

Natural solution of SUSY μ problem from modulus stabilization in modular flavor model

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We propose a solution to the supersymmetry (SUSY) μ problem within the framework of modular flavor symmetry. The explicit μ term is prohibited by modular symmetry, and an effective μ term is regenerated following the stabilization of the modulus field. We examine the stabilization mechanism of a single modulus field with the presence of SUSY breaking contributions described by the nonlinear SUSY realization scheme involving a nilpotent Goldstino $\mathbf{X}_{nl}$ superfield. A natural small μ_{eff} , significantly smaller than the SUSY scale, can result from either the expansion of typical modular forms using a small deviation parameter near the fixed point ω , or from the combined effects of suppression by powers of $q^{1/24}$ [or $(2\Im\tau)^{-1}$] along with the asymptotic suppression behavior of typical modular forms away from the fixed point $i\infty$, taking the form of appropriate power of the tiny deviation parameter. A natural small μ_{eff} can also be achieved by a weightonlike mechanism for $H_u H_d$ bilinear.

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I. INTRODUCTION

Low-energy supersymmetry (SUSY) is well motivated from a theoretical standpoint and is considered to be one of the most compelling extensions of the Standard Model. It not only address the quadratic divergence problem associated with the Higgs mass but also enables successful gauge coupling unification. Notably, the observed 125 GeV scalar, discovered by both the ATLAS [1] and CMS [2] collaborations of LHC, falls miraculously within the narrow 115–135 GeV “window” predicted by the minimal supersymmetric standard model (MSSM). This can be regarded as a strong support for low energy SUSY.

Although the framework of low energy SUSY is attractive, it still has several unsatisfactory features. In addition to the lack of conclusive signals for superparticles at the LHC, the MSSM is also bothered by the SUSY μ problem. The μ parameter, a SUSY preserving parameter in the superpotential, however, is required (without too much miraculous cancelation) to be roughly of the order of the soft SUSY breaking scale, as determined by the minimization of the MSSM Higgs potential. Many proposals have been put forward to address the μ problem, for example, by singlet extension of MSSM to the next-to-minimal supersymmetric standard model or by introducing

nonrenormalizable terms in the superpotential via Kim-Nilles mechanism [3] (or in the Kahler potential via the Giudice-Masiero mechanism [4]). Most solutions involve two steps [5]: First, the appearance of the μ parameter is prohibited, usually by some discrete or continuous symmetry. Second, the μ term is regenerated via some mechanism such as symmetry breaking. For detailed realizations of these two steps, one can refer to [5].

We propose to tackle the μ problem with a similar but different approach. The symmetry adopted to forbid the μ parameter is a modular flavor symmetry. The regeneration of the μ term is realized following the modular symmetry breaking, with the modulus stabilization mechanism accompanied by typical SUSY breaking contributions.

Modular symmetry is a geometrical symmetry of torus T^2 and it corresponds to a change of the corresponding cycle basis. Modular symmetry can naturally arise in string theory, for example, in the (exact) moduli spaces of Calabi-Yau threefolds like the quintic, and more generically in large classes of elliptic Calabi-Yau threefolds [6,7]. Recently, modular flavor symmetry was proposed in [8] to explain the flavor puzzles of the Standard Model (plus the massive neutrinos), which no longer necessitates the introduction of the flavon fields that may cause difficulties in flavor model buildings. For reviews, see [9–12]. In the modular flavor symmetry framework, the superpotential is tightly constrained by the modular symmetry, with the relevant Yukawa couplings being modular forms of a certain level N , which are functions of a modulus field. The matter superfields and the modular forms in the superpotential transform in some representations of certain

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finite modular group $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$, $N \leq 5$ (or its double cover Γ'_N). In this framework, the masses and mixings of quarks and leptons can be predicted with very few input parameters. Modular flavor models based on $\Gamma_2 \simeq S_3$ [13–15], $\Gamma_3 \simeq A_4$ [13,16–26], $\Gamma_4 \simeq S_4$ [16,27–31], $\Gamma_5 \simeq A_5$ [16,32–36], and nonholomorphic modular flavor models [37–40] had been studied in various works to explain the low energy flavor structures of the Standard Model. Modular symmetry can also be used to solve the strong CP problem [41,42]. Moreover, modular flavor symmetries can be combined with the GUT framework to further enhance the predictive power of the models, such as combining Γ_2, Γ_3 with $SU(5)$ [15,43–48], $SO(10)$ [49,50], or flipped $SU(5)$ [51–53] GUT models.

The vacuum expectation value (VEV) of the modulus field can be the only source of flavour symmetry breaking. It can also be the only source of CP symmetry breaking [42,54–57]. The VEV of the modulus field τ generally breaks the modular symmetry completely, except at three inequivalent fixed points $\tau = i\infty, i, e^{i2\pi/3}$ in the fundamental domain. The modular forms (thus the Yukawa couplings) and the fermion mass matrices are fixed once the modular flavor symmetry is fully or partially broken by the VEV of τ . From a top-down perspective, the value of modulus VEV can be determined dynamically with a modular-invariant scalar potential in 4d $N = 1$ supergravity. This is also crucial for giving nonzero masses to the modulus field, rendering it phenomenologically acceptable.

Experiences from heterotic string theory on moduli stabilization [58–60] indicate that the scalar potential favors minima at the boundary of the fundamental domain or at points inside the fundamental domain that are close to the fixed point $\omega = e^{i2\pi/3}$ with negative vacuum energy. In the

presence of additional “matter”-like fields, it is possible to obtain a dS vacua with additional uplifting contributions, such as the introduction of an anti-D3-brane that provides a positive uplifting potential, which would induce a SUSY breaking vacuum shift to a tiny positive vacuum energy. The presence of the antibranelike sources can be described in terms of a nilpotent Goldstino superfield [61] in a supersymmetric effective field theory. In the framework of non-linearly realized SUSY, the couplings of chiral, gauge and gravity superfields to the constrained nilpotent Goldstino superfield $\mathbf{X}_{nl}$ can be used to describe spontaneous SUSY breaking in supergravity [62–69] and the presence of the antibrane type sources with a supersymmetric action. Contributions from nilpotent chiral superfield can uplift the scalar potential responsible for the stabilization of the modulus field.

This paper is organized as follows. In Sec. II, we briefly review the Γ_N modular flavor symmetry. In Sec. III, we discuss the solution of SUSY μ problem using modular flavor symmetry. In Sec. IV, we discuss the stabilization of singlet modulus field in the presence of SUSY breaking contributions described by the nonlinear realized SUSY scheme with the nilpotent Goldstino $\mathbf{X}_{nl}$ superfield. In Sec. V, we discuss the regeneration of a naturally small μ_{eff} after modulus stabilization, when the modulus field is stabilized near some fixed points, such as $\omega \equiv e^{i2\pi/3}$ and $i\infty$. Section VI contains our conclusions.

II. MODULAR Γ_N SYMMETRY

The full modular group is defined as the set of linear fractional transformations acting on the complex modulus τ in the upper half complex plane

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}: \tau \rightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \text{with } a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \quad \text{Im } \tau > 0. \quad (2.1)$$

This group can be generated by two generators, S and T , which act as follows

$$S: \tau \mapsto -\frac{1}{\tau}, \quad T: \tau \mapsto \tau + 1. \quad (2.2)$$

These generators satisfy the relations

$$S^4 = 1, \quad (ST)^3 = 1, \quad S^2T = TS^2. \quad (2.3)$$

The S, T generators can be represented by the following $SL(2, \mathbb{Z})$ matrices,

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (2.4)$$

Finite modular groups Γ_N [8] are defined as $\Gamma_N = \bar{\Gamma}/\bar{\Gamma}(N)$, where $\bar{\Gamma} \equiv SL(2, \mathbb{Z})/\{\pm 1\}$ and $\bar{\Gamma}(N)$ is given by

$$\bar{\Gamma}(N) = \begin{cases} \Gamma(N), & N \geq 3 \\ \Gamma(N)/\{\pm 1\}, & N = 2 \end{cases}, \quad (2.5)$$

with $\Gamma(N)$ the principal congruence subgroup of level N

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad b = c = 0 \pmod{N}, \quad a = d = 1 \pmod{N} \right\}. \quad (2.6)$$

Finite modular groups Γ_N can be obtained by further imposing the condition $T^N = 1$ in addition to $S^2 = (ST)^3 = 1$.

