

Pion condensation and pion star from holographic QCD

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The properties of quantum chromodynamics (QCD) matter at finite isospin densities are investigated employing holographic hard-wall and soft-wall anti-de Sitter (AdS)/QCD models. It is confirmed that at high enough isospin densities, charged pions start to condense and the pion superfluid phase appears in the system. It is shown that the chiral condensate and the pion condensate can be transformed to each other and form a chiral circle in the superfluid phase. We derived the equation of state (EoS) for pionic matter, calculated the normalized trace anomaly Δ and $(\epsilon - 3p)/m_\pi^4$, and analyzed the sound speed and adiabatic index. Additionally, we provided data on the mass-radius relation and tidal deformability of pion stars. The results indicate that the holographic models align well with lattice QCD concerning isospin density, axial-vector condensation, EoS, and trace anomaly, though discrepancies in sound speed and adiabatic index emerge at higher isospin chemical potentials. The holographic models closely match the leading-order results from chiral perturbation theory (χ PT) in certain observables, suggesting that they may effectively capture aspects of χ PT dynamics within a five-dimensional framework under the probe approximation.

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I. INTRODUCTION

Under the condition of finite isospin chemical potential, quantum chromodynamics (QCD) exhibits a phenomenon: the condensation of charged pions. The onset of pion condensation spontaneously breaks the global $U(1)_I$ isospin symmetry. Such a symmetry-breaking mechanism is well known to drive systems into a superfluid phase. This phenomenon can occur in various physical environments, including heavy-ion collision experiments, the interiors of neutron stars, and the very early universe [1]. In the core of a neutron star, the excess of down quarks over up quarks generates an isospin chemical potential μ_I [2]. Although the potential is much smaller than the baryon chemical potential $\mu_I \ll \mu_B$, [3] the isospin asymmetry modifies the nuclear symmetry energy, which might possibly stiffen

the equation of state and thus help accommodate heavier neutron stars [3,4]. Furthermore, if the early Universe possessed a significant lepton flavor asymmetry, this asymmetry could be converted into a larger isospin chemical potential as the Universe expanded, triggering pion condensation. This condensation could influence not only the evolution of the early universe but also leave a distinctive imprint on the spectrum of primordial gravitational waves and the mass distribution of primordial black holes [5,6], offering valuable clues for studying the early Universe.

Pion condensation in the early universe may also lead to another phenomenon: the formation of pion stars [7–9]. As one type of bosonic stars [10], pion stars are formed due to pion condensation when the pion density in space region becomes high enough, thus can gravitationally attract each other to form a self-gravitating configuration overcoming their quantum-mechanical repulsion [11]. These hypothetical stars could reveal their existence through neutrino and photon signals released during their evaporation process, or through the observation of gravitational waves in binary systems. Although the stability and lifetime of pion stars remain an open question, their lifetime is relatively short, and they could impact big bang nucleosynthesis, potentially affecting the primordial abundance of nuclei.

Due to the sign problem in lattice QCD [12,13], direct calculations at finite baryon chemical potential are

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infeasible. Fortunately, lattice QCD calculations can be performed at finite isospin chemical potential without encountering the sign problem [14]. Significant progress has been made in lattice QCD simulations at finite isospin chemical potential, including confirming the second-order phase transition of Bose-Einstein condensation (BEC) of charged pions [15–22], studying pion dynamics, the pionic equation of state [8], and sound speed at low temperatures [23]. Beyond lattice QCD, various models like χ PT [9,24–27], Nambu-Jona-Lasinio [28–34], quark-meson [35,36], linear sigma [37,38], random matrix [39], and perturbative QCD [40] have been exploring QCD phase transitions and pionic matter at finite isospin chemical potential.

Inspired by the AdS/CFT correspondence principle [41–43], the bottom-up holographic QCD (HQCD) approach has become a powerful tool for studying strongly coupled gauge theories [44–55]. The application of HQCD to neutron stars offers new insights into their extreme conditions [56–58]. These models align with astrophysical observations and have broader applications, including superconductivity [59–61]. At finite isospin chemical potential, these models reveal that pion condensation occurs via a second-order transition when μ_I exceeds the pion mass, with enhanced chiral symmetry breaking characterized by a growing total condensate $\bar{\sigma}$ [62]. The phase structure features a pion condensed region and a tricritical point in the (T, μ_I) plane [63]. Spectral analyses show that charged pions become massless Goldstone bosons at the transition, while their damping and thermal widths follow universal relations [64,65]. In these models, the condensates are described through gauge and scalar fields in the bulk, which are dual to quark bilinear operators such as $\langle \bar{q}q \rangle$ and $\langle \bar{q}\gamma^5 q \rangle$. Therefore, the holographic models do not introduce hadronic degrees of freedom explicitly, and the condensed phases studied here should not be interpreted as composed of individual pions. Although pions appear as fluctuations around the vacuum of the same bulk fields, the condensate itself reflects a nonzero expectation value of quark bilinears in the presence of isospin chemical potential. Related studies further explored symmetry breaking and pion dynamics in this framework [66–73]. The Witten-Sakai-Sugimoto (WSS) model, as a top-down holographic construction, has also been applied to study pion condensation. Recent works incorporating finite pion mass and temperature show that the model reproduces the correct critical μ_I and second-order transition behavior, with results in good agreement with lattice QCD at low temperatures [74,75].

HQCD approach provides a framework to link the dynamics of an infinite number of QCD hadrons with low-energy effective theories, such as χ PT. For instance, by integrating out heavy Kaluza-Klein modes and retaining only the lowest-lying mesons, one can derive a four-dimensional chiral effective Lagrangian from HQCD, as shown in [76,77] and earlier work such as [78]. This

approach results in all low-energy constants of the χ PT being expressed in terms of holographic integrals, thereby directly connecting the holographic models with chiral effective theories used in QCD. Notably, the Gell-Mann-Oakes-Renner relation, which relates the pion mass to the quark condensate and quark masses, is naturally reproduced at the lowest order of the derivative expansion in this framework [77].

The structure of this paper is as follows: Sec. II introduces the five-dimensional holographic model used in our study. Section III details the calculation of the pion state equation. Section IV presents the calculation of the sound speed and the adiabatic index. Section V discusses the mass-radius relation and tidal deformability of pion stars. Finally, Sec. VI summarizes and discusses our findings.

