formalism on a compact manifold with temporal circumference β . Concretely, this approach involves the following steps [70]:

- (i) *Imaginary time (Wick rotation)*: We replace real time t by imaginary (Euclidean) time τ through the transformation

$$\tau \equiv it. \quad (14)$$

This step transforms the Minkowski spacetime into a Euclidean spacetime, making the field theory resemble a classical statistical system.

- (ii) *Euclidean Lagrangian*: Under Wick rotation, the Lagrangian transforms into its Euclidean counterpart. The Euclidean Lagrangian L_E is related to the original Minkowski Lagrangian by

$$L_E(\tau, x) \equiv -\mathcal{L}_M(t, x), \quad (15)$$

- (iii) *Periodic boundary conditions and temperature*: At finite temperature, the Euclidean time direction becomes compactified with period β , where

$$\tau \sim \tau + \beta, \quad \text{where } \beta \equiv \frac{1}{T}. \quad (16)$$

This periodicity reflects thermal equilibrium and emerges naturally from the trace over states in the partition function:

$$\mathcal{Z}(T) = \text{Tr}(e^{-\beta H}). \quad (17)$$

Following these steps, the Euclidean action for the scalar theory becomes

$$\begin{aligned} S_E &= S_0 + S_I \\ &= \int_0^\beta d\tau \int d^3\mathbf{x} \left[\frac{1}{2} (\partial\phi)^2 - \frac{c}{M^4} (\partial\phi)^4 + \dots \right] \\ &= \int_X \left[\frac{1}{2} (\partial\phi)^2 - \frac{c}{M^4} (\partial\phi)^4 + \dots \right], \end{aligned} \quad (18)$$

where we clearly separate the free action S_0 (quadratic in ϕ) from the interaction terms S_I (higher-order terms in ϕ).

Note that in the zero-temperature limit ($\beta \rightarrow \infty$), the periodicity condition disappears, and we naturally recover the standard Euclidean quantum field theory defined on infinite spacetime.

A crucial motivation for studying this theory at finite temperature is understanding how thermal entropy, a fundamental quantity describing the disorder and microscopic states of the system, constrains the properties of the theory. In particular, as we shall see from detailed calculations, consistency conditions related to the entropy demand the positivity of certain coupling constants. In this model, we will see that this implies

$$c > 0. \quad (19)$$

Thus, we can find that thermal field theory not only provides a rich physical framework to study quantum systems at finite temperature but also imposes powerful theoretical constraints on allowed interactions.

To quantify these ideas precisely, we construct the thermal partition function

$$\mathcal{Z}(T) = C \int \mathcal{D}\phi e^{-S_E} = C \int \mathcal{D}\phi e^{(-S_0 - S_I)}, \quad (20)$$

where the constant C contributes only to the vacuum energy. Given that the coupling $\frac{c}{M^4}$ is small (since M is large compared to other physical scales in the theory), we can expand the partition function perturbatively around the free theory. Explicitly, we obtain

$$\begin{aligned} \mathcal{Z}(T) &= C \int \mathcal{D}\phi e^{-S_0} \left[1 - S_I + \frac{1}{2} S_I^2 - \frac{1}{6} S_I^3 + \dots \right] \\ &= \mathcal{Z}_{(0)} \left[1 - \langle S_I \rangle_0 + \frac{1}{2} \langle S_I^2 \rangle_0 - \frac{1}{6} \langle S_I^3 \rangle_0 + \dots \right], \end{aligned} \quad (21)$$

where

$$\mathcal{Z}_{(0)} \equiv C \int \mathcal{D}\phi e^{-S_0}, \quad (22)$$

$$\langle \dots \rangle_0 \equiv \frac{\int \mathcal{D}\phi [\dots] e^{-S_0}}{\int \mathcal{D}\phi e^{-S_0}}. \quad (23)$$

The free energy density $f(T)$ is defined in terms of the thermodynamic free energy as

$$f(T) \equiv \lim_{V \rightarrow \infty} \frac{F(T, V)}{V}, \quad (24)$$

with $F(T, V) = -T \ln \mathcal{Z}$ and

$$\frac{F(T, V)}{V} = \frac{F_{(0)}}{V} - \frac{T}{V} \left[-\langle S_I \rangle_0 + \frac{1}{2} (\langle S_I^2 \rangle_0 - \langle S_I \rangle_0^2) - \dots \right]. \quad (25)$$

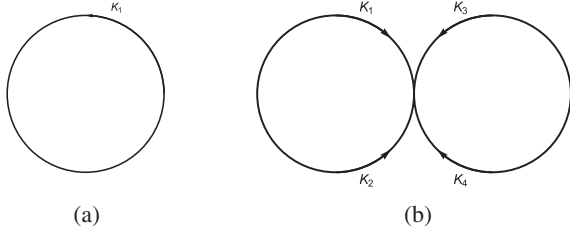


FIG. 1. Diagrams contributing to the free energy expansion. Panel (a) corresponds to the leading free theory contribution $f_{(0)}$. Panel (b) shows one representative Wick contraction contributing to $f_{(1)}$, in which K_1 contracts with K_2 ; the $K_1 - K_3$ and $K_1 - K_4$ contractions contribute analogously.

It is standard to separate the free energy into contributions from the free theory and corrections due to interactions:

$$f = f_0 + f_{(\geq 1)}. \quad (26)$$

For leading-order and first-order, we have

$$f_{(0)} = \lim_{V \rightarrow \infty} \frac{T}{V} S_0 = \lim_{V \rightarrow \infty} \frac{-T \ln \mathcal{Z}_0}{V}, \quad (27)$$

$$\begin{aligned} f_{(1)} &= \lim_{V \rightarrow \infty} \frac{T}{V} \langle S_I \rangle_0 \\ &= \lim_{V \rightarrow \infty} -\frac{T}{V} \int_X \frac{c}{M^4} \langle \partial_\mu \phi(X) \partial^\mu \phi(X) \partial_\nu \phi(X) \partial^\nu \phi(X) \rangle_0. \end{aligned} \quad (28)$$

These two quantities correspond to Fig. 1. Before carrying out the explicit calculation, it is useful to estimate their scaling. The leading term $f_{(0)}$ involves a single momentum integral and scales as T^4 . For $f_{(1)}$, Fig. 1(b) contains two independent loop momenta, and dimensional analysis gives $f_{(1)} \sim \frac{c}{M^4} T^8$.

