

High-precision lattice determination of the interaction potential of an SU(2) solitonic dipole and comparison with perturbative QED

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We determine the interaction potential of a solitonic dipole in the singlet state, modeled as an SU(2) field, using improved lattice simulations of two stationary solitons at varying separations. The potential is extracted from the energy of two-soliton configurations as a function of distance. At large separations, the interaction reproduces the classical Coulomb potential quantitatively up to an energy shift $\delta E_\infty \approx 9$ keV of the fitted asymptotic constant relative to $2m_e c_0^2$, assumed to be related to limited numerical precision on the lattice. At shorter distances, deviations from the Coulomb potential of pointlike charges appear, that are in qualitative agreement with the asymptotic formula of perturbative quantum electrodynamics, reflecting the running of the fine-structure constant, with the inverse fine-structure constant ($\alpha^{-1} \approx 137$) reproduced.

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I. INTRODUCTION

Previous work [1,2] has introduced a model for topological particles through a $3 + 1$ -dimensional generalization of the $1 + 1$ -dimensional Sine-Gordon model [3]. These models demonstrate how localized field excitations, characterized by topological quantum numbers, possess relativistic properties characteristic of particles, such as stability, Lorentz contraction, and relativistic mass increase, thereby resolving the distinction between particles and fields.

The aim of this work is to compare the interaction of stationary topological solitons in the $3 + 1$ -dimensional model, using precise numerical calculations with as few approximations as possible, with the behavior of interacting electrons at rest and the predictions of QED in order to identify possible discrepancies between solitons and electrons. However, within the limits of numerical accuracy, the results show consistent agreement extending to the strength of the interaction and even to the behavior of the fine-structure constant [4].

A. Mathematical model

To describe an SO(3)-valued field, we use the fundamental representation of SU(2),

$$Q(x) := e^{-i\alpha(x)\vec{\sigma}\cdot\vec{n}(x)} = \underbrace{\cos \alpha(x)}_{q_0(x)} - i\vec{\sigma} \cdot \underbrace{\vec{n}(x) \sin \alpha(x)}_{\vec{q}(x)}, \quad |\vec{n}| = 1,$$

with x in Minkowski space-time, Pauli matrices $\vec{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$, and arrows denoting components in the 3D Euclidean spaces, as in the $\mathfrak{su}(2)$ -algebra. Solitons are described by the Lagrangian

$$\mathcal{L}(x) := -\frac{\alpha_f \hbar c_0}{4\pi} \left(\frac{1}{4} \vec{R}_{\mu\nu}(x) \cdot \vec{R}^{\mu\nu}(x) + \Lambda(x) \right), \quad (1)$$

with Sommerfeld's fine structure constant

$$\alpha_f := \frac{e_0^2}{4\pi\epsilon_0 \hbar c_0}, \quad (2)$$

the potential

$$\Lambda(x) := \frac{q_0^6(x)}{r_0^4} \quad (3)$$

stabilizing the solitons core to a radius of r_0 and curvature tensor

$$\vec{R}_{\mu\nu} := \vec{\Gamma}_\mu \times \vec{\Gamma}_\nu, \quad (\partial_\mu Q)Q^\dagger := -i\vec{\sigma} \cdot \vec{\Gamma}_\mu, \quad (4)$$

that is,

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$$\begin{aligned} \vec{\Gamma}_\mu &\stackrel{(4)}{=} q_0 \partial_\mu \vec{q} - (\partial_\mu q_0) \vec{q} + \vec{q} \times \partial_\mu \vec{q} \\ &\stackrel{(1)}{=} (\partial_\mu \alpha) \vec{n} + \sin \alpha \cos \alpha \partial_\mu \vec{n} + \sin^2 \alpha \vec{n} \times \partial_\mu \vec{n}. \end{aligned} \quad (5)$$

B. Relation to physics

As shown in Refs. [1,2], the resulting nonlinear Euler-Lagrange equations admit four distinct stable single-soliton solutions

$$n_i(x) = \underbrace{Z}_{\pm 1} \frac{x^i}{|\vec{x}|}, \alpha(x) = \begin{cases} \arctan \frac{|\vec{x}|}{r_0}, \\ \pi - \arctan \frac{|\vec{x}|}{r_0}. \end{cases} \quad (6)$$

