

Nested ansatz method for Baker-Akhiezer functions

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We explain that the logic behind the derivation of the Noumi-Shiraishi function can be applied directly to the Baker-Akhiezer function (BAF). This amounts to changing an ansatz for BAF to a nested one, where the BAF of $N + 1$ variables is recursively expressed as a sum over BAFs of N variables. This may be seen as a generalization of symmetrization trick from [A. Mironov, A. Morozov, and A. Popolitov, [arXiv:2601.10500](#)] but for the generally nonsymmetric BAF. We demonstrate that, for usual nontwisted ($a = 1$) BAFs, this method correctly reproduces the Noumi-Shiraishi formula directly from linear equations, resolving the ambiguity related to nonsimple roots. For the first nontrivial twisted case ($N = 3$, $a = 2$) this method also fixes this ambiguity, moreover, answers for the first few layers of coefficients are in the form of *direct quantization* of the above article.

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I. INTRODUCTION

Multivariable Baker-Akhiezer functions (BAFs), introduced by O. Chalykh in [1,2] are peculiar common polynomial eigenfunctions of the Ruijsenaars-Schneider (RS) Hamiltonians possessing the bispectral symmetry. These latter Hamiltonians were recently realized [3] to be just one of infinitely many integrable systems, each corresponding to a ray of operators in the Ding-Iohara-Miki (DIM) algebra [4,5] with a certain slope (a_1, a_2) . The story actually has simpler incarnations at the level of the $W_{1+\infty}$ [6–9] and affine Yangian algebras [10,11], where it is most naturally related to superintegrable partition functions of matrix models [12–14]. Still, for BAFs most natural seems to be the fully (q, t) -deformed level of DIM algebra, where its connection to triad structure [15] (interrelating BAFs [1,16], Macdonald polynomials [17], and Noumi-Shiraishi (NS) functions [18]) is most clearly seen.

At this level, Hamiltonian systems for other (distinct from RS) rays (a_1, a_2) are solved by so-called twisted BAFs [16,19] which are, in turn, expected to belong to their respective (a_1, a_2) -triads, interrelating them with proper

(a_1, a_2) analogs of the Macdonald polynomials and Noumi-Shiraishi functions.

However, manifest expressions for the (a_1, a_2) -triads are not known (apart from the simplest $N = 2$, $a_1 = 1$, $a_2 = a = 2$ case) [20], the main complication being the gauge freedom in the definition of (twisted) BAF which, if not resolved correctly, completely obfuscates the triad structure. In this paper, we propose a method to fix this gauge freedom in a particular way that is consistent with the nontwisted ($a = 1$) case, with help of the nested ansatz.

In the original formulation by O. Chalykh [1], the Baker-Akhiezer function is defined as a peculiar ansatz

$$\Psi(\underline{z}, \underline{\lambda}) = q^{(\underline{\lambda} + m\underline{e}, \underline{z})} \sum_{k_{ij}=0}^m q^{-\sum_{i<j} k_{ij}(z_i - z_j)} \psi_{\underline{k}}(\underline{\lambda}) \quad (1)$$

satisfying a set of linear equations (periodicity conditions [21])

$$\Psi(z_k + j, \underline{\lambda}) = \Psi(z_l - j, \underline{\lambda}), \quad j = 1..m, k < l, \quad z_k = z_l \quad (2)$$

He proves that such defined $\Psi(\underline{z}, \underline{\lambda})$ exists and is unique up to normalization, as a function of $\underline{z}$ and $\underline{\lambda}$.

Inherent in this definition is ambiguity in choosing individual coefficients $\psi_{\underline{k}}$. This can be seen for the first time when $N = 3$, $m = 1$. In this case, there are two simple positive roots: $\alpha_{1,2}$ and $\alpha_{2,3}$, and one nonsimple root $\alpha_{1,3} = \alpha_{1,2} + \alpha_{2,3}$, therefore, the combination of coefficients $\tilde{\psi}_{1,1,0} = \tilde{\psi}_{1,1,0} + \tilde{\psi}_{0,0,1}$ enters all equations as a monolithic sum. Hence, the function Ψ is unambiguously

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defined, while the splitting of $\tilde{\psi}_k$ into individual ψ_k is essentially free.

In [22], we demonstrated that, in first simple cases, it is possible to resolve this ambiguity in a smart way by *additionally* requiring certain pole structure of individual items ψ_k . In [15], this initial observation further led to discovery of a remarkable *triad structure* that interlinks BAFs, Noumi-Shiraishi functions and Macdonald polynomials.

However, besides the triad relation to the NS function, which does have nicely factorized split formula for its coefficients, there seemed to be no reason in general why BAF groups of coefficients $\tilde{\psi}$ should split into nicely factorized q -Pochhammer expressions. At the same time, case analysis we performed in [23] suggests that this is the desired decomposition, as it is the one consistent with the *direct quantization* paradigm.

As we briefly explained in [24], and now rehearse in Sec. II, the technical reason behind the factorization of Macdonald (and NS) coefficients is that one is able to superimpose two independent structures:

- (i) one is the root lattice, consisting of $\alpha_{i,j} = e_i - e_j$ and, correspondingly, ratios $(\frac{x_i}{x_j})$, $i < j$
- (ii) another is nesting of permutation groups $S_N \supset S_{N-1} \supset \dots$ and, correspondingly, *individual* x_i , $i = 1..N$.

A natural question is, then, whether this superimposition of structures can be performed directly on the level of Baker-Akhiezer function, i.e. on the level of the corresponding ansatz and linear equation. This should be useful in situations where triadic picture is not fully worked out yet: twisted BAFs being one main example, and triads for other (than A_{N-1}) root systems being another.

The claim of this short letter is that this is, indeed, possible. We explain how this works in case of usual BAF in Sec. III and show that, indeed, the NS expansion is correctly reproduced this way.

In Sec. IV, we apply this reasoning to the first nontrivial twist case, $a = 2$. Already known [see [22] Eq. (49)] is the $N = 2$, $a = 2$ twisted BAF for an arbitrary m , and, in this section, we use the nested ansatz method to get manifest expressions for the first few layers of coefficients in the $N = 3$, $a = 2$ case. Unfortunately, solving even the nested linear system completely for high enough m [which is needed to restore/guess formulas for generic m , see Eqs. (38)–(46)] proves to be computationally challenging; difficulty being all the more exacting since here (in variance with [23]) we are interested in the answers for arbitrary $\underline{\lambda}$ and not just at the special point $\lambda = (0, m, \dots, (N-1)m)$.