Modular form $f(\tau)$ of weight $2k$ and level N is a holomorphic function of the complex modulus τ with well-defined transformation properties under the action of $\Gamma(N)$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} f(\tau), \quad \forall \gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N). \quad (2.7)$$

Modular forms of weight $2k$ and level N span a linear space of finite dimension. They can be arranged into some modular multiplets $Y_{\mathbf{r}}^{(2k)}$ that transform as irreducible representation $\mathbf{r}$ of the finite modular group Γ_N up to automorphy factor [8], i.e.,

$$Y_{\mathbf{r}}^{(2k)}(\gamma\tau) = (c\tau + d)^{2k} \rho_{\mathbf{r}}(\gamma) Y_{\mathbf{r}}^{(2k)}, \quad \forall \gamma \in \bar{\Gamma}, \quad (2.8)$$

where γ is the representative element of the coset $\gamma\bar{\Gamma}(N)$ in Γ_N , and $\rho_{\mathbf{r}}(\gamma)$ is the representation matrix of the element γ in the irreducible representation $\mathbf{r}$. The actions of the modular group on supermultiplets read

$$\phi_i(\gamma\tau) = (c\tau + d)^{-k_\phi} \rho_{ij}^{\mathbf{r}}(\gamma) \phi_j, \quad \forall \gamma \in \bar{\Gamma}, \quad (2.9)$$

where k_ϕ denotes the modular weight of ϕ , and ρ is a unitary representation of the quotient group $\bar{\Gamma}_N$.

III. NATURAL μ TERM IN MODULAR FLAVOR MODEL

As noted in [5], to solve the SUSY μ problem, the appearance of a μ term in the superpotential should be prohibited by some symmetry. We propose that the superfields H_u, H_d transform nontrivially under modular transformations, thereby prohibiting the ordinary $\mu H_u H_d$ term by modular symmetry.

The relevant Lagrangian can be given as

$$\mathcal{L} \supseteq \int d^4\theta \left[-3 \log(2\Im\tau) + \frac{H_u^\dagger H_u}{(2\Im\tau)^{k_u}} + \frac{H_d^\dagger H_d}{(2\Im\tau)^{k_d}} + \dots \right] + \int d^2\theta \left[\alpha \Lambda_{\text{SUSY}} \frac{Y_N^{(k_r)}(\tau)}{[\eta(\tau)]^{2k_3}} H_u H_d + \dots \right], \quad (3.1)$$

where the coefficient for bilinear $H_u H_d$ term is given by

$$\tilde{\mu} = \alpha \Lambda_{\text{SUSY}} \frac{Y_N^{(k_r)}(\tau)}{[\eta(\tau)]^{2k_3}}, \quad (3.2)$$

for $k_{H_u} + k_{H_d} = k_r - k_3 + 3$ to ensure the modular invariance of

$$e^G \equiv e^K |W|^2. \quad (3.3)$$

Here, $\alpha \sim \mathcal{O}(1)$ is a dimensionless parameter and Λ_{SUSY} is typically of order the SUSY scale. The Dedekind η function is a modular form with modular weight $1/2$ under modular transformation up to a phase. Its transformation under γ is given by

$$\eta(\gamma\tau) = \epsilon_\eta(\gamma) (c\tau + d)^{1/2} \eta(\tau), \quad \forall \gamma \in \bar{\Gamma}, \quad (3.4)$$

where $\epsilon_\eta(\gamma)$ depends on γ (but not on τ) and can be expressed as

$$\epsilon_\eta(\gamma) = \exp \left[\frac{\pi i}{12} \omega(a, b, c, d) - i \frac{\pi}{4} \right], \quad (3.5)$$

which satisfies $\epsilon_\eta(\gamma)^{24} = 1$ [70]. Here

$$\omega(a, b, c, d) = \left(\frac{a+d}{c} + 12s(-d, c) \right), \quad (3.6)$$

is an integer and $s(-d, c)$ is a Dedekind sum.

The expansion of the $\eta(\tau)$ function is

$$\begin{aligned} \eta(\tau) &= q^{1/24} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}, \\ &= q^{1/24} \left[1 + \sum_{n=1}^{\infty} (-1)^n (q^{n(3n-1)/2} + q^{n(3n+1)/2}) \right], \\ &= q^{1/24} [1 - q - q^2 + q^5 + q^7 - q^{12} - \dots], \quad q \equiv e^{i2\pi\tau}. \end{aligned} \quad (3.7)$$

It is evident that the Kahler potential for superfields with nontrivial modular weights, such as H_u and H_d superfields, is nonminimal. Therefore, after the modulus field τ acquires a VEV through the modulus stabilization mechanism, the H_u, H_d fields need to be normalized properly to get their canonical kinetic forms. This normalization corresponds to the following replacements

$$H_u \rightarrow H_u (2\Im\tau)^{k_{H_u}/2}, \quad H_d \rightarrow H_d (2\Im\tau)^{k_{H_d}/2}. \quad (3.8)$$

Thus, when the modulus field is stabilized, the effective μ parameter is regenerated as

$$\begin{aligned} \mu_{\text{eff}} &= \alpha \Lambda_{\text{SUSY}} \frac{Y_N^{(k_r)}(\tau_0)}{[\eta(\tau_0)]^{2k_3}} (2\Im\tau_0)^{(k_{H_u} + k_{H_d})/2} \\ &= \alpha \Lambda_{\text{SUSY}} \frac{Y_N^{(k_r)}(\tau_0)}{[\eta(\tau_0)]^{2k_3}} (2\Im\tau_0)^{(k_r - k_3 + 3)/2}. \end{aligned} \quad (3.9)$$

We need to explain why μ_{eff} is much smaller than Λ_{SUSY} and of order the electroweak scale or soft SUSY breaking scale m_{soft} . We will show that, the smallness of μ_{eff} is the consequence of large suppression factors, either originate from some power of deviation parameter that characterizes

the deviation of modulus VEV from typical fixed point, or from nontrivial power (determined by the choice of modular weights for H_u and H_d) of a small factor related to the modulus VEV. All these suppression factors are determined by the modulus stabilization mechanism, with the stabilized modulus VEVs being affected by SUSY breaking effects. The smallness of μ_{eff} can also be achieved by a weigtonlike mechanism [71] for $H_u H_d$ bilinear.

IV. STABILIZATION OF MODULUS FIELD WITH NILPOTENT GOLDSTINO $\mathbf{X}_{nl}$ SUPERFIELD

To stabilize the modulus field, we can utilize effective supergravity scalar potentials derived from a weight -3 superpotential to determine the minimum. The modulus VEV can always be restricted to the fundamental domain

$$D_H \equiv \left\{ \tau \in \mathcal{H} : -\frac{1}{2} \leq \Re \tau < \frac{1}{2}, |\tau| > 1 \right\} \cup \left\{ \tau \in \mathcal{H} : -\frac{1}{2} < \Re \tau \leq 0, |\tau| = 1 \right\}, \quad (4.1)$$

of the modular group, as any choice of the modulus VEV in the upper-half plane can be uniquely related to a $\tau \in D_H$ in the fundamental domain via a modular transformation, which is physically equivalent to such a choice. Discussions on modulus stabilization with a single modulus and multiple moduli can be found in [29,72–83].

To ensure the modular invariance and Kahler transformation invariance of

$$e^G \equiv e^K |W|^2, \quad (4.2)$$

the superpotential for the modulus should take the form

$$W(\tau) = c_1^3 \frac{H(\tau)}{[\eta(\tau)]^6}, \quad (4.3)$$

with the Kahler potential

$$K_{\text{mod}} = -3 \log(2\Im \tau), \quad (4.4)$$

in the unit $M_{\text{Pl}} = 1$. Here, $c_1 M_{\text{Pl}}$ denotes the typical mass scale for the superpotential, where c_1 is a dimensionless number.