The rest energy of solitons is given by the analytical expression

$$E_0 = \frac{\alpha_f \hbar c_0 \pi}{r_0 4}. \quad (7)$$

From the solutions in Eq. (6), it can be seen that the four topologically distinct field configurations have the same energy (7). The proportionality

$$*\vec{F}_{\mu\nu} := -\frac{e_0}{4\pi\epsilon_0 c_0} \vec{R}_{\mu\nu} \quad (8)$$

between the curvature tensor $\vec{R}_{\mu\nu}$ and the dual electromagnetic field strength tensor $*\vec{F}_{\mu\nu}$ relates the mathematical quantities to the physical properties of the solitonic solutions. As a result of this relationship, static solitons are merely sources of electric fields, i.e., topologically quantized charges. Inside the solitons, the electromagnetic fields are non-Abelian but become Abelian for distances much larger than the natural scale r_0 in Eq. (3), i.e., under the influence of the potential.

The four topologically stable solutions differ in their charge quantum number Z and in their coverings $\pm \frac{1}{2}$ of S^3 , whose absolute value defines the spin quantum number s of the solitons. Two-soliton configurations are classified according to their total spin S , which is defined by the combined covering numbers of the individual solitons. This results in singlet states with total spin $S = 0$ and triplet states with total spin $S = 1$, as must follow from the triangular inequality for spin quantum numbers. For the $r_0 = 0$ limit of a soliton-antisoliton pair, the singlet and triplet states are identical. Furthermore, for topological reasons, the $\vec{q} \equiv \vec{n}$ vectors defined in Eqs. (1), (5), and (6) are constant along the electric field lines. In this limiting case, the fields of the $\vec{q} := \vec{n} \sin \alpha$ vectors have singularities at the locations of the solitons because $\sin \alpha = 1$. For finite r_0 , however, $\sin \alpha = 0$ holds at these locations, so that the singularities are eliminated, as can be seen from the $\vec{q}$ vectors

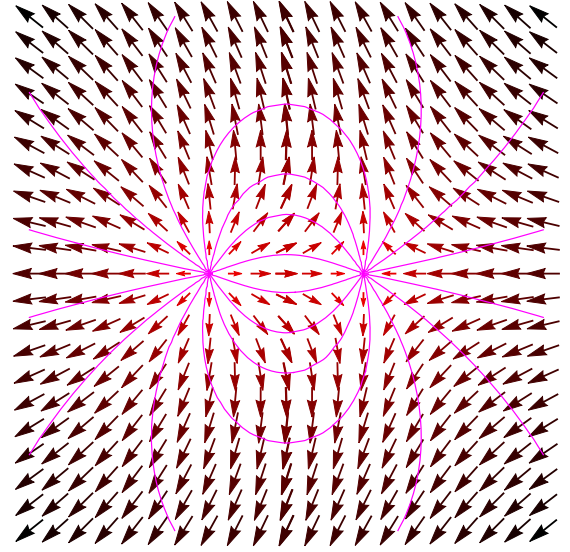


FIG. 1. Schematic representation of the $\vec{q} := \vec{n} \sin \alpha$ vectors for a soliton-antisoliton pair in the $S = 0$ spin state and comparison with the electric field lines of a dipole. For $\alpha \approx 0$, the $\vec{q}$ arrows are short and red; for $\alpha \approx \pi/2$, the arrows have length 1 and turn black.

shown in Fig. 1. As the distance from the soliton centers increases, α grows and, according to Eq. (6), rapidly approaches the value $\alpha = \pi/2$, i.e., $\vec{q} = \vec{n}$. This behavior is very advantageous for lattice calculations because the boundary conditions at the transition from the finite lattice to the continuous field outside the lattice can be easily taken into account.

This similarity to electrons motivated comparisons with perturbative QED by adjusting the solitons rest mass to $E_0 = m_e c_0^2 = 0.510998951$ MeV. This is done by setting

$$r_0 = 2.21320516 \text{ fm}, \quad (9)$$

which according to Eq. (7) is the classical electron radius multiplied by $\pi/4$.

Previous evaluations [5] of the solitonic dipole potential suffered from limited precision due to numeric instabilities of the energy minimization procedures. In this work, we present improved results for the solitonic dipole potential and a determination of a solitonic fine structure constant α_{sol} .

