

Scaling solutions for gauge invariant flow equations in dilaton quantum gravity

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We discuss the ultraviolet fixed point of asymptotically safe dilaton quantum gravity. It differs from the Reuter fixed point by the dependence of the Planck mass on a scalar field. The gauge invariant functional flow equation in the most general approximation with up to two derivatives strengthens the argument for the existence of this fixed point. The quantum effective action obtained from the scaling solution for dilaton quantum gravity can describe inflation for early cosmology and dynamical dark energy for late cosmology.

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I. INTRODUCTION

Extensions of general relativity to models for the metric and a scalar singlet field play a key role in modern cosmology. They are used for inflation [1–3] and dynamical dark energy [4–6]. The key ingredient is the quantum effective action for the metric coupled to the scalar. The field equations from its variation of the effective action with respect to the fields are exact and include all quantum effects. Cosmology is obtained as a solution to these field equations. The issue is to find the correct quantum effective action Γ . Employing diffeomorphism symmetry, one may use a derivative expansion with up to two derivatives

$$\Gamma = \int d^4x \sqrt{g} \left\{ U(\rho) + \frac{K(\rho)}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{F(\rho)}{2} R \right\}. \quad (1)$$

For the scalar field ϕ we assume a symmetry $\phi \rightarrow -\phi$, with invariant

$$\rho = \frac{1}{2} \phi^2. \quad (2)$$

The curvature scalar R involves two derivatives of the metric $g_{\mu\nu}$. The effective action will also include higher derivative terms, for example $\sim R^2$. Unless the coefficients

of such terms are huge they do not play an important role for most cosmological epochs.

The effective action (1) is the action of variable gravity [7–15], where the name reflects that the effective Planck mass M_p can vary with ϕ , $F(\rho) = M_p^2(\phi)$. Variable gravity has been employed for many interesting cosmological models, in particular, for dynamical dark energy, inflation, or quintessential inflation where both dynamical dark energy and inflation are described by the same scalar field ϕ . For a realistic model the effective action (1) has to be supplemented by a part involving fields for the particles of the standard model. Our quantum gravity computation will find that for large ρ the function F increases $\sim \phi^2$. If also the particle masses increase $\sim \phi$ the ratio of particle mass over Planck mass remains constant even if ϕ varies with cosmological time. This consequence of quantum scale symmetry [16] makes variable gravity compatible with the strong observational constraints on time varying couplings or apparent violations of the equivalence principle.

The effective action of variable gravity involves three free functions of the scalar field—the effective potential $U(\rho)$, the kinetic $K(\rho)$ and the squared Planck mass $F(\rho)$. Choosing them freely allows for a very wide variety of models. Predictions for cosmology depend however, on the choice of these functions. For a theory of quantum gravity one would like to predict these functions. The approach of asymptotic safety [17–19] is well suited for this purpose, since it can be based directly on a quantum field theory for the metric coupled to a scalar field, or more generally for gravitational fields coupled to fields for particles. The modern developments of asymptotic safety for gravity are based on functional flow equations of the effective action [20]. This approach introduces an infrared cutoff scale k such that only

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fluctuations with momentum larger than k are included. Then Γ_k becomes a k -dependent functional, and correspondingly U , K , and F are functions of ρ and k . The functional flow equation describes the dependence of these objects on k . The quantum effective action (1) is obtained by solving the flow equations for $k \rightarrow 0$.

The key ingredient of asymptotic safety describing a nonperturbatively renormalizable theory of quantum gravity is the presence of an ultraviolet fixed point of the flow. A solution valid for $k \rightarrow \infty$ permits the extension to arbitrarily short distances (ultraviolet completeness). Such a fixed point has been found for a large variety of truncations of the effective average action for pure gravity [21–69] and for gravity-matter systems [70–113]. It allows predictions for parameters of the standard model [114–126] and its extensions [127–140]. In our approximation (1) an ultraviolet fixed point requires a scaling solution for which U/k^4 , F/k^2 , and K become functions of a dimensionless scaling field without any explicit dependence on k . One can then start the flow at arbitrary short distances or arbitrarily large k and follow it to macroscopic distances or small k .

Most of the previous work on asymptotic safety concentrates on the Reuter fixed point and its extension to matter, with ρ -independent U , K , F for the scaling solution. For our purpose we are interested in a different fixed point, for which the scaling solution is characterized by $F \sim \rho$ for $\rho \rightarrow \infty$. This is named dilaton quantum gravity [141,142], since for field-independent F/ρ , K , and vanishing U , the effective action (1) is invariant under dilatation or scale transformations. For dilaton quantum gravity this holds for $k \rightarrow 0$ or $\rho \rightarrow \infty$ provided U vanishes $\sim k^4$. For nonzero ϕ the dilatation symmetry is spontaneously broken, producing the dilaton as a Goldstone boson. The aim of the present paper is to find scaling solutions for the approximation (1) for which U , F , and K are nontrivial functions of ρ . For the scaling solution of dilaton quantum gravity U has been found previously to be almost independent of ρ , while F approaches a constant for $\rho \rightarrow 0$ and is dominated by the dimensionless “nonminimal” coupling ξ for large ρ , corresponding to a term $\sim \xi \phi^2 R$ in the effective action.

While these qualitative features appear to be rather robust, a reliable computation of the kinetic $K(\rho)$ has to face certain challenges. The first concerns the fact that for $F = \xi \phi^2 = 2\xi\rho$ with constant ξ the kinetic can be negative. For constant K and ξ stability requires $K/\xi > -6$. This is due to a mixing in the kinetic terms for the scalar “cosmon”-field and the metric. Stability is most easily seen by a field transformation to the Einstein frame, where the coefficient of the scalar kinetic term is $\sim K + 6\xi$. A reliable computation should therefore cover also the range of negative K as long as stability is preserved. Second, for constant F/ρ and $K = -3F/\rho$ with $U = 0$ the effective action admits an enhanced symmetry, namely, conformal symmetry and local Weyl symmetry. In this limit the scalar field is no longer a dynamical degree of freedom. One

would like a computation of $K(\rho)$ which respects this symmetry. Third, for constants U and F and constant $K\rho$ the effective action is invariant under a multiplicative rescaling of ϕ , while keeping the metric invariant. Again, one would like to find a version of the renormalization flow that respects this additional symmetry.

The consistency with diffeomorphism symmetry, together with the additional symmetries mentioned above, poses a technical problem for the functional flow equations. If one wants to preserve these symmetries, the infrared cutoff term involves the macroscopic fields $g_{\mu\nu}$ and ϕ , for example, by the appearance of covariant derivatives. We base our computation on the setting of a gauge invariant flow equation [143]. The field dependence of the cutoff introduces correction terms to the simple one-loop form of the exact functional flow equation. In the setting of the simplified flow equation [144] these corrections have been estimated and found to be small. We omit them in the present work. The gauge invariant flow equation is, in this approximation, identical to the background field method with a particular “physical gauge fixing,” which removes the gauge modes in the metric without affecting the “physical fluctuations” [120].

II. FLOW EQUATION AND SCALING EQUATION

Our starting point is the flow equation for the effective average action [20]

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr}[(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} \partial_t \mathcal{R}_k], \quad (3)$$

where $\partial_t = k \partial_k$. Here $\mathcal{R}_k$ is the infrared cutoff, $\Gamma_k^{(2)}$ is the second functional derivatives of Γ_k , and $(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}$ is the propagator in the presence of fields and the infrared cutoff. For a discussion of the properties of the flow equation and its application in many fields we refer to Refs. [145–158]. We define dimensionless quantities by

$$\begin{aligned} u(\tilde{\rho}) &= U(\rho)/k^4, & w(\tilde{\rho}) &= F(\rho)/2k^2, \\ \kappa(\tilde{\rho}) &= K(\rho), & \tilde{\rho} &= \rho/k^2. \end{aligned} \quad (4)$$

The flow equation for u can be obtained by evaluating Eq. (3) for a constant scalar field and flat metric $g_{\mu\nu} = \eta_{\mu\nu}$. For w and κ , one adds small inhomogeneities in the macroscopic field ϕ and $g_{\mu\nu}$, for which we evaluate the flow equation by the use of the heat kernel method. One obtains

$$\partial_t u = \beta_U = -\partial_\tau u - 4u + \frac{1}{32\pi^2} M_U, \quad (5)$$

$$\partial_t w = \beta_F = -\partial_\tau w - 2w - \frac{1}{96\pi^2} M_F, \quad (6)$$

$$\partial_t \kappa = \beta_K = -\partial_\tau \kappa + M_K, \quad (7)$$

where M_U , M_F , and M_K are the flow kernels. The computation of M_U , M_F , and M_K is a main technical achievement of this work. We present this in various appendices. The switch from fixed ϕ to fixed ϕ/k or fixed $\tilde{\rho}$ introduces terms involving

$$-\tilde{\phi}\partial_{\tilde{\phi}} = -2\tilde{\rho}\partial_{\tilde{\rho}} = \partial_{\tau}, \quad (8)$$

where τ is the logarithm of the dimensionless field

$$\tau = -\log \sqrt{2\tilde{\rho}} = -\log |\tilde{\phi}| = -\log |\phi/k|. \quad (9)$$

The terms $-4u$ and $-2w$ reflect the dimension of U and F .

A possible scaling solution is obtained as a solution of Eqs. (5)–(7) for which u , w , and κ only depend on $\tilde{\rho}$ without any explicit dependence on k . This generalizes the notion of a fixed point for a finite number of couplings. The functions u , w , and κ encode an infinite number of couplings, defined, for example, as the coefficients of an expansion around $\tilde{\rho} = 0$. For the scaling solution, all these couplings take fixed values. We focus on this type of scaling solution. Further scaling solutions may be obtained by switching to renormalized dimensionless scaling fields reflecting the presence of an anomalous dimension. Such fields are related to $\tilde{\rho}$ by a nonlinear field transformation, see below.

The scaling solution for the functions $u(\tilde{\rho}, k)$ etc. requires

$$\partial_{\tilde{\rho}} u = \partial_{\tilde{\rho}} w = \partial_{\tilde{\rho}} \kappa = 0. \quad (10)$$

It has therefore to obey the “scaling equations”

$$\partial_{\tau} u(\tilde{\rho}) = -4u(\tilde{\rho}) + \frac{1}{32\pi^2} M_U, \quad (11)$$

$$\partial_{\tau} w(\tilde{\rho}) = -2w(\tilde{\rho}) - \frac{1}{96\pi^2} M_F, \quad (12)$$

$$\partial_{\tau} \kappa(\tilde{\rho}) = M_K. \quad (13)$$

This is a coupled system of three nonlinear differential equations. Its form is very similar to the flow equations for the dependence of k for the couplings $u_0 = u(\tilde{\rho} = 0)$, $w_0 = w(\tilde{\rho} = 0)$, $\kappa_0 = \kappa(\tilde{\rho} = 0)$. The scaling equations determine, however, three functions of $\tilde{\rho}$, and not only three couplings. This is of key importance for their possible solutions. They have to be defined for the whole range $0 \leq \tilde{\rho} < \infty$. As is common for differential equations, the corresponding boundary conditions pose severe constraints. The flow kernels M_U , M_F , M_K in Eqs. (5)–(7) involve first and second derivatives of $\tilde{\rho}$. The differential equations are second order. The general local solution therefore depends on six initial values $u(\tilde{\rho}_0)$, $\partial_{\tilde{\rho}} u(\tilde{\rho}_0)$, $w(\tilde{\rho}_0)$, $\partial_{\tilde{\rho}} w(\tilde{\rho}_0)$, $\kappa(\tilde{\rho}_0)$, $\partial_{\tilde{\rho}} \kappa(\tilde{\rho}_0)$ taken at some initial values $\tilde{\rho}_0$. These initial values define a family of local scaling

solutions. The boundary conditions for a global solution for all $\tilde{\rho}$ selects a subset of this family.

III. SCALING SOLUTION

A numerical solution of the scaling equations which extends to the whole range of $\tilde{\rho}$ is shown in Fig. 1. We observe simple characteristic features. The function $u(\tilde{\rho})$ interpolates between two constants $u_0 = u(\tilde{\rho} = 0)$ and $u_{\infty} = u(\tilde{\rho} \rightarrow \infty)$. We will see that u_{∞} is simply determined by the number of massless physical excitations, two for the graviton and one for scalar. For u_0 the gravitational contribution is somewhat modified which leads to a small shift. The function $w(\tilde{\rho})$ increases linearly for large $\tilde{\rho}$, reflecting the nonminimal gravitational coupling of the scalar. It approaches for $\tilde{\rho} \rightarrow \infty$ the form $w(\tilde{\rho}) \rightarrow \infty = \xi_{\infty} \tilde{\rho}$ with constant ξ_{∞} . The approximated form $w(\tilde{\rho}) = w_0 + \xi_{\infty} \tilde{\rho}$ shows a crossover for $\tilde{\rho}_c = w_0/\xi_{\infty}$, with $w_0 = w(\tilde{\rho} = 0)$. This value determines the location of the crossover for the potential $u(\tilde{\rho})$. We see in Fig. 1 that this form is a good approximation by plotting $\tilde{\xi}(\tilde{\rho}) = (w(\tilde{\rho}) - w_0)/\tilde{\rho}$, which corresponds to $w(\tilde{\rho}) = w_0 + \tilde{\xi}(\tilde{\rho})\tilde{\rho}$. The function $\tilde{\xi}(\tilde{\rho})$ varies only mildly between $\tilde{\rho} = 0$ and $\tilde{\rho} \rightarrow \infty$. Finally, the kinetic $\kappa(\tilde{\rho})$ also shows a crossover between two plateaus, $\kappa_0 = \kappa(\tilde{\rho} \rightarrow 0)$ and $\kappa_{\infty} = \kappa(\tilde{\rho} \rightarrow \infty)$, located near $\tilde{\rho}_c$. For the solution shown in Fig. 1, one has $\kappa_{\infty} < 0$, demonstrating our possibility to cover the range of negative K .

We can associate the asymptotic behavior with two fixed points. The infrared (IR) fixed point is approached for $\tilde{\rho} \rightarrow \infty$, or $k \rightarrow 0$ at fixed ϕ . The ultraviolet (UV) fixed point characterizes the limit $\tilde{\rho} \rightarrow 0$, or $k \rightarrow \infty$ at fixed ϕ . The scaling solution describes a crossover between the two fixed points as a function of $\tilde{\rho}$. The discussion of the UV completion of quantum gravity often considers only the couplings for $\tilde{\rho} \rightarrow 0$, e.g., u_0 , w_0 , and κ_0 . The functional form of the flow equations implies, however, that fixed points for the k flow of these couplings only exist if the infinitely many couplings encoded in the functions $u(\tilde{\rho})$, $w(\tilde{\rho})$, $\kappa(\tilde{\rho})$ also take fixed values. The UV completion of quantum gravity requires a scaling solution for these functions, which imposes constraints for the existence of the scaling solution for $\tilde{\rho} \rightarrow \infty$. This provides a deep connection between the IR and the UV. Quantum gravity becomes relevant for the behavior of the effective potential $U(\rho)$ for large ρ , which typically determines the late time behavior of dynamical dark energy within variable gravity models.

The qualitative behavior of the crossover for the functions $u(\tilde{\rho})$ and $w(\tilde{\rho})$ is rather robust and does not depend strongly on the behavior of $\kappa(\tilde{\rho})$. This may be demonstrated by solving the flow equations, (5) and (6), for a fixed value $\kappa(\tilde{\rho}) = 1$. The result is shown in Fig. 2. The initial values used here are $u(10^4) = u_{\infty} = 3/(128\pi^2) = 0.00237975$, $\partial_{\tilde{\rho}} u(10^4) \approx 0$, $w(10^4) = w_0 + \xi_{\infty} \tilde{\rho}_0 = 19.39$, and $\partial_{\tilde{\rho}} w(10^4) = \xi_{\infty} = 0.07$ as a boundary condition.