We now proceed to compute $\mathcal{Z}_0$ from which $f_{(0)}$ can be extracted. The partition function $\mathcal{Z}_0$ takes the form

$$\mathcal{Z}_0 = C \int \mathcal{D}\phi \exp \left[- \int_X \frac{1}{2} (\partial\phi)^2 \right]. \quad (29)$$

We adopt the following Fourier expansion for the scalar field:

$$\phi(X) = \sqrt{\beta V} \sum_{\omega_n} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \tilde{\phi}(K) e^{iK \cdot X} \equiv \sum_K \tilde{\phi}(K) e^{iK \cdot X}, \quad (30)$$

where $K \equiv (\omega_n, \mathbf{k})$, with bosonic Matsubara frequencies $\omega_n = 2\pi T n$, $n \in \mathbb{Z}$, imposed by the periodicity $e^{i\omega_n \beta} = 1$ [71]. Furthermore, for the real scalar field, we have the condition: $\phi^*(X) = \phi(X) \Rightarrow \tilde{\phi}^*(K) = \tilde{\phi}(-K)$. Substituting

this into the action gives

$$\begin{aligned} \frac{\mathcal{Z}_0}{C} &= \int \mathcal{D}\phi \exp \left[\int_X \frac{1}{2} \sum_{K, K'} (K_\mu K'^\mu) \right. \\ &\quad \times \tilde{\phi}(K) \tilde{\phi}(K') \times e^{iK \cdot X + iK' \cdot X} \left. \right] \\ &= \int \mathcal{D}\phi \exp \left[-\frac{1}{2} \beta V \sum_{\omega_n} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} K^2 \tilde{\phi}(-K) \tilde{\phi}(K) \beta \right] \\ &= \int \mathcal{D}\phi \exp \left[-\frac{1}{2} \beta^2 V \sum_{\omega_n} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} K^2 \tilde{\phi}^*(K) \tilde{\phi}(K) \right] \\ &= \int \mathcal{D}\phi \exp \left[\sum_K \left(-\frac{1}{2} \tilde{\phi}^*(K) K^2 \beta^2 \tilde{\phi}(K) \right) \right]. \end{aligned} \quad (31)$$

Here we used $\int_0^\beta d\tau e^{i(\omega_n + \omega'_n)\tau} = \beta \delta(\omega_n + \omega'_n)$ and $\int d^3 \mathbf{x} e^{i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{x}} = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}')$. In the last equality, $\sum_K$ denotes $V \sum_{\omega_n} \int \frac{d^3 \mathbf{k}}{(2\pi)^3}$. Using the Gaussian integral identity $\int d^D x e^{-\frac{1}{2} \mathbf{x}^T \hat{A} \mathbf{x}} = (2\pi)^{D/2} (\det \hat{A})^{-1/2}$, we obtain

$$\mathcal{Z}_0 = C \cdot \sqrt{\frac{1}{\det(K^2 \beta^2)}}. \quad (32)$$

Since the constant C is independent of temperature, we drop it in what follows. Therefore,

$$\begin{aligned} \ln \mathcal{Z}_0 &= -\frac{1}{2} \ln \det(K^2 \beta^2) \\ &= -\frac{1}{2} \sum_K \ln(K^2 \beta^2) = -\frac{1}{2} V \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \sum_{\omega_n} \ln(K^2 \beta^2) \\ &= V \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left[-\frac{1}{2} \beta |\mathbf{k}| - \ln(1 - e^{-\beta |\mathbf{k}|}) \right], \end{aligned} \quad (33)$$

where the last step uses the standard Matsubara sum formula (A16). Thus, the free energy density becomes

$$\int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left[\frac{1}{2} |\mathbf{k}| + T \cdot \ln(1 - e^{-|\mathbf{k}|/T}) \right]. \quad (34)$$

The leading temperature-independent term reflects the zero-temperature vacuum energy, which lacks thermal significance. As such, it can be consistently subtracted when evaluating thermodynamic quantities. This yields the renormalized result

$$\begin{aligned}
f_{(0)} &= \int \frac{d^3 \mathbf{k}}{(2\pi)^3} T \cdot \ln(1 - e^{-|\mathbf{k}|/T}) \\
&= \frac{T}{2\pi^2} \int dk \cdot k^2 \ln(1 - e^{-k/T}) \\
&= \frac{T^4}{2\pi^2} \int dx \cdot x^2 \ln(1 - e^{-x}) \\
&= -\frac{\pi^2 T^4}{90}.
\end{aligned} \tag{35}$$

In the third line we performed the change of variable $k/T \rightarrow x$. We now turn to the evaluation of $f_{(1)}$. From the last line of Eq. (28), $\langle \partial_\mu \phi(X) \partial^\mu \phi(X) \partial_\nu \phi(X) \partial^\nu \phi(X) \rangle_0$ is independent of spacetime coordinates by translational invariance, so the spacetime integral is trivial. Therefore,

$$f_{(1)} = -\frac{c}{M^4} \langle \partial_\mu \phi(0) \partial^\mu \phi(0) \partial_\nu \phi(0) \partial^\nu \phi(0) \rangle_0, \tag{36}$$

where the prefactor $\frac{T}{V}$ in Eq. (28) is canceled by $\frac{T}{V} \int_X = \frac{T}{V} \beta V = 1$. To handle the derivatives, we perform a Fourier transform of the fields

$$\begin{aligned}
f_{(1)} &= -\frac{8c}{M^4} \left\langle \sum_{K_1} iK_{1,\mu} \tilde{\phi}(K_1) \sum_{K_2} iK_2^\mu \tilde{\phi}(K_2) \right. \\
&\quad \times \sum_{K_3} iK_{3,\nu} \tilde{\phi}(K_3) \sum_{K_4} iK_4^\nu \tilde{\phi}(K_4) \\
&\quad \left. + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \right\rangle_0 \\
&= -\frac{8c}{M^4} \sum_{K_1, K_2, K_3, K_4} (K_{1,\mu} K_2^\mu K_{3,\nu} K_4^\nu + (2 \leftrightarrow 3) + (2 \leftrightarrow 4)) \\
&\quad \times \langle \tilde{\phi}(K_1) \tilde{\phi}(K_2) \tilde{\phi}(K_3) \tilde{\phi}(K_4) \rangle_0.
\end{aligned} \tag{37}$$