II. LATTICE FORMULATION

Exploiting the axial symmetry of the solitonic dipole, the field configuration $Q(q_r, z_k)$ is discretized on a static cylindrically symmetric lattice with spacing $a = \frac{r_0}{3}$,

$$\begin{aligned}
q_0(q_r, z_k) &= q_0(\vec{x})|_{\vec{x}=(q_r, 0, z_k)}, \\
q_\varrho(q_r, z_k) &= \vec{q}(\vec{x}) \cdot \vec{e}_x|_{\vec{x}=(q_r, 0, z_k)}, \\
q_z(q_r, z_k) &= \vec{q}(\vec{x}) \cdot \vec{e}_z|_{\vec{x}=(q_r, 0, z_k)},
\end{aligned} \quad (10)$$

where the discrete coordinates are related to the lattice indices by $q_r = ra$ for $r = 0, \dots, n_\varrho - 1$ and $z_k = (k - k_0)a$ for $k = 1, \dots, n_z$ with $k_0 = (n_z + 1)/2$.

The lattice size is chosen such that the distance of each soliton core to the nearest lattice boundary remains constant at $15r_0 = 33.2$ fm, while the separation between the soliton centers is varied. The lattice ranges from 45×111 (smallest) to 45×475 (largest).

It is initialized by embedding a single-soliton solution in each half of the lattice with the soliton centers separated by d lattice spacings. The equilibrium field configuration is then determined by minimizing the discretized energy functional corresponding to the Lagrangian density $\mathcal{L}(x)$ of Eq. (1),

$$E[q_0, q_\varrho, q_z] = \underbrace{H_{\text{pot}}[q_0]}_{\propto \Lambda} + \underbrace{H_{\text{curv}}[q_0, q_\varrho, q_z]}_{\propto \vec{R}_{\mu\nu} \cdot \vec{R}^{\mu\nu}} + \underbrace{H_{\text{out}}[q_0]}_{\text{exterior}}, \quad (11)$$

using a nonlinear conjugate-gradient method with a bracketing and golden-section line search along the descent direction.

The potential contribution takes the form

$$H_{\text{pot}}[q_0] \simeq \frac{\alpha_f \hbar c_0}{2r_0^4} a^2 \sum_{r=0}^{n_\varrho-1} \sum_{k=1}^{n_z} r q_0^6(q_r, z_k). \quad (12)$$

The curvature term H_{curv} is constructed from discrete approximations to the derivatives $\partial_\varrho q_\mu$ and $\partial_z q_\mu$ ($\mu \in \{0, \varrho, z\}$) entering the field strength tensor $\mathbf{R}_{\mu\nu}$. For interior lattice points, we employ fourth-order-accurate symmetric five-point stencils in both directions,

$$\begin{aligned}
\partial_\varrho q_\mu(q_r, z_k) &\approx \frac{-q_\mu(q_{r+2}, z_k) + 8q_\mu(q_{r+1}, z_k)}{12a} \\
&\quad - \frac{8q_\mu(q_{r-1}, z_k) - q_\mu(q_{r-2}, z_k)}{12a},
\end{aligned} \quad (13)$$

$$\begin{aligned}
\partial_z q_\mu(q_r, z_k) &\approx \frac{-q_\mu(q_r, z_{k+2}) + 8q_\mu(q_r, z_{k+1})}{12a} \\
&\quad - \frac{8q_\mu(q_r, z_{k-1}) - q_\mu(q_r, z_{k-2})}{12a},
\end{aligned} \quad (14)$$

while in the first and last lattice layers in ϱ and z we switch smoothly to three-point one-sided formulas of second order in a ,

$$\partial_\varrho q_\mu(q_0, z_k) \approx \frac{-3q_\mu(q_0, z_k) + 4q_\mu(q_1, z_k)}{2a} \quad (15)$$

$$- \frac{q_\mu(q_2, z_k)}{2a}, \quad (16)$$

$$\partial_\varrho q_\mu(q_{n_\varrho-1}, z_k) \approx \frac{q_\mu(q_{n_\varrho-3}, z_k) - 4q_\mu(q_{n_\varrho-2}, z_k)}{2a} \quad (17)$$

$$+ \frac{3q_\mu(q_{n_\varrho-1}, z_k)}{2a}, \quad (18)$$

and analogously for $\partial_z q_\mu$ at $z = z_1$ and $z = z_{n_z}$. Using these discrete derivatives, we construct the lattice analog of $\vec{R}_{\varrho z}$ and hence of the curvature energy density