II. MODEL SETUP

The (improved) soft-wall (SW) and hard-wall (HW) models provide a bottom-up holographic framework to explore QCD properties including hadron spectra [79–81], chiral phase transitions [82,83], and pion condensation [62,63,84]. In the models, the real part of the complex scalar field X represents sigma condensation, while its imaginary part describes the phase of the pion condensate under nonzero isospin chemical potential. These models are constructed using the probe approximation within the AdS₅ or AdS-Schwarzschild black hole spacetime.

The background metric is given as

$$ds^2 = \frac{L^2}{z^2} \left[-f(z)dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)} \right], \quad (1)$$

with AdS radius $L = 1$, blackening factor $f(z) = 1 - \frac{z_h^4}{z^4}$ and the horizon z_h .

The holographic QCD model with $N_f = 2$ is constructed from $SU(2)_L \times SU(2)_R \equiv SU(2)_V \times SU(2)_A$ flavor symmetry. Moreover, the $U(1)_I$ symmetry in our setup is defined as a subgroup of $SU(2)_V$. Thus, $U(1)_I$ is identified as a subgroup of the vector symmetry $SU(2)_V$ that governs the isospin dynamics. The Lagrangian density can be written as

$$\mathcal{L}_5 = \text{Tr} \left[|D_M X|^2 + m_5^2(z)|X|^2 + \lambda|X|^4 + \frac{1}{4g_5^2} (F_{L,MN}^2 + F_{R,MN}^2) \right], \quad (2)$$

with the coupling $g_5^2 = 12\pi^2/N_c = 4\pi^2$ [79]. The scalar field X and gauge fields $A_{L/R,M}$ are dual to the quark condensate and current operators, respectively. The trace Tr is taken over the color space. The scalar field X is a quark bilinear and transforms as a bifundamental under the chiral symmetry group. That is, under an $SU(2)_L \times SU(2)_R$

transformation,

$$X \rightarrow g_L X g_R^\dagger, \quad g_L \in SU(2)_L, \quad g_R \in SU(2)_R. \quad (3)$$

This transformation property reflects the fact that X is constructed from a left-handed quark and a right-handed antiquark. The m_5 is 5-dimensional mass and λ is quartic coupling. The vacuum of scalar can be given as

$$X = \frac{1}{2}(\Sigma \sigma^0 + i\Pi^a \sigma^a), \quad (4)$$

with Pauli matrices σ^a and identity matrix σ^0 . Here, Σ and Π corresponds to sigma condensate $\langle \bar{q}q \rangle$ and pion condensate $\langle \bar{q}\gamma^5 q \rangle$, respectively. The covariant derivative $D_M X$ and gauge field strength $F_{L/R}^{MN}$ are written as

$$D_M X = \partial_M X - iA_{L,M} X + iX A_{R,M}, \quad (5)$$

$$F_{L/R}^{MN} = \partial^M A_{L/R}^N - \partial^N A_{L/R}^M - i[A_{L/R}^M, A_{L/R}^N], \quad (6)$$

with $A_{L/R,M} = A_{L/R,M}^a t^a$ and generators t^a of $SU(2)$ group. The chiral gauge field can be redefined through vector and axial-vector fields,

$$A_{L,M} = V_M + A_M, \quad A_{R,M} = V_M - A_M, \quad (7)$$

then the covariant derivative and gauge field strength are rewritten as

$$D_M X = \partial_M X - i[V_M, X] - i\{A_M, X\}, \quad (8)$$

$$F_V^{MN} = \partial^M V^N - \partial^N V^M - i[V^M, V^N] - i[A^M, A^N], \quad (9)$$

$$F_A^{MN} = \partial^M A^N - \partial^N A^M - i[V^M, A^N] - i[A^M, V^N]. \quad (10)$$

In the following, the gauge $A_{L/R}^z = 0$ is considered.

As discussed in Refs. [62,63,84], the nonvanishing condensate can be chosen as $\Pi^{(1)} \equiv \Pi$ under $U(1)_I$ symmetry. Moreover, the nonzero fields of model are $A_0^{(2)} \equiv a_2$, $V_0^{(3)} \equiv v$, and Σ . From the holographic dictionary, the gauge fields can be expanded at the AdS boundary ($z \rightarrow 0$) as

$$v \rightarrow \mu_I - C_1 n_I z^2 + \mathcal{O}(z^2), \quad a_2 \rightarrow C_2 \langle \sigma_A \rangle z^2 + \mathcal{O}(z^2), \quad (11)$$

with isospin chemical potential μ_I , isospin density n_I , axial-vector condensate $\langle \sigma_A \rangle$, and normalized constants $C_{1,2}$. To compute isospin density n_I and axial-vector condensation $\langle \sigma_A \rangle$, we exploit the holographic dictionary: the on-shell 5D action with respect to boundary sources μ_I and μ_A yields the corresponding densities. Specifically, isospin density and axial-vector condensate are obtained by the integrated

expressions in Eqs. (28) and (29). For the (pseudo)scalar fields, the expansions are

$$\Sigma \rightarrow m_q \zeta z + \frac{\langle \sigma \rangle}{\zeta} z^3 + \mathcal{O}(z^3), \quad \Pi^a \rightarrow \frac{\langle \pi^a \rangle}{\zeta} z^3 + \mathcal{O}(z^3), \quad (12)$$

with normalization constant ζ , quark mass m_q , sigma condensate $\langle \sigma \rangle$, and pion condensate $\langle \pi^a \rangle$.