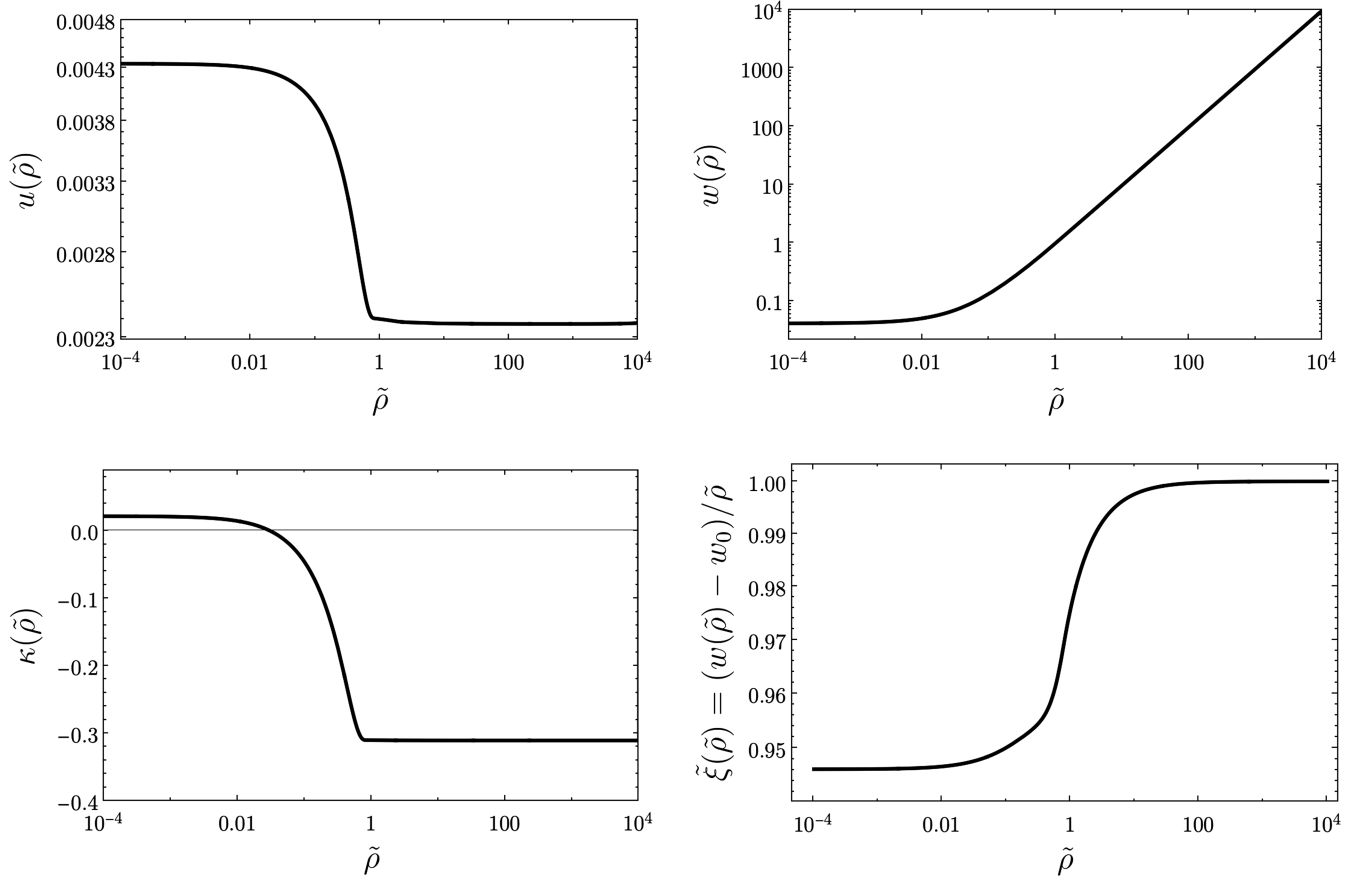


FIG. 1. Scaling solution for $u(\tilde{\rho})$ (top-left), $w(\tilde{\rho})$ (top-right), and $\kappa(\tilde{\rho})$ (bottom-left). The initial condition is set at $\tilde{\rho}_0 = 10^4$ to be $u(\tilde{\rho}_0) = u_\infty = 3/(128\pi^2) \approx 0.00238$, $\partial_{\tilde{\rho}} u(\tilde{\rho}_0) \approx 0$, $w(\tilde{\rho}_0) = w_0 + \xi_\infty \tilde{\rho}_0$, $\partial_{\tilde{\rho}} w(\tilde{\rho}_0) = \xi_\infty$ with $w_0 = 0.0277$, $\xi_\infty = 1$, $\kappa(\tilde{\rho}_0) = \kappa_\infty = -0.3$, and $\partial_{\tilde{\rho}} \kappa(\tilde{\rho}_0) \approx 0$. This initial condition corresponds to $B = 6 + \varepsilon_\infty = 6 + \kappa_\infty/\xi_\infty = 5.7$. The scaling solution for $\tilde{\xi}(\tilde{\rho}) = (w(\tilde{\rho}) - w_0)/\tilde{\rho}$ is displayed in the bottom-right panel. For large field value $\tilde{\rho} \rightarrow \infty$, this function converges to $\xi_\infty = 1$.

IV. INFRARED AND ULTRAVIOLET FIXED POINTS

The IR and UV-fixed point behavior for $\tilde{\rho} \rightarrow \infty$ and $\tilde{\rho} \rightarrow 0$ is seen clearly in Figs. 1 and 2. It can be understood

quantitatively. For the limits $\tilde{\rho} \rightarrow \infty$ and $\tilde{\rho} \rightarrow 0$ the flow generators can be expanded in powers of $\tilde{\rho}^{-1}$ or $\tilde{\rho}$, respectively. Keeping only few terms, we can write down explicit expressions. This permits an analytical treatment of the IR and UV-fixed points.

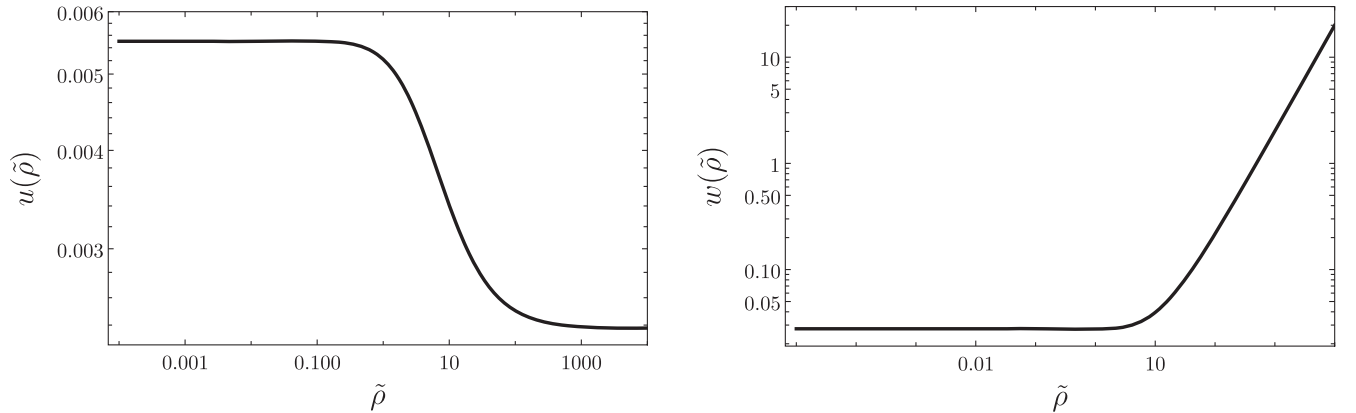


FIG. 2. Scaling solution for $u(\tilde{\rho})$ (left) and $w(\tilde{\rho})$ (right) with $\kappa(\tilde{\rho}) = 1$ fixed for all field values. The initial condition is set at $\tilde{\rho}_0 = 10^4$ to be $u(\tilde{\rho}_0) = u_\infty = 3/(128\pi^2) \approx 0.00238$, $\partial_{\tilde{\rho}} u(\tilde{\rho}_0) \approx 0$, $w(\tilde{\rho}_0) = w_0 + \xi_\infty \tilde{\rho}_0$ and $\partial_{\tilde{\rho}} w(\tilde{\rho}_0) = \xi_\infty$ with $w_0 = 0.0277$ and $\xi_\infty = 0.07$.

A. Large field expansion and IR fixed point

The properties of the IR fixed point are easy to understand. With dimensionless Planck mass $\sqrt{2\xi_\infty\tilde{\rho}}$ going to infinity, the gravitational coupling approaches zero. Thus, the metric fluctuations decouple, except for a constant contribution $M_U^{\text{gr}} = 2$, reflecting the 2 degrees of freedom of the massless graviton. For constant $u(\tilde{\rho} \rightarrow \infty) = u_\infty$ the scalar potential is flat and we are left with a massless scalar field. For $\kappa(\tilde{\rho} \rightarrow \infty) = \kappa_\infty$ there are no interactions from the kinetic term either. We are then left with a decoupled system of a free massless scalar and a massless graviton. The contribution of the massless scalar to the flow of the cosmological constant $U(\tilde{\rho} \rightarrow \infty)$ is given by $M_U^s = 1$. The scaling solution for $u(\tilde{\rho} \rightarrow \infty)$ is given by $u = M_U/(128\pi^2)$, resulting in

$$u_\infty = \frac{3}{128\pi^2} \approx 0.00238. \quad (14)$$

This is the value which is approached in Figs. 1 and 2. For the scaling solution the cosmological constant, $U = u_\infty k^4$ vanishes for $k \rightarrow 0$. Since gravity effectively decouples the nonminimal coupling, ξ and the kinetic κ become independent of $\tilde{\rho}$. For the IR fixed point they are free parameters. Since physics is not affected by a change of the normalization of the scalar field, only the ratio $\varepsilon_\infty = \kappa_\infty/\xi_\infty$ matters. We end with a continuous family of IR fixed points parametrized by ε_∞ . Stability requires $\varepsilon_\infty > -6$. Since $\kappa(\tilde{\rho})$ becomes a constant $k\partial_k K$ also becomes a constant. There is therefore no anomalous dimension for the scalar field, implying that ϕ/k is the appropriate dimensionless renormalized field for the IR fixed point.

For finite large $\tilde{\rho}$ the gravitational effects start to matter, being suppressed by powers of $\tilde{\rho}^{-1}$. The scalar field and the metric are no longer decoupled. In this large field region we can perform a series expansion. To this end, we define $y = 1/\tilde{\rho}$ and make the expansion ansatz

$$u(y) = u_\infty + \sum_{n=1}^{\infty} \frac{s_n}{n! \xi_\infty^n} y^n = u_\infty + \frac{s_1}{\xi_\infty} y + \dots, \quad (15)$$

$$w(y) = \xi_\infty y^{-1} + \sum_{n=0}^{\infty} \frac{t_n}{n! \xi_\infty^n} y^n = \xi_\infty y^{-1} + t_0 + \frac{t_1}{\xi_\infty} y + \dots, \quad (16)$$

$$\kappa(y) = \varepsilon_\infty \xi_\infty + \sum_{n=1}^{\infty} \frac{k_n}{n! \xi_\infty^{n-1}} y^n = \varepsilon_\infty \xi_\infty + k_1 y + \dots, \quad (17)$$

where $\varepsilon_\infty = \kappa_\infty/\xi_\infty$. Switching to the variable $x = y/\xi_\infty$ we expand the scaling solution in terms of x

$$\partial_\tau u(x) = 2x\partial_x u(x) = -4u(x) + A_u + \frac{C_u}{B}x = 2s_1x + \dots, \quad (18)$$

$$\begin{aligned} \partial_\tau w(x) &= 2x\partial_x w(x) = -2w(x) + \frac{A_w}{B} + \frac{C_w}{B^2}x + \dots \\ &= -\frac{2}{x} + 2t_1x + \dots, \end{aligned} \quad (19)$$

$$\partial_\tau \kappa(x) = 2x\partial_x \kappa(x) = \xi_\infty \frac{C_\kappa}{B^2}x + \dots = 2\xi_\infty k_1x + \dots. \quad (20)$$

Here $B = 6 + \varepsilon_\infty$ has to be positive for stability. For the conformal point $B = 0$ the expansion has to be modified. The coefficients of the IR expansion (18)–(20) read

$$A_u = \frac{3}{32\pi^2}, \quad (21)$$

$$C_u = \frac{1}{480\pi^2} (34Bt_0 + 75Bu_\infty - 54t_0 + 5k_1), \quad (22)$$

$$A_w = \frac{1}{1536\pi^2} (509B + 194), \quad (23)$$

$$\begin{aligned} C_w &= \frac{1}{322560\pi^2} (37950B^2t_0 + 85505B^2u_\infty - 20700Bt_0 \\ &\quad - 1680Bk_1 + 5124Bu_\infty + 245808t_0 - 41496k_1 \\ &\quad - 69804u_\infty), \end{aligned} \quad (24)$$

$$C_\kappa = -\frac{9}{16\pi^2} + \frac{21B}{32\pi^2} - \frac{5B^2}{64\pi^2} - \frac{B^3}{384\pi^2}. \quad (25)$$

The terms with factor B^{-1} involve the scalar fluctuations. There are no poles or other nonanalytic features as long as the stability condition $B > 0$ is met. This shows that our method can cover the whole stable range for K , including negative values.

The fixed point values for the coefficients t_0, s_1, t_1, k_1 are obtained from Eqs. (18)–(20) by equating the coefficients of x^n on both sides

$$t_0 = \frac{A_w}{2B} = \frac{509}{3072\pi^2} + \frac{97}{1536\pi^2 B}. \quad (26)$$

Also k_1 depends on B

$$k_1 = \frac{C_\kappa}{2B} = \frac{21}{64\pi^2 B} - \frac{9}{32\pi^2 B^2} - \frac{5}{128\pi^2} - \frac{B}{768\pi^2}. \quad (27)$$

For s_1 and t_1 we can then solve iteratively

$$s_1 = \frac{C_u}{6B}, \quad t_1 = \frac{C_w}{4B^2}. \quad (28)$$

These coefficients diverge for $B \rightarrow 0$, indicating that in the close vicinity of the conformal point $B = 0$, a different approximation may be more suited. For $B = 8$ ($B = 50$) we find $A_w = 0.281(1.692)$, $C_u = 0.046t_0 + 0.00106k_1 + 0.127u_\infty(0.347t_0 + 0.00106k_1 + 0.792u_\infty)$, and $C_\kappa = -0.167(-49.50)$. This can be compared with the values in Ref. [142], namely, $A_w = -4.419(-1025.99)$, $C_u = -5.0323t_0 - 2.516k_1 + 46.493u_\infty(3244.03t_0 + 73.728k_1 + 77139.1u_\infty)$, and $C_\kappa = -3.49(76629.8)$. A more detailed comparison can be found in Appendix G. While the overall features of the dilaton quantum gravity fixed point are rather robust, the quantitative details depend on the particular version of the functional flow equation.

We can use the expansion (15)–(17) in order to set initial conditions for numerical solutions of the scaling equations. We take x instead of $\tilde{\rho}$ and set initial conditions for some $x_{\text{in}} \ll 1$. With $u(x) = u_\infty + s_1 x$, fixed u_∞ , and s_1 a function of B , the initial conditions $u(x_{\text{in}})$ and $\partial_x u_{\text{in}}$ are fixed as functions of B . For $w = \frac{1}{x} + t_0 + t_1 x$ and t_0, t_1 functions of B , $w(x_{\text{in}})$ and $\partial_x w(x_{\text{in}})$ are also fixed functions of B . If we use $\varepsilon(x) = \kappa(x)/\xi_\infty = B - 6 + k_1 x$, the initial conditions for $\varepsilon(x_{\text{in}})$ and $\partial_x \varepsilon(x_{\text{in}})$ are determined as functions of B as well. We end with a one-parameter family of local scaling solutions parametrized by B . This feature persists if one continues the series to higher power of x . Formally, a second free parameter is ξ_∞ . This is, however, only fixing the overall normalization of the scalar field. Without loss of generality one may use a normalization defined by $\xi_\infty = 1$. In this case the scalar field ϕ denotes precisely the Planck mass in the IR limit.

B. Small field expansion and UV-fixed point

With initial conditions for some small x_{in} fixed by B , the scaling solution has to be continued to the UV-fixed point for $\tilde{\rho} \rightarrow 0$. The existence of such a global solution may select a specific value of B . The scaling solution has to exist for the whole range of an appropriate dimensionless “scaling field”. *A priori*, it is not obvious which is the form of the scaling field in the UV region. It may be a renormalized field in the case of a nonzero anomalous dimension. Because of the simple properties of the IR fixed point, which describes a free theory, the anomalous dimension vanishes in the IR limit, justifying the choice of ϕ/k as a dimensionless renormalized scaling field. The UV limit may be characterized by an interacting theory. An anomalous dimension is therefore possible. For example, if K diverges proportional ϕ^{-2} for $\phi \rightarrow 0$, the renormalized scaling field is ϕ instead of ϕ/k . In this case the anomalous dimension cancels the canonical dimension of the field. In the present paper we concentrate on scaling solutions for which the anomalous dimension of the scalar field vanishes in the UV. This is the case for solutions for which $\kappa(\tilde{\rho} \rightarrow 0)$ reaches a constant κ_0 , as in Fig. 1. The scaling solution is then a solution of the system of differential equations (11)–(13) which exists for the whole range of $\tilde{\rho}$.

Near the UV-fixed point we approximate the scaling solution by an expansion in power of $\tilde{\rho}$. A first approximation inserts on the rhs of the scaling equation the ansatz

$$u(\tilde{\rho}) = u_0, \quad w(\tilde{\rho}) = w_0, \quad \kappa(\tilde{\rho}) = \kappa_0. \quad (29)$$

If one can find a solution with this ansatz it would correspond to a full solution for all $\tilde{\rho}$ in our truncation, corresponding to the extended Reuter fixed point. The effective action would be invariant under the shift $\phi \rightarrow \phi + c$. Insertion of the ansatz (29) in the scaling equation yields

$$\begin{aligned} \partial_\tau u(\tilde{\rho}) = & -4u_0 - \frac{5u_0}{24\pi^2(u_0 - w_0)} - \frac{3u_0}{160\pi^2(u_0 - 4w_0)} \\ & + \frac{79}{480\pi^2}, \end{aligned} \quad (30)$$

$$\begin{aligned} \partial_\tau w(\tilde{\rho}) = & -2w_0 - \frac{25u_0}{64\pi^2(u_0 - w_0)} + \frac{5u_0^2}{1344\pi^2(u_0 - 4w_0)^2} \\ & - \frac{9u_0}{896\pi^2(u_0 - 4w_0)} + \frac{1207}{2688\pi^2}, \end{aligned} \quad (31)$$

$$\partial_\tau \kappa(\tilde{\rho}) = \frac{(5u_0 - 23w_0)}{240\pi^2(u_0 - 4w_0)^2} \kappa_0. \quad (32)$$

The lhs of these equations vanishes for $\tilde{\rho} = 0$, leaving us with three equations for u_0 , w_0 , and κ_0 . The first two equations do not involve κ_0 and fix

$$u_{0*} = 0.0055, \quad w_{0*} = 0.0277. \quad (33)$$

For these values Eq. (32) has no solution for $\kappa_0 > 0$. Requiring $\kappa_0 > 0$ as a condition for stability the system (30)–(32) is overdetermined. A constant solution (29) does not exist.

One concludes that for solutions with a vanishing anomalous dimension, the higher order terms in the expansion $u(\tilde{\rho}) = u_0 + c_1 \tilde{\rho} + \dots$ etc. have to contribute to the scaling equation for $\tilde{\rho} = 0$. This implies that a scaling solution with $\tilde{\rho}$ -independent u , w , and κ is not possible once the scaling solution for $\kappa(\tilde{\rho})$ is imposed. In general, the flow kernels involve derivatives $\partial_{\tilde{\rho}} u$ etc., such that the rhs of Eqs. (30)–(32) involve c_1 and similar terms. The system is not closed. The expansion of the scaling equation in powers of $\tilde{\rho}$ determines the coefficients of terms in higher order as functions of the lower order ones. This leaves free boundary values or initial values, as u_0 , w_0 , and κ_0 , for the local scaling solution in the vicinity of $\tilde{\rho}$. For global scaling solutions at least one these UV initial values has to match with the IR initial value B for the local scaling solution for large $\tilde{\rho}$.