The modular invariant holomorphic function $H(\tau)$ is given by

$$H(\tau)_{m,n} = (j(\tau) - 1728)^{m/2} j(\tau)^{n/3} \mathcal{P}(j(\tau)) \\ \equiv \left(\frac{G_6(\tau)}{[\eta(\tau)]^{12}} \right)^m \left(\frac{G_4(\tau)}{[\eta(\tau)]^8} \right)^n \mathcal{P}(j(\tau)), \quad (4.5)$$

where $\mathcal{P}(j(\tau))$ denotes a polynomial with respect to $j(\tau)$ and m, n are non-negative integers [58]. The definitions of $G_4(\tau)$ and $G_6(\tau)$ are given in the Appendix B. The Klein j

function is defined as a modular form of zero weight

$$j(\tau) \equiv \frac{3^6 5^3 G_4(\tau)^3}{\pi^{12} [\eta(\tau)]^{24}} \\ \approx 744 + q^{-1} + 196884q + 21493760q^2 \\ + 864299970q^3 + \mathcal{O}(q^4), \quad (4.6)$$

with $j(i\infty)$ diverges, $j(\omega) = 0$ and $j(i) = 1728$. Note that $j(\tau)$ has a triple zero at $\tau = \omega$, and $j(\tau) - 1728$ has a double zero at $\tau = i$. The derivative $j'(\tau)$ vanishes only at these values: $j'(\omega) = j'(i) = 0$ [84].

We should note that, in such a stabilization mechanism, the fixed points $z = i, \omega \equiv e^{i2\pi/3}$ are always the extrema of the scalar potential, although they may not be the global minimum. With proper SUSY breaking contributions, possibly from the “matter”-like sector, the stabilized modulus value, which is located at/near the fixed points $z_{FP} = \omega, i$, can slightly deviate from the original global minimum. The minimum of the scalar potential can also possibly be uplifted to dS (de Sitter) type. In fact, the classical 10d supergravity Maldacena-Nunez no-go result [85] suggests that any constructions leading to a dS vacuum must include nonclassical ingredients, such as D-branes, O-planes, or quantum corrections. To describe the scenario in which SUSY is linearly realized in a gravitational sector and nonlinearly realized in the uplifting SUSY breaking sector (for example, from anti-D-branes), the formalism of nonlinearly realized SUSY as formulated by [61] can be used for such a system. It was proved in [62] that, with the nonlinear SUSY realization scheme, the supergravity form is stable even if the supermultiplets in the SUSY breaking sector are split badly enough to allow some of its members to be integrated out while the others are not.

When SUSY is spontaneously broken in supergravity, the Goldstino is eaten by the gravitino via the super-Higgs mechanism. When such a breaking happens at low energies compared with the Planck scale, the Goldstino couplings can be conveniently described by a (constrained) independent superfield in the supergravity effective action. The Goldstino provides a way for an arbitrary nonsupersymmetric low-energy theory to realize SUSY nonlinearly. The emergence of the constraints may also be viewed as an effective field theory limit in which certain mass parameters become very large and the corresponding modes decouple.

The coupling of nonlinearly realized SUSY with a nilpotent chiral Goldstino superfield, subjected to constraints $\mathbf{X}_{nl}^2 = 0$, can be described by

$$\mathcal{L} \supseteq \int d^4\theta \mathbf{X}_{nl}^\dagger \mathbf{X}_{nl} + \left[\int d^2\theta f_0 \mathbf{X}_{nl} + \text{H.c.} \right]. \quad (4.7)$$

The nilpotent constraints for the chiral Goldstino superfield can be solved to be

$$\mathbf{X}_{nl} = \frac{\psi_X \psi_X}{2F_X} + \sqrt{2}\theta\psi_X + \theta^2 F_X, \quad (4.8)$$

where all the fields are the functions of $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar{\theta}$. The lowest component of $\mathbf{X}_{nl}$ is eliminated by the nilpotent constraints $\mathbf{X}_{nl}^2 = 0$ and is no longer an independent complex scalar field. This leaves only the Goldstino field ψ_X and an independent auxiliary field F_X in $\mathbf{X}_{nl}$ superfield. The Goldstino field ψ_X , which is the fermionic component of $\mathbf{X}_{nl}$, eventually mixes with the gravitino to give it a mass through the super-Higgs mechanism. The lowest component of $\mathbf{X}_{nl}$, which is denoted as X_{nl} for the Goldstino bilinear $\psi_X \psi_X$, obviously satisfies $X_{nl}^2 = 0$ due to its anti-commuting Grassman variable nature. The chiral superfield $\mathbf{X}_{nl}$ can be seen as the long-distance expression of the microscopic superfield $\mathbf{X}$ that satisfies $\bar{D}^\alpha \mathcal{J}_{\alpha\dot{\alpha}} = D_\alpha \mathbf{X}$ for Ferrara-Zumino multiplet $\mathcal{J}_{\alpha\dot{\alpha}}$ [61], which can be used even when SUSY is spontaneously broken and can be followed consistently along the renormalization group flow down to the deep IR. As noted in [63], for the anti-D3-brane in the Kachru-Kalosh-Linde-Trivedi [86] scenario, the representation in terms of $\mathbf{X}_{nl}$ is very convenient and enough to capture all the anti-D3-brane degrees of freedoms since it allows one to treat the effect of anti-D3-brane in terms of standard supergravity couplings of matter and moduli superfields to the nilpotent Goldstino superfield. From the perspective of low energy effective field theory, regardless of the UV origin of SUSY breaking, such a nonlinear realization scheme is also convenient for discussions on modulus stabilization and vacuum uplifting.

When coupled to supergravity, the most general form (at the two-derivative level) for the action in superspace is given by

$$\mathcal{L} \supseteq \int d^2\Theta d^2\bar{\Theta} \left[\frac{3}{8} (\bar{D}^2 - 8\mathcal{R}) e^{-K/3} + W \right] + \text{H.c.}, \quad (4.9)$$

with

$$K = \mathbf{X}_{nl}^\dagger \mathbf{X}_{nl}, \quad W = f_0 \mathbf{X}_{nl} + c_1^3, \quad (4.10)$$

where without loss of generality we take f_0 and c_1 to be real. Here $\mathcal{E}$ is the chiral density, $\mathcal{R}$ is the superspace curvature, and $\bar{D}^2 - 8\mathcal{R}$ is the covariant generalization of the chiral projection operators $D^\gamma D_\gamma$ in supergravity [87].

Integrating out the auxillary field F_X gives $F_X = -(W_{\mathbf{X}_{nl}})^* = -f_0$. To accommodate the modular flavor symmetry, the Kahler potential and superpotential need to be adapted to

$$K \supseteq -3 \log(2\Im\tau) + \frac{\mathbf{X}_{nl}^\dagger \mathbf{X}_{nl}}{(2\Im\tau)^{k_X}},$$

$$W \supseteq (f_0 \mathbf{X}_{nl} [\eta(\tau)]^{2k_X} + c_1^3) \frac{H(\tau)}{[\eta(\tau)]^6}. \quad (4.11)$$

To ensure the modular invariance of the superpotential, the modular weight k_X should satisfy $k_X \in 12\mathbb{Z}$. The supergravity scalar potential can be derived from

$$V = e^K [K^{\alpha\bar{\beta}} \nabla_\alpha W \nabla_{\bar{\beta}} \bar{W} - 3W\bar{W}], \quad (4.12)$$

where the Kahler-covariant derivative of the superpotential is

$$\nabla_\alpha W = W_\alpha + K_\alpha W. \quad (4.13)$$

From the Kahler potential, we can calculate the inverse Kahler metric

$$K^{\alpha\bar{\beta}} = \begin{pmatrix} \frac{(2\Im\tau)^2}{3} \left[1 - \frac{k_X}{3} \frac{X_{nl}^* X_{nl}}{(2\Im\tau)^{k_X}} \right] & \frac{i(2\Im\tau)k_X X_{nl}}{3} \\ -\frac{i(2\Im\tau)k_X X_{nl}^*}{3} & (2\Im\tau)^{k_X} \left[1 + \frac{k_X^2 (X_{nl}^* X_{nl})}{3(2\Im\tau)^{k_X}} \right] \end{pmatrix}. \quad (4.14)$$

Here the indices $\alpha, \beta \sim (\tau, X_{nl})$ and the nilpotency of X_{nl} is used in the calculation.