$$\mathcal{H}_{\text{curv}}(q_r, z_k) \propto \vec{R}_{\mu\nu}(q_r, z_k) \cdot \vec{R}^{\mu\nu}(q_r, z_k), \quad (19)$$

which is then integrated over the cylindrical volume by summing over r and k with the appropriate Jacobian factor,

$$H_{\text{curv}} \simeq \frac{\alpha_f \hbar c_0}{4\pi} a^2 \sum_{r=0}^{n_\varrho-1} \sum_{k=1}^{n_z} r \vec{R}_{\mu\nu}(q_r, z_k) \cdot \vec{R}^{\mu\nu}(q_r, z_k), \quad (20)$$

which reproduces the continuum expression from Eq. (1) up to the discretization errors of the finite-difference scheme.

The contribution H_{out} captures the energy outside the lattice by approximating it analytically via the Coloumb field of two charges at the location of the solitons cores at $z_\pm = \pm d/2$.

The radial and axial components of the electric field are given by

$$E_\varrho(q, z) = \frac{e_0}{4\pi\epsilon_0} \left[\frac{q}{(q^2 + z_+^2)^{3/2}} - \frac{q}{(q^2 + z_-^2)^{3/2}} \right], \quad (21)$$

$$E_z(q, z) = \frac{e_0}{4\pi\epsilon_0} \left[\frac{z_+}{(q^2 + z_+^2)^{3/2}} - \frac{z_-}{(q^2 + z_-^2)^{3/2}} \right]. \quad (22)$$

The exterior electric-energy contribution is given by

$$H_{\text{out}} = 2\pi \int_{\mathbb{R}^2 \setminus \text{lattice}} \varrho d\varrho dz \frac{\epsilon_0}{2} (E_\varrho^2(q, z) + E_z^2(q, z)), \quad (23)$$

with $d\varrho$ from 0 to $R = (n_\varrho - \frac{1}{2})a$ and dz from 0 to $Z = (n_z - \frac{1}{2})a$.

III. RESULTS

The energy $E(d)$ of solitonic dipoles at separation d is determined according to Eq. (11); see Fig. 2. Because of the finite precision of the lattice simulation the extrapolated value at infinity overestimates the analytical minimum $2E_0$ of Eq. (7) by a few keV, $E(\infty) = 2E_0 + \delta E_\infty$, leading to the definitions

$$E(d) := E[q_0, q_\varrho, q_z] - \delta E_\infty = 2m_e c_0^2 - \frac{\alpha_{\text{sol}} \hbar c_0}{d}. \quad (24)$$

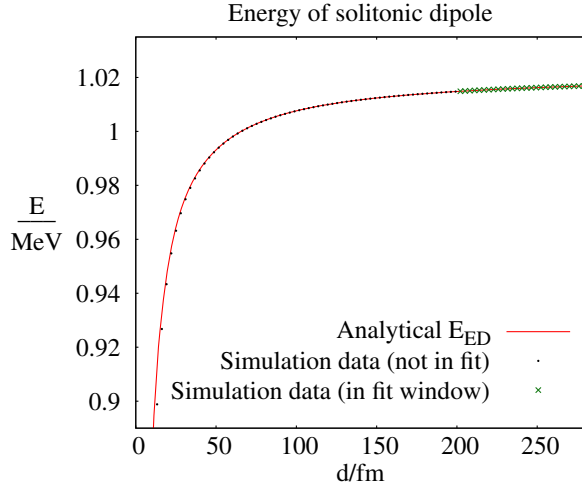


FIG. 2. The energies of solitonic dipoles with centers fixed at distance d are compared to the energy $E_{ED}(d)$ expected for a static dipole according to classical electrodynamics. Since discretization overestimates the energies of individual solitons, we shift the calculated energies by a corresponding constant δE_∞ , which is adjusted in the range $[200\text{--}280]$ fm. Light blue squares: points included in the fit. Red line: fit to Eq. (24). Black circles: lattice points outside the fit window $[0\text{--}200]$ fm. The markers are intentionally small to highlight deviations between analytical and numerical results at short distances.