One may ask if in our truncation a definite value of B is singled out, or if scaling solutions exist for continuous

ranges in B . For this purpose we solve the scaling equation for different initial values B . A requirement for an acceptable solution is stability of the scalar kinetic term for all values of $\tilde{\rho}$. In particular, $\kappa_0 = \kappa(\tilde{\rho} = 0)$ has to be positive. Stability requires positivity of the frame invariant combination $\hat{K}$,

$$\hat{K} = \frac{K}{F} + \frac{3}{2F^2} \left(\frac{\partial F}{\partial \phi} \right)^2, \quad \hat{K}k^2 = \frac{1}{2w} \left(\kappa + \frac{6\tilde{\rho}}{w} \left(\frac{\partial w}{\partial \tilde{\rho}} \right)^2 \right). \quad (34)$$

(In the Einstein frame one has $\partial F/\partial \phi = 0$ and $\hat{K} = K/M_{\text{pl}}^2$.) For $\tilde{\rho} \rightarrow 0$ the positivity of $\hat{K}k^2$ requires $\kappa_0 > 0$, while for $\tilde{\rho} \rightarrow \infty$ a positive $\hat{K}k^2$ is realized for

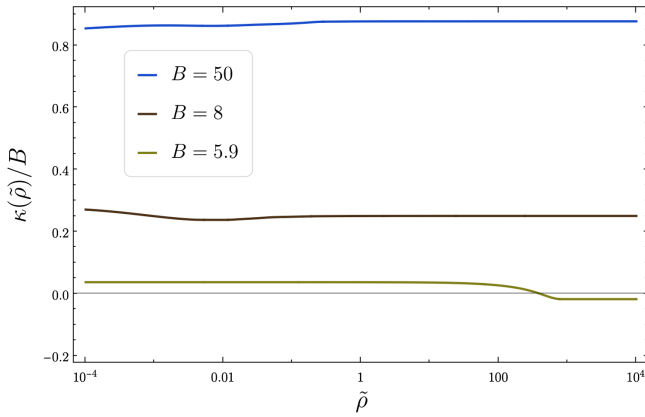
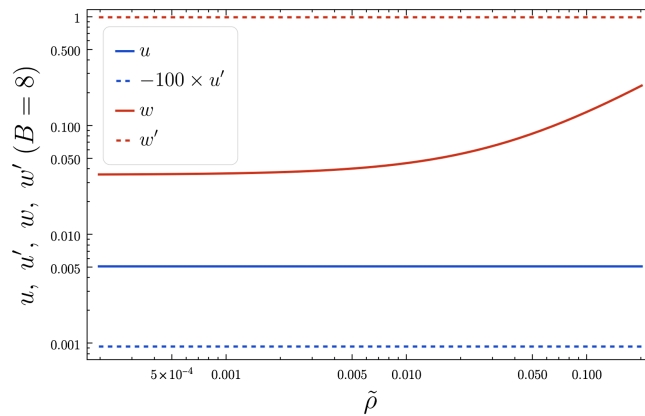


FIG. 3. Scaling solutions for $\kappa(\tilde{\rho})/B$ with several values of $B = 6 + \varepsilon_\infty = 6 + \kappa_\infty/\xi_\infty$. The initial conditions are set at $\tilde{\rho}_0 = 10^4$ to be $\kappa(\tilde{\rho}_0) = \kappa_\infty = \xi_\infty(B - 6)$, $u(\tilde{\rho}_0) = u_\infty = 3/(128\pi^2) \approx 0.00238$, $\partial_{\tilde{\rho}} u(\tilde{\rho}_0) \approx 0$, $w(\tilde{\rho}_0) = w_0 + \xi_\infty \tilde{\rho}_0$, and $\partial_{\tilde{\rho}} w(\tilde{\rho}_0) \approx \xi_\infty$ with $w_0 = 0.0277$ and $\xi_\infty = 1$. For all these values of B one finds only small shifts in the values of κ for large and small $\tilde{\rho}$.



$\kappa_\infty/\xi_\infty + 6 > 0$ or $B > 0$. In Fig. 3 we display $\kappa(\tilde{\rho})$ for different values of $B > B_c$, $B_c \approx 5.7$ for which $\hat{K}k^2$ remains positive for all $\tilde{\rho}$. For $B < B_c$, the stability criterion is violated and numerical instabilities develop.

Solutions with different B lead to different u_0 and w_0 . The derivatives $u'(\tilde{\rho} = 0)$ and $w'(\tilde{\rho} = 0)$ are nonzero and depend on B . In Fig. 4 we display $u(\tilde{\rho})$, $w(\tilde{\rho})$, $u'(\tilde{\rho})$, and $w'(\tilde{\rho})$ for two different values of B . This demonstrates the point made above that for the values u_0 and w_0 the derivatives of u and w play a role. At this stage it seems not excluded that within our truncation (and numerical accuracy) continuous families of scaling solutions exist.

V. SCALAR FIELD ANOMALOUS DIMENSION

For the extended Reuter fixed point with $\tilde{\rho}$ -independent u and w one has to change the scaling field by including an anomalous dimension. In this case κ_0 becomes a k -dependent function. The relation,

$$k \partial_k \kappa_0 = -\eta \kappa_0, \quad (35)$$

defines the anomalous dimension η . Switching to renormalized fields $\phi_R = K^{1/2} \phi$, $\tilde{\rho}_R = K \tilde{\rho}$ the flow equation (5) at fixed $\tilde{\rho}_R$ (instead of fixed $\tilde{\rho}$) reads

$$(\partial_t - (2 + \eta) \tilde{\rho} \partial_{\tilde{\rho}}) u = -4u + \frac{1}{32\pi^2} M_U, \quad (36)$$

and similar for the other flow equations. The form of the scaling equations remains the same if we use

$$\partial_\tau = -(2 + \eta) \tilde{\rho} \partial_{\tilde{\rho}}. \quad (37)$$

The scaling equations (30) and (31) yield the same relations for u_0 , w_0 as before since for $\tilde{\rho}_R = 0$ the lhs $\partial_\tau u$ or $\partial_\tau w$ vanishes. For Eq. (32) one replaces ∂_τ by $k \partial_k + \partial_\tau$ and extracts the anomalous dimension

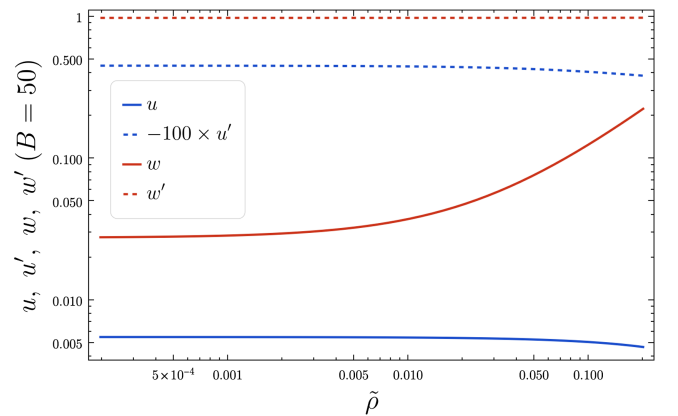


FIG. 4. Scaling solutions for $u(\tilde{\rho})$, $u'(\tilde{\rho})$, $w(\tilde{\rho})$, and $w'(\tilde{\rho})$ in cases of $B = 8$ (left) and $B = 50$ (right). The initial conditions are set at $\tilde{\rho}_0 = 10^4$ to be $\kappa(\tilde{\rho}_0) = \kappa_\infty = \xi_\infty(B - 6)$, $u(\tilde{\rho}_0) = u_\infty = 3/(128\pi^2) \approx 0.00238$, $\partial_{\tilde{\rho}} u(\tilde{\rho}_0) \approx 0$, $w(\tilde{\rho}_0) = w_0 + \xi_\infty \tilde{\rho}_0$, and $\partial_{\tilde{\rho}} w(\tilde{\rho}_0) \approx \xi_\infty$ with $w_0 = 0.0277$ and $\xi_\infty = 1$. The values of $\tilde{\rho} \rightarrow 0$ differ, most pronounced for $u'(0)$.

$$\eta = \frac{(23w_0 - 5u_0)}{240\pi^2(u_0 - 4w_0)^2} = 0.0116. \quad (38)$$

This value can be seen as the result of the gauge invariant flow equation for the gravity induced anomalous dimension for scalar fields for the extended Reuter fixed point. The extended Reuter fixed point is invariant under shifts of the renormalized field $\phi_R \rightarrow \phi_R + a$. This symmetry is preserved in our setting.

For dilaton quantum gravity with the IR fixed point $w = \xi_\infty \tilde{\rho}$, the modification of the scaling field by an anomalous dimension offers new possibilities for scaling solutions. The function $\kappa(\tilde{\rho})$ may now diverge for $\tilde{\rho} \rightarrow 0$, provided that this divergence is absorbed in the definition of a renormalized scaling field $\tilde{\rho}_R$. We will not pursue this possibility in the present paper. We mention, however, that for strongly diverging $\kappa(\tilde{\rho})$ the series expansion for $\tilde{\rho} \rightarrow 0$ may be rearranged. This is demonstrated by using a different ansatz for $\tilde{\rho} \rightarrow 0$

$$u(\tilde{\rho}) = u_0, \quad w(\tilde{\rho}) = w_0, \quad \kappa(\tilde{\rho}) = \frac{\kappa_{-1}}{\tilde{\rho}} + \kappa_0. \quad (39)$$

The scaling equations are now given by

$$\begin{aligned} \partial_\tau u(\tilde{\rho}) = & -4u_0 - \frac{5u_0}{24\pi^2(u_0 - w_0)} - \frac{3u_0}{160\pi^2(u_0 - 4w_0)} \\ & + \frac{1}{96\pi^2} + \frac{79}{480\pi^2}, \end{aligned} \quad (40)$$

$$\begin{aligned} \partial_\tau w(\tilde{\rho}) = & -2w_0 - \frac{25u_0}{64\pi^2(u_0 - w_0)} + \frac{5u_0^2}{1344\pi^2(u_0 - 4w_0)^2} \\ & - \frac{9u_0}{896\pi^2(u_0 - 4w_0)} - \frac{1}{192\pi^2} + \frac{1207}{2688\pi^2}. \end{aligned} \quad (41)$$

To find the UV-fixed point for u_0 and w_0 in the $\tilde{\rho} \rightarrow 0$ limit, we solve

$$\begin{aligned} -4u_0 - \frac{5u_0}{24\pi^2(u_0 - w_0)} - \frac{3u_0}{160\pi^2(u_0 - 4w_0)} \\ + \frac{1}{96\pi^2} + \frac{79}{480\pi^2} = 0, \end{aligned} \quad (42)$$

$$\begin{aligned} -2w_0 - \frac{25u_0}{64\pi^2(u_0 - w_0)} - \frac{17u_0^2 - 108u_0w_0}{2688\pi^2(u_0 - 4w_0)^2} \\ - \frac{1}{192\pi^2} + \frac{1207}{2688\pi^2} = 0, \end{aligned} \quad (43)$$

where $\partial_\tau u_0 = \partial_\tau w_0 = 0$ in the left-hand side of Eqs. (40) and (41). We find a UV-fixed point

$$u_{0*} = 0.00587, \quad w_{0*} = 0.0278. \quad (44)$$

For the flow of $\kappa(\tilde{\rho})$ the ansatz (39) yields

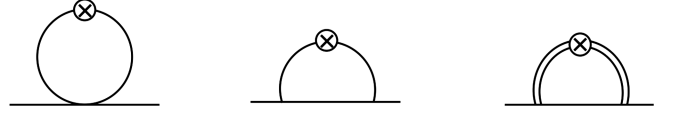


FIG. 5. Diagrams contributing to the flow equation for the kinetic. The single solid line denotes the scalar field ϕ , while the double solid line is the $\tilde{\sigma}$ component of the metric. The cross in a circle denotes the regulator insertion $\partial_t \mathcal{R}_k$. The tadpole type diagrams with metric fluctuations [$t_{\mu\nu}$ (traceless-transverse tensor), $\tilde{\sigma}$ (physical scalar metric fluctuation), v_μ (transverse vector), and u (measure scalar)] cancel due to the property (F3) in Appendix F.

$$\begin{aligned} (\partial_t + \partial_\tau) \kappa(\tilde{\rho}) = & \frac{1}{(4\pi)^2} \left[\left(\frac{9\kappa_0^2 \tilde{\rho}}{\kappa_{-1}^2} - \frac{25\kappa_0}{24\kappa_{-1}} + \frac{2}{3\tilde{\rho}} \right) \right. \\ & + \left(-\frac{5\kappa_0^2 \tilde{\rho}}{2\kappa_{-1}^2} + \frac{3\kappa_0}{2\kappa_{-1}} - \frac{2}{3\tilde{\rho}} \right) \\ & \left. + \frac{5u_0 - 23w_0}{15(u_0 - 4w_0)^2} \kappa_0 + \frac{65u_0 - 296w_0}{180(u_0 - 4w_0)^2} \frac{\kappa_{-1}}{\tilde{\rho}} \right]. \end{aligned} \quad (45)$$

The first two brackets on the rhs correspond to the two graphs in Fig. 5 with a scalar loop. The two contributions $2/(3\tilde{\rho})$ cancel, consistent with the absence of a term $\sim \tilde{\rho}^{-1}$ in a pure scalar theory [144]. The last term shows a gravitational contribution $\sim \tilde{\rho}^{-1}$. This leads to an anomalous dimension slightly different from two.

From Eq. (45) we extract the flow of κ_{-1} from the term $\sim \tilde{\rho}^{-1}$

$$(\partial_t + 2)\kappa_{-1} = -\frac{1}{(4\pi)^2} \frac{(296w_0 - 65u_0)}{180(u_0 - 4w_0)^2} \kappa_{-1}, \quad (46)$$

which yields

$$\tilde{\eta} = -\left(2 + \frac{1}{(4\pi)^2} \frac{(296w_0 - 65u_0)}{180(u_0 - 4w_0)^2} \right) \approx -2.026. \quad (47)$$

For dilaton quantum gravity with $w \sim \xi \tilde{\rho}$ for large $\tilde{\rho}$ a nonzero anomalous dimension for $\tilde{\rho} \rightarrow 0$ would have to be combined with a zero anomalous dimension for $\tilde{\rho} \rightarrow \infty$. Thus the anomalous dimension should depend on $\tilde{\rho}$. This requires a nonlinear relation between the renormalized scalar field ϕ_R and ϕ . Nonlinear scaling fields are possible. Nonlinear field transformations induce, however, additional terms in the flow equation [94,151,159–166].

VI. CONCLUSIONS

We have investigated the scaling solution for dilaton quantum gravity in the variable gravity approximation. It defines an ultraviolet complete quantum field theory for a scalar singlet coupled to the metric, as used for models of inflation or dynamical dark energy. We employ the gauge

invariant flow equation, which focuses on physical fluctuations, respecting diffeomorphism symmetry and global dilatation symmetry.

Our results for the overall features of the dimensionless scalar potential u and the dimensionless squared Planck mass w are in accordance with earlier investigations. As a function of the squared dimensionless scalar field $\tilde{\rho}$ one finds an almost flat potential $u(\tilde{\rho})$ and a Planck mass increasing for large values of the scalar field, $w(\tilde{\rho}) \approx w_0 + \xi_\infty \tilde{\rho}$. These central properties of the dilaton quantum gravity fixed point do not depend on the particular version of the functional flow equation or on the precise truncation. This enhances the robustness of the arguments in favor of the existence of such an ultraviolet fixed point.

The use of the gauge invariant flow equation may contribute to a better understanding of the field-dependence of the kinetic—the coefficient in front of the scalar kinetic term. Because of its focus on the physical fluctuations it can take well into account the mixing of the kinetic terms for the scalar field and the physical scalar fluctuation in the metric. In particular, it allows us to explore a negative sign of the kinetic in the region of large $\tilde{\rho}$, as long as it remains consistent with a stability criterion requiring a positive kinetic term for the eigenstate corresponding to the inflaton or cosmon.

Within our approximation we find two different types of scaling solutions, corresponding to two different ultraviolet fixed points. For the extended Reuter fixed point the potential $u(\tilde{\rho})$ and the squared Planck mass $w(\tilde{\rho})$ are independent of $\tilde{\rho}$. Such a scaling solution employs a renormalized scaling field. The dependence of K on k involves a nonvanishing anomalous field dimension η . We compute the quantum gravity contribution to η .

The second UV-fixed point is the dilaton quantum gravity fixed point with $w(\tilde{\rho})$ increasing $\sim \tilde{\rho}$ for large $\tilde{\rho}$. Our truncation and numerical solution seem to be compatible with a continuous family of scaling solutions parametrized by B . This family includes the possibility of a negative kinetic for large $\tilde{\rho}$. For the scaling solutions found in the present paper the anomalous field dimension η vanishes for all $\tilde{\rho}$. It remains to be seen if further scaling solutions can be found with η depending on $\tilde{\rho}$, vanishing only for $\tilde{\rho} \rightarrow \infty$. Such scaling solutions require a scaling field which is related to the original or microscopic scalar field by a nonlinear field transformation. On the other hand, more extended truncations may reduce the continuous family of scaling solutions to a discrete set, leave only a single scaling solution, or even exclude the existence of scaling solutions with $w(\tilde{\rho}) \sim \tilde{\rho}$ for large $\tilde{\rho}$.

At the present stage the argument for the existence of a dilaton quantum gravity fixed point different from the extended Reuter fixed point is strengthened by our investigation. This fixed point realizes the exact quantum scale symmetry in the infrared limit for $\tilde{\rho} \rightarrow \infty$. The issue of quantum scale symmetry for the ultraviolet limit $\tilde{\rho} \rightarrow 0$ will depend on the anomalous field dimension.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: EFFECTIVE AVERAGE ACTION FOR NONMINIMALLY COUPLED SCALAR-GRAVITY SYSTEM

The central method in this work is the functional renormalization group (FRG) [167]. For quantum field theory we employ a functional differential equation for the effective average action [20] (see also [145,168,169])

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \text{Tr}[(\Gamma_k^{(2)}[\Phi] + \mathcal{R}_k)^{-1} \partial_t \mathcal{R}_k]. \quad (\text{A1})$$

Here $\partial_t = k \partial_k$ is a dimensionless scale derivative and “STr” in Eq. (A1) is summing over all internal degrees of freedom and momenta of the superfield Φ . The matrix of second functional derivatives $\Gamma_k^{(2)}$ corresponds to the full two-point function, i.e., full inverse propagator of Φ in which an infrared cutoff profile function $\mathcal{R}_k$ is introduced.