Still, even the answers that *are* available, demonstrate the tip of the emerging structure:

- (i) just as in the $N = 2$, $a = 2$ case, the answers are sums of fully factorized q -Pochhammer-like expressions (which are similar to the ones appearing in original Noumi-Shiraishi function);

- (ii) dependence on $\Lambda_i = q_i^\lambda$ is fully encapsulated (hidden) inside peculiar homogeneous combinations $g_{i,j}(k)$ [see (25)];
- (iii) apart from $g_{i,j}(k)$ factors, all other factors are symmetric q -numbers, where m enters linearly [see, for instance, (30)];
- (iv) these two latter properties combined mean that the answers adhere to the general spirit of *direct quantization*, advocated in [23]. Moreover, answers analytically continue to arbitrary $t = q^{-m}$ straightforwardly.

These notable properties of the answers give some merit to the method they were obtained by: to the nested ansatz method. Demonstrating usefulness and advantages of this method is the main goal of this short letter. We conclude in Sec. V with some plausible directions for future applications.

Notation We denote with an underline the sequences of variables, for instance

$$\underline{x} = x_1, \dots, x_N; \quad \underline{\lambda} = \lambda_1, \dots, \lambda_N \quad (3)$$

$(x; q)_n$ is the q -Pochhammer symbol

$$(x; q)_n = \prod_{k=0}^{n-1} (1 - q^k x) = \frac{(x; q)_\infty}{(q^n x; q)_\infty} \quad (4)$$

$[n]$ and $[n]!$ (and, respectively, $[[n]]$, $[[n]]!$) are the q -numbers and q -factorials in usual (and, respectively, symmetric) conventions

$$[n] = \frac{q^n - 1}{q - 1} \quad [n]! = \prod_{k=1}^n [k];$$

$$[[n]] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \quad [[n]]! = \prod_{k=1}^n [[k]] \quad (5)$$

II. NESTED ANSATZ FOR MACDONALD POLYNOMIALS AND NS FUNCTION

Let us recall the reasoning behind original Macdonald's explicit formula [17] [Eqs. (6.20), (6.24), (7.14')] and its Noumi-Shiraishi generalization [18] [Eqs. (1.10), (1.11)] to the case of noninteger partitions λ .

Key observation is that it is possible to expand Macdonald polynomial of N variables $x_1, \dots, x_N$ into Macdonald polynomials of $N-1$ variables $x_2, \dots, x_N$, with x_1 -dependent coefficients

$$M_\lambda(x_1, \dots, x_N) = \sum_{\mu} c_{\lambda, \mu}(x_1) M_\mu(x_2, \dots, x_N) \quad (6)$$

where the expansion coefficients $c_{\lambda, \mu}(x_1)$ are nothing but one-variable skew Macdonald polynomials

$$c_{\lambda,\mu}(x_1) = M_{\lambda/\mu}(x_1) = \prod_{1 \leq i < j \leq n} \frac{(q^{\mu_j - \lambda_i + 1} t^{j-i-1}; q)_{\lambda_i - \mu_i}}{(q^{\mu_i - \lambda_j} t^{j-i}; q)_{\lambda_i - \mu_i}} \prod_{1 \leq i \leq j < n} \frac{(q^{\mu_i - \mu_j} t^{j-i+1}; q)_{\lambda_i - \mu_i}}{(q^{\mu_i - \mu_j + 1} t^{j-i}; q)_{\lambda_i - \mu_i}} \cdot x_1^{|\lambda| - |\mu|} \quad (7)$$

Here, crucially, the formula is very explicit and manifest in terms of λ, μ and x_1 .

The observation of Noumi-Shiraishi is then that the formula (7) makes sense not only for integral $\lambda \in \mathbb{Z}_{>0}$, but for arbitrary complex λ . Correspondingly, the Noumi-Shiraishi function (which now features infinite summation over the space of upper-triangular N by N matrices $\mathcal{M}_N$, with entries $\theta_{i,j}$)

$$\begin{aligned} \mathfrak{P}_{q,t}(\underline{x}, \underline{\lambda}) &= \prod_{i=1}^N \left(x_i^{\lambda_i - \beta \rho_i} \right) \sum_{\theta_{ij}=0}^{\infty} \left(\prod_{n=2}^N \prod_{1 \leq i < n} \prod_{s=0}^{\theta_{in}-1} \frac{(1 - q^{s+1-\theta_{in}} t^{-1}) (1 - t q^{s+\lambda_n - \lambda_i + \sum_{b>n} (\theta_{ib} - \theta_{nb})})}{(1 - q^{s-\theta_{in}}) (1 - q^{s+1+\lambda_n - \lambda_i + \sum_{b>n} (\theta_{ib} - \theta_{nb})})} \right) \\ &\times \prod_{n=2}^N \prod_{1 \leq i < j < n} \prod_{s=0}^{\theta_{in}-1} \frac{(1 - t q^{s+\lambda_j - \lambda_i + \sum_{b>n} (\theta_{ib} - \theta_{jb})}) (1 - t^{-1} q^{s+1+\lambda_j - \lambda_i - \theta_{jn} + \sum_{b>n} (\theta_{ib} - \theta_{jb})})}{(1 - q^{s+1+\lambda_j - \lambda_i + \sum_{b>n} (\theta_{ib} - \theta_{jb})}) (1 - q^{s+\lambda_j - \lambda_i - k_{jn} + \sum_{b>n} (\theta_{ib} - \theta_{jb})})} \right) \prod_{1 \leq i < j \leq N} \left(\frac{x_j}{x_i} \right)^{\theta_{ij}}, \quad (8) \end{aligned}$$

$\beta = \log_q(t)$, is shown in [18] to be the simultaneous eigenfunction of all Ruijsenaars-Schneider operators. In terms of generating function for the RS operators [18] [Eq. (1.3)] this can be stated as follows

$$D^x(u) \mathfrak{P}_{q,t}(\underline{x}, \underline{\lambda}) = \mathfrak{P}_{q,t}(\underline{x}, \underline{\lambda}) \cdot \prod_{i=1}^N (1 - u s_i) \quad (9)$$

Crucially, the whole line-of-thought builds upon on the possibility of expansion (6)—which is possible due to Macdonald polynomials being the basis in the space of symmetric polynomials in *any* number of variables x .