From the general expression of the supergravity scalar potential in (4.12), the scalar potential for the modulus field τ and X_{nl} can be calculated as

$$V(X_{nl}, X_{nl}^*, \tau, \tau^*) = \frac{e^{\frac{|X_{nl}|^2}{(2\Im\tau)^{k_X}}}}{(2\Im\tau)^3 |\eta(\tau)|^{12}} \left\{ \frac{(2\Im\tau)^2}{3} \left(1 - \frac{k_X}{3} \frac{|X_{nl}|^2}{(2\Im\tau)^{k_X}} \right) |\tilde{W}_M \tilde{c}_M + H(\tau) \tilde{W}_T|^2 \right. \\ \left. + (2\Im\tau)^{k_X} \left[1 + \frac{k_X^2 |X_{nl}|^2}{3(2\Im\tau)^{k_X-1}} \right] |H(\tau) \tilde{W}_X|^2 + 2\Re \left[\frac{i(2\Im\tau)k_X X_{nl}}{3} (\tilde{W}_X H(\tau))^* (\tilde{W}_M \tilde{c}_M + H(\tau) \tilde{W}_T) \right] \right. \\ \left. - 3|\tilde{c}_M|^2 |H(\tau)|^2 \right\}, \quad (4.15)$$

where

$$\begin{aligned}
\tilde{W}_M &\equiv H'(\tau) - \frac{3iH(\tau)}{2\pi} \hat{G}_2(\tau, \tau^*), \\
\tilde{W}_T &\equiv k_X f_0 X_{nl} [\eta(\tau)]^{2k_X} \left[\frac{i}{2\pi} \hat{G}_2(\tau, \tau^*) + \frac{i}{2\Im\tau} + \frac{ic_1^3 X_{nl}^*}{f_0 (2\Im\tau)^{k_X+1} [\eta(\tau)]^{2k_X}} \right], \\
\tilde{c}_M &\equiv f_0 X_{nl} [\eta(\tau)]^{2k_X} + c_1^3, \\
\tilde{W}_X &\equiv f_0 [\eta(\tau)]^{2k_X} \left(1 + \frac{|X_{nl}|^2}{(2\Im\tau)^{k_X}} \right) + \frac{X_{nl}^* c_1^3}{(2\Im\tau)^{k_X}}.
\end{aligned} \tag{4.16}$$

The constraint $X_{nl}^2 = 0$ for the lowest component of $\mathbf{X}_{nl}$ is used to eliminate some of the high order X_{nl} terms. We also use the following expression for the derivative of the Dedekind η function

$$\frac{\eta'(\tau)}{\eta(\tau)} = \frac{i}{4\pi} G_2(\tau), \tag{4.17}$$

where the holomorphic Eisenstein series G_2 , which is a quasimodular form

$$G_2(\gamma\tau) = (c\tau + d)^2 G_2(\tau) - 2\pi ic(c\tau + d), \tag{4.18}$$

is related to its weight two nonholomorphic counterpart $\hat{G}_2(\tau, \bar{\tau})$ as

$$\hat{G}_2(\tau, \tau^*) = G_2(\tau) - \frac{\pi}{\Im\tau}. \tag{4.19}$$

From the definition of $H(\tau)$, its derivative with respect to τ can be calculated to be

$$\begin{aligned}
H'(\tau) &= \{j(\tau)^{n/3} [j(\tau) - 1728]^{m/2} \mathcal{P}(j(\tau))\}', \\
&= \left\{ \frac{n}{3} j(\tau)^{n/3-1} [j(\tau) - 1728]^{m/2} \mathcal{P}(j(\tau)) \right. \\
&\quad + \frac{m}{2} j(\tau)^{n/3} [j(\tau) - 1728]^{m/2-1} \mathcal{P}(j(\tau)) \\
&\quad \left. + j(\tau)^{n/3} [j(\tau) - 1728]^{m/2} \mathcal{P}'(j(\tau)) \right\} j'(\tau).
\end{aligned} \tag{4.20}$$

For nonvanishing values of $H(\tau)$, we have

$$\begin{aligned}
H'(\tau) &= H(\tau) j'(\tau) \left[\frac{n}{3} \frac{1}{j(\tau)} + \frac{m}{2} \frac{1}{j(\tau) - 1728} \right. \\
&\quad \left. + \frac{1}{\mathcal{P}(j(\tau))} \frac{\partial \mathcal{P}(j(\tau))}{\partial j(\tau)} \right],
\end{aligned} \tag{4.21}$$

making obvious that $H(\tau)$ can be factored out from $\tilde{W}_M$ as

$$\begin{aligned}
\tilde{W}_M &= H(\tau) \left(j'(\tau) \left[\frac{n}{3} \frac{1}{j(\tau)} + \frac{m}{2} \frac{1}{j(\tau) - 1728} \right. \right. \\
&\quad \left. \left. + \frac{1}{\mathcal{P}(j(\tau))} \frac{\partial \mathcal{P}(j(\tau))}{\partial j(\tau)} \right] - \frac{3i}{2\pi} \hat{G}_2(\tau, \tau^*) \right).
\end{aligned}$$

Substituting the factorized form of $\tilde{W}_M$ back into the scalar potential, it becomes apparent that the $|H(\tau)|^2$ factor can be extracted from the expression when $H(\tau) \neq 0$. This observation plays an important role in locating the extrema of the scalar potential.

For $m = n = 0$, the trivial case

$$H'(\tau) = \mathcal{P}'(j(\tau)) j'(\tau), \tag{4.22}$$

vanishes at $z = \omega, i$ because $j'(\omega) = j'(i) = 0$.

For $m = 1$ or $n = 1, 2$, from the results

$$\begin{aligned}
\frac{j'(i)}{[j(i) - 1728]^{1/2}} &= -i2\pi \frac{[E_4(i)]^2}{[\eta(i)]^{12}}, & \frac{j'(\omega)}{[j(\omega)]^{1/3}} &= -i2\pi \frac{E_4(\omega)E_6(\omega)}{[\eta(\omega)]^{16}} = 0, \\
\frac{j'(\omega)}{[j(\omega)]^{2/3}} &= -i2\pi \frac{E_6(\omega)}{[\eta(\omega)]^8},
\end{aligned} \tag{4.23}$$

we can see that $H'(i)$ and $H'(\tau)$ are finite with such choices of m, n , within which we use

$$\frac{\partial}{\partial \tau} j(\tau) = -i2\pi \frac{E_6(\tau)}{E_4(\tau)} j(\tau) = -i2\pi \frac{E_6(\tau)[E_4(\tau)]^2}{[\eta(\tau)]^{24}}, \tag{4.24}$$

and the relations [58]

$$j(\tau) - 1728 = \left(\frac{E_6(\tau)}{[\eta(\tau)]^{12}} \right)^2, \quad j(\tau) = \left(\frac{E_4(\tau)}{[\eta(\tau)]^8} \right)^3, \tag{4.25}$$

as well as the values of η function at $z = i, \omega$

$$\eta(i) = \frac{\Gamma(\frac{1}{4})}{2\pi^{3/4}}, \quad \eta(\omega) = e^{-i\frac{\pi}{24}} \frac{3^{\frac{1}{24}} [\Gamma(\frac{1}{3})]^{\frac{3}{2}}}{2\pi}, \quad (4.26)$$

and some special value for $E_4(\tau)$ and $E_6(\tau)$ [70]

$$E_4(i) = \frac{48[\Gamma(\frac{5}{4})]^4}{\pi^2[\Gamma(\frac{3}{4})]^4}, \quad E_4(\omega) = 0, \quad E_6(\omega) = \frac{729[\Gamma(\frac{4}{3})]^6}{2\pi^3[\Gamma(\frac{2}{3})]^6}. \quad (4.27)$$

For $m > 1$ and $n > 2$, it is obvious that $H'(\tau)$ is not singular at zero points of $H(\tau)$, therefore it vanishes at i and ω , as $j'(i) = j'(\omega) = 0$.