1. Setup

In this work, we set up the system of a singlet scalar field ϕ nonminimally coupled to gravity using the FRG equation (A1). In such a system, we give the effective action as

$$\Gamma_k = \Gamma_k^{\text{SG}} + \Gamma_{\text{gf}} + \Gamma_{\text{gh}}. \quad (\text{A2})$$

Here, our truncated scalar-gravity part Γ_k^{SG} is given by

$$\Gamma_k^{\text{(SG)}} = \int_x \sqrt{g} \left\{ U(\rho) + \frac{K(\rho)}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{F(\rho)}{2} R \right\}. \quad (\text{A3})$$

We have defined the field dependent Planck mass $F(\rho)$, the scalar effective potential $U(\rho)$, and the kinetic potential $K(\rho)$ as functions of the scalar invariant $\rho = \phi^2/2$.

To specify the physical gauge fixing for diffeomorphism acting on the metric fluctuation field $h_{\mu\nu}$, we expand the macroscopic metric around a constant background metric $\bar{g}_{\mu\nu}$ such that

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \quad (\text{A4})$$

In this work, we employ the maximally symmetric background space for simplicity, i.e.,

$$\bar{R}_{\mu\rho\nu\sigma} = \frac{\bar{R}}{12}(\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}), \quad \bar{R}_{\mu\nu} = \frac{\bar{R}}{4}\bar{g}_{\mu\nu}. \quad (\text{A5})$$

In the current setup for the scalar-gravity sector (A3), this choice provides the same result of the beta functions as the case with a general homogeneous background field. The gauge fixing for diffeomorphism is given by

$$\Gamma_{\text{gf}} = \frac{1}{2\alpha} \int_x \sqrt{\bar{g}} \bar{g}^{\mu\nu} \Sigma_\mu \Sigma_\nu. \quad (\text{A6})$$

Here, the general gauge fixing reads

$$\Sigma_\mu = \bar{D}^\nu h_{\nu\mu} - \frac{\beta+1}{4} \bar{D}_\mu h, \quad (\text{A7})$$

with $h = \bar{g}^{\mu\nu} h_{\mu\nu}$ the trace mode of metric fluctuations. Bars on operators denote that covariant derivatives and index contractions are formed with the background metric. The ghost action associated with the gauge fixing (A7) is given by

$$\Gamma_{\text{gh}} = - \int_x \sqrt{\bar{g}} \bar{C}_\mu \left[\bar{g}^{\mu\rho} \bar{D}^2 + \frac{1-\beta}{2} \bar{D}^\mu \bar{D}^\rho + \bar{R}^{\mu\rho} \right] C_\rho, \quad (\text{A8})$$

where C and $\bar{C}$ are ghost and antighost fields. Equations (A6) and (A7) constitute a general family of gauge fixings for diffeomorphism symmetry, specified by two parameters α and β . The choice $\beta = -1$ and $\alpha \rightarrow 0$ is a “physical gauge fixing” that acts only on the gauge modes in $h_{\mu\nu}$, leaving the physical fluctuations unaffected. In this work, we use this gauge choice. Nevertheless, the following subsection treats general α and β in order to demonstrate explicitly the particular role of the physical gauge fixing.

2. Physical metric decomposition

A key quantity for the flow equation is the inverse propagator or one-particle irreducible two-point function, as given by the matrix of second functional derivatives of Γ_k . To derive the explicit form for the metric two-point function, we split the metric fluctuations into physical and gauge fluctuations [170]. Accordingly, we decompose the metric fluctuations into

$$h_{\mu\nu} = f_{\mu\nu} + a_{\mu\nu}, \quad (\text{A9})$$

where $f_{\mu\nu}$ are the physical metric fluctuations, while $a_{\mu\nu}$ are the gauge fluctuations or gauge modes. The physical metric fluctuations satisfy the transverse constraint $\bar{D}^\mu f_{\mu\nu} = 0$. In turn, the physical metric fluctuations can be decomposed into two independent fields as

$$f_{\mu\nu} = t_{\mu\nu} + s_{\mu\nu}, \quad (\text{A10})$$

where the graviton $t_{\mu\nu}$ is the transverse and traceless (TT) tensor, i.e., $\bar{D}^\mu t_{\mu\nu} = \bar{g}^{\mu\nu} t_{\mu\nu} = 0$. The tensor $s_{\mu\nu}$ is given as a linear function of a scalar field σ such that

$$s_{\mu\nu} = \hat{S}_{\mu\nu} \sigma = \frac{1}{3} P_{\mu\nu} \sigma. \quad (\text{A11})$$

For a background geometry with constant curvature, the projection operator can be found explicitly as

$$\begin{aligned} P_{\mu\nu} &= (\bar{g}_{\mu\nu} \bar{\Delta}_S + \bar{D}_\mu \bar{D}_\nu - \bar{R}_{\mu\nu}) \left(\bar{\Delta}_S - \frac{\bar{R}}{3} \right)^{-1} \\ &= \left[\bar{g}_{\mu\nu} \left(\bar{\Delta}_S - \frac{\bar{R}}{4} \right) + \bar{D}_\mu \bar{D}_\nu \right] \left(\bar{\Delta}_S - \frac{\bar{R}}{3} \right)^{-1}, \end{aligned} \quad (\text{A12})$$

with $\bar{\Delta}_S = -\bar{D}^2 = -\bar{D}^\mu \bar{D}_\mu$ the covariant Laplacian acting on scalar fields (spin-0 fields). Here, in the second equality, we have used the maximally symmetric space (A5).

Similarly, the gauge modes or unphysical metric fluctuations $a_{\mu\nu}$ are decomposed into a transverse vector mode v_μ , satisfying $\bar{D}^\mu v_\mu = 0$, and a scalar mode u . In summary, the metric fluctuations (A9) are parametrized by

$$\begin{aligned} f_{\mu\nu} &= t_{\mu\nu} + \frac{1}{3} P_{\mu\nu} \sigma, \\ a_{\mu\nu} &= \bar{D}_\mu v_\nu + \bar{D}_\nu v_\mu - \bar{D}_\mu \bar{D}_\nu u. \end{aligned} \quad (\text{A13})$$

Note here that the trace mode of the metric fluctuations $h = \bar{g}^{\mu\nu} h_{\mu\nu}$ is given by

$$\begin{aligned} h &= \bar{g}^{\mu\nu} \left(t_{\mu\nu} + \frac{1}{3} P_{\mu\nu} \sigma + \bar{D}_\mu v_\nu + \bar{D}_\nu v_\mu - \bar{D}_\mu \bar{D}_\nu u \right) \\ &= \sigma + \bar{\Delta}_S u. \end{aligned} \quad (\text{A14})$$

The ghost fields can be decomposed similarly into vector and scalar fields

$$C_\mu = C_\mu^\perp + \bar{D}_\mu C, \quad \bar{C}_\mu = \bar{C}_\mu^\perp + \bar{D}_\mu \bar{C}, \quad (\text{A15})$$

where $C_\mu^\perp$ ($\bar{C}_\mu^\perp$) is the transverse (anti)ghost field and C ($\bar{C}$) is the scalar (anti)ghost field.

3. Jacobians

The decompositions of the metric fluctuation (A13) and the ghost fields (A15) yield Jacobians of the path integral measures for different modes. We start with the Gaussian integral for the metric fluctuations $h_{\mu\nu}$. In the maximally symmetric space (A5), the Gaussian integrals for each mode in the physical metric decomposition read

$$\begin{aligned}
1 &= \int \mathcal{D}h_{\mu\nu} \exp\left(-\frac{1}{2}h_{\mu\nu}h^{\mu\nu}\right) \\
&= J_{\text{grav}} \int \mathcal{D}t_{\mu\nu} \mathcal{D}\sigma \mathcal{D}v_\mu \mathcal{D}u \exp\left[-\frac{1}{2} \int_x \sqrt{\bar{g}} \{t_{\mu\nu}t^{\mu\nu} + \sigma \hat{S}^{\alpha\beta} \hat{S}_{\alpha\beta} \sigma + 2v_\mu (\bar{\mathcal{D}}_1)^{\mu\nu} v_\nu + 2u \bar{\mathcal{D}}_0 \bar{\Delta}_S u\}\right] \\
&= J_{\text{grav}} [\det'_{(1T)}(\bar{\mathcal{D}}_1)]^{-1/2} [\det'_{(0)}(\bar{\mathcal{D}}_0) \bar{\Delta}_S]^{-1/2} [\det'_{(0)}(\hat{S}^{\alpha\beta} \hat{S}_{\alpha\beta})]^{-1/2}, \tag{A16}
\end{aligned}$$

where $E_{(t)}$ is the identical matrix acting on the space satisfying the TT condition. We have in the maximal symmetric space and $\bar{g}^{\mu\nu}t_{\mu\nu} = 0$. One finds the Jacobian arising from the decomposition of the metric fluctuation,

$$\begin{aligned}
J_{\text{grav}} &= [\det'_{(1T)}(\bar{\mathcal{D}}_1)]^{1/2} [\det'_{(0)}(\bar{\mathcal{D}}_0) \bar{\Delta}_S]^{1/2} [\det'_{(0)}(\hat{S}^{\alpha\beta} \hat{S}_{\alpha\beta})]^{1/2} \\
&\equiv J_v \cdot J_u \cdot J_\sigma. \tag{A17}
\end{aligned}$$

with differential operators

$$\bar{\mathcal{D}}_1 = \bar{\Delta}_V - \frac{\bar{R}}{4}, \quad \bar{\mathcal{D}}_0 = \bar{\Delta}_S - \frac{\bar{R}}{4}. \tag{A18}$$

Here $\bar{\Delta}_V = -\bar{D}^2$ is the Laplacian acting on vector fields (spin-1 fields). A prime denotes a subtraction of a zero eigenmode. This subtraction, however, does not contribute to the beta functions within the present truncation, so that we hereafter neglect it. The Jacobian for σ in the maximally symmetric space (A5) can be computed as

$$\begin{aligned}
J_\sigma &= [\det_{(0)}(\hat{S}^{\alpha\beta} \hat{S}_{\alpha\beta})]^{1/2} \\
&= \left[\det_{(0)} \frac{1}{3} \left(\bar{\Delta}_S - \frac{\bar{R}}{3} \right)^{-1} \bar{\mathcal{D}}_0 \right]^{1/2}. \tag{A19}
\end{aligned}$$

For the evaluation of fluctuations of σ , it is convenient to remove $(\bar{\Delta}_S - \frac{\bar{R}}{3})^{-1}$. This can be done by redefining the field σ such that

$$\tilde{\sigma} = \left(\bar{\Delta}_S - \frac{\bar{R}}{3} \right)^{-1} \sigma, \tag{A20}$$

for which the Jacobian (A19) reads

$$J_{\tilde{\sigma}} = \left[\det_{(0)}(\bar{\mathcal{D}}_0) \left(\bar{\Delta}_S - \frac{\bar{R}}{3} \right) \right]^{1/2}. \tag{A21}$$

In the same manner, the Gauss integral for the ghost fields is

$$\begin{aligned}
1 &= \int \mathcal{D}C_\mu \mathcal{D}\bar{C}_\mu \exp(\bar{C}_\mu \bar{C}^\mu) \\
&= J_{\text{gh}} \int \mathcal{D}C_\mu^\perp \mathcal{D}\bar{C}_\mu^\perp \mathcal{D}C \mathcal{D}\bar{C} \exp(\bar{C}_\mu^\perp \bar{C}^{\perp\mu} + \bar{C} \bar{\Delta}_S C) \\
&= J_{\text{gh}} [\det_{(0)}(\bar{\Delta}_S)], \tag{A22}
\end{aligned}$$

namely, one has

$$J_{\text{gh}} = [\det_{(0)}(\bar{\Delta}_S)]^{-1}. \tag{A23}$$

To summarize, the decompositions yield Jacobians are found to be

$$\begin{aligned}
J_v &= [\det'_{(1T)}(\bar{\mathcal{D}}_1)]^{1/2}, \quad J_{\tilde{\sigma}} = \left[\det'_{(0)}(\bar{\mathcal{D}}_0) \left(\bar{\Delta}_S - \frac{\bar{R}}{3} \right) \right]^{1/2}, \\
J_u &= [\det'_{(0)}(\bar{\mathcal{D}}_0) \bar{\Delta}_S]^{1/2}, \quad J_{\text{gh}} = [\det''_{(0)}(\bar{\Delta}_S)]^{-1}. \tag{A24}
\end{aligned}$$

By the same reason in Eq. (A17), we neglect the primes denoting the subtraction of zero modes. For the regularized Jacobians we replace the relevant differential operator $\bar{\mathcal{D}}$ by $P_k(\bar{\mathcal{D}})$ to avoid the Jacobians to induce strong nonlocalities. Then, these Jacobians are written in forms of flow equations as

$$J_v = -\frac{1}{2} \text{Tr}_{(1T)} \frac{\partial_t \mathcal{R}_k(\bar{\mathcal{D}}_1)}{P_k(\bar{\mathcal{D}}_1)}, \tag{A25}$$

$$J_{\tilde{\sigma}} = -\frac{1}{2} \text{Tr}_{(0)} \frac{\partial_t \mathcal{R}_k(\bar{\mathcal{D}}_0)}{P_k(\bar{\mathcal{D}}_0)} - \frac{1}{2} \text{Tr}_{(0)} \frac{\partial_t \mathcal{R}_k(\bar{\Delta}_S - \frac{\bar{R}}{3})}{P_k(\bar{\Delta}_S - \frac{\bar{R}}{3})}, \tag{A26}$$

$$J_u = -\frac{1}{2} \text{Tr}_{(0)} \frac{\partial_t \mathcal{R}_k(\bar{\mathcal{D}}_0)}{P_k(\bar{\mathcal{D}}_0)} - \frac{1}{2} \text{Tr}_{(0)} \frac{\partial_t \mathcal{R}_k(\bar{\Delta}_S)}{P_k(\bar{\Delta}_S)}, \tag{A27}$$

$$J_{\text{gh}} = \text{Tr}_{(0)} \frac{\partial_t \mathcal{R}_k(\bar{\Delta}_S)}{P_k(\bar{\Delta}_S)}. \tag{A28}$$

APPENDIX B: VARIATIONS

To obtain the Hessian, we need to compute the variations for the effective action (A2). Hence, we expand the effective action around background fields into polynomials of fluctuation fields:

$$\begin{aligned}
\Gamma_k[\Phi] &= \Gamma_k[\bar{\Phi}] + \Gamma_k^{(1)}[\bar{\Phi}](x) \cdot \delta\Phi(x) \\
&\quad + \frac{1}{2} \delta\Phi(x) \cdot \Gamma_k^{(2)}[\bar{\Phi}](x, y) \cdot \delta\Phi(y) + \dots, \tag{B1}
\end{aligned}$$

from which the Hessian is identified as $\Gamma_k^{(2)}[\bar{\Phi}](x, y) = \frac{\delta^2 \Gamma_k}{\delta\Phi(x) \delta\Phi(y)}|_{\Phi=\bar{\Phi}}$. Here, the metric field and the scalar field are linearly expanded around their respective background

fields $\bar{g}_{\mu\nu}$ and $\bar{\phi}$:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad \phi = \bar{\phi} + \varphi. \quad (\text{B2})$$

For our effective action

$$\Gamma_k^{(\text{SG})}[g_{\mu\nu}, \phi] = \int_x \sqrt{g} g^{\mu\nu} \frac{K(\rho)}{2} \partial_\mu \phi \partial_\nu \phi + \int_x \sqrt{g} U(\rho) - \int_x \frac{F(\rho)}{2} \sqrt{g} R, \quad (\text{B3})$$

the second variation reads

$$\begin{aligned} \delta^2 \Gamma_k^{(\text{SG})}[\bar{\Phi}] &= \int_x \sqrt{\bar{g}} \{2\bar{g}^{\mu\nu} K(\bar{\rho})\} \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi + \int_x \sqrt{\bar{g}} \left[\{\bar{g}^{\mu\nu} (\delta^2 K(\rho))|_{\phi=\bar{\phi}}\} \frac{1}{2} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} + \{4\bar{g}^{\mu\nu} \delta K(\rho)|_{\phi=\bar{\phi}}\} \frac{1}{2} \partial_\mu \varphi \partial_\nu \bar{\phi} \right] \\ &+ \int_x \sqrt{\bar{g}} H_2^{\mu\nu} K(\bar{\rho}) \frac{1}{2} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} + \int_x \sqrt{\bar{g}} 2H_1^{\mu\nu} \left(2K(\bar{\rho}) \frac{1}{2} \partial_\mu \varphi \partial_\nu \bar{\phi} + (\delta K(\rho))|_{\phi=\bar{\phi}} \frac{1}{2} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} \right) \\ &+ \int_x \sqrt{\bar{g}} [G_2 U(\bar{\rho}) + 2G_1 \delta U(\rho) + \delta^2 U(\rho)]|_{\phi=\bar{\phi}} - \int_x \sqrt{\bar{g}} \frac{1}{2} [I_2 F(\bar{\rho}) + 2I_1 \delta F(\rho) + \delta^2 F(\rho) \bar{R}]|_{\phi=\bar{\phi}}. \end{aligned} \quad (\text{B4})$$

Here, we have defined

$$H_1^{\mu\nu} = \delta(\sqrt{g} g^{\mu\nu}) / \sqrt{g} = \frac{1}{2} h \bar{g}^{\mu\nu} - h^{\mu\nu}, \quad (\text{B5})$$

$$H_2^{\mu\nu} = \delta^2(\sqrt{g} g^{\mu\nu}) / \sqrt{g} = \left(-\frac{1}{2} h^{\alpha\beta} h_{\alpha\beta} + \frac{1}{4} h^2 \right) \bar{g}^{\mu\nu} + 2h^\mu{}_\lambda h^{\nu\lambda} - h h^{\mu\nu}, \quad (\text{B6})$$

$$G_1 = \delta\sqrt{g} / \sqrt{g} = \frac{1}{2} h, \quad (\text{B7})$$

$$G_2 = \delta^2\sqrt{g} / \sqrt{g} = -\frac{1}{2} h^{\alpha\beta} h_{\alpha\beta} + \frac{1}{4} h^2 + \bar{D}_\mu \bar{D}_\nu h^{\mu\nu} + \bar{\Delta}_S h, \quad (\text{B8})$$

$$I_1 = \delta(\sqrt{g} R) / \sqrt{g} = \frac{1}{2} \bar{R} h - \bar{R}^{\alpha\beta} h_{\alpha\beta}, \quad (\text{B9})$$

$$\begin{aligned} I_2 &= \delta^2(\sqrt{g} R) / \sqrt{g} = -\frac{1}{2} h^{\alpha\beta} \bar{\Delta}_T h_{\alpha\beta} - \frac{1}{2} \bar{R} h^{\alpha\beta} h_{\alpha\beta} + h^{\alpha\beta} \bar{R}_\alpha{}^\rho h_{\rho\beta} + h \bar{D}^\mu \bar{D}^\nu h_{\mu\nu} + \bar{D}^\mu h_{\mu\alpha} \bar{D}^\nu h_\nu{}^\alpha \\ &+ \frac{1}{2} h \bar{\Delta}_S h + h^{\alpha\beta} \bar{R}_\alpha{}^\mu{}_\beta{}^\nu h_{\mu\nu} - h \bar{R}^{\alpha\beta} h_{\alpha\beta} + \frac{1}{4} \bar{R} h^2, \end{aligned} \quad (\text{B10})$$

and the variations for the scalar potentials K , F , and U are computed as

$$\delta K(\rho)|_{\phi=\bar{\phi}} = (K'(\bar{\rho}) \bar{\phi}) \varphi, \quad \delta^2 K(\rho)|_{\phi=\bar{\phi}} = (K'(\bar{\rho}) + 2\bar{\rho} K''(\bar{\rho})) \varphi^2, \quad (\text{B11})$$

$$\delta U(\rho)|_{\phi=\bar{\phi}} = (U'(\bar{\rho}) \bar{\phi}) \varphi, \quad \delta^2 U(\rho)|_{\phi=\bar{\phi}} = (U'(\bar{\rho}) + 2\bar{\rho} U''(\bar{\rho})) \varphi^2, \quad (\text{B12})$$

$$\delta F(\rho)|_{\phi=\bar{\phi}} = (F'(\bar{\rho}) \bar{\phi}) \varphi, \quad \delta^2 F(\rho)|_{\phi=\bar{\phi}} = (F'(\bar{\rho}) + 2\bar{\rho} F''(\bar{\rho})) \varphi^2, \quad (\text{B13})$$

where primes denote derivatives with respect to $\bar{\rho} = \bar{\phi}^2/2$.