III. NESTED ANSATZ FOR NONTWISTED BAF

In [15] we discovered that O.Chalykh's Baker-Akhiezer function (1), (2) while defined from completely different considerations, has striking relation to the Noumi-Shiraishi function (8)

$$\begin{aligned} \Psi(\underline{x}, \underline{\lambda}) &= \mathcal{N}_{\underline{\lambda}} \mathfrak{P}_{q,q^{-m}}(\underline{x}, \underline{\lambda}) \\ \mathcal{N}_{\underline{\lambda}} &= \prod_{k>l} \prod_{j=1}^m \left(q^{\frac{\lambda_k - \lambda_l}{2} - j} - q^{-\frac{\lambda_k - \lambda_l}{2}} \right) \quad (10) \end{aligned}$$

which, furthermore, is a part of more complex triad structure, interrelating BAFs, Macdonald polynomials and NS functions [16, fig.1].¹

¹Note that the coefficient $\mathcal{N}_{\underline{\lambda}}$ occurs when one is interested in bi-spectral dual Ψ : $\Psi(\underline{x}, \underline{\lambda}) \stackrel{\text{def}}{=} \Psi(\underline{\lambda}, \underline{x})$. In practice one often works in nonsymmetric normalization where the first coefficient of Ψ is unity: $\psi_0(\underline{\lambda}) = 1$. In such normalization one simply has $\Psi(\underline{x}, \underline{\lambda}) = \mathfrak{P}_{q,q^{-m}}(\underline{x}, \underline{\lambda})$.

In this triadic logic, a natural question is then, whether the reasoning from the previous section, that relies on the decomposition into basis in $x_2, \dots, x_N$, which is present for Macdonald and NS “vertices” of the triad, may be somehow adapted and applied directly to the BAF “vertex.”

Recall that (nontwisted) Baker-Akhiezer function is defined (following [1]) to be a function of variables $x_1, \dots, x_N$ and $\lambda_1, \dots, \lambda_N$ of the form

$$\Psi_m^{[N]}(\underline{z}, \underline{\lambda}) = q^{(\underline{\lambda} + m \underline{\rho}, \underline{z})} \sum_{k_{ij}=0}^m q^{-\sum_{i<j} k_{ij} (z_i - z_j)} \psi_{\underline{k}}(\underline{\lambda}), \quad q^{z_i} = x_i \quad (11)$$

where we have explicitly specified m and N dependence of Ψ , that satisfies a set of linear difference equations

$$\Psi_m^{[N]}(z_k + j, \underline{\lambda}) = \Psi_m^{[N]}(z_l - j, \underline{\lambda}), \quad j = 1..m, \quad k < l, \quad z_k = z_l \quad (12)$$

Let us try to apply the previous section logic to these circumstances.

Observe that when going from $N-1$ variables to N variables a large portion of equations (12) stays the same (with suitable change $x_i \rightarrow x_{i+1}$, $\lambda_i \rightarrow \lambda_{i+1}$) and added are only the equations for roots $\alpha_{1,k}$, $k = 2..N$.

It makes sense then to decompose $\Psi_m^{[N]}(\underline{x}, \underline{\lambda})$ as a sum over $\Psi_m^{[N-1]}(\underline{x}, \underline{\mu})$ with suitable μ : then the “old” equations for roots $\alpha_{i,j}$, $i < j = 2..N$ would automatically hold.

Accurate degree counting of x_i and $x_i^{\lambda_i}$ in the original ansatz (1) shows that the nested ansatz for BAF should be

$$\Psi_m^{[N]}(x_1, \dots, x_N | \lambda_1, \dots, \lambda_N) = x_1^{\lambda_1 + m(N-1)/2} \sum_{k_{12}, \dots, k_{1N}=0}^m \frac{\hat{\Psi}_{m, k_{12}, \dots, k_{1N}}}{x_1^{\sum_{i=2}^N k_{1i}}} \Psi_m^{[N-1]} \left(x_2, \dots, x_N | \lambda_2 + k_{12} - \frac{m}{2}, \dots, \lambda_N + k_{1N} - \frac{m}{2} \right) \quad (13)$$

that is, we sum over $\underline{\mu} = \underline{\lambda} + \underline{\Delta}$ where the integral shift $\underline{\Delta}$ compensates for the power of x_1 in each item.

Starting now with $N = 2$ BAF [for which directly solving linear equations (12) with original ansatz (11) has no ambiguity—but it similarly can be expressed as a nested ansatz over $\Psi^{[1]}(x, \lambda)$]

$$\begin{aligned} \Psi_m^{[2]}(x_1, x_2 | \lambda_1, \lambda_2) &= x_1^{\lambda_1 + m/2} \sum_{k_{12}=0}^m \frac{\hat{\Psi}_{m, k_{12}}}{x_1^{k_{12}}} \cdot \underbrace{x_2^{\lambda_2 + k_{12} - m/2}}_{\Psi_m^{[1]}(x_2 | \lambda_2 + k_{12} - \frac{m}{2})} \\ \hat{\Psi}_{m, k_{12}} &= (-1)^{k_{12}} q^{-mk_{12} + k_{12}(k_{12}-1)/2} \frac{[m]!}{[m - k_{12}]! [k_{12}]!} \prod_{i=1}^{k_{12}} \frac{(\frac{\Lambda_1}{\Lambda_2} q^{m-i+1} - 1)}{(\frac{\Lambda_1}{\Lambda_2} q^{-i} - 1)}; \quad \Lambda_i = q^{\lambda_i} \end{aligned} \quad (14)$$

In symmetric q -numbers this can be put into even nicer form [which would continue to persist in the $a = 2$ case, see (26)]; it is straightforward to see that this expression is equivalent to the NS formula coefficients (8)

$$\hat{\Psi}_{m, k_{12}} = (-1)^{k_{12}} \frac{[m]!}{[m - k_{12}]! [k_{12}]!} \prod_{i=0}^{k_{12}-1} \frac{[\lambda_2 - \lambda_1 - m + i]}{[\lambda_2 - \lambda_1 + 1 + i]} \quad (15)$$