So, we have

$$H'(i) = \begin{cases} -i\pi(12)^n \frac{[E_4(i)]^2}{[\eta(i)]^{12}} P(1728), & m = 1 \\ 0, & m \neq 1 \end{cases},$$

$$H'(\omega) = \begin{cases} -\frac{i2\pi}{3} (-12)^{3m/2} \frac{E_6(\omega)}{[\eta(\omega)]^8} P(0), & n = 1 \\ 0, & n \neq 1 \end{cases}. \quad (4.28)$$

The condition $H(\tau) = 0$ can be reduced to

$$j(\tau)^{n/3} = 0, \quad \text{or} \quad [j(\tau) - 1728]^{m/2} = 0, \quad \text{or} \quad \mathcal{P}(j(\tau)) = 0. \quad (4.29)$$

The corresponding solutions are $\tau = i, \omega$ (for $m, n \neq 0$) and $\tau = \tilde{\tau}_i$ that satisfies $P(j(\tilde{\tau}_i)) = 0$. Whereas $H'(\tau)$ typically vanishes at $\tau = i$ and $\tau = \omega$ (barring special choices of m and n), it generally does not vanish at $\tau = \tilde{\tau}_i$, the point where $P(j(\tilde{\tau}_i)) = 0$. However, for some special choices of $P(j(\tilde{\tau}))$, $H'(\tau)$ can still vanish at $\tau = \tilde{\tau}_0$. For example, when

$$P(j(\tau)) = \prod_i (j(\tau) - j(\tilde{\tau}_i))^2, \quad (4.30)$$

then $\tau = \tilde{\tau}_0$ acts as a multiple root of $P(j(\tilde{\tau})) = 0$ and $H'(\tau)$ vanishes at such roots. We will discuss such interesting possibilities in the next subsection.

The generic, complicated form of the scalar potential may obscure the discussions regarding the stabilization mechanism. Therefore, we would like to concentrate on its typical simplification forms.

A. The case with $k_X = 0$

When $k_X = 0$, the scalar potential can be simplified significantly. It is evident that, due to the presence of the $K_\alpha W$ terms within the Kahler-covariant derivative of the superpotential $\nabla_\alpha W$, the scalar potential still includes nontrivial interference terms between the X_{nl} sector and the modulus field τ sector, despite the fact that the Kahler metric $K^{\alpha\bar{\beta}}$ is diagonal in this case.

The potential can then be written as

$$V(X_{nl}, X_{nl}^*, \tau, \tau^*) = \frac{e^{|X_{nl}|^2}}{(2\Im\tau)^3 |\eta(\tau)|^{12}} \left\{ \frac{(2\Im\tau)^2}{3} |\tilde{W}_M \tilde{c}_M|^2 + |H(\tau)|^2 [f_0(1 + |X_{nl}|^2) + X_{nl}^* c_1^3]^2 - 3|\tilde{c}_M|^2 |H(\tau)|^2 \right\}$$

$$= \frac{1}{(2\Im\tau)^3 |\eta(\tau)|^{12}} \left\{ \frac{(2\Im\tau)^2}{3} |\tilde{W}_M|^2 (f_0^2 |X_{nl}|^2 + c_1^6 + 2\Re[f_0 X_{nl} c_1^3]) + |H(\tau)|^2 (f_0^2 + 2f_0^2 |X_{nl}|^2 \right.$$

$$+ 2\Re[f_0 c_1^3 X_{nl}^*] + |X_{nl}|^2 c_1^6) - 3|H(\tau)|^2 (f_0^2 |X_{nl}|^2 + c_1^6 + 2\Re[f_0 X_{nl} c_1^3]) \left. \right\}$$

$$+ \frac{|X_{nl}|^2}{(2\Im\tau)^3 |\eta(\tau)|^{12}} \left\{ \frac{(2\Im\tau)^2}{3} |\tilde{W}_M|^2 c_1^6 + |H(\tau)|^2 f_0^2 - 3c_1^6 |H(\tau)|^2 \right\}, \quad (4.31)$$

where we adopt

$$\tilde{W}_M = H'(\tau) - \frac{3iH(\tau)}{2\pi} \hat{G}_2(\tau, \tau^*),$$

$$\tilde{c}_M = f_0 X_{nl} + c_1^3, \quad (4.32)$$

from the definitions in (4.16).

In this scenario, the modulus VEV can be determined by the extreme conditions of the scalar potential

$$\frac{\partial}{\partial \tau} V(X_{nl}, X_{nl}^*, \tau, \tau^*) = \frac{\partial}{\partial \tau^*} V(X_{nl}, X_{nl}^*, \tau, \tau^*) = 0, \quad (4.33)$$

$$\frac{\partial}{\partial X_{nl}} V(X_{nl}, X_{nl}^*, \tau, \tau^*) = \frac{\partial}{\partial X_{nl}^*} V(X_{nl}, X_{nl}^*, \tau, \tau^*) = 0. \quad (4.34)$$

Given that the scalar potential is real, the two equations on the right follow directly from those on the left via complex conjugation. To determine if a local extremum is a minimum, we need to assess the positive definiteness of the 4×4 Hessian matrix

$$(H_f)_{ij} \equiv \frac{\partial^2 V}{\partial x_i \partial x_j}, \quad \text{for } x_i = \Re\tau, \Im\tau, \Re X_{nl}, \Im X_{nl}, \quad (4.35)$$

at that extremum. The elements of the Hessian matrix can be rewritten by

$$\begin{aligned}\frac{\partial^2}{\partial x_1^2} V &= 2 \left(\frac{\partial^2}{\partial \tau \partial \tau^*} + \Re \frac{\partial^2}{\partial \tau \partial \tau} \right) V(X_{nl}, X_{nl}^*, \tau, \tau^*), \\ \frac{\partial^2}{\partial x_2^2} V &= 2 \left(\frac{\partial^2}{\partial \tau \partial \tau^*} - \Re \frac{\partial^2}{\partial \tau \partial \tau} \right) V(X_{nl}, X_{nl}^*, \tau, \tau^*), \\ \frac{\partial^2}{\partial x_1 \partial x_2} V &= -2 \Im \frac{\partial^2}{\partial \tau \partial \tau} V(X_{nl}, X_{nl}^*, \tau, \tau^*),\end{aligned}\tag{4.36}$$