δE_∞ and α_{sol} are determined for sufficiently large separations (typically $d \in [200, 280]$ fm) by a common fit leading to

$$\delta E_\infty = 9.432(3) \text{ keV}, \quad \alpha_{sol} \hbar c_0 = 1.4387(8) \text{ MeV fm} \quad (25)$$

with uncertainties propagated from the nonlinear least-squares fit. In Fig. 2, the numerical results for $E(d)$ of Eq. (24) are compared to the interaction of two-point charges $\pm e_0$ at rest as described in electrodynamics by the electron rest energy and the Coulomb potential

$$E_{ED}(d) = 2m_e c_0^2 - \frac{\alpha_f \hbar c_0}{d}. \quad (26)$$

The good agreement of $\alpha_{sol} \hbar c_0$ with $\frac{e_0^2}{4\pi\epsilon_0} = 1.43996 \text{ MeV fm}$ is evidence that solitons interact like point charges for distances d much larger than r_0 of Eq. (9).

When adjusting the two parameters in Eq. (25) in the fit region $[d_<, 280 \text{ fm}]$, one observes that δE_∞ remains essentially unchanged over practically the entire range of reasonable $d_<$ -values, whereas $\alpha_{sol} \hbar c_0$ notes an interesting dependence. Since $1/\alpha_f$ is a famous number, we compare it in Fig. 3 with the dependence of $1/\alpha_{sol}$ of Eq. (24) on $d_<$. For $d_<$ less than 240 fm, $1/\alpha_{sol}$ begins to stabilize and then tends to decrease slightly, but from 80 fm downward, a

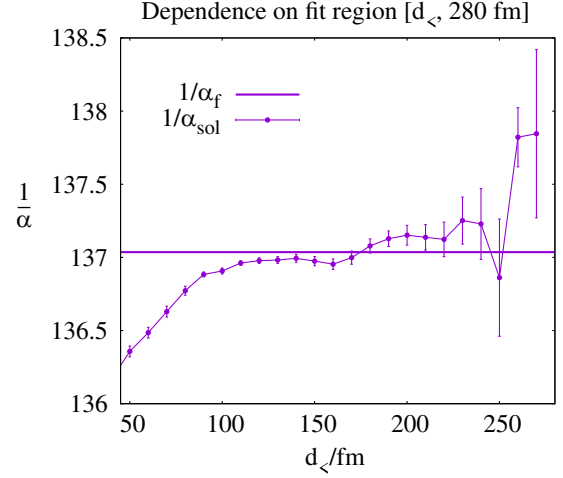


FIG. 3. Dependence of the fit parameter α_{sol} of Eq. (24) on the size of the fit region $[d_<, 280 \text{ fm}]$.

strong decrease in $1/\alpha_{sol}$ becomes apparent which can no longer be attributed to numerical inaccuracies.

At their center, due to their formulation by the non-Abelian scalar field $Q(x)$ of Eq. (1), solitons are non-Abelian and have a finite radius. The extent of the non-Abelian field's reach can be determined from the d dependence of the fine-structure constant. We therefore introduce the quantity $\alpha_{sol}(d)$ into Eq. (24) and define

$$\alpha_{sol}(d) = \frac{d}{\hbar c_0} [2m_e c_0^2 - E(d)]. \quad (27)$$

In Fig. 4, we compare $\alpha_{sol}(d)$ with the large distance approximation of perturbative QED [see Ref. [4], Eq. (7.95)],

$$\alpha(d) = \alpha_f \left\{ 1 + \frac{\alpha_f}{\sqrt{2\pi}} \frac{e^{-2d/\bar{\lambda}}}{(2d/\bar{\lambda})^{3/2}} \right\}, \quad (28)$$

with $\bar{\lambda} = \hbar/(m_e c_0)$ being the reduced Compton wavelength of the electron and the distance d between the two solitons.

Figure 4 shows the running coupling and compares it to Eq. (28) of perturbative QED. The good agreement of $\alpha_{sol}(d)$ as $d \rightarrow \infty$ shows that the long-range Coulomb field is well reproduced in the numerical calculations. The strong d dependence of $\alpha_{sol}(d)$ is, for sufficiently precise calculations, independent of the numerical approximation, as is also shown by the comparison with earlier, less accurate calculations; see Fig. 5 in Ref. [5]. Its reason is the finite extent of the solitons. It is interesting how surprisingly well this dependence agrees with the predictions of perturbation theory. Since the behavior shown in Fig. 4 is a direct consequence of the model, requiring no adjustments whatsoever, any discrepancy with the behavior of electrons would immediately indicate a flaw in the model.