One can now solve the “new” equations for $\alpha_{1,k}, k = 2..N$ for $N = 3$ to manifestly and *unambiguously* obtain

$$\begin{aligned} \hat{\Psi}_{m, k_{12}, k_{13}} &= (-1)^{k_{12} + k_{13}} \frac{[m]!}{[m - k_{12}]! [k_{12}]!} \frac{[m]!}{[m - k_{13}]! [k_{13}]!} \prod_{i=0}^{k_{12}-1} \frac{[\lambda_2 - \lambda_1 + i - m]}{[\lambda_2 - \lambda_1 + i + 1]} \prod_{i=0}^{k_{13}-1} \frac{[\lambda_3 - \lambda_1 + i - m]}{[\lambda_3 - \lambda_1 + i + 1]} \\ &\times \prod_{i=0}^{k_{13}-1} \frac{[\lambda_3 - \lambda_2 + i - m] [\lambda_3 - \lambda_2 + i + m + 1 - k_{12}]}{[\lambda_3 - \lambda_2 + i + 1] [\lambda_3 - \lambda_2 + i - k_{12}]} \end{aligned} \quad (16)$$

Similarly explicit and fully factorized formula takes place for $N = 4$, that is for $\hat{\Psi}_{m, k_{12}, k_{13}, k_{14}}$. Observe how the first line in (16) directly corresponds to the first line of (8) and the second one, respectively, to the second. One just has to perform the trivial change of λ -variables $\lambda_i \leftrightarrow -\lambda_{N-i+1}$, which, in other words, corresponds to x_N , and not x_1 , being the distinguished variable in (13).

To summarize, the Noumi-Shiraishi formula is correctly reproduced with the help of nested ansatz.

IV. NESTED ANSATZ FOR TWISTED BAF

Having checked the consistency of our method in the nontwisted $a = 1$ case, we now turn our attention to the more interesting case of twisted BAFs (tBAFs)

As a base for our method we have the previously obtained $N = 2$ $a = 2$ tBAF

$$\begin{aligned} \Psi_m^{(2)[2]}(x_1, x_2 | \lambda_1, \lambda_2) &= x_1^{\lambda_1} x_2^{\lambda_2} \cdot x_1^m x_2^{-m} \cdot \sum_{k=0}^{2m} \psi_m^{(2)[2]}(\lambda_1, \lambda_2) \cdot \left(\frac{x_2}{x_1} \right)^k \\ \psi_m^{(2)[2]}(\lambda_1, \lambda_2) &= \left(\sum_{r=\max(0, k-m)}^{\lfloor \frac{k}{2} \rfloor} (-1)^r q^{(k-2r)(-m+\frac{1}{2}k)+\frac{1}{2}r(r+1)} (1-q)^{k-2r} \frac{[m+k-2r]!}{[m-k+r]! [k-2r]! [r]!} \right. \\ &\quad \left. (\Lambda_1 \Lambda_2)^{k-2r} \frac{\prod_{i=1}^r \text{ff}_{1,2}(i-1-m) \prod_{i=0}^{r-1} \text{ff}_{1,2}(k-i)}{\prod_{i=1}^k \text{ff}_{1,2}(i)} \right), \end{aligned} \quad (17)$$

where $x_i = q^{z_i/a}$ and $\Lambda_i = q^{\lambda_i}$ —as compared to the original Chalykh's definition. The peculiar factor is

$$\text{ff}_{i,j}(k) = (\Lambda_i^a - q^k \Lambda_j^a) \quad (18)$$

The analog of nested ansatz (13) reads

$$\Psi_m^{(a)[N]}(x_1, \dots, x_N | \lambda_1, \dots, \lambda_N) = x_1^{\lambda_1 + am(N-1)/2} \sum_{k_{1,2}, \dots, k_{1,N}=0}^{ma} \frac{\tilde{\Psi}_{m,k_{1,2}, \dots, k_{1,N}}^{(a)}}{x_1^{\sum_{i=2}^N k_{1,i}}} \Psi_m^{(a)[N-1]} \left(x_2, \dots, x_N | \lambda_2 + k_{1,2} - \frac{ma}{2}, \dots, \lambda_N + k_{1,N} - \frac{ma}{2} \right) \quad (19)$$

and the linear equations are (only the ones new to N variables)

$$\Psi_m^{(a)[N]} \left(x_i \epsilon q^{-\frac{j}{2a}}, \dots, x_i q^{\frac{j}{2a}}, \dots \right) = \epsilon^j \Psi_m^{(a)[N]} \left(x_i \epsilon q^{\frac{j}{2a}}, \dots, x_i q^{-\frac{j}{2a}}, \dots \right) \quad (20)$$

A. Peculiarities of the $N=3$ $a=2$ case

The simplest nontrivial case, where nested ansatz is applicable, is the $N=3$, $a=2$ case. Here we discuss the structure of answers that already starts to be visible in this case.²

The $N=3$ unknown coefficients have two indices: corresponding to multiplicities of roots $\alpha_{1,2}$ and $\alpha_{1,3}$.

As in (17) some coefficients are factorized, and some are not, and we describe them now in the order of increasing complexity.

The first nonfactorizable coefficient is $\tilde{\Psi}_{m,2,0}$. For subsequent m it is equal to