Using Wick's theorem, we have

$$\begin{aligned}
f_{(1)} &= -\frac{c}{M^4} \sum_{K_1, K_2, K_3, K_4} K_{1,\mu} K_2^\mu K_{3,\nu} K_4^\nu \\
&\quad \times [\langle \tilde{\phi}(K_1) \tilde{\phi}(K_2) \rangle_0 \langle \tilde{\phi}(K_3) \tilde{\phi}(K_4) \rangle_0 \\
&\quad + \langle \tilde{\phi}(K_1) \tilde{\phi}(K_3) \rangle_0 \langle \tilde{\phi}(K_2) \tilde{\phi}(K_4) \rangle_0 \\
&\quad + \langle \tilde{\phi}(K_1) \tilde{\phi}(K_4) \rangle_0 \langle \tilde{\phi}(K_2) \tilde{\phi}(K_3) \rangle_0].
\end{aligned} \tag{39}$$

With $\langle \tilde{\phi}(K_1) \tilde{\phi}(K_2) \rangle_0 = \frac{T^2}{V} \delta_{\omega_1 + \omega_2, 0} (2\pi)^d \delta^{(d)}(\mathbf{k}_1 + \mathbf{k}_2) \frac{1}{K_1^2}$, we obtain

$$\begin{aligned}
f_{(1)} &= -\frac{8c}{M^4} T^2 \sum_{\omega_1, \omega_3} \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_3}{(2\pi)^3} K_1^2 K_3^2 \frac{1}{K_1^2} \frac{1}{K_3^2} \\
&\quad - \frac{16c}{M^4} T^2 \sum_{\omega_1, \omega_2} \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} (K_1 \cdot K_2)^2 \frac{1}{K_1^2} \frac{1}{K_2^2} \\
&= -\frac{8c}{M^4} T^2 \sum_{\omega_n \omega'_n} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{d^3 \mathbf{k}'}{(2\pi)^3} \cdot \mathbf{1} \\
&\quad - \frac{16c}{M^4} T^2 \sum_{\omega_n \omega'_n} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{d^3 \mathbf{k}'}{(2\pi)^3} (K \cdot K')^2 \frac{1}{\omega_n^2 + \mathbf{k}^2} \frac{1}{\omega_n'^2 + \mathbf{k}'^2},
\end{aligned} \tag{40}$$

where we relabeled the four external momenta into two independent ones, $K = (\omega_n, \mathbf{k})$ and $K' = (\omega'_n, \mathbf{k}')$, as shown in Fig. 1(b).

The first term in Eq. (40) is proportional to the constant sum-integral

$$\mathcal{I} \equiv T^2 \sum_{\omega_n, \omega'_n} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} \mathbf{1} = \left[T \sum_{\omega_n} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \mathbf{1} \right]^2. \tag{41}$$

Although this expression carries an explicit factor of T^2 , it does not correspond to a genuine thermal contribution. In thermal field theory, temperature dependence arises from the difference between the discrete Matsubara sum and the corresponding zero-temperature frequency integral. For a sufficiently regular function $F(\omega)$ one may write

$$T \sum_n F(\omega_n) = \int \frac{d\omega}{2\pi} F(\omega) + \Delta_T[F], \tag{42}$$

where the first term, which is independent of T , represents the vacuum contribution and $\Delta_T[F]$ contains the genuine thermal correction.

For the present case the integrand is constant, $F(\omega) = 1$, so the thermal part vanishes. A convenient derivation follows from the Poisson summation formula. To this end, we introduce a regulator $F_\alpha(\omega) = e^{-\alpha|\omega|}$ with $\alpha > 0$, which approaches unity in the limit $\alpha \rightarrow 0^+$. Applying the Poisson summation formula (B8) gives

$$T \sum_{n \in \mathbb{Z}} F_\alpha(\omega_n) = \sum_{m \in \mathbb{Z}} \int \frac{d\omega}{2\pi} e^{im\beta\omega} F_\alpha(\omega). \tag{43}$$

The $m = 0$ term reproduces the vacuum integral $\int \frac{d\omega}{2\pi} F_\alpha(\omega)$. The remaining $m \neq 0$ terms are proportional to $\alpha/(\alpha^2 + m^2\beta^2)$ and vanish as $\alpha \rightarrow 0^+$. Hence $\Delta_T[1] = 0$. (See Appendix B for more details.)

Therefore, the sum-integral in Eq. (41) contains only the vacuum contribution. Since physical thermodynamic quantities are defined after subtracting the zero-temperature

vacuum contribution (equivalently, by absorbing it into a vacuum energy counterterm), this term does not contribute to the renormalized free energy or entropy and can be discarded. We thus retain only the second term in Eq. (40):

$$\begin{aligned}
 f_{(1)} &= -\frac{16c}{M^4} T^2 \sum_{\omega_n, \omega'_n} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{d^3 \mathbf{k}'}{(2\pi)^3} (K \cdot K')^2 \frac{1}{\omega_n^2 + \mathbf{k}^2} \frac{1}{\omega_n'^2 + \mathbf{k}'^2} \\
 &= -\frac{16c}{M^4} T^2 \int \frac{d^3 \mathbf{k} d^3 \mathbf{k}'}{(2\pi)^6} \sum_{\omega_n, \omega'_n} \frac{\omega_n^2 \omega_n'^2 + 2\omega_n \omega'_n \mathbf{k} \cdot \mathbf{k}' + (\mathbf{k} \cdot \mathbf{k}')^2}{(\omega_n^2 + \mathbf{k}^2)(\omega_n'^2 + \mathbf{k}'^2)} \\
 &= -\frac{16c}{M^4} T^2 \int \frac{d^3 \mathbf{k} d^3 \mathbf{k}'}{(2\pi)^6} \sum_{\omega_n, \omega'_n} \frac{\omega_n^2 \omega_n'^2 + (\mathbf{k} \cdot \mathbf{k}')^2}{(\omega_n^2 + \mathbf{k}^2)(\omega_n'^2 + \mathbf{k}'^2)}. \quad (44)
 \end{aligned}$$