APPENDIX C: HESSIANS

In this section, we focus on the evaluation of the quantum corrections to $F(\bar{\rho})$ and $U(\bar{\rho})$. We assume a constant background scalar field $\bar{\phi}$ and therefore drop the term $\partial_\mu \bar{\phi}$ in the Hessians. The quantum corrections to $K(\bar{\rho})$ are discussed in Appendix F. In the following, to make computations simple, we employ the maximal symmetric space for the background

metric (A5). We present the explicit Hessians for each mode:

$$\begin{aligned} \delta^2 \Gamma_k^{(\text{SG})}[\bar{\Phi}] &= \int_x \sqrt{\bar{g}} [\bar{g}^{\mu\nu} K(\bar{\rho}) \varphi (-\partial_\mu \partial_\nu) \varphi] |_{\phi=\bar{\phi}} + \int_x \sqrt{\bar{g}} [G_2 U(\bar{\rho}) + 2G_1 \delta U(\rho) + \delta^2 U(\rho)] |_{\phi=\bar{\phi}} \\ &\quad - \frac{1}{2} \int_x \sqrt{\bar{g}} [I_2 F(\bar{\rho}) + 2I_1 \delta F(\rho) + \delta^2 F(\rho) \bar{R}] |_{\phi=\bar{\phi}}. \end{aligned} \quad (\text{C1})$$

Then, the two-point functions (Hessians) $\Gamma_k^{(2)}$ is given by

$$\begin{aligned} \delta^2 \Gamma_k[\bar{\Phi}] &= \delta\Phi \cdot \Gamma_k^{(2)}[\bar{\Phi}] \cdot \delta\Phi = \int_x \sqrt{\bar{g}} \begin{pmatrix} h_{\mu\nu} & \varphi \end{pmatrix} \begin{pmatrix} (\Gamma_{(hh)}^{(2)})^{\mu\nu\rho\sigma} & (\Gamma_{(h\varphi)}^{(2)})^{\mu\nu} \\ (\Gamma_{(\varphi h)}^{(2)})^{\rho\sigma} & \Gamma_{(\varphi\varphi)}^{(2)} \end{pmatrix} \begin{pmatrix} h_{\rho\sigma} \\ \varphi \end{pmatrix} \\ &= \int_x \sqrt{\bar{g}} \begin{pmatrix} t_{\mu\nu} & v_\mu & \sigma & u & \varphi \end{pmatrix} \begin{pmatrix} (\Gamma_{(tt)}^{(2)})^{\mu\nu\rho\sigma} & 0 & 0 & 0 & 0 \\ 0 & (\Gamma_{(vv)}^{(2)})^{\mu\nu} & 0 & 0 & 0 \\ 0 & 0 & \Gamma_{(\sigma\sigma)}^{(2)} & \Gamma_{(\sigma u)}^{(2)} & \Gamma_{(\sigma\varphi)}^{(2)} \\ 0 & 0 & \Gamma_{(u\sigma)}^{(2)} & \Gamma_{(uu)}^{(2)} & \Gamma_{(u\varphi)}^{(2)} \\ 0 & 0 & \Gamma_{(\varphi\sigma)}^{(2)} & \Gamma_{(\varphi u)}^{(2)} & \Gamma_{(\varphi\varphi)}^{(2)} \end{pmatrix} \begin{pmatrix} t_{\rho\sigma} \\ v_\nu \\ \sigma \\ u \\ \varphi \end{pmatrix}. \end{aligned} \quad (\text{C2})$$

For our decomposition of the metric fluctuations, the matrix of the Hessians becomes block diagonal for each degree of freedom or spin. Note that the Hessians depend on only the background fields $\bar{g}_{\mu\nu}$ and $\bar{\phi}$.

1. Transverse-traceless tensor

For the $t_{\mu\nu}$ mode, we obtain the Hessian as

$$(\Gamma_{(tt)}^{(2)})^{\mu\nu\rho\sigma} = \left[\frac{F}{2} \bar{\mathcal{D}}_T - U \right] P^{(t)\mu\nu\rho\sigma}, \quad (\text{C3})$$

where we define the derivative operator

$$\bar{\mathcal{D}}_T = \bar{\Delta}_T + \frac{2\bar{R}}{3}, \quad (\text{C4})$$

with the Laplacian $\bar{\Delta}_T = -\bar{D}^2$ acting on transverse traceless tensor fields (spin-2 fields). The TT-projection operator is given by

$$P^{(t)\mu\nu\rho\sigma} = \frac{1}{2} (P^{\mu\rho} P^{\nu\sigma} + P^{\mu\sigma} P^{\nu\rho}) - \frac{1}{3} P^{\mu\nu} P^{\rho\sigma}, \quad (\text{C5})$$

with $P^{\mu\nu}$ defined by Eq. (A12).

2. Transverse vector

The Hessian for the spin-1 gauge mode v_μ is given by

$$(\Gamma_{(vv)}^{(2)})^{\mu\nu} = \frac{1}{\alpha} F \bar{\mathcal{D}}_1 \left[\bar{\mathcal{D}}_1 + \alpha \frac{\bar{R}}{2} - \alpha \frac{U}{F} \right] P^{(v)\mu\nu}, \quad (\text{C6})$$

with $P^{(v)}$ the projection operator on the vector mode, $P^{(v)}_\mu{}^\mu = 3$. Note that $\bar{\mathcal{D}}_1$ is defined in Eq. (A18). For $\alpha \rightarrow 0$ the inverse propagator for the gauge-vector mode becomes independent of U

$$\lim_{\alpha \rightarrow 0} \alpha (\Gamma_k^{(2)})_{vv}^{\mu\nu} = F(\bar{\mathcal{D}}_1)^2 P^{(v)\mu\nu}. \quad (C7)$$

Note that the overall factor $1/\alpha$ cancels with that in the regulator function $\mathcal{R}_k^{vv}(\bar{\Delta}_V)$ within the flow equation.

3. Scalars

We next turn to the Hessian for the scalar modes. In the (σ, u, φ) -field basis, we obtain

$$\Gamma_{(00)}^{(2)} = \begin{pmatrix} (\Gamma_{(00)}^{(2)})_{\text{grav}} & \frac{1}{2} \left[-F' \left(\bar{\Delta}_S + \frac{\bar{R}}{4} \right) + U' \right] \bar{\phi} & \frac{1}{4} (-F' \bar{R} + 2U') \bar{\phi} \bar{\Delta}_S \\ \frac{1}{2} \left[-F' \left(\bar{\Delta}_S + \frac{\bar{R}}{4} \right) + U' \right] \bar{\phi} & \frac{1}{4} (-F' \bar{R} + 2U') \bar{\phi} \bar{\Delta}_S & K \bar{\Delta}_S + m^2(\bar{\rho}) - \frac{1}{2} \xi(\bar{\rho}) \bar{R} \end{pmatrix}, \quad (C8)$$

where we define $m^2(\bar{\rho}) = U' + 2\bar{\rho}U''$ and $\xi(\bar{\rho}) = F' + 2\bar{\rho}F''$. Here the spin-0 gravitational part is given by the following 2×2 matrix:

$$(\Gamma_{(00)}^{(2)})_{\text{grav}} = \begin{pmatrix} -\frac{F}{6} \left[\left(\bar{\Delta}_S - \frac{3}{8} \bar{R} \right) \bar{\Delta}_S - \frac{U}{2F} \left(\bar{\Delta}_S - \frac{\bar{R}}{2} \right) \right] \left(\bar{\Delta}_S - \frac{\bar{R}}{3} \right)^{-1} + \frac{(\beta+1)^2}{16\alpha} \bar{\Delta}_S & \frac{F}{4} \left(\frac{U}{F} - \frac{\bar{R}}{4} \right) \bar{\Delta}_S + \frac{(\beta+1)(\beta-3)}{16\alpha} \left(\bar{\Delta}_S + \frac{\bar{R}}{\beta-3} \right) \bar{\Delta}_S \\ \frac{F}{4} \left(\frac{U}{F} - \frac{\bar{R}}{4} \right) \bar{\Delta}_S + \frac{(\beta+1)(\beta-3)}{16\alpha} \left(\bar{\Delta}_S + \frac{\bar{R}}{\beta-3} \right) \bar{\Delta}_S & \frac{F}{16} \left(\bar{\Delta}_S - \frac{\bar{R}}{2} \right) \left(\bar{R} - \frac{4U}{F} \right) \bar{\Delta}_S + \frac{(\beta-3)^2}{16\alpha} \left(\bar{\Delta}_S + \frac{\bar{R}}{\beta-3} \right)^2 \bar{\Delta}_S \end{pmatrix}. \quad (C9)$$

For general α and β the Hessian in the scalar sector is rather complicated. It simplifies considerably for a physical gauge fixing that corresponds to $\beta = -1$. For $\beta = -1$ the factor $1/\alpha$ remains only in the Hessian for the spin-0 gauge mode $\Gamma_{(uu)}^{(0)}$. In the limit $\alpha \rightarrow 0$ we can further neglect in the matrix (C8) the elements mixing u with σ and φ . They do not diverge for $\alpha \rightarrow 0$ and drop out in the propagator that is the inverse of $\Gamma^{(2)}$. As a result, for a physical gauge fixing the physical fluctuations and the gauge modes decouple. The Hessian becomes block diagonal in physical and gauge modes. For the physical modes one obtains the 2×2 matrix in the (σ, φ) basis

$$(\Gamma_{(00)}^{(2)})_{\text{ph}} = \begin{pmatrix} -\frac{F}{6} \left[\left(\bar{\Delta}_S - \frac{3}{8} \bar{R} \right) \bar{\Delta}_S - \frac{U}{2F} \left(\bar{\Delta}_S - \frac{\bar{R}}{2} \right) \right] \left(\bar{\Delta}_S - \frac{\bar{R}}{3} \right)^{-1} & \frac{1}{2} \left[-F' \left(\bar{\Delta}_S + \frac{\bar{R}}{4} \right) + U' \right] \bar{\phi} \\ \frac{1}{2} \left[-F' \left(\bar{\Delta}_S + \frac{\bar{R}}{4} \right) + U' \right] \bar{\phi} & K \bar{\Delta}_S + m^2(\bar{\rho}) - \frac{1}{2} \xi(\bar{\rho}) \bar{R} \end{pmatrix}. \quad (C10)$$

In the basis with the redefinition of σ as in Eq. (A20), i.e., $(\tilde{\sigma}, \varphi)$, the Hessian is given by

$$(\Gamma_{(00)}^{(2)})_{\text{ph}} = \begin{pmatrix} -\frac{F}{6} \left[\left(\bar{\Delta}_S - \frac{3}{8} \bar{R} \right) \bar{\Delta}_S - \frac{U}{2F} \left(\bar{\Delta}_S - \frac{\bar{R}}{2} \right) \right] \left(\bar{\Delta}_S - \frac{\bar{R}}{3} \right)^{-1} & \frac{\bar{\phi}}{2} \left[-F' \left(\bar{\Delta}_S + \frac{\bar{R}}{4} \right) + U' \right] \left(\bar{\Delta}_S - \frac{\bar{R}}{3} \right) \\ \frac{\bar{\phi}}{2} \left[-F' \left(\bar{\Delta}_S + \frac{\bar{R}}{4} \right) + U' \right] \left(\bar{\Delta}_S - \frac{\bar{R}}{3} \right) & K \bar{\Delta}_S + m^2(\bar{\rho}) - \frac{1}{2} \xi(\bar{\rho}) \bar{R} \end{pmatrix}. \quad (C11)$$

The Hessian for the gauge scalar mode reads

$$(\Gamma_{(00)}^{(2)})_{\text{gauge}} = \Gamma_{(uu)}^{(2)} = \left(\bar{\Delta}_S - \frac{\bar{R}}{4} \right)^2 \bar{\Delta}_S. \quad (C12)$$

4. Ghost fields

Finally, the Hessians for the ghost field obtains from Eq. (A8) as

$$\begin{aligned} (\Gamma_{(\bar{C}^\perp C^\perp)}^{(2)})^{\mu\nu} &= \bar{D}_1 P^{(v)\mu\nu}, \\ \Gamma_{(\bar{C}C)}^{(2)} &= \left(\bar{\Delta}_S + \frac{\bar{R}}{\beta - 3} \right) \bar{\Delta}_S. \end{aligned} \quad (C13)$$

Note that $\bar{D}_1$ is defined in Eq. (A18).

APPENDIX D: HEAT KERNEL EVALUATION OF THE FLOW GENERATOR

Let us briefly summarize the heat kernel formulas used in the gravitational setting. Suppose that the flow generator ζ_k takes

$$\zeta_k = \sum_i a_i \text{tr} W(\bar{\Delta}_i), \quad (D1)$$

with $\bar{\Delta}_i = -\bar{g}^{\mu\nu} \bar{D}_\mu \bar{D}_\nu - \mathbf{U}$ appropriate differential operators acting on i fields such as spin-2 traceless-transverse tensor, etc., and a_i weight factors. The right-hand side of Eq. (D1) is

$$\begin{aligned} \text{tr} W(\bar{\Delta}) &= \sum_m W(\lambda_m) = \sum_m \int_0^\infty dz \delta(z - \lambda_m) W(z) \\ &= \int_0^\infty dz W(z) \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{ds}{2\pi i} e^{sz} \text{tr} e^{-s\bar{\Delta}}, \end{aligned} \quad (D2)$$

where we use the identity $\sum_m \exp(-s\lambda_m) = \text{tr} \exp(-s\bar{\Delta})$.

Next, one expands the trace of $e^{-s\bar{\Delta}}$ in terms of s such that

$$\begin{aligned} \text{tr} e^{-s\bar{\Delta}} &= \frac{1}{16\pi^2} \int_x \sqrt{\bar{g}} \{ \text{tr}[c_0(\bar{\Delta})] s^{-2} + \text{tr}[c_2(\bar{\Delta})] s^{-1} \\ &\quad + \text{tr}[c_4(\bar{\Delta})] + \dots \}. \end{aligned} \quad (D3)$$

The coefficients $\text{tr}[c_n(\bar{\Delta})]$ are well known as the heat kernel coefficients. Inserting Eq. (D3) into Eq. (D2) yields

$$\text{tr} W(\bar{\Delta}) = \frac{1}{16\pi^2} \sum_{n=0}^{\infty} Q_{2-n} \int_x \sqrt{\bar{g}} \text{tr}[c_{2n}(\bar{\Delta})], \quad (D4)$$

where we have performed the inverse Laplace transformation (Mellin's inverse formula) such that

$$\begin{aligned} Q_n &= \int_0^\infty dz W(z) \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{ds}{2\pi i} e^{sz} s^{-n} \\ &= \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} W(z) \quad (n \geq 1), \end{aligned} \quad (D5)$$

with $\Gamma(n)$ being the Gamma function. For the first three terms ($n = 0, 1, 2$), Eq. (D5) becomes

$$Q_2 = \int_0^\infty dz z W(z), \quad Q_1 = \int_0^\infty dz W(z), \quad Q_0 = W(z=0). \quad (D6)$$

For negative n , one has

$$Q_{-n} = (-1)^n \left. \frac{\partial^n W}{\partial z^n} \right|_{z=0}, \quad n \geq 0. \quad (D7)$$

For $[\bar{\Delta}, \mathbf{U}] = 0$, the heat kernel coefficients are [171]

$$c_0(\bar{\Delta}) = \mathbf{1}, \quad (D8)$$

$$c_2(\bar{\Delta}) = \frac{\bar{R}}{6} \mathbf{1} - \mathbf{U}, \quad (D9)$$

$$\begin{aligned} c_4(\bar{\Delta}) &= \left(\frac{1}{180} \bar{R}_{\mu\nu\rho\sigma}^2 - \frac{1}{180} \bar{R}_{\mu\nu}^2 + \frac{1}{72} \bar{R}^2 - \frac{1}{30} \bar{\square} \bar{R} \right) \mathbf{1} \\ &\quad + \frac{1}{2} \mathbf{U}^2 + \frac{1}{6} \bar{\square} \mathbf{U} + \frac{1}{12} \Omega_{\mu\nu} \Omega^{\mu\nu} - \frac{\bar{R}}{6} \mathbf{U}. \end{aligned} \quad (D10)$$

In this work, it is sufficient to take $\text{tr}[c_0]$ and $\text{tr}[c_2]$ which are given by

$$\text{tr}[c_0] = b_0, \quad \text{tr}[c_2] = b_2 \bar{R}. \quad (D11)$$

We list the value of b_0 and b_2 for $\bar{\Delta}_i$ with $\mathbf{U} = 0$ acting on several fields in Table I.