A. Hard-wall model

The zero temperature is considered in the hard-wall model. So, the blacken factor is $f(z) = 1$ and the geometry remains confining. The full action can be written as [62,84]

$$S_5 = - \int d^4x \int_\epsilon^{z_m} dz \sqrt{-g} (\mathcal{L}_5 + \mathcal{L}_{BD}), \quad (13)$$

$$\mathcal{L}_{BD} = \text{Tr} \{ \lambda_4 z_m |X|^4 - m_2^2 z_m |X|^2 \} \delta(z - z_m), \quad (14)$$

with ultraviolet (UV) and infrared (IR) cutoff ϵ and z_m . The UV cutoff can be viewed as a small number that tends to zero. Reference [80] introduces the no-dimensional parameters λ_4 and m_2 at the boundary. The additional boundary term in (14) is introduced specifically to allow a nonzero quark condensate in the chiral limit. By the scalar potential at the IR, this term modifies the IR boundary conditions in (20), ensuring a stable vacuum solution with $\langle \bar{q}q \rangle \neq 0$. The 5D mass square $m_5^2(z)$ is viewed as a constant in the hard-wall model. According to AdS/CFT correspondence, the 5D mass is $m_5^2 = \Delta(\Delta - 4) = -3$ with conformal dimension $\Delta = 3$. The values of the parameters are chosen as Refs. [62,84] and shown in Table I. The parameters in the hard-wall model are chosen to reproduce pion mass m_π , rho mass m_ρ , and pion decay constant f_π . The equations of motion are given as

$$a_2 \Pi v - \left(\frac{3}{z^2} + a_2^2 \right) \Sigma + \frac{3}{z} \Sigma' - \Sigma'' = 0, \quad (15)$$

$$v \Sigma a_2 - \left(\frac{3}{z^2} + v^2 \right) \Pi + \frac{3}{z} \Pi' - \Pi'' = 0, \quad (16)$$

$$\frac{g_5^2 \Sigma}{z^2} (v \Pi - a_2 \Sigma) - \frac{a_2'}{z} + a_2'' = 0, \quad (17)$$

$$\frac{g_5^2 \Pi}{z^2} (a_2 \Sigma - v \Pi) - \frac{v'}{z} + v'' = 0. \quad (18)$$

The following boundary conditions are considered for solving the coupled second-order differential equations. At the UV ($z \rightarrow \epsilon$), the boundary conditions are

TABLE I. The parameters of the hard-wall (HW) model and soft-wall (SW) model, taken from [62,63,65,84].

Parameters	m_q (MeV)	λ	z_m^{-1} (MeV)	λ_4	m_2^2	μ_g (MeV)	μ_c (MeV)	γ_m	κ
HW	2.29	0	323	4.4	5.39	...	...	...	...
SW-I	3.22	80	...	...	...	440	1450	...	...
SW-II	3.58	14.7	...	...	...	440	...	3.7	1

$$\left. \frac{\Sigma}{z} \right|_\epsilon = m_q \zeta, \quad v|_\epsilon = \mu_I, \quad \left. \frac{\Pi}{z} \right|_\epsilon = 0, \quad a_2|_\epsilon = 0. \quad (19)$$

with normalization constant $\zeta = 1$. At the IR ($z \rightarrow z_m$), the boundary conditions are

$$\begin{aligned} \partial_z \Sigma|_{z_m} &= -\frac{\Sigma}{2z_m} (\lambda_4 (\Sigma^2 + \pi^2) - 2m_2^2) \Big|_{z_m}, & \partial_z v|_{z_m} &= 0, \\ \partial_z \Pi|_{z_m} &= -\frac{\Pi}{2z_m} (\lambda_4 (\Sigma^2 + \pi^2) - 2m_2^2) \Big|_{z_m}, & \partial_z a_2|_{z_m} &= 0. \end{aligned} \quad (20)$$

By applying UV and IR boundary conditions, combined with the Chebyshev spectral method, the equations of motion (15)–(18) can be solved. Furthermore, through UV expansion (12), we can obtain the condensations of sigma and pion. The condensates as a function of isospin chemical potential μ_I are shown in Fig. 1.

B. Soft-wall model

In the soft-wall model, we consider black hole backgrounds at low temperatures (2, 5, 10, and 15 MeV) to simulate finite-temperature effects. This choice is motivated by the fact that, in real QCD at vanishing baryon and isospin chemical potentials, the QCD phase transition is a

smooth crossover. In bottom-up holographic QCD models, black hole backgrounds are often used as effective geometries to describe such crossover behavior. We are aware that, in pure AdS_5 gravity, black holes at low temperatures are thermodynamically unstable. However, in more realistic holographic QCD constructions where the background involves coupled gravity-dilaton(-Maxwell) dynamics [44–47,85,86], it has been shown that black hole solutions can remain stable even at low temperatures, depending on the choice of dilaton potential and gauge coupling profile. Therefore, our use of black hole geometries in this regime is phenomenologically justified as an approximation. We emphasize that the appearance of stable low-temperature black-hole solutions in our probe-approximation setup should be regarded as model artifacts rather than physical predictions of holographic QCD. Numerical calculations show negligible differences in the results at different temperatures. Therefore, we have chosen the 10 MeV (with horizon $z_h = 1/\pi T \simeq 31.8 \text{ GeV}^{-1}$) temperature for presentation. This temperature is small compared to the pion mass $m_\pi \simeq 140 \text{ MeV}$ and can be approximated as zero temperature. The full action is given as

$$S_5 = - \int d^4x \int_\epsilon^{z_h} dz \sqrt{-g} e^{-\Phi(z)} \mathcal{L}_5, \quad (21)$$

where $\Phi(z) = \mu_g^2 z^2$ is dilaton field with constant μ_g that match the Regge slope of light mesons. To describe the mesons spectra and chiral condensate simultaneously, we

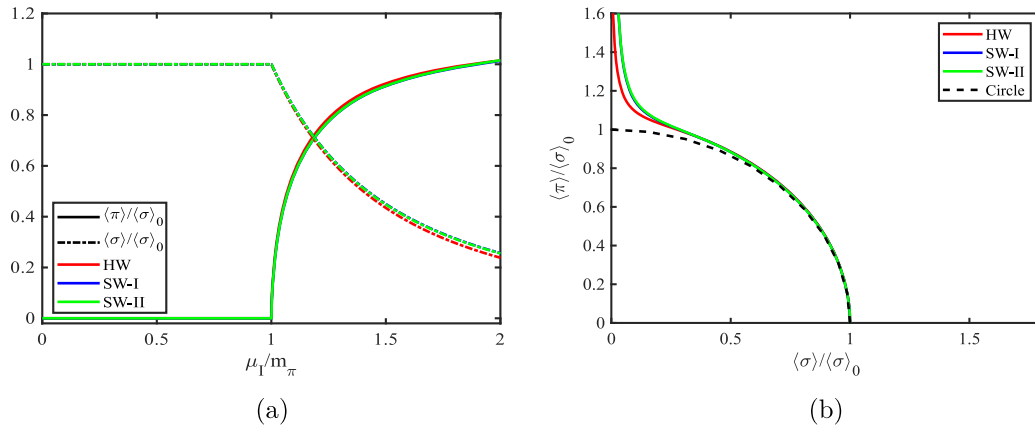


FIG. 1. Panel (a) illustrates how the sigma and pion condensates vary with the isospin chemical potential μ_I in both the HW and SW models. Panel (b) reveals the interrelationship between the sigma and pion condensates, where the black dashed line represents the unit circle.