The middle cross term in the second line vanishes because it is odd under the integral. The remaining two numerator terms are treated separately: we first carry out the Matsubara sums and then perform the momentum integrals. For the first term in the numerator,

$$\begin{aligned}
 &\sum_{\omega_n, \omega'_n} \frac{\omega_n^2}{(\omega_n^2 + \mathbf{k}^2)} \frac{\omega_n'^2}{(\omega_n'^2 + \mathbf{k}'^2)} \\
 &= \sum_{\omega_n, \omega'_n} \left[1 - \frac{\mathbf{k}^2}{(\omega_n^2 + \mathbf{k}^2)} \right] \left[1 - \frac{\mathbf{k}'^2}{(\omega_n'^2 + \mathbf{k}'^2)} \right] \\
 &= \sum_{\omega_n, \omega'_n} \mathbf{1} - \sum_{\omega_n, \omega'_n} \mathbf{1} \cdot \frac{\mathbf{k}^2}{\omega_n^2 + \mathbf{k}^2} \\
 &\quad - \sum_{\omega_n, \omega'_n} \mathbf{1} \cdot \frac{\mathbf{k}'^2}{\omega_n'^2 + \mathbf{k}'^2} + \sum_{\omega_n, \omega'_n} \frac{\mathbf{k}^2}{\omega_n^2 + \mathbf{k}^2} \frac{\mathbf{k}'^2}{\omega_n'^2 + \mathbf{k}'^2}, \quad (45)
 \end{aligned}$$

where the first three terms vanish after vacuum subtraction, for the same reason as above. The only nonzero contribution comes from the last part of Eq. (45), and the result is

$$\frac{\mathbf{k}^2}{2|\mathbf{k}|T} [1 + 2n_B(|\mathbf{k}|)] \frac{\mathbf{k}'^2}{2|\mathbf{k}'|T} [1 + 2n_B(|\mathbf{k}'|)], \quad (46)$$

using the standard Matsubara sum (A10), where $n_B(E_k) \equiv \frac{1}{e^{\beta E_k} - 1}$ is the Bose distribution. For the second part of the numerator, we use isotropic tensor reduction. First,

$$(\mathbf{k} \cdot \mathbf{k}')^2 = (\delta^{ij} k_i k'_j) (\delta^{mn} k_m k'_n) = \delta^{ij} \delta^{mn} k_i k_m k'_j k'_n. \quad (47)$$

Then,

$$\begin{aligned}
 &\int \frac{d^3 \mathbf{k} d^3 \mathbf{k}'}{(2\pi)^6} \sum_{\omega_n, \omega'_n} \frac{(\mathbf{k} \cdot \mathbf{k}')^2}{(\omega_n^2 + \mathbf{k}^2)(\omega_n'^2 + \mathbf{k}'^2)} \\
 &= \delta^{ij} \delta^{mn} \sum_{\omega_n, \omega'_n} \left(\int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{k_i k_m}{\omega_n^2 + \mathbf{k}^2} \right) \left(\int \frac{d^3 \mathbf{k}'}{(2\pi)^3} \frac{k'_j k'_n}{\omega_n'^2 + \mathbf{k}'^2} \right). \quad (48)
 \end{aligned}$$

For any rotationally invariant function $H(\mathbf{k}^2)$, one has

$$\int d^3 \mathbf{k} k_i k_j H(\mathbf{k}^2) = \frac{\delta_{ij}}{3} \int d^3 \mathbf{k} \mathbf{k}^2 H(\mathbf{k}^2), \quad (49)$$

since the left-hand side is an isotropic rank-two tensor and must therefore be proportional to δ_{ij} ; contracting both sides with δ^{ij} fixes the coefficient to be $1/3$. Applying this identity to both momentum integrals, we obtain

$$\begin{aligned}
 &\int \frac{d^3 \mathbf{k} d^3 \mathbf{k}'}{(2\pi)^6} \sum_{\omega_n, \omega'_n} \frac{(\mathbf{k} \cdot \mathbf{k}')^2}{(\omega_n^2 + \mathbf{k}^2)(\omega_n'^2 + \mathbf{k}'^2)} \\
 &= \frac{1}{9} \delta^{ij} \delta^{mn} \delta_{im} \delta_{jn} \int \frac{d^3 \mathbf{k} d^3 \mathbf{k}'}{(2\pi)^6} \sum_{\omega_n, \omega'_n} \frac{\mathbf{k}^2 \mathbf{k}'^2}{(\omega_n^2 + \mathbf{k}^2)(\omega_n'^2 + \mathbf{k}'^2)} \\
 &= \frac{1}{3} \int \frac{d^3 \mathbf{k} d^3 \mathbf{k}'}{(2\pi)^6} \sum_{\omega_n, \omega'_n} \frac{\mathbf{k}^2 \mathbf{k}'^2}{(\omega_n^2 + \mathbf{k}^2)(\omega_n'^2 + \mathbf{k}'^2)}, \quad (50)
 \end{aligned}$$

where we used $\delta^{ij} \delta^{mn} \delta_{im} \delta_{jn} = 3$. Using the same Matsubara sum (A10),

$$\begin{aligned}
 &\sum_{\omega_n, \omega'_n} \frac{\mathbf{k}^2}{\omega_n^2 + \mathbf{k}^2} \frac{\mathbf{k}'^2}{\omega_n'^2 + \mathbf{k}'^2} \\
 &= \frac{\mathbf{k}^2}{2|\mathbf{k}|T} [1 + 2n_B(|\mathbf{k}|)] \frac{\mathbf{k}'^2}{2|\mathbf{k}'|T} [1 + 2n_B(|\mathbf{k}'|)], \quad (51)
 \end{aligned}$$

we obtain from Eq. (40)

$$\begin{aligned}
 f_{(1)} &= -\frac{16c}{M^4} T^2 \int \frac{d^3 \mathbf{k} d^3 \mathbf{k}'}{(2\pi)^6} \left(\frac{|\mathbf{k}| |\mathbf{k}'|}{4T^2} + \frac{1}{3} \frac{|\mathbf{k}| |\mathbf{k}'|}{4T^2} \right) \\
 &\quad \times [1 + 2n_B(|\mathbf{k}|)] [1 + 2n_B(|\mathbf{k}'|)], \quad (52)
 \end{aligned}$$

i.e.,

$$f_{(1)} = -\frac{16c}{3M^4} T^2 \int \frac{d^3\mathbf{k} d^3\mathbf{k}'}{(2\pi)^6} \frac{|\mathbf{k}||\mathbf{k}'|}{T^2} \times \left[1 + 2 \frac{1}{e^{\beta|\mathbf{k}|} - 1} \right] \left[1 + 2 \frac{1}{e^{\beta|\mathbf{k}'|} - 1} \right]. \quad (53)$$