APPENDIX E: FLOW GENERATOR

In this appendix, we calculate the flow equations. In the current setup, the flow generator consists of three parts

$$\partial_t \Gamma_k = \pi_2 + \pi_0 + \eta_k. \quad (E1)$$

TABLE I. Heat kernel coefficients for the individual fields in maximally symmetric four dimensional spacetime. ‘‘T’’ and ‘‘TT’’ denote ‘‘transverse’’ and ‘‘transverse-traceless,’’ respectively.

	Tensor ($h_{\mu\nu}$)	T Tensor ($f_{\mu\nu}$)	TT Tensor ($t_{\mu\nu}$)	T Vector ($A_\mu^T, v_\mu, C_\mu^\perp$)	Scalar (u, σ, φ, C)
b_0	10	9	5	3	1
b_2	$\frac{5}{3}$	$\frac{3}{2}$	$-\frac{5}{6}$	$\frac{1}{4}$	$\frac{1}{6}$

Here, π_2 denotes the contribution from fluctuations of the TT-tensor field $t_{\mu\nu}$, while π_0 involves both the scalar field φ and the physical scalar mode in the metric field. These contributions correspond to the physical modes, while the measure modes such as the ghost fields and the vector and unphysical scalar modes in the metric field are collected in η_k .

1. Physical metric fluctuations

We first evaluate the contributions from the physical fluctuations. There are the TT mode ($t_{\mu\nu}$) and the physical spin-0 modes (φ and $\tilde{\sigma}$), whose forms of the flow generators are given by

$$\pi_2 = \frac{1}{2} \text{Tr}_{(2)} \frac{\partial_t \mathcal{R}_k}{\Gamma_k^{(2)} + \mathcal{R}_k} \Big|_{tt}, \quad (\text{E2})$$

$$\pi_0 = \frac{1}{2} \text{Tr}_{(0)} \frac{\partial_t \mathcal{R}_k}{\Gamma_k^{(2)} + \mathcal{R}_k} \Big|_{\text{ph}} + J_{\tilde{\sigma}}, \quad (\text{E3})$$

respectively. The two-point functions $\Gamma_k^{(2)}$ for the TT mode and the physical spin-0 modes are shown in Eqs. (C3) and (C10), respectively. The Jacobian $J_{\tilde{\sigma}}$ is given in Eq. (A26).

To extract the explicit form of the beta functions, we define the dimensionless quantities

$$\begin{aligned} \tilde{\rho} &= \frac{\bar{\phi}^2}{2k^2} = \frac{\bar{\rho}}{k^2}, & u(\tilde{\rho}) &= \frac{U(\bar{\rho})}{k^4}, & w(\tilde{\rho}) &= \frac{F(\bar{\rho})}{2k^2}, & \kappa(\tilde{\rho}) &= K(\bar{\rho}), \\ \tilde{\xi}(\tilde{\rho}) &= \xi(\bar{\rho}) = F' + 2\bar{\rho}F'' = 2w' + 4\tilde{\rho}w'', & \tilde{m}^2(\tilde{\rho}) &= \frac{m^2(\bar{\rho})}{k^2} = \frac{U' + 2\bar{\rho}U''}{k^2} = u' + 2\tilde{\rho}u''. \end{aligned} \quad (\text{E4})$$

Primes acting on the dimensionless quantities denote the derivative with respect to $\tilde{\rho}$. Hereafter we introduce the τ derivative as $\partial_\tau = -2\tilde{\rho}\partial_{\tilde{\rho}}$.

a. Spin-2 TT mode

We can evaluate the contributions (E2). For the TT mode, our choice of IR cutoff function corresponds to the type-II cutoff function defined in Ref. [172], with

$$W_2(z) = \frac{\partial_t(wk^2R_k(z))}{(wk^2P_k(z) - uk^4)}. \quad (\text{E5})$$

The heat kernel method yields the flow generator for the TT mode fluctuation as

$$\pi_2 = \frac{\Omega}{(4\pi)^2} \left[\frac{10}{3} k^4 \frac{w + \frac{\partial_t w + \partial_\tau w}{8}}{w - u} - \frac{25}{4} k^2 \frac{w + \frac{\partial_t w + \partial_\tau w}{6}}{w - u} \bar{R} \right], \quad (\text{E6})$$

where $\Omega = \int_x \sqrt{g}$ is the spacetime volume where we use

$$\frac{\partial_t(w(\tilde{\rho})k^2)}{k^2} = 2w(\tilde{\rho}) + \partial_t w(\tilde{\rho}) - 2\tilde{\rho}\partial_{\tilde{\rho}} w(\tilde{\rho}) = 2w(\tilde{\rho}) + \partial_t w(\tilde{\rho}) + \partial_\tau w(\tilde{\rho}). \quad (\text{E7})$$

b. Spin-0 physical modes

We now calculate the contributions from the spin-0 physical scalar fields. Within the current truncation, it suffices to deal with terms up to linear order in $\bar{R}$.

$$\Gamma_k^{\tilde{\sigma}\tilde{\sigma}}(z) = -\frac{1}{3} \left[\left(\frac{F}{2} z^3 - \frac{U}{4} z^2 \right) - \frac{\bar{R}}{24} \left(\frac{17F}{2} z^2 - 5U^2 z \right) + \mathcal{O}(\bar{R}^2) \right], \quad (\text{E8})$$

$$\Gamma_k^{\varphi\varphi}(z) = K(\bar{\rho})z + m^2(\bar{\rho}) - \frac{1}{2} \xi(\bar{\rho}) \bar{R}, \quad (\text{E9})$$

$$\Gamma_k^{\tilde{\sigma}\varphi}(z) = \Gamma_k^{\varphi\tilde{\sigma}}(z) = -\bar{\phi} \left[\left(\frac{F'}{2} z^2 - \frac{U'}{2} z \right) - \frac{1}{12} \left(\frac{F'}{2} z - 2U' \right) \bar{R} + \mathcal{O}(\bar{R}^2) \right]. \quad (\text{E10})$$

Hereafter we neglect higher operator terms $\mathcal{O}(\bar{R}^2)$. We employ the cutoff function for the physical scalar modes as

$$\mathcal{R}_k^{\text{ph}}(z) = \begin{pmatrix} \mathcal{R}_k^{\tilde{\sigma}\tilde{\sigma}}(z) & \mathcal{R}_k^{\tilde{\sigma}\varphi}(z) \\ \mathcal{R}_k^{\varphi\tilde{\sigma}}(z) & \mathcal{R}_k^{\varphi\varphi}(z) \end{pmatrix} = \begin{pmatrix} \Gamma_k^{\tilde{\sigma}\tilde{\sigma}}(P_k) - \Gamma_k^{\tilde{\sigma}\tilde{\sigma}}(z) & \Gamma_k^{\tilde{\sigma}\varphi}(P_k) - \Gamma_k^{\tilde{\sigma}\varphi}(z) \\ \Gamma_k^{\varphi\tilde{\sigma}}(P_k) - \Gamma_k^{\varphi\tilde{\sigma}}(z) & \Gamma_k^{\varphi\varphi}(P_k) - \Gamma_k^{\varphi\varphi}(z) \end{pmatrix} \Big|_{\bar{R} \rightarrow 0}. \quad (\text{E11})$$

For this cutoff function, the Laplacian $z = \bar{\Delta}_S$ only in the curvature-scalar independent terms is replaced with $P_k(z) = z + R_k(z)$. The full two-point function for the physical scalar modes reads

$$\begin{aligned} (\Gamma_k^{(2)} + \mathcal{R}_k)|_{\text{ph}} &= \begin{pmatrix} -\frac{1}{3}\left(\frac{F}{2}P_k^3 - \frac{U}{4}P_k^2\right) + \frac{\bar{R}}{72}\left(\frac{17F}{2}z^2 - 5U^2\right) & -\bar{\phi}\left[\left(\frac{F'}{2}P_k^2 - \frac{U'}{2}P_k\right) - \frac{1}{12}\left(\frac{F'}{2}z - 2U'\right)\bar{R}\right] \\ -\bar{\phi}\left[\left(\frac{F'}{2}P_k^2 - \frac{U'}{2}P_k\right) - \frac{1}{12}\left(\frac{F'}{2}z - 2U'\right)\bar{R}\right] & KP_k + m^2 - \frac{1}{2}\xi\bar{R} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{3}\left(\frac{F}{2}P_k^3 - \frac{U}{4}P_k^2\right) & -\bar{\phi}\left(\frac{F'}{2}P_k^2 - \frac{U'}{2}P_k\right) \\ -\bar{\phi}\left(\frac{F'}{2}P_k^2 - \frac{U'}{2}P_k\right) & KP_k + m^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{72}\left(\frac{17F}{2}z^2 - 5U^2\right) & \frac{\bar{\phi}}{12}\left(\frac{F'}{2}z - 2U'\right) \\ \frac{\bar{\phi}}{12}\left(\frac{F'}{2}z - 2U'\right) & -\frac{1}{2}\xi \end{pmatrix} \bar{R} \\ &\equiv G_{\text{ph}}^{-1} + I_{\text{ph}}\bar{R}. \end{aligned} \quad (\text{E12})$$

Here, the propagator matrix for the physical scalar fields is given by

$$\begin{aligned} G_{\text{ph}} &= \frac{1}{-\frac{1}{3}\left(\frac{F}{2}P_k^3 - \frac{U}{4}P_k^2\right)(KP_k + m^2) - 2\bar{\rho}\left(\frac{F'}{2}P_k^2 - \frac{U'}{2}P_k\right)^2} \begin{pmatrix} KP_k + m^2 & \bar{\phi}\left(\frac{F'}{2}P_k^2 - \frac{U'}{2}P_k\right) \\ \bar{\phi}\left(\frac{F'}{2}P_k^2 - \frac{U'}{2}P_k\right) & -\frac{1}{3}\left(\frac{F}{2}P_k^3 - \frac{U}{4}P_k^2\right) \end{pmatrix} \\ &= \frac{1}{\left(\frac{F}{2}P_k^3 - \frac{U}{4}P_k^2\right)(KP_k + m^2) + 6\bar{\rho}\left(\frac{F'}{2}P_k^2 - \frac{U'}{2}P_k\right)^2} \begin{pmatrix} -3(KP_k + m^2) & -3\bar{\phi}\left(\frac{F'}{2}P_k^2 - \frac{U'}{2}P_k\right) \\ -3\bar{\phi}\left(\frac{F'}{2}P_k^2 - \frac{U'}{2}P_k\right) & \frac{F}{2}P_k^3 - \frac{U}{4}P_k^2 \end{pmatrix} \\ &\equiv \frac{1}{\left(\frac{F}{2}P_k^3 - \frac{U}{4}P_k^2\right)(KP_k + m^2) + 6\bar{\rho}\left(\frac{F'}{2}P_k^2 - \frac{U'}{2}P_k\right)^2} \tilde{G}_{\text{ph}} \equiv \begin{pmatrix} \mathcal{P}_{\tilde{\sigma}\tilde{\sigma}} & \mathcal{P}_{\tilde{\sigma}\varphi} \\ \mathcal{P}_{\varphi\tilde{\sigma}} & \mathcal{P}_{\varphi\varphi} \end{pmatrix}. \end{aligned} \quad (\text{E13})$$

Note that

$$\partial_t \mathcal{R}_k^{\text{ph}}(z) = \begin{pmatrix} -\frac{1}{3}\partial_t\left(wk^2(P_k^3 - z^3) - \frac{wk^4}{4}(P_k^2 - z^2)\right) & -\bar{\phi}\partial_t\left(w'(P_k^2 - z^2) - \frac{w'k^2}{2}(P_k - z)\right) \\ -\bar{\phi}\partial_t\left(w'(P_k^2 - z^2) - \frac{w'k^2}{2}(P_k - z)\right) & \partial_t(\kappa(P_k - z)) \end{pmatrix}. \quad (\text{E14})$$

The inverse form of Eq. (E12) is then computed in polynomials of $\bar{R}$ as

$$(\Gamma_k^{(2)} + \mathcal{R}_k)|_{\text{ph}}^{-1} = (G_{\text{ph}}^{-1} + I_{\text{ph}}\bar{R})^{-1} = (1 + G_{\text{ph}}I_{\text{ph}}\bar{R})^{-1}G_{\text{ph}} = G_{\text{ph}} - G_{\text{ph}}I_{\text{ph}}G_{\text{ph}}\bar{R} + \dots, \quad (\text{E15})$$

from which the flow kernel becomes

$$\frac{1}{2}\text{Tr}_{(0)}\frac{\partial_t \mathcal{R}_k}{\Gamma_k^{(2)} + \mathcal{R}_k}|_{\text{ph}} = \frac{1}{2}\text{Tr}_{(0)}[G_{\text{ph}}\partial_t \mathcal{R}_k^{\text{ph}}] - \frac{1}{2}\text{Tr}_{(0)}[G_{\text{ph}}I_{\text{ph}}G_{\text{ph}}\partial_t \mathcal{R}_k^{\text{ph}}]\bar{R} + \dots \quad (\text{E16})$$

Here, the first term in the right-hand side is

$$\frac{1}{2}\text{Tr}_{(0)}[G_{\text{ph}}\partial_t \mathcal{R}_k^{\text{ph}}] = \frac{\Omega}{2(4\pi)^2}b_0^{(0)}\mathcal{Q}_2[G_{\text{ph}}(z)\partial_t \mathcal{R}_k^{\text{ph}}(z)] + \frac{\Omega}{2(4\pi)^2}b_2^{(0)}\bar{R}\mathcal{Q}_1[G_{\text{ph}}(z)\partial_t \mathcal{R}_k^{\text{ph}}(z)], \quad (\text{E17})$$

$$-\frac{1}{2}\text{Tr}_{(0)}[G_{\text{ph}}I_{\text{ph}}G_{\text{ph}}\partial_t \mathcal{R}_k^{\text{ph}}]\bar{R} = -\frac{\Omega}{2(4\pi)^2}b_0^{(0)}\mathcal{Q}_2[G_{\text{ph}}(z)I_{\text{ph}}(z)G_{\text{ph}}(z)\partial_t \mathcal{R}_k^{\text{ph}}(z)]\bar{R}. \quad (\text{E18})$$

Finally, we evaluate the contribution from the Jacobian (A26) for $\tilde{\sigma}$ which evaluates to

$$J_{\tilde{\sigma}} = -\frac{1}{2} \text{Tr}_{(0)} \frac{\partial_t \mathcal{R}_k(\bar{\mathcal{D}}_0)}{P_k(\bar{\mathcal{D}}_0)} - \frac{1}{2} \text{Tr}_{(0)} \frac{\partial_t \mathcal{R}_k(\bar{\Delta}_S - \frac{\bar{R}}{3})}{P_k(\bar{\Delta}_S - \frac{\bar{R}}{3})} \\ = -\frac{\Omega}{2(4\pi)^2} \left(k^4 + \frac{5}{6} k^2 \bar{R} \right). \quad (\text{E19})$$

To summarize, the flow equations for U and F are

$$\pi_0|_{\bar{R}^0} = \frac{\Omega}{2(4\pi)^2} b_0^{(0)} Q_2[G_{\text{ph}}(z) \partial_t \mathcal{R}_k^{\text{ph}}(z)] - \frac{\Omega k^4}{2(4\pi)^2}, \quad (\text{E20})$$

$$\pi_0|_{\bar{R}^1} = \frac{\Omega}{2(4\pi)^2} b_2^{(0)} Q_1[G_{\text{ph}}(z) \partial_t \mathcal{R}_k^{\text{ph}}(z)] \\ - \frac{\Omega}{2(4\pi)^2} b_0^{(0)} Q_2[G_{\text{ph}}(z) I_{\text{ph}}(z) G_{\text{ph}}(z) \partial_t \mathcal{R}_k^{\text{ph}}(z)] \\ - \frac{\Omega k^2}{2(4\pi)^2} \frac{5}{6}. \quad (\text{E21})$$

2. Measure contribution

We now calculate the measure contribution η_k . This contribution takes the simple form [143]

$$\eta_k = -\frac{1}{2} \text{Tr}_{(1)} \frac{\partial_t P_k(\bar{\mathcal{D}}_1)}{P_k(\bar{\mathcal{D}}_1)} - \frac{1}{2} \text{Tr}_{(0)} \frac{\partial_t P_k(\bar{\mathcal{D}}_0)}{P_k(\bar{\mathcal{D}}_0)}, \quad (\text{E22})$$

with

$$\bar{\mathcal{D}}_1 = \bar{\Delta}_V - \frac{\bar{R}}{4}, \quad \bar{\mathcal{D}}_0 = \bar{\Delta}_S - \frac{\bar{R}}{4}. \quad (\text{E23})$$

To see this, we evaluate each of the contributions from measure modes which arise from the gauge fluctuations, the ghost fluctuations, and the regularization of the Jacobian.