have considered two possible forms of the five-dimensional mass squared $m_5^2(z)$ that depend on the fifth dimension, as follows:

$$m_5^2(z) = -3 - \mu_c^2 z^2, \quad (\text{SW-I}) \quad (22a)$$

$$m_5^2(z) = -3(1 + \gamma_m \tanh[\kappa\Phi(z)]), \quad (\text{SW-II}) \quad (22b)$$

with free parameter μ_c , γ_m , and κ . The leading term of them are consistent with the hard-wall model $m_5^2 = -3$. The

modified 5D mass profile is adopted so that the model can better describe chiral restoration and reproduce the meson spectra. The values of the parameters are given from Refs. [63,65] and shown in Table I. The parameters in soft-wall model I are global fitted to the masses of the pion, rho, and a_1 mesons, while those in soft-wall model II are determined by the pion mass, the vacuum chiral condensate, and the critical temperature. The equations of motion of the soft-wall model are obtained as

$$\frac{\lambda\Sigma^3}{2z^2} + \frac{v\Pi a_2}{f} + \left(-\frac{a_2^2}{f} + \frac{\lambda\Pi^2 + 2m_5^2(z)}{2z^2}\right)\Sigma + \left(-f' + f\left(\frac{3}{z} + \Phi'\right)\right)\Sigma' - f\Sigma'' = 0, \quad (23)$$

$$\frac{\lambda\Pi^3}{2z^2} + \frac{v\Sigma a_2}{f} + \left(-\frac{v^2}{f} + \frac{\lambda\Sigma^2 + 2m_5^2(z)}{2z^2}\right)\Pi + \left(-f' + f\left(\frac{3}{z} + \Phi'\right)\right)\Pi' - f\Pi'' = 0, \quad (24)$$

$$\frac{g_5^2\Sigma}{z^2}(v\Pi - a_2\Sigma) + \left(-\frac{f}{z} - f\Phi'\right)a_2' + fa_2'' = 0, \quad (25)$$

$$\frac{g_5^2\Pi}{z^2}(a_2\Sigma - v\Pi) + \left(-\frac{f}{z} - f\Phi'\right)v' + fv'' = 0. \quad (26)$$

At the UV, the boundary conditions are the same as Eq. (19). At the IR ($z \rightarrow z_h$), the expansions of Σ , Π , a_2 , and v have the following form

$$\begin{aligned} \Sigma(z) &\rightarrow \Sigma_0 + \frac{6\Sigma_0 + 2z_h^2\mu_c^2\Sigma_0 - \lambda\Pi_0^2\Sigma_0 - \lambda\Sigma_0^3}{8z_h}(z - z_h) + \mathcal{O}(z - z_h), \\ \Pi(z) &\rightarrow \Pi_0 + \frac{6\Pi_0 + 2z_h^2\mu_c^2\Pi_0 - \lambda\Sigma_0^2\Pi_0 - \lambda\Pi_0^3}{8z_h}(z - z_h) + \mathcal{O}(z - z_h), \\ a_2(z) &\rightarrow a_{2,1}(z - z_h) + \mathcal{O}(z - z_h), \\ v(z) &\rightarrow v_1(z - z_h) + \mathcal{O}(z - z_h), \end{aligned} \quad (27)$$

with integral constants Σ_0 , Π_0 , $a_{2,1}$, and v_1 .

Similarly, by applying boundary conditions to solve the equations of motion, we derive the condensates as a function of the isospin chemical potential μ_I . The outcomes are presented in Fig. 1. As illustrated in Fig. 1(a), when the isospin chemical potential exceeds the mass threshold of the pion meson m_π , the condensates undergo significant changes. With further increases in the isospin chemical potential, sigma condensation gradually decreases, while pion condensation correspondingly increases. The combination of these two condensates $\tilde{\sigma} = \sqrt{\sigma^2 + \pi^2}$ forms an approximate “chiral circle,” as intuitively displayed in Fig. 1(b). Both the SW and HW models predict similar condensation patterns.

It should be noted that the results obtained in this paper differ from the results in Ref. [63]. In that paper, at high

enough isospin densities, the pion meson condensation tends to vanish, showing a similar effect with the baryon density effect on chiral condensate. At the same time, the shape of the formed “chiral circle” also differs. The reason for this discrepancy lies in the different approximations used in this paper compared to [63]. Specifically, in Ref. [63], the gauge field V describing the isospin current is incorporated into the background, resulting in the geometric structure of an AdS-RN black hole. In this paper, however, we follow the method of [62,84], solving the probe action with the gauge field V included, while keeping the background as an AdS-Schwarzschild black hole with isospin charge. Of course, one might check the validities of the two approximations by doing a full backreaction analysis, which is out of the scope of the current work and will be left for the future.

III. PIONIC EQUATION OF STATE

As discussed in the previous section, pion condensate emerges in the system when the nonzero isospin chemical potential reaches and exceeds the pion mass threshold. In this section, we explore the equation of state of the system with pion condensate through holographic QCD models. Based on the holographic duality principle, the partition function in gravitational theory is equivalent to quantum field theory on the boundary, that is, $\mathcal{Z}_{\text{Gra}} = \mathcal{Z}_{\text{QCD}}$. As a consequence of this equivalence, we can obtain the isospin density n_I and the axial vector condensate, as follows

$$n_I = \int dz \frac{\partial \mathcal{L}_5}{\partial \mu_I} = \int dz \frac{1}{z} (v \Pi^2 - a_2 \Pi \Sigma), \quad (28)$$

$$\langle \sigma \rangle_A = \int dz \frac{\partial \mathcal{L}_5}{\partial \mu_A} = \int dz \frac{1}{z} (a_2 \Sigma^2 - v \Pi \Sigma), \quad (29)$$

where we introduce the source term μ_A for the axial vector field at the AdS boundary. During the calculation of axial-vector condensate, μ_A is incorporated into the model, and after completing the calculation steps it is reset to 0.