Then, by the same argument of Eq. (40), only the purely thermal piece remains:

$$\begin{aligned} f_{(1)} &= -\frac{16c}{3M^4} \int \frac{d^3\mathbf{k} d^3\mathbf{k}'}{(2\pi)^6} \frac{4|\mathbf{k}||\mathbf{k}'|}{[e^{\beta|\mathbf{k}|} - 1][e^{\beta|\mathbf{k}'|} - 1]} \\ &= -\frac{16c}{3M^4 \pi^4} \int dk dk' \frac{k^3 k'^3}{[e^{\beta k} - 1][e^{\beta k'} - 1]} \\ &= -\frac{16c\pi^4}{675M^4 \beta^8}. \end{aligned} \quad (54)$$

Finally, the free energy density, including both the leading-order and first-order contributions, is given by

$$f = f_{(0)} + f_{(1)} = -\frac{\pi^2 T^4}{90} - \frac{16c\pi^4 T^8}{675M^4}. \quad (55)$$

At this point, having incorporated the leading and first-order terms, one might naturally ask whether higher-order corrections, such as the second-order term $f_{(2)}$, should also be considered. To assess this, we estimate the magnitude of subsequent terms in the perturbative expansion. For example,

$$f_{(2)} \sim \frac{c^2}{M^8} T^{12} \sim c^2 \left(\frac{T}{M} \right)^8 T^4, \quad (56)$$

$$f_{(3)} \sim \frac{c^3}{M^{12}} T^{16} \sim c^3 \left(\frac{T}{M} \right)^{12} T^4, \quad (57)$$

and so on. Each term is increasingly suppressed by higher powers of the expansion parameter $(T/M)^4$, whether arising from higher-loop corrections or higher-dimensional operators.

The entropy density is therefore

$$s = -\frac{\partial f}{\partial T} = \frac{2\pi^2 T^3}{45} + \frac{128c\pi^4 T^7}{675M^4}. \quad (58)$$

For our positivity analysis, we work with thermal entropy density (entropy per unit volume) in the canonical ensemble at fixed T and V , and compare two theories: the low-energy EFT obtained after integrating out the heavy sector, which includes the induced higher-derivative interaction proportional to c , and the free light theory obtained by removing the heavy sector and setting $c = 0$.

The physical requirement we impose is that, in a consistent ultraviolet completion, integrating out additional microscopic degrees of freedom should not reduce the

thermal entropy of the resulting low-energy description in the regime $m \ll T \ll M$. Importantly, this statement is not a claim of monotonicity under arbitrary interactions at fixed T , but rather a comparison between two different theories: one obtained from a UV-complete model by integrating out a heavy field, and one in which that heavy sector is absent altogether.

Evaluating the entropy density in the effective theory perturbatively in $T/M \ll 1$, we find

$$s_{\text{EFT}}(T) = s_{\text{free}}(T) + \frac{128\pi^4 c T^7}{675 M^4} + \mathcal{O}\left(\frac{T^{11}}{M^8}\right). \quad (59)$$

Requiring that the entropy density of the EFT exceed that of the free light theory,

$$s_{\text{EFT}}(T) > s_{\text{free}}(T) \quad (m \ll T \ll M), \quad (60)$$

immediately implies

$$c > 0. \quad (61)$$

This criterion can be viewed as a thermodynamic consistency condition on the low-energy description induced by heavy physics, rather than a general statement about interacting systems at fixed temperature.

IV. ILLUSTRATIVE ULTRAVIOLET COMPLETIONS

A. A global $U(1)$ UV completion

To make the comparison leading to $c > 0$ more explicit, it is useful to exhibit a simple UV completion in which the Goldstone EFT arises by integrating out a heavy field. A minimal example is the global $U(1)$ linear sigma model,

$$\mathcal{L}_{\text{UV}} = -|\partial_\mu \Phi|^2 - \lambda \left(|\Phi|^2 - \frac{v^2}{2} \right)^2, \quad \lambda > 0, \quad (62)$$

where Φ is a complex scalar and the vacuum spontaneously breaks the global $U(1)$ symmetry. Parametrizing fluctuations in polar variables,

$$\Phi(x) = \frac{v + h(x)}{\sqrt{2}} e^{i\phi(x)/v}, \quad (63)$$

the Lagrangian becomes

$$\mathcal{L}_{\text{UV}} = -\frac{1}{2}(\partial h)^2 - \frac{1}{2} \left(1 + \frac{h}{v} \right)^2 (\partial \phi)^2 - \frac{1}{2} m_h^2 h^2 + \dots, \quad (64)$$

$$m_h^2 = 2\lambda v^2, \quad (65)$$

where h is the heavy radial mode and ϕ is the (massless) Goldstone boson. At energies and temperatures well below

the radial mass, $E, T \ll m_h$, we may integrate out h to obtain a local EFT for ϕ . At tree level, solving the equation of motion for h gives

$$h \simeq -\frac{1}{vm_h^2}(\partial\phi)^2 + \mathcal{O}\left(\frac{\partial^4}{m_h^4}\right), \quad (66)$$

which upon substitution into (65) yields

$$\mathcal{L}_{\text{EFT}} = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{2v^2m_h^2}(\partial\phi)^4 + \mathcal{O}\left(\frac{\partial^6}{m_h^6}\right). \quad (67)$$

Matching to the form used in the main text,

$$\mathcal{L}_{\text{EFT}} = -\frac{1}{2}(\partial\phi)^2 + \frac{c}{M^4}(\partial\phi)^4 + \dots, \quad (68)$$

and identifying $M \equiv m_h$, we obtain

$$\frac{c}{M^4} = \frac{1}{2v^2m_h^2} \Rightarrow c = \frac{m_h^2}{2v^2} = \lambda > 0. \quad (69)$$

Thus, in this explicit UV completion the positivity of c follows directly from the stability condition $\lambda > 0$ of the scalar potential.