$$\begin{aligned} \tilde{\Psi}_{1,2,0} &= \frac{[m]}{\text{ff}_{1,2}(1)\text{ff}_{1,2}(2)} \cdot (-q) \cdot \text{ff}_{1,2}(-1)\text{ff}_{1,2}(2) \\ \tilde{\Psi}_{2,2,0} &= \frac{[m]}{\text{ff}_{1,2}(1)\text{ff}_{1,2}(2)} \cdot \left((-q) \cdot \text{ff}_{1,2}(-2)\text{ff}_{1,2}(2) + \frac{[4][3](q-1)^2}{[2]q^2} (\Lambda_1 \Lambda_2)^2 \right) \\ \tilde{\Psi}_{3,2,0} &= \frac{[m]}{\text{ff}_{1,2}(1)\text{ff}_{1,2}(2)} \cdot \left((-q) \cdot \text{ff}_{1,2}(-3)\text{ff}_{1,2}(2) + \frac{[4][5](q-1)^2}{q^4} (\Lambda_1 \Lambda_2)^2 \right) \\ \tilde{\Psi}_{4,2,0} &= \frac{[m]}{\text{ff}_{1,2}(1)\text{ff}_{1,2}(2)} \cdot \left((-q) \cdot \text{ff}_{1,2}(-4)\text{ff}_{1,2}(2) + \frac{[6][5][3](q-1)^2}{[2]q^6} (\Lambda_1 \Lambda_2)^2 \right) \\ \tilde{\Psi}_{5,2,0} &= \frac{[m]}{\text{ff}_{1,2}(1)\text{ff}_{1,2}(2)} \cdot \left((-q) \cdot \text{ff}_{1,2}(-5)\text{ff}_{1,2}(2) + \frac{[6][7][4](q-1)^2}{[2]q^8} (\Lambda_1 \Lambda_2)^2 \right), \end{aligned} \quad (21)$$

where from, in general, we conclude

$$\tilde{\Psi}_{m,2,0} = \frac{[m]}{\text{ff}_{1,2}(1)\text{ff}_{1,2}(2)} \cdot \left((-q) \cdot \text{ff}_{1,2}(-m)\text{ff}_{1,2}(2) + \frac{[m-1][m+1][m+2](q-1)^2}{[2]q^{2(m-1)}} (\Lambda_1 \Lambda_2)^2 \right) \quad (22)$$

which is in accordance with the structure, of (17).

²At present, even in the $N=3$, $a=2$ we are unable to obtain all answers for *generic* m . Still, the answers we can obtain have peculiar structure, which we want to describe. To obtain general m answers and to move to $N>3$ and/or $a>2$ cases arguably requires some crossing of ideas from the present paper and [23]—which should be possible since answers seem to adhere to direct quantization prescription.

Going through similar motions, one obtains for $\tilde{\psi}_{m,n_1,0}$

$$\tilde{\psi}_{m,n_1,0} = \left(\sum_{r=\max(0,n_1-m)}^{\lfloor \frac{n_1}{2} \rfloor} (-1)^r q^{\frac{n_1^2}{2} + \frac{r^2}{2} - n_1 r + \frac{r}{2}} q^{-(n_1-2r)m} (1-q)^{n_1-2r} \frac{[m+n_1-2r]!}{[m-n_1+r]![n_1-2r]![r]!} \right. \\ \left. (\Lambda_1 \Lambda_2)^{n_1-2r} \frac{\prod_{i=1}^r \text{ff}_{1,2}(i-1-m) \prod_{i=0}^{r-1} \text{ff}_{1,2}(n_1-i)}{\prod_{i=1}^{n_1} \text{ff}_{1,2}(i)} \right) \quad (23)$$

that is, at the end of the day the formula is *literally* the same as (17), with trivial change $\Lambda_{2,3} \rightarrow \Lambda_{1,2}$.

Dequantization This formula (23) can be, in fact, made much simpler by, first, using *symmetric* q -numbers

$$\llbracket n \rrbracket := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \quad (24)$$

and, second, considering *symmetric* root-related quantities instead of not quite symmetric $\text{ff}_{1,2}(k)$

$$\text{gg}_{1,2}(k) := \frac{q^{-k/2} \text{ff}_{1,2}(k)}{(q^{-1/2} - q^{1/2}) \Lambda_1 \Lambda_2} = \frac{(q^{-k/2} \frac{\Lambda_1}{\Lambda_2} - q^{k/2} \frac{\Lambda_2}{\Lambda_1})}{(q^{-1/2} - q^{1/2})} \Big|_{\Lambda_i = q^{\lambda_i/2}} \llbracket \lambda_2 - \lambda_1 + k \rrbracket =_{q \rightarrow 1} (\lambda_2 - \lambda_1 + k) \quad (25)$$

With help of these two devices the formula becomes (note that explicit Λ_i entirely disappears from the formula—only the “root-related” quantities $\text{gg}_{i,j}(k)$ enter)

$$(23) = \left(\sum_{r=\max(0,n_1-m)}^{\lfloor \frac{n_1}{2} \rfloor} (-1)^r \frac{\llbracket m+n_1-2r \rrbracket!}{\llbracket m-n_1+r \rrbracket! \llbracket n_1-2r \rrbracket! \llbracket r \rrbracket!} \frac{\prod_{i=1}^r \text{gg}_{1,2}(i-1-m) \prod_{i=0}^{r-1} \text{gg}_{1,2}(n_1-i)}{\prod_{i=1}^{n_1} \text{gg}_{1,2}(i)} \right) \quad (26)$$

moreover there are no *ad hoc* q -factors; The *classical limit* $q \rightarrow 1$ is, accordingly, very simple

$$(23) \Big|_{\substack{\Lambda_i = q^{\lambda_i/2} \\ q \rightarrow 1}} = \left(\sum_{r=\max(0,n_1-m)}^{\lfloor \frac{n_1}{2} \rfloor} (-1)^r \frac{(m+n_1-2r)!}{(m-n_1+r)!(n_1-2r)!(r)!} \frac{\prod_{i=1}^r (\lambda_2 - \lambda_1 + i - 1 - m) \prod_{i=0}^{r-1} (\lambda_2 - \lambda_1 + n_1 - i)}{\prod_{i=1}^{n_1} (\lambda_2 - \lambda_1 + i)} \right) \quad (27)$$

The relation between (27) and (26) is then, naturally a *dequantization prescription*—to be applied in more complicated cases. Or, put another way, should one somehow obtain classical answer first, representing it in the form (27) allows for its straightforward promotion to the full quantum answer (26). This is exactly in spirit of *direct quantization* paradigm, first signs of which are observed in [23]; now it receives further evidence.