The computational results of the holographic QCD model are presented in Fig. 2. The circular markers in the figure represent the lattice QCD data [8], while the solid lines correspond to the predicted results of HW, SW-I, and SW-II, respectively. The error bars on the lattice data are smaller than or comparable to the symbol size and therefore would not be visually discernible in these plots, so we have omitted them to keep the figure uncluttered. We retain only the central values for clarity. The dashed lines in the figure depict the results derived from chiral perturbation theory (χ PT). At zero temperature, according to χ PT [25], the expressions for isospin density and axial-vector condensation can be simplified to

$$n_I = f_\pi^2 \mu_I \left(1 - \frac{m_\pi^4}{\mu_I^4} \right) \Theta(\mu_I - m_\pi), \quad (30)$$

$$\langle \sigma \rangle_A = \frac{f_\pi^2 m_\pi^2}{\mu_I} \sqrt{1 - \frac{m_\pi^4}{\mu_I^4}} \Theta(\mu_I - m_\pi), \quad (31)$$

with pion decay constant f_π and Heaviside step function Θ . In the paper, we clarify that m_π is computed from our model by considering the fluctuations of the Π field, which represents the pseudo-Goldstone boson. The different dashed lines in the figure correspond to the results obtained from the pion decay constants of different models [65,87].

From Fig. 2, it can be observed that the predictions of the HW model and the SW-II model match the data of the lattice QCD quite well, while the results of the SW-I model are approximately half of the lattice data. This discrepancy is because the value of the pion decay constant obtained in the SW-I model is 70.7 MeV, which is significantly lower than the results of lattice simulations. In comparison, the predicted values of the HW model and the SW-II model are closer to the experimental observation 92.4 MeV. The difference in the pion decay constants leads to a deviation between the model prediction and the data.

At zero temperature, the pionic pressure p and energy density ϵ can be given by thermodynamic relations

$$p = \frac{\log \mathcal{Z}_{\text{QCD}}}{V} = \int_0^{\mu_I} d\mu'_I n_I(\mu'_I), \quad \epsilon = -p + \mu_I n_I. \quad (32)$$

The solid lines in Fig. 3 show the predicted results of our model, while the data from lattice QCD are presented in the circles. Additionally, the Figs. 3(b) and 3(c) display the normalized trace anomalies $\Delta = 1/3 - p/\epsilon$ and $(\epsilon - 3p)/m_\pi^4$, respectively. In these subfigures, the black dashed lines correspond to the predictions of χ PT [23].

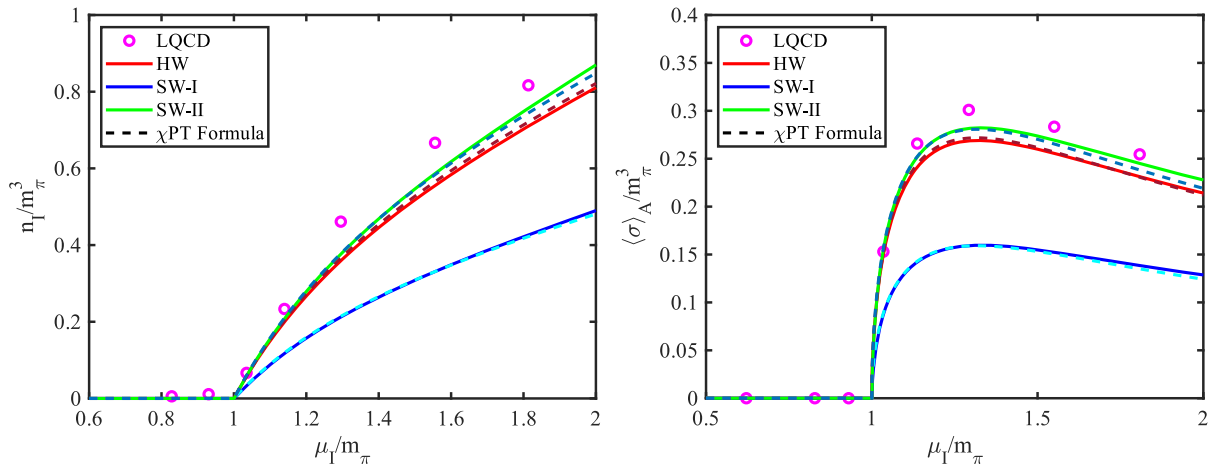


FIG. 2. The isospin density and axial vector condensate as a function of isospin chemical potential μ_I in HW and SW model. The black dashed lines are calculations from χ PT formulas (30) and (31).

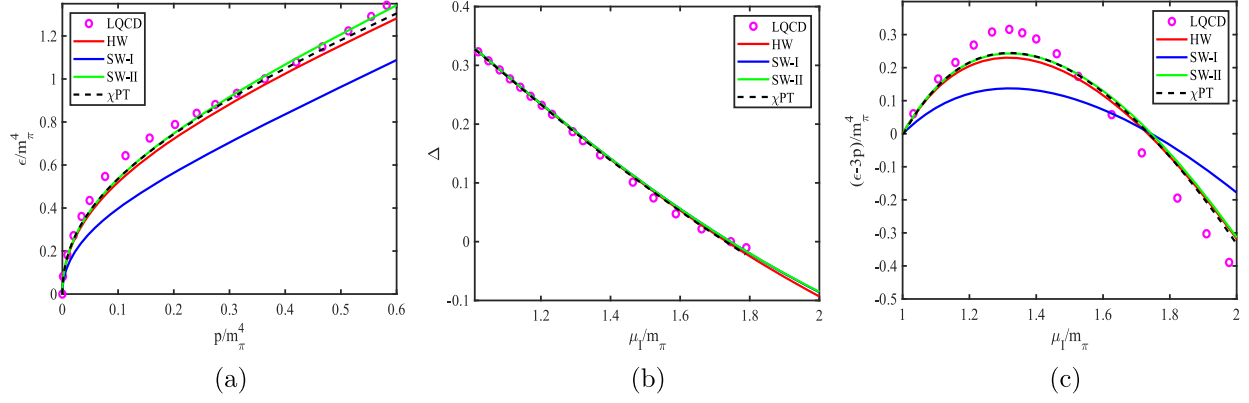


FIG. 3. The equation of state, normalized trace anomalies $\Delta = 1/3 - p/\epsilon$ and $(\epsilon - 3p)/m_\pi^4$ in HW and SW model. The magenta circles represent lattice QCD data [8]. The black dashed lines are calculations from χ PT [23].