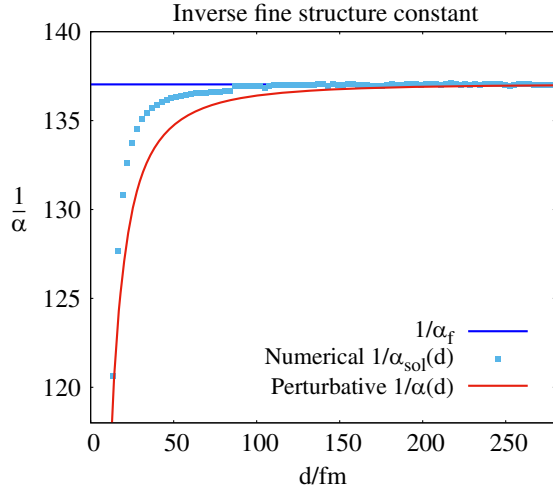


FIG. 4. Comparison between the running coupling for the solitonic dipole and perturbative QED according to Eq. (28). For the numerical determination of $\alpha_{\text{sol}}(d)$ according to Eq. (27), only the asymptotic dipole energy δE_{∞} is adjusted in the interval $[d_{<}, 280 \text{ fm}]$ with $d_{<} = 200 \text{ fm}$, whereby the choice of $d_{<}$ does not significantly affect the position of the data points.

IV. DISCUSSION

The soliton model investigated in this work can be viewed as a further development of a dual formulation of Dirac's magnetic monopoles [6,7]. Dirac's magnetic monopoles have two singularities: the central singularity at the center, which is characteristic of all point charges, and the famous Dirac string, which leads to a quantization of the magnetic charge due to the requirement that it is invisible. Following a dual transformation, Dirac's formulation thus explains the quantization of electric charge. In Wu and Yang's non-Abelian formulation of magnetic monopoles [8,9], they succeed in avoiding the Dirac string. As mentioned in Ref. [10], their ansatz is based on a suggestion by Hsu. Although they do not mention it, their gauge field can be derived from a hedgehog field consisting of unit vectors pointing in radial directions and thus makes use of the property that an S^2 can be described without singularities using radial vectors.

Our soliton model [1] is based on an additional step to eliminate the singularity at the origin; see also Fig. 1. Instead of unit vectors, radial vectors are used [see the imaginary part of Q in Eq. (1)], whose lengths approach zero at the origin and increase monotonically to a length of one as the distance from the center increases, thereby eliminating singularities. The topological solitons of this model are thus Dirac monopoles whose two singularities have been eliminated by the $SU(2)$ and $SO(3)$ formulations, respectively. The idea of defining electric and magnetic field strengths as area densities in the target space [see Eq. (4) and (8)], was already formulated by Pisello in 1977 for an S^2 field [11,12], whereas our work employs area densities on S^3 .

The Skyrme model, which describes only short-range interactions, was proposed by Skyrme to account for nuclear forces and nucleons. It is also based on an $SU(2)$ field [13,14]. Since Gauss's law does not apply to the nuclear force, the concept of field strength cannot be meaningfully defined in the Skyrme model. In contrast to the Skyrmions, topological solitons reproduce the long range behavior of QED.

Extended magnetic monopoles—regular field configurations of finite energy with long-range interactions, known as 't Hooft–Polyakov monopoles [15,16]—were discovered in the Georgi–Glashow model [17]. Forces between these monopoles are influenced by the charged Higgs field. In the Prasad–Sommerfield limit, Manton finds that the force between monopoles of different charges follows an inverse quadratic law, but the forces between monopoles of equal charge vanish [18], which distinguishes them from topological solitons as well as charged particles in general.

Electrons, as described in conventional QED, are treated as pointlike elementary particles without resolved internal structure. By contrast, the present topological solitons arise as extended, non-Abelian field configurations with finite core size. Nevertheless, within the precision of the observables investigated here, in particular the singlet-channel interaction potential and the extracted effective fine-structure constant, no measurable deviation has yet been identified that would permit a sharp phenomenological distinction between the solitonic configurations and electrons.