with similar terms for X_{nl} and X_{nl}^* in terms of ρ and θ

$$\begin{aligned}\frac{\partial^2}{\partial x_3^2} V &= 2 \left(\frac{\partial^2}{\partial X_{nl} \partial X_{nl}^*} + \Re \frac{\partial^2}{\partial X_{nl} \partial X_{nl}} \right) V(X_{nl}, X_{nl}^*, \tau, \tau^*) \\ &= \left(\cos^2 \theta \frac{\partial^2}{\partial \rho^2} - 2 \frac{\cos \theta \sin \theta}{\rho} \frac{\partial^2}{\partial \rho \partial \theta} + \frac{\sin^2 \theta}{\rho^2} \frac{\partial^2}{\partial \theta^2} + \frac{\sin^2 \theta}{\rho} \frac{\partial}{\partial \rho} + 2 \frac{\cos \theta \sin \theta}{\rho^2} \frac{\partial}{\partial \theta} \right) V, \\ \frac{\partial^2}{\partial x_4^2} V &= 2 \left(\frac{\partial^2}{\partial X_{nl} \partial X_{nl}^*} - \Re \frac{\partial^2}{\partial X_{nl} \partial X_{nl}} \right) V(X_{nl}, X_{nl}^*, \tau, \tau^*) \\ &= \left(\sin^2 \theta \frac{\partial^2}{\partial \rho^2} + 2 \frac{\sin \theta \cos \theta}{\rho} \frac{\partial^2}{\partial \rho \partial \theta} + \frac{\cos^2 \theta}{\rho^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cos^2 \theta}{\rho} \frac{\partial}{\partial \rho} - 2 \frac{\sin \theta \cos \theta}{\rho^2} \frac{\partial}{\partial \theta} \right) V, \\ \frac{\partial^2}{\partial x_3 \partial x_4} V &= -2 \Im \frac{\partial^2}{\partial X_{nl} \partial X_{nl}} V(X_{nl}, X_{nl}^*, \tau, \tau^*) \\ &= \left(\sin \theta \cos \theta \frac{\partial^2}{\partial \rho^2} + \frac{\cos 2\theta}{\rho} \frac{\partial^2}{\partial \rho \partial \theta} - \frac{\sin \theta \cos \theta}{\rho^2} \frac{\partial^2}{\partial \theta^2} - \frac{\sin \theta \cos \theta}{\rho} \frac{\partial}{\partial \rho} - \frac{\cos 2\theta}{\rho^2} \frac{\partial}{\partial \theta} \right) V.\end{aligned}\tag{4.37}$$

Other second derivatives of $V(X_{nl}, X_{nl}^*, \tau, \tau^*)$ with respect to τ, τ^* and ρ, θ are given by

$$\begin{aligned}\frac{\partial^2}{\partial x_1 \partial x_3} V &= \left(\frac{\partial^2}{\partial \tau \partial X_{nl}} + \frac{\partial^2}{\partial \tau^* \partial X_{nl}^*} + \frac{\partial^2}{\partial \tau \partial X_{nl}^*} + \frac{\partial^2}{\partial \tau^* \partial X_{nl}} \right) V(X_{nl}, X_{nl}^*, \tau, \tau^*) \\ &= \left[\cos \theta \left(\frac{\partial^2}{\partial \tau \partial \rho} + \frac{\partial^2}{\partial \tau^* \partial \rho} \right) - \frac{\sin \theta}{\rho} \left(\frac{\partial^2}{\partial \tau \partial \theta} + \frac{\partial^2}{\partial \tau^* \partial \theta} \right) \right] V(X_{nl}, X_{nl}^*, \tau, \tau^*), \\ \frac{\partial^2}{\partial x_2 \partial x_4} V &= - \left(\frac{\partial^2}{\partial \tau \partial X_{nl}} + \frac{\partial^2}{\partial \tau^* \partial X_{nl}^*} - \frac{\partial^2}{\partial \tau \partial X_{nl}^*} - \frac{\partial^2}{\partial \tau^* \partial X_{nl}} \right) V(X_{nl}, X_{nl}^*, \tau, \tau^*) \\ &= - \left[i \sin \theta \left(\frac{\partial^2}{\partial \tau \partial \rho} - \frac{\partial^2}{\partial \tau^* \partial \rho} \right) + i \frac{\cos \theta}{\rho} \left(\frac{\partial^2}{\partial \tau \partial \theta} - \frac{\partial^2}{\partial \tau^* \partial \theta} \right) \right] V(X_{nl}, X_{nl}^*, \tau, \tau^*), \\ \frac{\partial^2}{\partial x_1 \partial x_4} V &= -i \left(\frac{\partial^2}{\partial \tau \partial X_{nl}} - \frac{\partial^2}{\partial \tau^* \partial X_{nl}^*} + \frac{\partial^2}{\partial \tau \partial X_{nl}^*} - \frac{\partial^2}{\partial \tau^* \partial X_{nl}} \right) V(X_{nl}, X_{nl}^*, \tau, \tau^*) \\ &= \left[\sin \theta \left(\frac{\partial^2}{\partial \tau \partial \rho} + \frac{\partial^2}{\partial \tau^* \partial \rho} \right) + \frac{\cos \theta}{\rho} \left(\frac{\partial^2}{\partial \tau \partial \theta} + \frac{\partial^2}{\partial \tau^* \partial \theta} \right) \right] V(X_{nl}, X_{nl}^*, \tau, \tau^*), \\ \frac{\partial^2}{\partial x_2 \partial x_3} V &= i \left(\frac{\partial^2}{\partial \tau \partial X_{nl}} - \frac{\partial^2}{\partial \tau^* \partial X_{nl}^*} - \frac{\partial^2}{\partial \tau \partial X_{nl}^*} + \frac{\partial^2}{\partial \tau^* \partial X_{nl}} \right) V(X_{nl}, X_{nl}^*, \tau, \tau^*) \\ &= \left[i \cos \theta \left(\frac{\partial^2}{\partial \tau \partial \rho} - \frac{\partial^2}{\partial \tau^* \partial \rho} \right) - i \frac{\sin \theta}{\rho} \left(\frac{\partial^2}{\partial \tau \partial \theta} - \frac{\partial^2}{\partial \tau^* \partial \theta} \right) \right] V(X_{nl}, X_{nl}^*, \tau, \tau^*).\end{aligned}\tag{4.38}$$

The analytical expression for each element of the Hessian matrix is highly complex and only takes a simplified form at specific points, such as the fixed points ω and i . Since this study does not focus on the fixed points, we evaluate the positivity of the eigenvalues for the Hessian matrix at the extrema numerically, rather than through analysis of its analytical form, in the subsequent discussions.

The extreme condition (4.33) $V_\tau = 0$ can be calculated as

$$\begin{aligned} \frac{\partial}{\partial \tau} V = & \frac{\frac{3i}{2\pi} \hat{G}_2(\tau, \tau^*)}{(2\Im\tau)^3 |\eta(\tau)|^{12}} \left\{ \frac{(2\Im\tau)^2}{3} |\tilde{W}_M \tilde{c}_M|^2 + |H(\tau)|^2 |f_0(1 + |X_{nl}|^2) + X_{nl}^* c_1^3|^2 - 3|\tilde{c}_M|^2 |H(\tau)|^2 \right\} \\ & + \frac{1}{(2\Im\tau)^3 |\eta(\tau)|^{12}} \left\{ \frac{-4i\Im\tau}{3} |\tilde{W}_M \tilde{c}_M|^2 + \frac{(2\Im\tau)^2}{3} |\tilde{c}_M|^2 \tilde{W}_M^* \tilde{W}_M' + H(\tau)^* H'(\tau) (|f_0(1 + |X_{nl}|^2) + X_{nl}^* c_1^3|^2 - 3|\tilde{c}_M|^2) \right\} \\ & + \frac{\frac{3i}{2\pi} \hat{G}_2(\tau, \tau^*) |X_{nl}|^2}{(2\Im\tau)^3 |\eta(\tau)|^{12}} \left\{ \frac{(2\Im\tau)^2}{3} |\tilde{W}_M \tilde{c}_M|^2 + |H(\tau)|^2 (f_0^2 - 3|\tilde{c}_M|^2) \right\} + \frac{|X_{nl}|^2}{(2\Im\tau)^3 |\eta(\tau)|^{12}} \left\{ \frac{-4i\Im\tau}{3} |\tilde{W}_M \tilde{c}_M|^2 \right. \\ & \left. + \frac{(2\Im\tau)^2}{3} |\tilde{c}_M|^2 \tilde{W}_M^* \tilde{W}_M' + H(\tau)^* H'(\tau) (f_0^2 - 3|\tilde{c}_M|^2) \right\}, \end{aligned} \quad (4.39)$$

and its complex conjugate for V_{τ^*} .