Slightly more complicated are the coefficients $\tilde{\psi}_{m,0,n_2}$, where good basis functions start to get extra factors, typical for Noumi-Shiraishi answers (the “core” of the formula is still about the root $e_{1,3}$, but extra factors are related to the “old” $N=2$ root $e_{2,3}$)

$$\tilde{\psi}_{m,0,n_2} = \left(\sum_{r=\max(0,n_2-m)}^{\lfloor \frac{n_2}{2} \rfloor} (-1)^r q^{\frac{n_2^2}{2} + \frac{r^2}{2} - n_2 r + \frac{r}{2}} q^{-(n_2-2r)m} (1-q)^{n_2-2r} \frac{[m+n_2-2r]!}{[m-n_2+r]![n_2-2r]![r]!} \right. \\ \left. (\Lambda_1 \Lambda_3)^{n_2-2r} \frac{\prod_{i=1}^r \text{ff}_{1,3}(i-1-m) \prod_{i=0}^{r-1} \text{ff}_{1,3}(n_1-i)}{\prod_{i=1}^{n_2} \text{ff}_{1,3}(i)} \right) \\ \times \left(\prod_{i=0}^{n_2-1} \frac{\text{ff}_{2,3}(-m+i)}{\text{ff}_{2,3}(i)} \right) \left(\prod_{i=1}^{n_2} \frac{\text{ff}_{2,3}(m+i)}{\text{ff}_{2,3}(i)} \right) \quad (28)$$

which, with help of similarly introduced $gg_{1,3}(k)$ and $gg_{2,3}(k)$ becomes

$$\begin{aligned} \tilde{\psi}_{m,0,n_2} = & \left(\sum_{r=\max(0,n_2-m)}^{\lfloor \frac{n_2}{2} \rfloor} (-1)^r \frac{[m+n_2-2r]!}{[m-n_2+r]![n_2-2r]![r]!} \frac{\prod_{i=1}^r gg_{1,3}(i-1-m) \prod_{i=0}^{r-1} gg_{1,3}(n_2-i)}{\prod_{i=1}^{n_2} gg_{1,3}(i)} \right) \\ & \times \left(\prod_{i=0}^{n_2-1} \frac{gg_{2,3}(-m+i)}{gg_{2,3}(i)} \right) \left(\prod_{i=1}^{n_2} \frac{gg_{2,3}(m+i)}{gg_{2,3}(i)} \right) \end{aligned} \quad (29)$$

Note how the structure of this formula parallels the general structure of $a = 1$ Noumi-Shiraishi formulas (8), (16): the $\alpha_{2,3}$ -root has the associated factors in the numerator, involving m and $-m$, while $\alpha_{1,3}$ -root has one factor with $-m$, one with $+1$, and the λ -independent q -multinomial factor. This structure seems, therefore, to be essential to the whole Noumi-Shiraishi construction, as it persists to the twisted case here, and in more complicated examples below. In future, the reason behind this phenomenon needs to be understood.

The homogeneous symmetric quantities (24) and (25) prove to be the right terms indeed, as the more complicated answers are equally simple. For instance $\tilde{\psi}_{m,1,1}$ equals

$$\begin{aligned} \tilde{\psi}_{m,1,1} = & [m]^2 [m+1] \frac{1}{gg_{1,2}(1)} \cdot \frac{gg_{1,3}(-m)}{1} \cdot \frac{1}{gg_{2,3}(-1)gg_{2,3}(1)} + [m]^2 [m+1] \frac{gg_{1,2}(-m)}{1} \cdot \frac{1}{gg_{1,3}(1)} \cdot \frac{1}{gg_{2,3}(-1)gg_{2,3}(1)} \\ & + [m]^3 [m+1] \frac{1}{gg_{1,2}(1)} \cdot \frac{1}{gg_{1,3}(1)} \cdot \frac{gg_{2,3}(-m-1)gg_{2,3}(m+1)}{gg_{2,3}(-1)gg_{2,3}(1)} \end{aligned} \quad (30)$$

where now the answer stops being monomial in $gg_{2,3}(\bullet)$.

The R -operators At this point it is convenient to introduce the following “raising” operators $R_{i,j}$, for the convenience of writing formulas like (29) and (30). Namely, the operator $R_{1,2}$ (and, analogously, $R_{1,3}$) acts by successfully removing $gg_{1,2}(k)$ with the *highest* k from the denominator and putting the corresponding $gg_{1,2}(-m+k_{\max}-k)$ to the numerator. For instance

$$\frac{1}{gg_{1,2}(1)gg_{1,2}(2)gg_{1,2}(3)} \xrightarrow{R_{1,2}} \frac{gg_{1,2}(-m)}{gg_{1,2}(1)gg_{1,2}(2)} \xrightarrow{R_{1,2}} \frac{gg_{1,2}(-m)gg_{1,2}(-m+1)}{gg_{1,2}(1)} \xrightarrow{R_{1,2}} \frac{gg_{1,2}(-m)gg_{1,2}(-m+1)gg_{1,2}(-m+2)}{1} \quad (31)$$

The operator $R_{2,3}$ is slightly more tricky. It depends on additional shift δ and, “determining” the $gg_{2,3}$ with the lowest (negative) and highest (positive) arguments in the denominator, multiplies the corresponding monomial by

$$gg_{2,3}(-m-k_{\min}+\delta)gg_{2,3}(m+k_{\max}-\delta) \quad (32)$$

With help of this notation $\tilde{\psi}_{m,0,n_2}$ becomes simply

$$\tilde{\psi}_{m,0,n_2} = \left(\prod_{i=0}^{n_2-1} \frac{gg_{2,3}(-m+i)}{gg_{2,3}(i)} \right) \left(\prod_{i=1}^{n_2} \frac{gg_{2,3}(m+i)}{gg_{2,3}(i)} \right) \left(\sum_{r=\max(0,n_2-m)}^{\lfloor \frac{n_2}{2} \rfloor} (-1)^r \frac{[m+n_2-2r]!}{[m-n_2+r]![n_2-2r]![r]!} R_{1,3}^r \right) \frac{1}{\prod_{i=1}^{n_2} gg_{1,3}(i)} \quad (33)$$

while the more complicated $\tilde{\psi}_{m,1,1}$ is written as

$$\tilde{\psi}_{m,1,1} = ([m]^2 [m+1] R_{1,2} + [m]^2 [m+1] R_{1,3} + [m]^3 [m+1] R_{2,3}(0)) \frac{1}{gg_{1,2}(1)gg_{1,3}(1)gg_{2,3}(-1)gg_{2,3}(1)} \quad (34)$$

Note how direct quantization/dequantization continues to hold.