Thermal free energy and entropy at $T \ll M$. In the full UV theory (62), the heavy radial mode contributes to thermal quantities only through Boltzmann-suppressed terms $\sim e^{-m_h/T}$. Therefore, for $T \ll m_h$ the temperature-dependent part of the free energy is dominated by the light Goldstone sector and is captured by the EFT (67). Expanding perturbatively in $T/m_h \ll 1$ reproduces the same leading correction found in the main text,

$$f_{\text{EFT}}(T) = f_{\text{free}}(T) - \frac{16\pi^4}{675} \frac{cT^8}{M^4} + \mathcal{O}\left(\frac{T^{12}}{M^8}\right), \quad (70)$$

and hence,

$$s_{\text{EFT}}(T) - s_{\text{free}}(T) = +\frac{128\pi^4}{675} \frac{cT^7}{M^4} + \mathcal{O}\left(\frac{T^{11}}{M^8}\right). \quad (71)$$

Requiring that the low-energy description obtained by integrating out the heavy field have larger thermal entropy density than the free light theory in the regime $m \ll T \ll M$ then implies $c > 0$, in agreement with the UV matching (69). We emphasize that this is a comparison between two different theories (with and without the heavy sector), rather than a general monotonicity statement for arbitrary interactions at fixed temperature.

B. $\lambda\phi^4$ interaction and its UV completion

We begin by considering a scalar field theory in four dimensions with a quartic self-interaction,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \lambda\phi^4, \quad (72)$$

where we follow the sign convention appropriate to our thermodynamic analysis. With this convention, stability of the vacuum requires

$$\lambda > 0. \quad (73)$$

This sign choice does not contradict our entropy bound, since the operator ϕ^4 is marginal in four dimensions and should be interpreted as a self-interaction of the light field, rather than as an operator generated by integrating out heavy degrees of freedom. In general, such marginal interactions are not constrained by our entropy-based argument.

However, in certain restricted situations the quartic interaction can arise from integrating out a heavy field. In such cases, the sign of λ is not arbitrary but is fixed by the structure of the ultraviolet completion. A simple toy model illustrating this point is provided by a tree-level exchange of a heavy scalar field.

Consider a UV theory in which the light scalar ϕ interacts with a heavy scalar Φ of mass M through a cubic coupling,

$$\mathcal{L}_{\text{UV}} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}(\partial_\mu\Phi)^2 - \frac{1}{2}M^2\Phi^2 - \frac{g}{2}\phi^2\Phi. \quad (74)$$

At energies well below the heavy mass scale, $E \ll M$, the heavy field Φ can be integrated out. At tree level, this amounts to solving its classical equation of motion,

$$(\square + M^2)\Phi = -\frac{g}{2}\phi^2, \quad (75)$$

which in the low-energy limit reduces to

$$\Phi(x) \simeq -\frac{g}{2M^2}\phi^2(x). \quad (76)$$

Substituting this solution back into the Lagrangian (74), one obtains the effective quartic interaction for ϕ ,

$$\begin{aligned} V_{\text{eff}}(\phi) &\simeq \frac{1}{2}M^2\left(-\frac{g}{2M^2}\phi^2\right)^2 + \frac{g}{2}\phi^2\left(-\frac{g}{2M^2}\phi^2\right) \\ &= -\frac{g^2}{8M^2}\phi^4. \end{aligned} \quad (77)$$

Comparing with (72), we identify the Wilson coefficient

$$\lambda = -\frac{g^2}{8M^2} < 0. \quad (78)$$

Since $g^2 > 0$ and $M^2 > 0$, the quartic coupling generated by the tree-level exchange of a heavy scalar is necessarily

negative. This reflects the general fact that single-boson exchange mediates an attractive interaction, corresponding to a negative contribution to the effective potential.

In this sense, when a $\lambda\phi^4$ interaction arises from integrating out heavy degrees of freedom, its sign is fixed and agrees with the expectation from the underlying UV physics. Our entropy bound is consistent with this structure: it constrains leading higher-dimensional operators generated by heavy fields, while allowing marginal self-interactions such as ϕ^4 to have either sign unless they are tied to a specific UV completion.

V. CONNECTION TO UNITARITY AND CAUSALITY

Most existing positivity bounds are derived from the analytic structure of $2 \rightarrow 2$ scattering amplitudes, where fundamental principles such as unitarity and causality play a central role. Unitarity implies the optical theorem, which enforces the non-negativity of the imaginary part of forward scattering amplitudes, while causality leads to analyticity properties of amplitudes in the complex energy plane. Combined with dispersion relations, these properties translate into positivity constraints on Wilson coefficients in the low-energy effective action.

The entropy-based bound derived in this work follows a different logical route and does not rely directly on scattering amplitudes. Instead, it is based on thermodynamic considerations and general properties of quantum entropy. It is therefore appropriate to clarify the limited and indirect sense in which unitarity and causality enter the discussion.

- (a) *Unitarity and entropy inequalities.* The entropy inequalities employed in this work, in particular the Araki-Lieb inequality [68,69], are kinematic statements about von Neumann entropies of density matrices. They hold for any bipartite quantum system and do not require unitary time evolution or specific dynamical assumptions. In particular, unitarity does not play a role in the derivation of the Araki-Lieb inequality itself.

In our context, unitarity is assumed only at a minimal level: namely, that the quantum theory admits a well-defined density matrix with a positive spectrum and conserved trace,

$$\frac{d}{dt} \text{Tr}(\rho) = 0. \quad (79)$$

These basic consistency conditions ensure that von Neumann entropy is well-defined and that general entropy inequalities can be meaningfully applied. Beyond this assumption of probability conservation and positivity, no further use of dynamical unitarity is made in our argument.

- (b) *Causality and relativistic consistency.* Causality enters our analysis only insofar as the thermal quantum field theory under consideration is assumed to be a consistent relativistic quantum field theory (QFT). In the Euclidean formulation used to compute the thermal partition function, there is no notion of real-time evolution or signal propagation. The connection to causality arises only upon analytic continuation to the Lorentzian signature.

In axiomatic quantum field theory, relativistic causality is encoded in the principle of locality, formulated, for example, in the Haag-Kastler framework as the commutativity of observables at spacelike separation [72],

$$[\phi(x), \phi(y)] = 0 \quad \text{for } (x - y)^2 > 0. \quad (80)$$

This microcausality condition follows from the analytic structure of correlation functions and is consistent with the reconstruction of Lorentzian QFT from Euclidean correlation functions satisfying the Osterwalder-Schrader axioms, including reflection positivity [73,74].