It can be observed from Fig. 3 that the predictions of the HW and the SW-II models are in good agreement with the lattice QCD data [8], whereas the results of the SW-I model are significantly different from the data, which is mainly due to the difference in the isospin density. For the equation of state, the holographic QCD models' predictions show an approximately linear growth trend at high isospin chemical potential, while the lattice QCD data exhibit different growth characteristics. This discrepancy may arise from the fact that we neglected the contribution of the gravitational background in our calculations and only considered the probe approximation.

For the trace anomaly Δ , the results of the holographic models are similar to the results of χ PT, both showing a linearly decreasing trend. However, the lattice QCD data show a change in slope when the isospin chemical potential is around $1.7 m_\pi$. For the trace anomaly $(\epsilon - 3p)/m_\pi^4$, the predictions of models are closer to the results of the χ PT. However, at the isospin chemical potential of about $1.3 m_\pi$,

neither of the models can reach the maximum value shown by the lattice data.

IV. THE SPEED OF SOUND

From the equation of state derived in the previous section, we are able to calculate two quantities for astrophysical observations: the speed of sound c_s and the adiabatic index γ . The square of the speed of sound is defined as

$$c_s^2 = \frac{\partial p}{\partial \epsilon}, \quad (33)$$

and the adiabatic index is given as

$$\gamma = \frac{d \log p}{d \log \epsilon} = \frac{\epsilon}{p} c_s^2. \quad (34)$$

Figure 4 illustrates the results of the speed of sound and the adiabatic index for the isospin chemistry potential and compares them with the lattice QCD data and the χ PT calculations. The results indicate that the predictions of the holographic model are very close to the χ PT calculations,

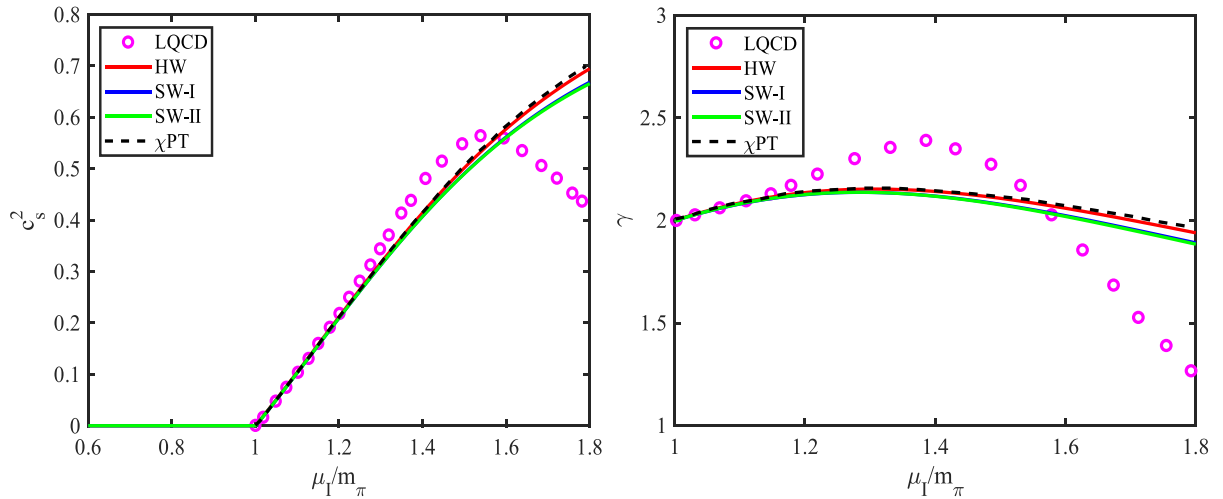


FIG. 4. The speed of sound and the adiabatic index as a function of the isospin chemical potential in the HW and SW models. The black dashed lines are calculations from χ PT and the magenta circles are the lattice QCD data from Ref. [23].

and at low chemical potentials, the holographic model agrees well with the lattice data. However, when the isospin chemical potential exceeds $1.2 m_\pi$, significant differences begin to emerge between the holographic model and lattice data. Specifically, when the chemical potential reaches $1.5 m_\pi$, lattice data show that the speed of sound reaches a maximum and then decreases with increasing chemical potential, exhibiting nonmonotonic behavior, while the holographic model predicts a continuous increase in the speed of sound.

In the conformal limit, the adiabatic index should approach $\gamma = 1$, but spontaneous chiral symmetry breaking can cause it to increase to about $\gamma \simeq 2$, which is discussed in detail in Ref. [88]. The adiabatic index value of 1.75 is considered the quark matter bound for distinguishing between two core matter states of neutron stars. Although a critical adiabatic index of 1.75 has been suggested in studies of high-density baryonic matter, in our analysis at zero baryon density this value is used only as a qualitative reference. The behavior of the adiabatic index in our model may indicate a gradual transition from a meson-dominated (BEC-like) phase to one where quarklike (BCS-like) degrees of freedom begin to emerge. The predictions of the holographic model show that the adiabatic index reaches a peak at an isospin chemical potential of about $1.25 m_\pi$, then gradually decreases to approach 2, suggesting that the system may always be near the chiral symmetry broken phase. In contrast, lattice QCD data show an extreme value of the adiabatic index at an isospin chemical potential of about $1.4 m_\pi$, approximately 2.4, and then it drops to 1.3, suggesting that the system may have undergone a crossover from Bose-Einstein condensate (BEC) to Bardeen-Cooper-Schrieffer (BCS) pairing at sufficiently large μ_I . In this regime, the initially tightly bound pionlike states gradually become more weakly bound quark pairs, analogous to the Cooper pairing mechanism in superconductors. This difference may be due to the fact that the holographic models do not take into account the contribution of the gluon background in the calculations. If gravitational background were included in the model, similar phase transition behavior might be observed. Furthermore, as mentioned above, within the approximation used in Ref. [63], at high isospin density, both the chiral condensate and pion condensate will vanish, which in some sense turns the system from a tightly combined BEC status (meson degree of freedom) to a loosely correlated BCS status (quark degree of freedom). Therefore, similar behavior of crossover from BEC to BCS might appear in that scenario, and in the full back-reaction study as well. However, in this work, we will stick to the current scenario and leave the full back-reaction analysis to the future.