Within this expression, $\tilde{W}_M'$ is given by

$$\tilde{W}_M' = H''(\tau) - \frac{3iH'(\tau)}{2\pi} \hat{G}_2(\tau, \tau^*) - \frac{3iH(\tau)}{2\pi} \hat{G}_2'(\tau, \tau^*). \quad (4.40)$$

When the value of τ satisfies $H(\tau) \neq 0$, $\tilde{W}_M'$ can be rewritten as

$$\begin{aligned} \tilde{W}_M' = & H(\tau) [j'(\tau)]^2 \left[\frac{n}{3} \frac{1}{j(\tau)} + \frac{m}{2} \frac{1}{j(\tau) - 1728} + \frac{1}{\mathcal{P}(j(\tau))} \frac{\partial \mathcal{P}(j(\tau))}{\partial j(\tau)} \right]^2 \\ & + H(\tau) \left\{ j'(\tau) \left[\frac{n}{3} \frac{1}{j(\tau)} + \frac{m}{2} \frac{1}{j(\tau) - 1728} + \frac{1}{\mathcal{P}(j(\tau))} \frac{\partial \mathcal{P}(j(\tau))}{\partial j(\tau)} \right] \right\}' \\ & - H(\tau) \frac{3i}{2\pi} \hat{G}_2(\tau, \tau^*) [j'(\tau)]^2 \left[\frac{n}{3} \frac{1}{j(\tau)} + \frac{m}{2} \frac{1}{j(\tau) - 1728} + \frac{1}{\mathcal{P}(j(\tau))} \frac{\partial \mathcal{P}(j(\tau))}{\partial j(\tau)} \right] - \frac{3iH(\tau)}{2\pi} \hat{G}_2'(\tau, \tau^*), \end{aligned} \quad (4.41)$$

from (4.20). The derivative of $\hat{G}_2(\tau, \tau^*)$ with respect to τ is given as follows

$$\hat{G}_2'(\tau, \tau^*) = \frac{i\pi^3}{18} (E_2(\tau)^2 - E_4(\tau)) - i \frac{2\pi}{(2\Im\tau)^2}. \quad (4.42)$$

From Eqs. (4.22) and (4.41), one can check that $|H(\tau)|^2$ acts as a positive multiplicative factor in V_τ when $H(\tau) \neq 0$. In addition to the solutions of $V_\tau/|H(\tau)|^2 = 0$ (for $H(\tau) \neq 0$), the constraints $\tilde{W}_M' = 0$ can be satisfied also by some solutions of $H(\tau) = 0$, which we need to check point by point.

From Eq. (4.20), we have the following discussions on the condition $H'(\tau) = 0$ when $H(\tau) = 0$:

(i) The case $\tau = i$ is a solution of $H(\tau) = 0$:

When $\tau = i$ is a solution of $H(\tau) = 0$ for $m > 0$, the condition $H'(\tau) = 0$ can be satisfied automatically for $m \neq 1$, see Eq. (4.28). It can also be satisfied when $\mathcal{P}(1728) = 0$ for $m = 1$, requiring $\mathcal{P}(j(\tau))$ to take the form

$$\mathcal{P}(j(\tau)) \propto (j(\tau) - 1728)^k, \quad k \in \mathbb{N}. \quad (4.43)$$

Such factor of $\mathcal{P}(j(\tau))$ can be absorbed into the $(j(\tau) - 1728)^{m/2}$ term in $H(\tau)$ after the redefinition of “ m ,” rendering the redefined $\mathcal{P}(1728) \neq 0$. So, we conclude that $H'(i) = 0$ can be satisfied for $m > 1$. Note that $H'(i) = 0$ for $m = 0$, although $\tau = i$ is in general not a solution of $H(\tau) = 0$ when $m = 0$.

(ii) The case $\tau = \omega$ is a solution of $H(\tau) = 0$:

Similarly, when $\tau = \omega$ is a solution of $H(\tau) = 0$ for $n > 0$, the condition $H'(\tau) = 0$ can be satisfied automatically for $n \neq 1$. It can also be satisfied when $\mathcal{P}(0) = 0$ for $n = 1$, which requires $\mathcal{P}(j(\tau))$ to take the form

$$\mathcal{P}(j(\tau)) \propto [j(\tau)]^k, \quad k \in \mathbb{N}. \quad (4.44)$$

After the redefinition of “ n ,” we conclude that the condition $H'(\omega) = 0$ can be satisfied for $n > 1$. Note also $H'(\omega) = 0$ when $n = 0$, although $\tau = \omega$ is in general not a solution of $H(\tau) = 0$ when $n = 0$.

(iii) The case $\tau = \tilde{\tau}_i$ with $\mathcal{P}(j(\tilde{\tau}_i)) = 0$:

The condition $H'(\tau) = 0$ can be satisfied for $\mathcal{P}'(j(\tilde{\tau}_i)) = 0$, which requires $\tau = \tilde{\tau}_i$ being a root of

$P(j(\tau))$ with multiplicity two or greater. Therefore,

$$P(j(\tau)) \propto \pm g(j(\tau)) \prod_i (j(\tau) - j(\tilde{\tau}_i))^{k_i}, \quad k_i \in \mathbb{N}, \quad (4.45)$$

with $k_i \geq 2$ and $g(j(\tau))$ an arbitrary positive polynomial function of $j(\tau)$.

Before we discuss the condition $H''(\tau) = 0$ when $H(\tau) = 0$, we need to know the values of $H''(\tau)$ at the fixed points. It can be calculated that

$$H''(i) = \begin{cases} -\frac{2\pi^2}{[\eta(i)]^{24}} [E_4(i)]^4 \left[\frac{n}{3} P(1728)(12)^{n-3} + (12)^n P'(1728) \right] & m = 0 \\ 0 & m = 1 \\ -\frac{2\pi^2}{[\eta(i)]^{24}} [E_4(i)]^4 (12)^n P(1728) & m = 2 \\ 0 & m > 2 \end{cases},$$

$$H''(\omega) = \begin{cases} 0 & n = 0, 1, 3, 4, 5 \\ -\frac{8\pi^2}{9} \frac{[E_6(\omega)]^2}{[\eta(\omega)]^{16}} (-12)^{\frac{n}{2}} P(0) & n = 2 \\ 0 & n > 5 \end{cases}. \quad (4.46)$$

Here we use the following relations

$$\frac{j''(\omega)}{[j(\omega)]^{\frac{1}{3}}} = -\frac{8\pi^2}{3} \frac{[E_6(\omega)]^2}{[\eta(\omega)]^{16}}, \quad \frac{j''(\omega)}{[j(\omega)]^{\frac{1}{3}}} = -\frac{8\pi^2}{3[\eta(\omega)]^8} \frac{[E_6(\omega)]^2}{E_4(\omega)} \rightarrow \infty,$$

$$\frac{j''(i)}{[j(i) - 1728]^{\frac{1}{2}}} = -\frac{2\pi^2}{[\eta(\omega)]^{12}} \frac{[E_4(i)]^4}{E_6(i)} \rightarrow \infty, \quad (4.47)$$

and

$$\frac{[j'(i)]^2}{[j(i) - 1728]^{2-\frac{m}{2}}} = -\frac{4\pi^2}{[\eta(\omega)]^{12m}} \frac{[E_6(i)]^2 [E_4(i)]^4}{[E_6(i)]^{4-m}},$$

$$\frac{[j'(\omega)]^2}{[j(\omega)]^{2-\frac{n}{3}}} = -\frac{4\pi^2}{[\eta(\omega)]^{8n}} \frac{[E_6(\omega)]^2 [E_4(\omega)]^4}{[E_4(\omega)]^{6-n}}, \quad (4.48)$$

to cancel each divergent term among the $j''(\tau)$ and $(j'(\tau))^2$ related terms within $H''(\tau)$ at the fixed points when $m = 1$ or $n = 1$.