The general structure of $\tilde{\psi}_{m,n_1,n_2}$

In general, looking at many specific examples, one can conclude that $\tilde{\psi}_{m,n_1,n_2}$ has the following structure

$$\tilde{\psi}_{m,n_1,n_2} = \text{Pol}_{m,n_1,n_2}(R_{1,2}, R_{1,3}, R_{2,3}(\delta)) \frac{1}{\text{Denom}(m, n_1, n_2)} \quad (35)$$

where $\text{Pol}_{m,n_1,n_2}(R_{1,2}, R_{1,3}, R_{2,3})$ is a certain polynomial in symbols $R_{1,2}, R_{1,3}, R_{2,3}$ with m -dependent coefficients (which are assembled from symmetric q -numbers).

$\text{Denom}(m, n_1, n_2)$ is simple and manifest expression

$$\text{Denom}(m, n_1, n_2) = \prod_{i=1}^{n_1} \text{gg}_{1,2}(i) \prod_{j=1}^{n_2} \text{gg}_{1,3}(j) \prod_{k=0}^{n_2-1} (\text{gg}_{2,3}(-n_1 + k) \text{gg}_{2,3}(n_2 - k)) \quad (36)$$

Answers for level $n_2 = 1$: $\tilde{\psi}_{m,n_1,1}$

Clearly, the Denom function in this case equals

$$\text{Denom}(m, n_1, 1) = \left(\prod_{i=1}^{n_1} \text{gg}_{1,2}(i) \right) \cdot \text{gg}_{1,3}(1) \cdot (\text{gg}_{2,3}(-n_1) \text{gg}_{2,3}(1)) \quad (37)$$

And the corresponding operator polynomials $\text{Pol}_{m,n_1,1}$ turn out to be equal, in the first few cases, to

$$\text{Pol}_{m,0,1} = \llbracket m \rrbracket \llbracket m+1 \rrbracket R_{2,3}(0) \quad (38)$$

$$\text{Pol}_{m,1,1} = \llbracket m \rrbracket^2 \llbracket m+1 \rrbracket R_{1,2} + \llbracket m \rrbracket^2 \llbracket m+1 \rrbracket R_{1,3} + \llbracket m \rrbracket^3 \llbracket m+1 \rrbracket R_{2,3}(0) \quad (39)$$

$$\begin{aligned} \text{Pol}_{m,2,1} = & (-1) \llbracket m \rrbracket^2 \llbracket m+1 \rrbracket R_{1,2} R_{2,3}(1) \\ & + \llbracket m-1 \rrbracket \llbracket m \rrbracket^3 \llbracket m+1 \rrbracket R_{1,2} + \llbracket m-1 \rrbracket \llbracket m \rrbracket^2 \llbracket m+1 \rrbracket^2 R_{1,3} + \frac{\llbracket m-1 \rrbracket \llbracket m \rrbracket^3 \llbracket m+1 \rrbracket^2}{\llbracket 2 \rrbracket} R_{2,3}(0) \end{aligned} \quad (40)$$

$$\begin{aligned} \text{Pol}_{m,3,1} = & (-1) \llbracket m-1 \rrbracket \llbracket m \rrbracket^2 \llbracket m+1 \rrbracket R_{1,2}^2 + (-1) \llbracket m-1 \rrbracket \llbracket m \rrbracket^2 \llbracket m+1 \rrbracket R_{1,2} R_{1,3} + (-1) \llbracket m-1 \rrbracket \llbracket m \rrbracket^3 \llbracket m+1 \rrbracket R_{1,2} R_{2,3}(1) \\ & + \frac{\llbracket m-2 \rrbracket \llbracket m-1 \rrbracket \llbracket m \rrbracket^3 \llbracket m+1 \rrbracket^2}{\llbracket 2 \rrbracket} R_{1,2} + \frac{\llbracket m-2 \rrbracket \llbracket m-1 \rrbracket \llbracket m \rrbracket^2 \llbracket m+1 \rrbracket^2 \llbracket m+2 \rrbracket}{\llbracket 2 \rrbracket} R_{1,3} \\ & + \frac{\llbracket m-2 \rrbracket \llbracket m-1 \rrbracket \llbracket m \rrbracket^3 \llbracket m+1 \rrbracket^2 \llbracket m+2 \rrbracket}{\llbracket 2 \rrbracket \llbracket 3 \rrbracket} R_{2,3}(0) \end{aligned} \quad (41)$$

$$\begin{aligned} \text{Pol}_{m,4,1} = & \frac{\llbracket m-1 \rrbracket \llbracket m \rrbracket^2 \llbracket m+1 \rrbracket}{\llbracket 2 \rrbracket} R_{1,2}^2 R_{2,3}(2) \\ & + (-1) \llbracket m-2 \rrbracket \llbracket m-1 \rrbracket \llbracket m \rrbracket^3 \llbracket m+1 \rrbracket R_{1,2}^2 + (-1) \llbracket m-2 \rrbracket \llbracket m-1 \rrbracket \llbracket m \rrbracket^2 \llbracket m+1 \rrbracket^2 R_{1,2} R_{1,3} \\ & + (-1) \frac{\llbracket m-2 \rrbracket \llbracket m-1 \rrbracket \llbracket m \rrbracket^3 \llbracket m+1 \rrbracket^2}{\llbracket 2 \rrbracket} R_{1,2} R_{2,3}(1) \\ & + \frac{\llbracket m-3 \rrbracket \llbracket m-2 \rrbracket \llbracket m-1 \rrbracket \llbracket m \rrbracket^3 \llbracket m+1 \rrbracket^2 \llbracket m+2 \rrbracket}{\llbracket 2 \rrbracket \llbracket 3 \rrbracket} R_{1,2} \\ & + \frac{\llbracket m-3 \rrbracket \llbracket m-2 \rrbracket \llbracket m-1 \rrbracket \llbracket m \rrbracket^2 \llbracket m+1 \rrbracket^2 \llbracket m+2 \rrbracket \llbracket m+3 \rrbracket}{\llbracket 2 \rrbracket \llbracket 3 \rrbracket} R_{1,3} \\ & + \frac{\llbracket m-3 \rrbracket \llbracket m-2 \rrbracket \llbracket m-1 \rrbracket \llbracket m \rrbracket^3 \llbracket m+1 \rrbracket^2 \llbracket m+2 \rrbracket \llbracket m+3 \rrbracket}{\llbracket 2 \rrbracket \llbracket 3 \rrbracket \llbracket 4 \rrbracket} R_{2,3}(0) \end{aligned} \quad (42)$$

At this point we already see that the smaller m -dependent q -number for $\tilde{\psi}_{m,n_1,1}$ is $\llbracket m-n_1+1 \rrbracket$, therefore, for the following ones we cannot really restore all the coefficients from our available data.