In practical terms, this structure is reflected in the standard $i\epsilon$ prescription of the Feynman propagator,

$$G_F(x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon}, \quad (81)$$

which specifies how Lorentzian correlators are obtained as boundary values of Euclidean correlation functions. In the present work, causality plays no direct role in deriving the entropy inequality, but it underlies the assumption that the Euclidean thermal path integral admits a consistent continuation to a relativistic quantum theory.

- (c) *Summary.* In summary, the entropy-based positivity bound presented here does not rely on unitarity and causality as direct inputs in the way amplitude-based approaches do. Instead, these principles enter only indirectly, through the assumption that the underlying quantum field theory is internally consistent, admits a well-defined thermal partition function, and satisfies general quantum information inequalities. From this perspective, our result should be viewed as complementary to existing positivity bounds derived from scattering amplitudes, rather than as an alternative derivation of them.

VI. CONCLUSION AND DISCUSSION

In this work we computed the thermal entropy of a shift-symmetric scalar EFT and showed that thermodynamic consistency yields a nontrivial sign constraint on the effective interactions. In particular, in the regime $m \ll T \ll M$, the entropy comparison between the

heavy-induced EFT and the free light theory implies that the leading dimension-eight Wilson coefficient is strictly positive.

The derivation is based on finite-temperature field theory and entropy inequalities, rather than on scattering amplitudes and dispersion relations. Viewed this way, the result provides a complementary perspective on EFT consistency and clarifies how thermodynamic data can encode constraints usually associated with analyticity, unitarity, and causality.

The same strategy can be extended in several directions, including EFTs with fermions, gauge fields, and gravity. These settings are often difficult for standard amplitude-based positivity bounds because of massless exchanges and, in some backgrounds, the lack of a conventional S -matrix. Since our method does not depend on a flat-space S -matrix, it may provide a useful alternative diagnostic in such cases.

Note added. Recently, we became aware of a closely related preprint by Fernández-Sarmiento, Penco, and Rosen [75] through correspondence from Riccardo Penco. Their analysis formulates an entropy-positivity condition in terms of a negative thermal shift of the grand potential and tests it across several systems, together with a discussion of regimes where the bound may not apply. Our approach starts from a different argument: by beginning with the Araki-Lieb inequality and an explicit leading-order thermal QFT computation, we demonstrate this condition in a Lorentz-invariant scalar EFT and obtain the positive sign of the leading dimension-eight operator. We also outline the assumptions of our derivation and its regime of validity within this setup.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: USEFUL MATSUBARA SUM FORMULAS

In this appendix we collect and derive several Matsubara sum identities used in the main text. We focus on bosonic Matsubara frequencies

$$\omega_n = 2\pi nT, \quad T = \beta^{-1}, \quad n \in \mathbb{Z}, \quad (\text{A1})$$

which arise naturally in the imaginary-time formalism of finite-temperature field theory.

Throughout this appendix we consider a massless scalar field for simplicity, since this is the case relevant for the EFT discussed in the main text.

1. The contour integral method

The standard technique for evaluating sums of the form $S = T \sum_{n=-\infty}^{\infty} g(i\omega_n)$ is to convert the sum into a contour integral in the complex energy plane (z -plane). We introduce the Bose-Einstein distribution function as a weighting function:

$$n_B(z) = \frac{1}{e^{\beta z} - 1}. \quad (\text{A2})$$

The function $n_B(z)$ has simple poles strictly on the imaginary axis at $z = i\omega_n = i \frac{2\pi n}{\beta}$ with residue T :

$$\text{Res}[n_B(z), z = i\omega_n] = \lim_{z \rightarrow i\omega_n} \frac{z - i\omega_n}{e^{\beta z} - 1} = \frac{1}{\beta e^{i2\pi n}} = T. \quad (\text{A3})$$

By Cauchy's residue theorem, the sum can be written as a contour integral enclosing the imaginary axis:

$$T \sum_{n=-\infty}^{\infty} g(i\omega_n) = \oint_C \frac{dz}{2\pi i} g(z) n_B(z), \quad (\text{A4})$$

where the contour C consists of small counterclockwise circles around each pole $i\omega_n$. Assuming $g(z)$ is analytic except for specific poles on the real axis (and decays sufficiently fast at infinity), we can deform the contour C into two integration lines running parallel to the imaginary axis (one slightly to the right, one slightly to the left) and effectively wrap them around the physical poles of $g(z)$.

This leads to the “Master Formula” for Matsubara summation:

$$T \sum_{n=-\infty}^{\infty} g(i\omega_n) = - \sum_{z_k} \text{Res}[g(z) n_B(z)]_{z=z_k}, \quad (\text{A5})$$

where z_k are the poles of the function $g(z)$.

2. Summation for the propagator

We compute the sum appearing in the perturbation diagrams:

$$S_1 = T \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + \mathbf{k}^2}. \quad (\text{A6})$$

Here, $g(i\omega_n) = \frac{1}{-(i\omega_n)^2 + \mathbf{k}^2} = \frac{1}{-z^2 + \mathbf{k}^2} = -\frac{1}{z^2 - \mathbf{k}^2}$. The function $g(z) = -\frac{1}{(z - |\mathbf{k}|)(z + |\mathbf{k}|)}$ has two simple poles on the real axis

at $z = \pm|\mathbf{k}|$. Applying the residue formula,

$$S_1 = -\left(\text{Res}\left[\frac{-n_B(z)}{z^2 - \mathbf{k}^2}\right]_{z=|\mathbf{k}|} + \text{Res}\left[\frac{-n_B(z)}{z^2 - \mathbf{k}^2}\right]_{z=-|\mathbf{k}|}\right) \quad (\text{A7})$$

$$= \frac{n_B(|\mathbf{k}|)}{2|\mathbf{k}|} + \frac{n_B(-|\mathbf{k}|)}{-2|\mathbf{k}|} \quad (\text{A8})$$

$$= \frac{1}{2|\mathbf{k}|} [n_B(|\mathbf{k}|) - n_B(-|\mathbf{k}|)]. \quad (\text{A9})$$

Using the identity $n_B(-|\mathbf{k}|) = -(1 + n_B(|\mathbf{k}|))$, we obtain the standard result:

$$T \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + \mathbf{k}^2} = \frac{1}{2|\mathbf{k}|} [1 + 2n_B(|\mathbf{k}|)]. \quad (\text{A10})$$

The two terms correspond to different physics: The first term $1/(2k)$ corresponds to the zero-temperature (vacuum) contribution. The second term $n_B(k)/k$ represents the thermal excitation of bosonic modes. In thermal calculations, one often subtracts the vacuum piece, keeping only the $n_B(k)$ terms' contribution.