a. Spin-1 measure modes

The spin-1 measure contribution is given by

$$\eta_1 = \delta_k^{(1)} - \epsilon_k^{(1)}, \quad (\text{E24})$$

with the spin-1 vector gauge mode (C7) and the Jacobians (A25) for the spin-1 gauge field

$$\delta_k^{(1)} = \lim_{\alpha \rightarrow 0} \frac{1}{2} \text{Tr}_{(1)} \frac{\partial_t \mathcal{R}_k}{\Gamma_k^{(2)} + \mathcal{R}_k} \Big|_{vv} + J_v \\ = \text{Tr}_{(1T)} \frac{\partial_t P_k(\bar{\mathcal{D}}_1)}{P_k(\bar{\mathcal{D}}_1)} - \frac{1}{2} \text{Tr}_{(1T)} \frac{\partial_t P_k(\bar{\mathcal{D}}_1)}{P_k(\bar{\mathcal{D}}_1)} \\ = \frac{1}{2} \text{Tr}_{(1T)} \frac{\partial_t P_k(\bar{\mathcal{D}}_1)}{P_k(\bar{\mathcal{D}}_1)}, \quad (\text{E25})$$

and the spin-1 ghost mode (C13)

$$\epsilon_k^{(1)} = \text{Tr}_{(1T)} \frac{\partial_t \mathcal{R}_k}{\Gamma_k^{(2)} + \mathcal{R}_k} \Big|_{\bar{c}^\perp c^\perp} = \text{Tr}_{(1T)} \frac{\partial_t P_k(\bar{\mathcal{D}}_1)}{P_k(\bar{\mathcal{D}}_1)}. \quad (\text{E26})$$

This sums up to the measure contribution from the spin-1 modes

$$\eta_1 = -\frac{1}{2} \text{Tr}_{(1T)} \frac{\partial_t P_k(\bar{\mathcal{D}}_1)}{P_k(\bar{\mathcal{D}}_1)} \\ = -\frac{1}{2(4\pi)^2} \int_x \sqrt{\bar{g}} [3k^4 + 2k^2 \bar{R}]. \quad (\text{E27})$$

b. Spin-0 measure modes

Next we discuss the spin-0 measure contribution. The spin-0 gauge mode coming from the metric fluctuation (C12) and the corresponding Jacobian (A27) is

$$\delta_k^{(0)} = \lim_{\alpha \rightarrow 0} \frac{1}{2} \text{Tr}_{(0)} \frac{\partial_t \mathcal{R}_k}{\Gamma_k^{(2)} + \mathcal{R}_k} \Big|_{uu} + J_u \\ = \left(\text{Tr}_{(0)} \frac{\partial_t P_k(\bar{\mathcal{D}}_0)}{P_k(\bar{\mathcal{D}}_0)} + \frac{1}{2} \text{Tr}_{(0)} \frac{\partial_t P_k(\bar{\Delta}_S)}{P_k(\bar{\Delta}_S)} \right) \\ - \frac{1}{2} \left(\text{Tr}_{(0)} \frac{\partial_t P_k(\bar{\mathcal{D}}_0)}{P_k(\bar{\mathcal{D}}_0)} + \text{Tr}_{(0)} \frac{\partial_t P_k(\bar{\Delta}_S)}{P_k(\bar{\Delta}_S)} \right) \\ = \frac{1}{2} \text{Tr}_{(0)} \frac{\partial_t P_k(\bar{\mathcal{D}}_0)}{P_k(\bar{\mathcal{D}}_0)}, \quad (\text{E28})$$

while the spin-0 ghost mode (C13) is

$$-\epsilon_k^{(0)} = -\text{Tr}_{(0)} \frac{\partial_t \mathcal{R}_k}{\Gamma_k^{(2)} + \mathcal{R}_k} \Big|_{\bar{c}c} + J_{\text{gh}} \\ = -\left(\text{Tr}_{(0)} \frac{\partial_t P_k(\bar{\mathcal{D}}_0)}{P_k(\bar{\mathcal{D}}_0)} + \text{Tr}_{(0)} \frac{\partial_t P_k(\bar{\Delta}_S)}{P_k(\bar{\Delta}_S)} \right) \\ + \text{Tr}_{(0)} \frac{\partial_t P_k(\bar{\Delta}_S)}{P_k(\bar{\Delta}_S)} \\ = -\text{Tr}_{(0)} \frac{\partial_t P_k(\bar{\mathcal{D}}_0)}{P_k(\bar{\mathcal{D}}_0)}, \quad (\text{E29})$$

where J_{gh} is given in Eq. (A28). Thus, the measure contribution of the spin-0 measure modes becomes

$$\begin{aligned}\eta_0 &= \delta_k^{(0)} - \epsilon_k^{(0)} = -\frac{1}{2} \text{Tr}_{(0)} \frac{\partial_t P_k(\bar{\mathcal{D}}_0)}{P_k(\bar{\mathcal{D}}_0)} \\ &= -\frac{1}{2(4\pi)^2} \int_x \sqrt{\bar{g}} \left[k^4 + \frac{5}{6} k^2 \bar{R} \right].\end{aligned}\quad (\text{E30})$$

Together with Eq. (E27), the contributions from the measure modes are given by

$$\begin{aligned}\eta_k &= \eta_1 + \eta_0 \\ &= -\frac{1}{2} \text{Tr}_{(1T)} \frac{\partial_t P_k(\bar{\mathcal{D}}_1)}{P_k(\bar{\mathcal{D}}_1)} - \frac{1}{2} \text{Tr}_{(0)} \frac{\partial_t P_k(\bar{\mathcal{D}}_0)}{P_k(\bar{\mathcal{D}}_0)} \\ &= -\frac{1}{2(4\pi)^2} \int_x \sqrt{\bar{g}} \left(4k^4 + \frac{17}{6} k^2 \bar{R} \right).\end{aligned}\quad (\text{E31})$$

APPENDIX F: QUANTUM CORRECTIONS TO THE SCALAR KINETIC TERM

In this section, we show the detailed derivation of the flow equation for K which is the coefficient of the kinetic term of the scalar field. For a derivation in a simple scalar field theory, see e.g., Refs. [147,173] for the derivation.

In the kinetical sector, background curvature operators such as $\bar{R}$ and $\bar{R}_{\mu\nu}$ do not appear. We therefore work on a flat background $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$ for which $\sqrt{\bar{g}} = 1$, $\bar{R} = 0$ and $\tilde{\sigma} = (\bar{\Delta}_S)^{-1} \sigma = (-\partial^2)^{-1} \sigma$. In this case, the kinetic term for the scalar field is

$$\Gamma_k^{\text{kin}}[\phi] = \frac{1}{2} \int_x K(\bar{\rho}) \delta^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} \equiv \frac{1}{2} \int_x K(\bar{\rho}) \mathcal{O}_K. \quad (\text{F1})$$

From the kinetic term (F1), we obtain the flow of the kinetical K as

$$\frac{\partial_t K(\bar{\rho})}{2} = \frac{1}{\Omega} \frac{\delta}{\delta \mathcal{O}_K} \partial_t \Gamma_k, \quad (\text{F2})$$

where $\Omega = \int_x = (2\pi)^4 \delta^4(p=0)$ is the four dimensional spacetime volume.

Here, thanks to the properties of the metric fluctuations and tensorial structures of variations, we can reduce the computation of the flow kernel of K . First, since we are interested in the corrections to the p^2 term for the two-point functions of $\bar{\phi}$, we symmetrize external momenta according to $\partial_\mu \partial_\nu \rightarrow \frac{1}{4} p^2 \delta_{\mu\nu}$. Then, the contraction between the Kronecker delta and $H_2^{\mu\nu}$ becomes

$$\begin{aligned}\delta_{\mu\nu} H_2^{\mu\nu} &= \delta_{\mu\nu} \left[\left(-\frac{1}{2} h^{\alpha\beta} h_{\alpha\beta} + \frac{1}{4} h^2 \right) \delta^{\mu\nu} + 2h^\mu{}_\lambda h^{\nu\lambda} - h h^{\mu\nu} \right] \\ &= 0.\end{aligned}\quad (\text{F3})$$

This implies that the tadpole diagrams arising from the metric fluctuations do not induce quantum corrections to the flow kernel of K . Second, the threshold functions in the tadpole diagram do not carry the external momentum p_μ , and therefore the loop effects arising from the four-point vertices without momentum dependence do not give contributions to the flow kernel of K . Here, we have

$$\begin{aligned}H_1^{\mu\nu} &= \frac{1}{2} h \bar{g}^{\mu\nu} - h^{\mu\nu} = \frac{1}{2} \delta^{\mu\nu} \bar{\Delta}_S \tilde{\sigma} - \frac{1}{3} (\delta^{\mu\nu} \bar{\Delta}_S - \partial^\mu \partial^\nu) \tilde{\sigma} + \dots \\ &= \frac{1}{6} \delta^{\mu\nu} \bar{\Delta}_S \tilde{\sigma} + \frac{1}{3} \partial^\mu \partial^\nu \tilde{\sigma} + \dots.\end{aligned}\quad (\text{F4})$$

The relevant functional derivatives of the scalar kinetic term are

$$\begin{aligned}&\frac{\delta^2}{\delta \phi(x) \delta \phi(y)} \int_z \sqrt{g} \frac{K(\rho)}{2} g^{\mu\nu} \partial_\mu^z \phi \partial_\nu^z \phi \Big|_{g=\delta, \phi=\bar{\phi}} \\ &= \left[-K \partial^2 - \frac{1}{2} (K' + 2\bar{\rho} K'') \mathcal{O}_K - (\bar{\phi} K') \partial^2 \bar{\phi} - (\bar{\phi} K') \partial_\mu \bar{\phi} \partial^\mu \right] \delta(x-y),\end{aligned}\quad (\text{F5})$$

$$\begin{aligned}&\frac{\delta^2}{\delta \tilde{\sigma}(x) \delta \tilde{\sigma}(y)} \int_z \frac{K(\rho)}{2} \left(\frac{1}{6} \delta^{\mu\nu} \bar{\Delta}_S^z \tilde{\sigma} + \frac{1}{3} \partial^{z\mu} \partial^{z\nu} \tilde{\sigma} \right) \partial_\mu^z \phi \partial_\nu^z \phi \Big|_{g=\delta, \phi=\bar{\phi}} \\ &= \left[\left(\frac{\bar{\phi} K'}{4} \mathcal{O}_K + \frac{K}{3} \partial^2 \bar{\phi} \right) \partial^2 + \frac{K}{6} \partial_\mu \bar{\phi} \partial^2 \partial^\mu + \left(\frac{\bar{\phi} K'}{6} (\partial_\mu \bar{\phi}) (\partial_\nu \bar{\phi}) + \frac{K}{3} (\partial_\mu \partial_\nu \bar{\phi}) \right) \partial^\mu \partial^\nu + \dots \right] \delta(x-y),\end{aligned}\quad (\text{F6})$$

$$\begin{aligned}&\frac{\delta^2}{\delta \phi(x) \delta \tilde{\sigma}(y)} \int_z \frac{K(\rho)}{2} \left(\frac{1}{6} \delta^{\mu\nu} \bar{\Delta}_S^z \tilde{\sigma} + \frac{1}{3} \partial^{z\mu} \partial^{z\nu} \tilde{\sigma} \right) \partial_\mu^z \phi \partial_\nu^z \phi \Big|_{g=\delta, \phi=\bar{\phi}} \\ &= \left[\left(\frac{\bar{\phi} K'}{12} \mathcal{O}_K + \frac{K}{6} (\partial^2 \bar{\phi}) \right) \partial^2 - \frac{K}{6} \partial_\mu \bar{\phi} \partial^2 \partial^\mu - \left(\frac{\bar{\phi} K'}{6} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} + \frac{K}{3} (\partial_\mu \partial_\nu \bar{\phi}) \right) \partial^\mu \partial^\nu + \dots \right] \delta(x-y),\end{aligned}\quad (\text{F7})$$

where $\dots$ includes operators irrelevant for the computation of the flow equation for the kinetic. The term $\sim \frac{\delta^2}{\delta\sigma\delta\sigma}$ acting on the scalar kinetic term does not contribute due to the property (F3). Thus the Hessian in momentum space for computing the flow equation for K is given by

$$(\Gamma_k^{(2)} + \mathcal{R}_k)_{\text{ph}} = \begin{pmatrix} -\frac{1}{6}\left(FP_k - \frac{U}{2}\right)P_k^2 & \frac{\bar{\phi}}{2}(-F'P_k + U')P_k \\ \frac{\bar{\phi}}{2}(-F'P_k + U')P_k & KP_k + m^2(\bar{\rho}) \end{pmatrix} + \begin{pmatrix} 0 & \mathcal{X}\partial^2 + \mathcal{Y}^\mu\partial^2\partial_\mu + \mathcal{Z}^{\mu\nu}\partial_\mu\partial_\nu \\ \mathcal{X}'\partial^2 - \mathcal{Y}^\nu\partial^2\partial_\nu - \mathcal{Z}^{\mu\nu}\partial_\mu\partial_\nu & \mathcal{A}^\mu\partial_\mu + \mathcal{B} \end{pmatrix} \\ = G_{\text{ph}}^{-1} + \mathcal{V}_{\text{ph}}, \quad (\text{F8})$$

where we have defined

$$\begin{aligned} \mathcal{A}^\mu &= -\bar{\phi}K'(\partial^\mu\bar{\phi}), & \mathcal{B} &= -\frac{1}{2}(K' + 2\bar{\rho}K'')\mathcal{O}_K - \bar{\phi}K'(\partial^2\bar{\phi}), \\ \mathcal{X} &= \frac{\bar{\phi}K'}{4}\mathcal{O}_K + \frac{K}{3}\partial^2\bar{\phi}, & \mathcal{X}' &= \frac{\bar{\phi}K'}{12}\mathcal{O}_K + \frac{K}{6}\partial^2\bar{\phi}, \\ \mathcal{Y}^\mu &= \frac{K}{6}(\partial^\mu\bar{\phi}), & \mathcal{Z}^{\mu\nu} &= \frac{\bar{\phi}K'}{6}(\partial^\mu\bar{\phi})(\partial^\nu\bar{\phi}) + \frac{K}{3}(\partial^\mu\partial^\nu\bar{\phi}). \end{aligned} \quad (\text{F9})$$

Next, we expand the full propagator into polynomials of the vertex operator $\mathcal{V}_{\text{ph}}$ as

$$(\Gamma_k^{(2)} + \mathcal{R}_k)|_{\text{ph}}^{-1} = (G_{\text{ph}}^{-1} + \mathcal{V}_{\text{ph}})^{-1} = (1 + G_{\text{ph}}\mathcal{V}_{\text{ph}})^{-1}G_{\text{ph}} = G_{\text{ph}} - G_{\text{ph}}\mathcal{V}_{\text{ph}}G_{\text{ph}} + G_{\text{ph}}\mathcal{V}_{\text{ph}}G_{\text{ph}}\mathcal{V}_{\text{ph}}G_{\text{ph}} - \dots, \quad (\text{F10})$$

from which the flow equation for kinetic is obtained as

$$\begin{aligned} & -\frac{1}{2}\text{Tr}_{(0)}[G_{\text{ph}}\mathcal{V}_{\text{ph}}G_{\text{ph}}\partial_i\mathcal{R}_k^{\text{ph}}] + \frac{1}{2}\text{Tr}_{(0)}[G_{\text{ph}}\mathcal{V}_{\text{ph}}G_{\text{ph}}\mathcal{V}_{\text{ph}}G_{\text{ph}}\partial_i\mathcal{R}_k^{\text{ph}}] \\ & = -\frac{1}{2}\text{Tr}_{(0)}[G_{\text{ph}}\mathcal{V}_{\text{ph}}^{(2)}G_{\text{ph}}\partial_i\mathcal{R}_k^{\text{ph}}] + \frac{1}{2}\text{Tr}_{(0)}[G_{\text{ph}}\mathcal{V}_{\text{ph}}^{(1)}G_{\text{ph}}\mathcal{V}_{\text{ph}}^{(1)}G_{\text{ph}}\partial_i\mathcal{R}_k^{\text{ph}}] + \dots, \end{aligned} \quad (\text{F11})$$

where we have defined

$$\mathcal{V}_{\text{ph}}^{(2)} = \begin{pmatrix} 0 & \mathcal{X}\partial^2 + \mathcal{Z}^{\mu\nu}\partial_\mu\partial_\nu \\ \mathcal{X}'\partial^2 - \mathcal{Z}^{\mu\nu}\partial_\mu\partial_\nu & \mathcal{B} \end{pmatrix}, \quad \mathcal{V}_{\text{ph}}^{(1)} = \begin{pmatrix} 0 & \mathcal{Y}^\mu\partial^2\partial_\mu \\ -\mathcal{Y}^\nu\partial^2\partial_\nu & \mathcal{A}^\mu\partial_\mu \end{pmatrix}. \quad (\text{F12})$$

The first term in Eq. (F11) corresponds to the first diagram in Fig. 5, while the second two diagrams involve in the second term in Eq. (F11). The symmetrization $\partial_\mu\partial_\nu \rightarrow \delta_{\mu\nu}\partial^2/4$ in the functional trace ($\text{Tr}_{(0)}$) yields

$$\mathcal{X}\partial^2 + \mathcal{Z}^{\mu\nu}\partial_\mu\partial_\nu \rightarrow \left(\frac{\bar{\phi}K'}{4}\mathcal{O}_K + \frac{K}{3}\partial^2\bar{\phi}\right)\partial^2 + \frac{1}{4}\left(\frac{\bar{\phi}K'}{6}\mathcal{O}_K + \frac{K}{3}\partial^2\bar{\phi}\right)\partial^2 = \left(\frac{7}{24}\bar{\phi}K'\mathcal{O}_K + \frac{5K}{12}\partial^2\bar{\phi}\right)\partial^2 = (\mathcal{V}_{\text{ph}}^{(2)})^{12}\partial^2, \quad (\text{F13a})$$

$$\mathcal{X}'\partial^2 - \mathcal{Z}^{\mu\nu}\partial_\mu\partial_\nu \rightarrow \left(\frac{\bar{\phi}K'}{12}\mathcal{O}_K + \frac{K}{6}\partial^2\bar{\phi}\right)\partial^2 - \frac{1}{4}\left(\frac{\bar{\phi}K'}{6}\mathcal{O}_K + \frac{K}{3}\partial^2\bar{\phi}\right)\partial^2 = \left(\frac{1}{24}\bar{\phi}K'\mathcal{O}_K + \frac{K}{12}\partial^2\bar{\phi}\right)\partial^2 = (\mathcal{V}_{\text{ph}}^{(2)})^{21}\partial^2, \quad (\text{F13b})$$

and

$$\mathcal{A}^\mu\mathcal{A}^\nu\partial_\mu\partial_\nu \rightarrow \frac{2\bar{\rho}K'^2}{4}\mathcal{O}_K\partial^2, \quad \mathcal{A}^\mu\mathcal{Y}^\nu\partial^2\partial_\mu\partial_\nu \rightarrow -\frac{\bar{\phi}K'K}{6 \times 4}\mathcal{O}_K\partial^4, \quad \mathcal{Y}^\mu\mathcal{Y}^\nu\partial^4\partial_\mu\partial_\nu \rightarrow \frac{K^2}{36 \times 4}\mathcal{O}_K\partial^6. \quad (\text{F14})$$