Some parts of the expression, however, *can* be fixed. Namely,

$$\begin{aligned} \text{Pol}_{m,5,1} = & \frac{[[m-2]] [[m-1]] [[m]]^2 [[m+1]]}{[[2]]} R_{1,2}^3 + \frac{[[m-2]] [[m-1]] [[m]]^2 [[m+1]]}{[[2]]} R_{1,2}^2 R_{1,3} \\ & + \frac{[[m-2]] [[m-1]] [[m]]^3 [[m+1]]}{[[2]]} R_{1,2}^2 R_{2,3}(2) + (-1) \frac{[[m-3]] [[m-2]] [[m-1]] [[m]]^3 [[m+1]]^2}{[[2]]} R_{1,2}^2 \\ & + (-1) \frac{[[m-3]] [[m-2]] [[m-1]] [[m]]^2 [[m+1]]^2 [[m+2]]}{[[2]]} R_{1,2} R_{1,3} \\ & + (-1) \frac{[[m-3]] [[m-2]] [[m-1]] [[m]]^3 [[m+1]]^2 [[m+2]]}{[[2]] [[3]]} R_{1,2} R_{2,3}(1) + [[m-4]] \cdot \text{smth}_4 \end{aligned} \quad (43)$$

where we denote with smth_4 the piece we cannot restore/verify.

Further,

$$\begin{aligned} \text{Pol}_{m,6,1} = & (-1) \frac{[[m-2]] [[m-1]] [[m]]^2 [[m+1]]}{[[2]] [[3]]} R_{1,2}^3 R_{2,3}(3) + \frac{[[m-3]] [[m-2]] [[m-1]] [[m]]^3 [[m+1]]}{[[2]]} R_{1,2}^3 \\ & + \frac{[[m-3]] [[m-2]] [[m-1]] [[m]]^2 [[m+1]]^2}{[[2]]} R_{1,2}^2 R_{1,3} + \frac{[[m-3]] [[m-2]] [[m-1]] [[m]]^3 [[m+1]]^2}{[[2]]^2} R_{1,2}^2 R_{2,3}(2) \\ & + [[m-4]] \cdot \text{smth}_4 + [[m-5]] [[m-4]] \cdot \text{smth}_5 \end{aligned} \quad (44)$$

$$\begin{aligned} \text{Pol}_{m,7,1} = & (-1) \frac{[[m-3]] [[m-2]] [[m-1]] [[m]]^2 [[m+1]]}{[[2]] [[3]]} R_{1,2}^4 + (-1) \frac{[[m-3]] [[m-2]] [[m-1]] [[m]]^2 [[m+1]]}{[[2]] [[3]]} R_{1,2}^3 R_{1,3} \\ & + (-1) \frac{[[m-3]] [[m-2]] [[m-1]] [[m]]^3 [[m+1]]}{[[2]] [[3]]} R_{1,2}^3 R_{2,3}(3) \\ & + [[m-4]] \cdot \text{smth}_4 + [[m-5]] [[m-4]] \cdot \text{smth}_5 + [[m-6]] [[m-5]] [[m-4]] \cdot \text{smth}_6 \end{aligned} \quad (45)$$

$$\begin{aligned} \text{Pol}_{m,8,1} = & \frac{[[m-3]] [[m-2]] [[m-1]] [[m]]^4 [[m+1]]}{[[2]] [[3]] [[4]]} R_{1,2}^4 R_{2,3}(4) \\ & + [[m-4]] \cdot \text{smth}_4 + [[m-5]] [[m-4]] \cdot \text{smth}_5 + [[m-6]] [[m-5]] [[m-4]] \cdot \text{smth}_6 \\ & + [[m-7]] [[m-6]] [[m-5]] [[m-4]] \cdot \text{smth}_7 \end{aligned} \quad (46)$$

All these answers continue to confirm the principle of direct quantization. Moreover, observe, how generalization from integer m to generic $t = q^{-m}$, which is done separately for each $[[m + \bullet]]$ and $\text{gg}_{i,j}(k)$ factor, is evident. At the moment we refrain from writing/conjecturing general formula even for $\text{Pol}_{n_1,1}$, and even more so for Pol_{n_1,n_2} until we are able to gather more experimental data. Still, even the presented partial answers illustrate the main point of the letter: the nested ansatz method:

- (1) can be applied directly to (twisted) BAF;
- (2) leads to unambiguous answers (in the sense that splitting of $\tilde{\psi}$ to $\hat{\psi}$ is resolved);
- (3) which are manifestly a direct quantization, even in the stronger sense than that of [23], since no insertion of

ad hoc $q^\bullet$ -factors is needed: everything is accounted for by considering symmetric q -numbers.

V. CONCLUSIONS

In this paper, we presented the nested ansatz method for calculating Baker-Akhiezer functions in the sense of O. Chalykh [1] in various setups. The method is particularly useful in situations when the triad picture is not fully established, which we demonstrated by obtaining first few answers for coefficients of twisted BAF in the first unknown nontrivial case of $N = 3$, $a = 2$.

Crucially, these answers continue to confirm the original educated guess of [23] that full quantum answers in the

proper basis are restored from their classical counterparts by simply changing numbers to q -numbers, and Pochhammer symbols to q -Pochhammer symbols: the so-called *direct quantization*.

Generally speaking, the method is an improvement upon the initial ansatz (1), and may be used with success in all situations where analogs of BAF appear. Immediate and most fruitful research directions seem to be

- (i) generalization of the NS formulas to other root systems;
- (ii) constraining the form of elliptic linear equations from the form of the elliptic NS function;
- (iii) understanding the underlying reason why the peculiar appearing combinations of $gg_{i,j}(k)$ factors are the right terms for direct quantization;
- (iv) combination of key ideas from [23] and the present text in order to obtain general formula for the tBAF.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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