3. Sum of logarithms: $\sum_n \ln [(\omega_n^2 + \mathbf{k}^2)\beta^2]$

In the evaluation of the free partition function, one encounters sums of the form

$$S_2 = \sum_{n=-\infty}^{\infty} \ln [(\omega_n^2 + \mathbf{k}^2)\beta^2]. \quad (\text{A11})$$

Since $\ln(z)$ has a branch cut rather than simple poles, the direct residue method is subtle. Instead, differentiating with respect to $|\mathbf{k}|$ gives

$$\frac{dS_2}{d|\mathbf{k}|} = \sum_n \frac{2|\mathbf{k}|}{\omega_n^2 + \mathbf{k}^2} = 2|\mathbf{k}| \sum_n \frac{1}{\omega_n^2 + \mathbf{k}^2}. \quad (\text{A12})$$

Using Eq. (A10),

$$\frac{dS_2}{d|\mathbf{k}|} = 2|\mathbf{k}| \cdot \beta \frac{1}{2|\mathbf{k}|} [1 + 2n_B(|\mathbf{k}|)] = \beta [1 + 2n_B(|\mathbf{k}|)]. \quad (\text{A13})$$

Integrating with respect to $|\mathbf{k}|$,

$$S_2(|\mathbf{k}|) = \beta|\mathbf{k}| + 2 \int d|\mathbf{k}'| \beta n_B(|\mathbf{k}'|) + \text{const.} \quad (\text{A14})$$

Using

$$\beta n_B(|\mathbf{k}|) = \frac{d}{d|\mathbf{k}|} \ln(1 - e^{-\beta|\mathbf{k}|}), \quad (\text{A15})$$

we obtain

$$\sum_n \ln [(\omega_n^2 + \mathbf{k}^2)\beta^2] = \beta|\mathbf{k}| + 2 \ln(1 - e^{-\beta|\mathbf{k}|}) + \text{const.} \quad (\text{A16})$$

The additive constant is independent of $|\mathbf{k}|$ and contributes only to a temperature-independent vacuum term, which can be dropped when computing thermodynamic quantities.

APPENDIX B: POISSON SUMMATION AND THERMAL SUM DECOMPOSITION

In this appendix we derive the Poisson summation formula and show how it leads to the decomposition of Matsubara sums into vacuum and thermal contributions. This is used in the main text to justify that constant sum-integrals contain only vacuum contributions.

1. Derivation of the poisson summation formula

Let $f(x)$ be a function on the real line with Fourier transform

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x). \quad (\text{B1})$$

We define a sum:

$$P(x) \equiv \sum_{n \in \mathbb{Z}} f(x + na). \quad (\text{B2})$$

It is immediate that $P(x)$ is periodic with period a : $P(x + a) = P(x)$. Any sufficiently regular periodic function admits a Fourier series expansion, so

$$P(x) = \sum_{m \in \mathbb{Z}} c_m e^{i\frac{2\pi m}{a}x}, \quad (\text{B3})$$

with Fourier coefficients

$$c_m = \frac{1}{a} \int_0^a dx P(x) e^{-i\frac{2\pi m}{a}x}. \quad (\text{B4})$$

Substituting Eq. (B2) into Eq. (B4) gives

$$c_m = \frac{1}{a} \int_0^a dx \sum_{n \in \mathbb{Z}} f(x + na) e^{-i\frac{2\pi m}{a}x}. \quad (\text{B5})$$

Equivalently,

$$\begin{aligned}
c_m &= \frac{1}{a} \int_0^a dx \sum_{n \in \mathbb{Z}} f(x + na) e^{-i\frac{2\pi m}{a}x} \\
&= \frac{1}{a} \sum_{n \in \mathbb{Z}} \int_0^a dx f(x + na) e^{-i\frac{2\pi m}{a}x} \\
&= \frac{1}{a} \int_{-\infty}^{\infty} dx f(x) e^{-i\frac{2\pi m}{a}x} \\
&= \frac{1}{a} \tilde{f}\left(\frac{2\pi m}{a}\right). \tag{B6}
\end{aligned}$$

Substituting into Eq. (B3) gives

$$P(x) = \frac{1}{a} \sum_{m \in \mathbb{Z}} \tilde{f}\left(\frac{2\pi m}{a}\right) e^{i\frac{2\pi m}{a}x}. \tag{B7}$$

Evaluating at $x = 0$ yields the Poisson summation formula

$$\sum_{n \in \mathbb{Z}} f(na) = \frac{1}{a} \sum_{m \in \mathbb{Z}} \tilde{f}\left(\frac{2\pi m}{a}\right). \tag{B8}$$

2. Constant integrand and absence of thermal correction

To analyze the constant integrand, we introduce a regulator

$$F_\alpha(\omega) = e^{-\alpha|\omega|}, \quad \alpha > 0, \tag{B9}$$

and take $\alpha \rightarrow 0^+$ at the end. Using Eq. (B8),

$$T \sum_{n \in \mathbb{Z}} F_\alpha(\omega_n) = \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{im\beta\omega} e^{-\alpha|\omega|}. \tag{B10}$$

For $m = 0$,

$$\int \frac{d\omega}{2\pi} e^{-\alpha|\omega|}, \tag{B11}$$

the integrand does not depend on β and therefore produces only a temperature-independent contribution. In thermal field theory, such contributions correspond to vacuum normalization and are removed by the standard subtraction of the zero-temperature free energy.

While the $m \neq 0$ terms encode genuine thermal corrections, the integral evaluates to

$$\frac{1}{\pi} \frac{\alpha}{\alpha^2 + (m\beta)^2}, \tag{B12}$$

which vanishes as $\alpha \rightarrow 0^+$. Therefore, the thermal correction vanishes:

$$\Delta_T[1] = 0, \quad \Rightarrow \left[T \sum_n - \int \frac{d\omega}{2\pi} \right] 1 = 0. \tag{B13}$$

Thus, constant sum-integrals contain only vacuum contributions and are removed by vacuum subtraction in thermal free energy.

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