Evaluating Eq. (F11) with the propagator (E14), the regulator (E11) and the vertices (F13) and (F14), we have

$$\begin{aligned}
& -\frac{1}{2}\text{Tr}_{(0)}[G_{\text{ph}}\mathcal{V}_{\text{ph}}^{(2)}G_{\text{ph}}\partial_t\mathcal{R}_k^{\text{ph}}] + \frac{1}{2}\text{Tr}_{(0)}[G_{\text{ph}}\mathcal{V}_{\text{ph}}^{(1)}G_{\text{ph}}\mathcal{V}_{\text{ph}}^{(1)}G_{\text{ph}}\partial_t\mathcal{R}_k^{\text{ph}}] \\
& = -\frac{1}{2}\frac{1}{(4\pi)^2}\int_x \text{tr}\left[\begin{pmatrix}\mathcal{P}_{\tilde{\sigma}\tilde{\sigma}} & \mathcal{P}_{\tilde{\sigma}\varphi} \\ \mathcal{P}_{\varphi\tilde{\sigma}} & \mathcal{P}_{\varphi\varphi}\end{pmatrix}\begin{pmatrix}0 & (\mathcal{V}_{\text{ph}}^{(2)})^{12}(\partial^2) \\ (\mathcal{V}_{\text{ph}}^{(2)})^{21}(\partial^2) & \mathcal{B}\end{pmatrix}\begin{pmatrix}\mathcal{P}_{\tilde{\sigma}\tilde{\sigma}} & \mathcal{P}_{\tilde{\sigma}\varphi} \\ \mathcal{P}_{\varphi\tilde{\sigma}} & \mathcal{P}_{\varphi\varphi}\end{pmatrix}\begin{pmatrix}\partial_t\mathcal{R}_k^{\tilde{\sigma}\tilde{\sigma}} & \partial_t\mathcal{R}_k^{\tilde{\sigma}\varphi} \\ \partial_t\mathcal{R}_k^{\varphi\tilde{\sigma}} & \partial_t\mathcal{R}_k^{\varphi\varphi}\end{pmatrix}\right] \\
& + \frac{1}{2}\frac{1}{(4\pi)^2}\int_x \text{tr}\left[\begin{pmatrix}\mathcal{P}_{\tilde{\sigma}\tilde{\sigma}} & \mathcal{P}_{\tilde{\sigma}\varphi} \\ \mathcal{P}_{\varphi\tilde{\sigma}} & \mathcal{P}_{\varphi\varphi}\end{pmatrix}\begin{pmatrix}0 & \mathcal{Y}^\mu\partial^2\partial_\mu \\ -\mathcal{Y}^\nu\partial^2\partial_\nu & \mathcal{A}^\mu\partial_\mu\end{pmatrix}\begin{pmatrix}\mathcal{P}_{\tilde{\sigma}\tilde{\sigma}} & \mathcal{P}_{\tilde{\sigma}\varphi} \\ \mathcal{P}_{\varphi\tilde{\sigma}} & \mathcal{P}_{\varphi\varphi}\end{pmatrix}\begin{pmatrix}0 & \mathcal{Y}^\mu\partial^2\partial_\mu \\ -\mathcal{Y}^\nu\partial^2\partial_\nu & \mathcal{A}^\mu\partial_\mu\end{pmatrix}\right. \\
& \left.\times\begin{pmatrix}\mathcal{P}_{\tilde{\sigma}\tilde{\sigma}} & \mathcal{P}_{\tilde{\sigma}\varphi} \\ \mathcal{P}_{\varphi\tilde{\sigma}} & \mathcal{P}_{\varphi\varphi}\end{pmatrix}\begin{pmatrix}\partial_t\mathcal{R}_k^{\tilde{\sigma}\tilde{\sigma}} & \partial_t\mathcal{R}_k^{\tilde{\sigma}\varphi} \\ \partial_t\mathcal{R}_k^{\varphi\tilde{\sigma}} & \partial_t\mathcal{R}_k^{\varphi\varphi}\end{pmatrix}\right]. \tag{F15}
\end{aligned}$$

By extracting the term proportional to $(\partial_\mu\tilde{\phi})^2$, we obtain the flow equation for K .

$$u_0=0.0006460, \quad w_0=0.002757, \quad \eta=-\partial_t\log\kappa_0=1.6669. \tag{G5}$$

APPENDIX G: COMPARISON OF FLOW KERNELS

In this appendix, we compare the flow kernels between the present work and Ref. [142] within the large and small field expansions.

1. Small field limit

The small-field expansion for the flow kernels are

$$u(\tilde{\rho}) = \sum_{n=0}^{\infty} \frac{u_n}{n!} \tilde{\rho}^n = u_0 + u_1\tilde{\rho} + \dots, \tag{G1}$$

$$w(\tilde{\rho}) = \sum_{n=0}^{\infty} \frac{w_n}{n!} \tilde{\rho}^n = w_0 + w_1\tilde{\rho} + \dots, \tag{G2}$$

$$\kappa(\tilde{\rho}) = \sum_{n=0}^{\infty} \frac{\kappa_n}{n!} \tilde{\rho}^n = \kappa_0 + \kappa_1\tilde{\rho} + \dots. \tag{G3}$$

In our work we find a UV-fixed point

$$u_0=0.0055, \quad w_0=0.0277, \quad \eta=-\partial_t\log\kappa_0=0.0116, \tag{G4}$$

while Ref. [142] finds

2. Large field limit

Next we consider the large-field expansion of the flow kernels in terms of $x = 1/(\xi_\infty\tilde{\rho})$. To compare our results with those of Ref. [142], we adopt the ansatz

$$u(x) = u_\infty + \sum_{n=0}^{\infty} \frac{u_n}{n!} x^n, \tag{G6}$$

$$w(x) = x^{-1} + \sum_{n=0}^{\infty} \frac{t_n}{n!} x^n, \tag{G7}$$

$$\kappa(x) = \varepsilon_\infty\xi_\infty + \xi_\infty \sum_{n=1}^{\infty} \frac{k_n}{n!} x^n, \tag{G8}$$

for which we have

$$\partial_t u = -4u - 2x \frac{\partial u}{\partial x} + \frac{A_V}{B^3} + \frac{C_V}{B^4} x, \tag{G9}$$

$$\partial_t w = -2w - 2x \frac{\partial w}{\partial x} + \frac{A_F}{B^3} + \frac{C_F}{B^4} x, \tag{G10}$$

$$\partial_t \kappa = -2x \frac{\partial \kappa}{\partial x} + \frac{C_K}{B^4} x. \tag{G11}$$

The coefficients A_V, C_V etc. are related to Eqs. (18)–(20) by suitable factors of B and ξ_∞ . Our study gives

$$A_V = \frac{1}{32\pi^2} (3\epsilon^3 + 54\epsilon^2 + 324\epsilon + 648),$$

$$\begin{aligned}
C_V = & \frac{1}{480\pi^2} (34t_0\epsilon^4 + 762t_0\epsilon^3 + 6372t_0\epsilon^2 + 23544t_0\epsilon + 32400t_0 + 5k_1\epsilon^3 + 90k_1\epsilon^2 + 540k_1\epsilon + 1080k_1 + 75u_\infty\epsilon^4 \\
& + 1800u_\infty\epsilon^3 + 16200u_\infty\epsilon^2 + 64800u_\infty\epsilon + 97200u_\infty),
\end{aligned}$$

$$\begin{aligned}
A_F &= \frac{1}{1536\pi^2} (509\epsilon^3 + 9356\epsilon^2 + 57300\epsilon + 116928), \\
C_F &= \frac{1}{322560\pi^2} (37950t_0\epsilon^4 + 890100t_0\epsilon^3 + 8070408t_0\epsilon^2 + 33502896t_0\epsilon + 53561088t_0 - 1680k_1\epsilon^3 - 71736k_1\epsilon^2 \\
&\quad - 679392k_1\epsilon - 1856736k_1 + 85505u_\infty\epsilon^4 + 2057244u_\infty\epsilon^3 + 18491508u_\infty\epsilon^2 + 73592064u_\infty\epsilon + 109408320u_\infty), \\
C_K &= \frac{1}{32\pi^2} (-5\epsilon^4 - 69\epsilon^3 - 360\epsilon^2 - 648\epsilon).
\end{aligned} \tag{G12}$$

Ref. [142] finds

$$\begin{aligned}
A_V &= \frac{1}{192\pi^2} (9\epsilon^3 + 82\epsilon^2 + 612\epsilon + 2760), \\
C_V &= \frac{1}{2592\pi^2} (1080t_0\epsilon^3 - 3240t_0\epsilon^2 - 62208t_0\epsilon + 1080k_1\epsilon^2 - 3240k_1\epsilon - 62208k_1 \\
&\quad + 383u_\infty\epsilon^4 + 5004u_\infty\epsilon^3 + 51120u_\infty\epsilon^2 + 278208u_\infty\epsilon + 382320u_\infty), \\
A_F &= \frac{1}{3456\pi^2} (-253\epsilon^3 - 6094\epsilon^2 - 36240\epsilon - 51840), \\
C_F &= \frac{1}{5184\pi^2} (2148t_0\epsilon^3 - 2916t_0\epsilon^2 - 98712t_0\epsilon + 2310k_1\epsilon^2 - 972k_1\epsilon - 92880k_1 \\
&\quad - 345u_\infty\epsilon^4 - 23498u_\infty\epsilon^3 - 213492u_\infty\epsilon^2 - 546552u_\infty\epsilon - 263520u_\infty), \\
C_K &= \frac{1}{36\pi^2} (-\epsilon^4 + 90\epsilon^3 + 12636\epsilon^2 - 26244\epsilon).
\end{aligned} \tag{G13}$$

If we use the ansatz (18)–(20), i.e.,

$$\partial_\tau u(x) = 2x\partial_x u(x) = -4u(x) + A_u + \frac{C_u}{B}x = 2s_1x + \dots, \tag{G14}$$

$$\partial_\tau w(x) = 2x\partial_x w(x) = -2w(x) + \frac{A_w}{B} + \frac{C_w}{B^2}x + \dots = -\frac{2}{x} + 2t_1x + \dots, \tag{G15}$$

$$\partial_\tau \kappa(x) = 2x\partial_x \kappa(x) = \xi_\infty \frac{C_\kappa}{B}x + \dots = 2\xi_\infty k_1x + \dots, \tag{G16}$$

the coefficients in Ref. [142] are

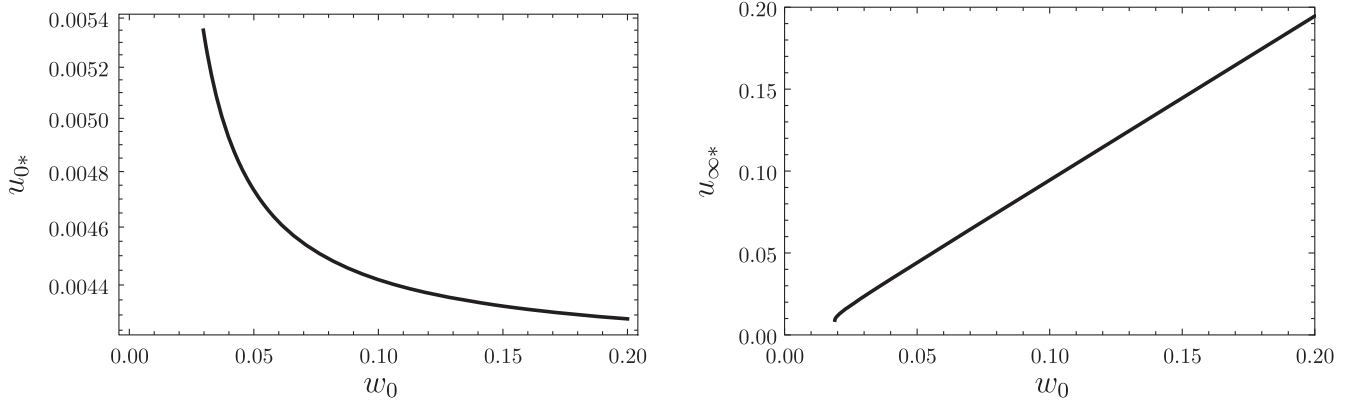
$$A_u = \frac{1}{93312\pi^2} (6210 - 1728\epsilon + 531\epsilon^2 - 130\epsilon^3) \tag{G17}$$

$$\begin{aligned}
C_u &= \frac{1}{46656\pi^2} (-5184k_1 + 31860u_\infty - 5184t_0\epsilon + 2322k_1\epsilon + 7254u_\infty\epsilon + 2322t_0\epsilon^2 - 639k_1\epsilon^2 - 2022u_\infty\epsilon^2 \\
&\quad - 639t_0\epsilon^3 + 150k_1\epsilon^3 + 676u_\infty\epsilon^3)
\end{aligned} \tag{G18}$$

$$A_w = \frac{1}{2239488\pi^2} (-933120 - 341280\epsilon + 29988\epsilon^2 - 5070\epsilon^3) \tag{G19}$$

$$\begin{aligned}
C_w &= \frac{1}{1119744\pi^2} (-557280k_1 - 1581120u_\infty - 592272t_0\epsilon + 179928k_1\epsilon - 2752272u_\infty\epsilon + 179928t_0\epsilon^2 - 30636k_1\epsilon^2 \\
&\quad - 319608u_\infty\epsilon^2 - 30636t_0\epsilon^3 + 5214k_1\epsilon^3 + 42000u_\infty\epsilon^3 + 5214t_0\epsilon^4 - 887k_1\epsilon^4 - 7192u_\infty\epsilon^4)
\end{aligned} \tag{G20}$$

$$C_\kappa = \frac{1}{7776\pi^2} (-26244\epsilon + 25758\epsilon^2 - 10602\epsilon^3 + 3275\epsilon^4). \tag{G21}$$

FIG. 6. UV (left) and IR (right) fixed point values of u_0 and $u_{\infty*}$ varying w_0 .

APPENDIX H: FIXED POINT STRUCTURE AND SCALING SOLUTIONS FOR $u(\tilde{\rho})$ WITH FIXED $w(\tilde{\rho})$ AND $\kappa(\tilde{\rho})$

In this appendix we demonstrate a simple case where we fix $\kappa(\tilde{\rho}) = 1$ and $w(\tilde{\rho}) = w_0$ (constant). In this case, the flow equation for $u(\tilde{\rho})$ is given by

$$\partial_t u(\tilde{\rho}) = -\partial_{\tilde{\rho}} u(\tilde{\rho}) - 4u(\tilde{\rho}) - \frac{5u(\tilde{\rho})}{24\pi^2(u(\tilde{\rho}) - w_0)} - \frac{3u(\tilde{\rho})}{160\pi^2(u(\tilde{\rho}) - 4w_0)} + \frac{79}{480\pi^2}. \quad (\text{H1})$$

The scaling solution thus reads

$$\partial_{\tilde{\rho}} u(\tilde{\rho}) = -4u(\tilde{\rho}) - \frac{5u(\tilde{\rho})}{24\pi^2(u(\tilde{\rho}) - w_0)} - \frac{3u(\tilde{\rho})}{160\pi^2(u(\tilde{\rho}) - 4w_0)} + \frac{79}{480\pi^2}. \quad (\text{H2})$$

Setting $u(\tilde{\rho}) = u_0$ and solving

$$0 = -4u_0 - \frac{5u_0}{24\pi^2(u_0 - w_0)} - \frac{3u_0}{160\pi^2(u_0 - 4w_0)} + \frac{79}{480\pi^2}. \quad (\text{H3})$$

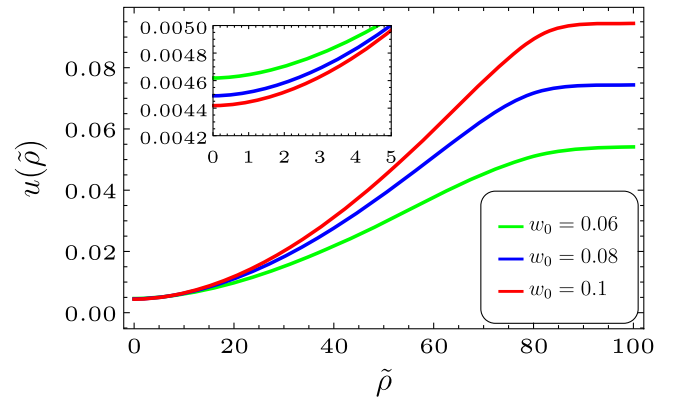
One typically finds two fixed points. One is approached for $\tilde{\rho} \rightarrow \infty$ and denoted by $u_{\infty*}$ (IR fixed point). The other is approached for $\tilde{\rho} \rightarrow 0$ and denoted by u_{0*} (UV-fixed point). For several fixed values of w_0 , we obtain the IR fixed point $u_{\infty*}$ and the UV-fixed point u_{0*} , respectively, as

$$(\text{IR}): u_{\infty*} = \begin{cases} 0.0542678 & \text{for } w_0 = 0.06, \\ 0.0743981 & \text{for } w_0 = 0.08, \\ 0.0944702 & \text{for } w_0 = 0.1, \end{cases} \quad (\text{H4})$$

and

$$(\text{UV}): u_{0*} = \begin{cases} 0.00461833 & \text{for } w_0 = 0.06, \\ 0.00448945 & \text{for } w_0 = 0.08, \\ 0.00441818 & \text{for } w_0 = 0.1. \end{cases} \quad (\text{H5})$$

The w_0 dependence of the IR and UV-fixed point values is depicted in Fig. 6. The scaling solution for $u(\tilde{\rho})$ with several values of w_0 is displayed in Fig. 7, illustrating the crossover between the UV and IR fixed points.

FIG. 7. Scaling solutions for $u(\tilde{\rho})$ with $\kappa(\tilde{\rho}) = 1$ and several fixed values of $w(\tilde{\rho}) = w_0$.

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