V. CONCLUSION

We have presented high-precision lattice simulations of an $SO(3)$ solitonic dipole. From the fitted Coulomb coefficient we extract an effective inverse solitonic fine-structure constant $\alpha_{\text{sol}}^{-1} = 137.1(1)$, within the numerical precision of the lattice consistent with the CODATA value [19] $\alpha_f^{-1} = 137.035999177(21)$. Furthermore, the effective coupling $\alpha_{\text{sol}}(d)$ is derived from the lattice interaction energies and compared with the perturbative estimate Eq. (28). $\alpha_{\text{sol}}(d)$ reproduces the expected QED vacuum polarization at intermediate separations astonishingly well and recovers the asymptotic Coulomb behavior.

The solitonic dipole model reproduces key Coulomb and QED-scale features once the electron mass scale is related to the natural solitonic length scale through r_0 , supporting the interpretation of the $SO(3)$ solitons as effective electron-like degrees of freedom in this framework.

Further investigations, including analysis of the triplet channel, are required to establish clear distinctions between solitonic and electronic configurations. Ongoing simulations aim to extract a solitonic analog of the hyperfine splitting arising from the relative orientations of the spins of the electron and the positron.

In this work, we were able to account for the results regarding the interaction of stationary solitons. A

comparison with high-energy electron scattering experiments is only possible using dynamic calculations that take into account the Lorentz contraction and the kinetic energy of moving solitons. The solitonic nature of electrons is supported primarily by the fact that electrons are extremely stable and cannot be excited. Topological solitons are topologically quantized and characterized by topological quantum numbers. The topological structure can only be resolved by solitons with opposite quantum numbers.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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- [1] M. Faber, A model for topological fermions, *Few Body Syst.* **30**, 149 (2001).
 - [2] M. Faber, A geometric model in 3+1D space-time for electrodynamic phenomena, *Universe* **8**, 73 (2022).
 - [3] M. Remoissenet, *Waves Called Solitons: Concepts and Experiments* (Springer, New York, 1996).
 - [4] Michael E. Peskin and Daniel V. Schroeder, *An Introduction To Quantum Field Theory (1st ed.)* (Perseus Books Publishing, Reading, Massachusetts, 1995), p. 842.
 - [5] J. Wabnig, J. Resch, D. Theuerkauf, F. Anmasser, and M. Faber, Numerical evaluation of a soliton pair with long-range interaction, *Universe* **11**, 113 (2025).
 - [6] P. A. M. Dirac, Quantised singularities in the electromagnetic field, *Proc. R. Soc. A* **133**, 60 (1931).
 - [7] P. A. M. Dirac, The theory of magnetic poles, *Phys. Rev.* **74**, 817 (1948).
 - [8] T. T. Wu and C. N. Yang, Some solutions of the classical isotopic gauge field equations, in *Properties of Matter Under Unusual Conditions*, edited by H. Mark and S. Fernbach (Wiley-Interscience, New York, 1969), pp. 349–354.
 - [9] T. T. Wu and C. N. Yang, Some remarks about unquantized non-abelian gauge fields, *Phys. Rev. D* **12**, 3843 (1975).
 - [10] T. T. Wu and C. N. Yang, Static sourceless gauge field, *Phys. Rev. D* **13**, 3233 (1976).
 - [11] D. Pisello, Nonlinear classical theory of electromagnetism, *Int. J. Theor. Phys.* **16**, 863 (1977).
 - [12] D. Pisello, Unified field theory with homotopic charge, *Int. J. Theor. Phys.* **17**, 143 (1978).
 - [13] T. H. R. Skyrme, A Nonlinear theory of strong interactions, *Proc. R. Soc. A* **247**, 260 (1958).
 - [14] T. H. R. Skyrme, A nonlinear field theory, *Proc. R. Soc. A* **260**, 127 (1961).
 - [15] A. M. Polyakov, Particle spectrum in quantum field theory, *JETP Lett.* **20**, 194 (1974).
 - [16] G. 't Hooft, Magnetic monopoles in unified gauge theories, *Nucl. Phys.* **B79**, 276 (1974).
 - [17] H. Georgi and S. L. Glashow, Unity of all elementary-particle forces, *Phys. Rev. Lett.* **32**, 438 (1974).
 - [18] N. Manton, The force between 't Hooft-Polyakov monopoles, *Nucl. Phys.* **B126**, 525 (1977).
 - [19] National Institute of Standards and Technology, CODATA Internationally Recommended 2022 Values of the Fundamental Physical Constants: Inverse Fine-Structure Constant (2024), https://physics.nist.gov/cgi-bin/cuu/Value?alphinv/search_for=alphainv.