Overall, the HW and SW models provide us with profound insights into strongly coupled pionic matter, but the comparison with lattice QCD data also reveals

the limitations of these models. Future research can focus on more precise model parametrization or the development of gravitational frameworks to more comprehensively describe these complex phenomena.

The results from Sec. III and IV indicate that both the holographic soft-wall and hard-wall models yield results similar to those of χ PT. In particular, the speed of sound and the adiabatic index from these models align closely and show a consistent discrepancy with lattice data. This suggests a potential equivalence between holographic models and χ PT. As noted in Ref. [77], integrating out the heavy Kaluza-Klein modes results in a four-dimensional chiral effective action. A similar discussion is found in Ref. [72], where the holographic model is used to derive and reproduce the formula (30) of χ PT. These findings imply that holographic models share a common ground with χ PT in capturing the physics of strong interactions.

V. PION STARS

In the previous discussion, we studied the pionic equation of state and observed its consistency with lattice QCD data, as well as similar behavior to the results of χ PT. This section will further explore the physical properties of pion stars based on the equation of state.

In the very early universe, asymmetry in lepton flavors led to the phenomenon of pion condensation, which eventually resulted in the formation of pion stars. Pion stars are a type of Bose stars [10,89,90], notable for their existence without the need for new physics beyond the Standard Model. These celestial bodies maintain electrical neutrality and beta equilibrium through the presence of electrons, muons, and their neutrinos. According to Ref. [91], leptons make a significant contribution to the star's equation of state. However, in this discussion, we will temporarily disregard the impact of leptons on the overall equation of state and focus on other characteristics of pion stars.

A. Mass-radius relation

The Tolman-Oppenheimer-Volkoff (TOV) equations originate from general relativity, describing the hydrostatic equilibrium state under the assumption of spherical symmetry. Combined with the equation of state, we can calculate the mass M and radius R of pion stars. The form of the TOV equations are as follow,

$$\frac{d}{dr}p(r) + [p(r) + \epsilon(r)] \frac{G[m(r) + 4\pi r^3 P(r)]}{r^2 \left[1 - 2 \frac{Gm(r)}{r}\right]} = 0, \quad (35)$$

$$\frac{d}{dr}m(r) - 4\pi r^2 \epsilon(r) = 0, \quad (36)$$

with radial coordinate r and 4D Newtonian gravitational constant G . The pressure p and the energy density ϵ both depend on the radial coordinate. Furthermore, $m(r)$ represents the gravitational mass within the radius r . The solution of the

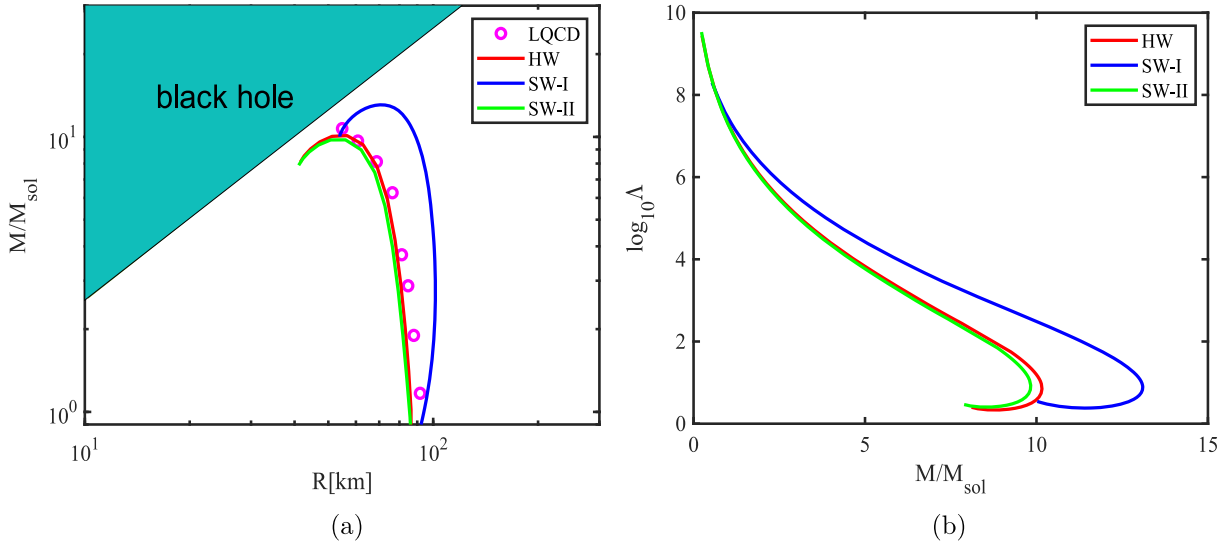


FIG. 5. The mass-radius relation and tidal deformability of pion stars. The magenta circles are the mass-radius relation obtained through the equation of state from lattice QCD data Ref. [23].

equations employed the following boundary conditions:

$$m(r=0) = 0, \quad p(r=r_B) = 0, \quad (37)$$

with the radius of the star $r_B = R$. The total mass of the pion star is $M = m(r_B)$.

Using the pionic equation of state obtained in the previous section, we derived the mass-radius relation for pion stars, as shown in Fig. 5(a). The solid lines of different colors in the figure represent the results of the HW and SW models, while the magenta circles are the mass-radius relations obtained through the equation of state from lattice data. It can be seen from the figure that, due to the proximity of the equation of state to the lattice data, the HW and SW-II models are similar to the lattice results, while the SW-I model predicts a larger upper limit of mass. Overall, the estimated radius of pion stars is between 50-100 kilometers, with the maximum mass reaching 10 times the mass of the sun, which is significantly different from the neutron stars with a radius of 10 kilometers and a maximum mass of 2 times the mass of the sun. If electrons and muons, as well as β equilibrium, are further considered, different mass-radius relations for pion stars will be obtained, as detailed in Ref. [8].

B. Tidal deformability

Finally, we investigated the dimensionless tidal deformability Λ ,

$$\Lambda = \frac{2k_2}{3C^5}, \quad (38)$$

where

$$\begin{aligned} k_2 = & \frac{8C^5}{5} (1-2C)^2 [2 + 2C(y-1) - y] \\ & \times \{2C[6-3y+3C(5y-8)] \\ & + 4C^3[13-11y+C(3y-2)+2C^2(1+y)] \\ & + 3(1-2C)^2[2-y+2C(y-1)] \ln(1-2C)\}^{-1}, \end{aligned} \quad (39)$$

and

$$C = \frac{GM}{r_B}, \quad y = \frac{r_B \beta(r_B)}{H(r_B)}. \quad (40)$$

The functions $H(r)$ and $\beta(r)$ satisfy the following equations,

$$\begin{aligned} \frac{d}{dr} \beta(r) - \frac{2H(r)}{\left[1 - 2\frac{Gm(r)}{r}\right]} \left\{ -2\pi G \left[5\epsilon(r) + 9P(r) + \frac{d}{dr} \epsilon(r) \left[P(r) + \epsilon(r) \right] \right] + \frac{3}{r^2} + \frac{2}{1 - 2\frac{Gm(r)}{r}} G^2 \left[\frac{M}{r^2} + 4\pi r P(r) \right]^2 \right\} \\ - \frac{2\beta(r)}{r \left[1 - 2\frac{Gm(r)}{r} \right]} \left\{ -1 + \frac{Gm(r)}{r} + 2\pi G r^2 [\epsilon(r) - P(r)] \right\} = 0, \end{aligned} \quad (41)$$

and

$$\beta(r) - \frac{d}{dr}H(r) = 0, \quad (42)$$

with the boundary conditions

$$H(r \rightarrow 0) \rightarrow a_0 r^2, \quad \beta(r \rightarrow 0) \rightarrow 2a_0 r. \quad (43)$$

The choice of the constant a_0 has no effect on the final result.

The numerical results are shown in Fig. 5(b). It can be seen from the figure that for a mass of 5 solar masses, the tidal deformability is approximately 10^4 .

VI. CONCLUSION AND DISCUSSION

This study uses holographic hard-wall and soft-wall models to explore the relation between pion condensation, isospin density, axial-vector condensation, and isospin chemical potential. We successfully derived the EoS for pionic matter using thermodynamic relations. Additionally, we calculated the normalized trace anomaly Δ and $(\epsilon - 3p)/m_\pi^4$. Using the EoS, we analyzed the sound speed and adiabatic index, which are crucial for understanding the physical characteristics of pion stars. We also obtained the mass-radius relation and tidal deformability of pion stars, providing quantitative results for studying these bosonic stars.

The results from the holographic models show good agreement with lattice QCD in terms of isospin density, axial-vector condensation, EoS, and normalized trace anomaly. This indicates that holographic models are highly reliable in describing the QCD phase diagram. However, discrepancies were found in the calculations of the sound speed and adiabatic index. At low isospin chemical potentials, the results of the holographic models and lattice QCD coincide. Still, when the isospin chemical potential exceeds $1.5 m_\pi$, the sound speed in lattice data reaches a maximum and then starts to decrease, while it continues to rise in the holographic model. Similarly, when the isospin chemical potential exceeds $1.4 m_\pi$, the adiabatic index in lattice QCD reaches a maximum and then rapidly decreases, while it remains around 2 in the holographic model. These differences might indicate a crossover from BEC to BCS at high isospin chemical potentials, which the holographic model fails to capture.

Further analysis shows that the results from the holographic models are very similar to those from χ PT. In the calculations of isospin density and axial-vector condensation, the results of the holographic models are highly consistent with the approximate formulas of χ PT. In addition, the calculations of the trace anomaly, sound speed, and adiabatic index also align with the predictions of χ PT. Notably, the approximate linear decrease in the trace anomaly, the continuous increase in sound speed with

isospin chemical potential, and the adiabatic index remaining around 2 are all consistent with χ PT calculations. This suggests that the holographic model reproduces several leading-order results of chiral perturbation theory and may provide a five-dimensional framework that mimics certain features of χ PT under the probe approximation. We emphasize that in our setup the condensates are modeled through homogeneous background fields in the bulk, which are dual to quark bilinear operators. Dynamical pions are not included as explicit degrees of freedom on the gravity side. As such, the observed agreement with chiral perturbation theory is restricted to a few leading-order results and does not imply a full equivalence. Moreover, including the backreaction is expected to change the background geometry and may modify the results significantly, potentially breaking the observed agreement with chiral perturbation theory. These issues require a more complete holographic treatment and are left for future study.

We emphasize that, while our present analysis shows good agreement with certain LO (and partially two-flavor NLO) features, it remains unclear how consistently this holographic construction would match the full three-flavor NLO chiral expansion [92]. In particular, the influence of the strange quark mass and associated low-energy constants has yet to be systematically addressed in our framework. Extending this model may reveal whether the holographic results can align well with three-flavor NLO analyses, especially at higher isospin chemical potentials. We plan to explore these refinements in future work.

In this study, we employed the holographic approach to the pionic equation of state to calculate the mass-radius relation and tidal deformability of pion stars. Our calculations revealed that the typical radius of a pion star is approximately 100 kilometers, with a mass around $10M_\odot$. The tidal deformability, Λ , ranges from 10 to 10^8 . The presence of lepton flavor asymmetry in the early Universe [5,6] could potentially drive the formation of pion condensation. Additionally, when the energy density in regions of high isospin density reaches a critical threshold, gravitational collapse may occur, leading to the formation of pion stars. However, in this paper, we have not fully explored the formation mechanism or lifetime of pion stars, which are topics that require further investigation.

Our approach differs slightly from the scheme in Ref. [63], leading to qualitative differences in the results. Further research is needed to clarify the reasons for these differences and their impact on physical phenomena. The current study neglects the contribution of gluon dynamics, which may be one of the reasons for the inconsistencies between the holographic model and lattice QCD data. Future studies could consider coupling the gravitational background with the system to reproduce lattice QCD results, thereby gaining a more comprehensive understanding of the $T - \mu_f$ phase diagram and properties of strongly interacting pionic matter.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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