When $H(\tau) = 0$, we have the following discussions on the condition $H''(\tau) = 0$:

- (i) The case $\tau = i$ is a solution of $H(\tau) = 0$:

When $\tau = i$ is a solution of $H(\tau) = 0$ for $m > 0$, the condition $H''(\tau) = 0$ can be satisfied automatically for $m \neq 2$. For other choices of m , it can also be satisfied when $\mathcal{P}(1728) = \mathcal{P}'(1728) = 0$, requiring $P(j(\tau))$ to take the form

$$P(j(\tau)) \propto (j(\tau) - 1728)^{k+1}, \quad k \in \mathbb{N}. \quad (4.49)$$

After the redefinition of m , we conclude that $H''(i) = 0$ can be satisfied for $m \neq 0, 2$.

- (ii) The case $\tau = \omega$ is a solution of $H(\tau) = 0$:

Similarly, when $\tau = \omega$ is a solution of $H(\tau) = 0$ for $n > 0$, the condition $H''(\tau) = 0$ can be satisfied

automatically for $n \neq 2$. For other choices of m , it can also be satisfied when $\mathcal{P}(0) = 0$, requiring $P(j(\tau))$ to take the form

$$P(j(\tau)) \propto [j(\tau)]^k, \quad k \in \mathbb{N}. \quad (4.50)$$

After the redefinition of n , we conclude that $H''(\omega) = 0$ can be satisfied for $n \neq 0, 2$.

- (iii) The case $\tau = \tilde{\tau}_i$ with $P(j(\tilde{\tau}_i)) = 0$:

The condition $H''(\tau) = 0$ can be satisfied for $P'(j(\tilde{\tau}_i)) = P''(j(\tilde{\tau}_i)) = 0$, which requires $\tau = \tilde{\tau}_i$ being a root of $P(j(\tau))$ with multiplicity three or greater. Therefore, the most general form of $P(j(\tau))$ should be

$$P(j(\tau)) = \pm g(j(\tau)) \prod_i (j(\tau) - j(\tilde{\tau}_i))^{k_i}, \quad k_i \in \mathbb{N}, \quad (4.51)$$

with $k_i \geq 3$ and $g(j(\tau))$ an arbitrary positive polynomial function of $j(\tau)$.

It is obvious from (4.39) that the stationary condition for τ is determined by the solutions of $W_M^* W'_M = 0$ and $|W_M|^2 = 0$ when $H(\tau) = 0$. From (4.28) and (4.46), it is obvious that $W_M^* W'_M = 0$ always vanish at $\tau = i$ except when $m = 1$ (or $\tau = \omega$ except when $n = 1$). However, they will not vanish at $\tau = \tilde{\tau}_i$ in general, where $P(j(\tilde{\tau}_i)) = 0$. Therefore, solutions of equation $V_\tau / |H(\tau)|^2 = 0$ (for

$H(\tau) \neq 0$) and fixed points $\tau = i, \omega$ are always stationary points with respect to τ and τ^* . As the solutions of combined equations $V_\tau/|H(\tau)|^2 = 0$ (for $H(\tau) \neq 0$) and (4.34) in general can not be expressed in analytical forms and need numerical calculations, it is hard to get the desired location of the modulus VEV for our model buildings. So, we would like to concentrate on other solutions that can be determined analytically, with the corresponding modulus VEVs deviating slightly from the fixed points. Such nonfixed point solutions of modulus VEVs can come from $H(\tau) = 0$.

The value $\tau = \tilde{\tau}_i$ with $P(j(\tilde{\tau}_i)) = 0$ generally fails to be a solution of $V_\tau = 0$ because the derivatives $H'(\tau)$ [or the expression $[H''(\tau) - \frac{3iH'(\tau)}{2\pi}\hat{G}_2(\tau, \tau^*)]]$ are typically nonzero at this point. In order for $H'(\tilde{\tau}_i) = 0$, $P(j(\tau))$ needs to take the form (4.45). In such a case, nonfixed point values of τ with $P(j(\tau)) = 0$ can also satisfy $P'(j(\tau)) = 0$ to guarantee $\tilde{W}'_M \tilde{W}_M^* = 0$. We will concentrate on such interesting choices of $P(j(\tau))$ in our subsequent studies unless otherwise specified. As the $H(\tau)$ (with $m, n \neq 1$) factor can be effectively factored out from $\tilde{W}'_M$ and $\tilde{W}_M$, any nonfixed point value of τ satisfying $|H(\tau)|^2 = 0$ is an extremum of the potential, as long as the condition $V_{X_{nl}} = 0$ is met.

It was argued in [58,78,79] that modular invariance and $\tau \leftrightarrow -\tau^*$ symmetry of the scalar potential ensure that the first derivative of scalar potential on τ along certain directions at the boundary of the fundamental domain vanishes

$$\left. \frac{\partial}{\partial x_1} V \right|_{x_1=\pm 1/2, 0} = 0, \quad \left. \frac{\partial}{\partial \rho_\tau} V \right|_{\rho_\tau=1} = 0, \quad (4.52)$$

where $\tau \equiv x_1 + ix_2 \equiv \rho_\tau e^{i\theta_\tau}$. As the two independent directional derivatives vanish at the finite fixed points $\tau = \omega$ and $\tau = i$, the scalar potential is always stationary with respect to τ at such finite fixed points [79,83], agreeing with our previous discussions.

The extrema conditions (4.34) can be rewritten as

$$\frac{\partial}{\partial \rho} V = \frac{\partial}{\partial \theta} V = 0, \quad (4.53)$$

with $X_{nl} \equiv \rho e^{i\theta}$. We have the following discussions for some special parameter choices:

(1) Case $f_0 = 0$:

The extrema of the scalar potential are located at parameter points where $X_{nl} = 0$. The modulus VEV associated with these extrema is identical to the case involving only a single modulus field τ , which had been extensively studied in previous works [58,59,75]. It has been demonstrated that the minima of the scalar potential can slightly deviate from the left (right) cusp symmetric point and the boundary when $m \neq 0$ and $n = 0$ for $H(\tau)$, resulting in negative vacuum energy.

(2) Case $c_1 = 0$:

The extrema of the scalar potential are also located at the parameter points with $\rho = 0$ (that is, $X_{nl} = 0$). The corresponding potential energy is given by

$$V|_{X_{nl}=0} = \frac{1}{(2\Im\tau)^3 |\eta(\tau)|^{12}} |H(\tau)|^2 f_0^2 \geq 0, \quad (4.54)$$

for all values of τ within the fundamental region. The parameter points for local minimum of the scalar potential should satisfy both $H(\tau) = 0$ and $X_{nl} = 0$. As $|H(\tau)|^2$ can be effectively factored out from V_τ with $P(j(\tau))$ taking the form (4.51), solutions of $H(\tau) = 0$ indeed satisfy $V_\tau = 0$ in general (except for typical choices of $m, n \leq 2$ at the fixed points). Therefore, in this case, the stabilized modulus VEVs are located at the solutions of $H(\tau) = 0$, corresponding to vanishing vacuum energy.

(3) Case $f_0 \neq 0, c_1 \neq 0$:

We now focus on the general case with $f_0, c_1 \neq 0$. It can be calculated that the extremum condition for θ is

$$\frac{\partial}{\partial \theta} V = -\frac{2f_0 c_1^3}{(2\Im\tau)^3 |\eta(\tau)|^{12}} \rho \sin \theta \left[\frac{(2\Im\tau)^2}{3} |\tilde{W}_M|^2 - 2|H(\tau)|^2 \right] = 0, \quad (4.55)$$

while the extremum condition for ρ is

$$\begin{aligned} \frac{\partial}{\partial \rho} V = \frac{1}{(2\Im\tau)^3 |\eta(\tau)|^{12}} \left\{ 2\rho \left[\frac{(2\Im\tau)^2}{3} |\tilde{W}_M|^2 (f_0^2 + c_1^6) - 2c_1^6 |H(\tau)|^2 \right] \right. \\ \left. + 2f_0 c_1^3 \cos \theta \left[\frac{(2\Im\tau)^2}{3} |\tilde{W}_M|^2 - 2|H(\tau)|^2 \right] \right\} = 0. \end{aligned} \quad (4.56)$$

It should be noted that the point $\tau = i\infty$ does not satisfy the extremum condition for θ since $[(2\Im\tau)|\eta(\tau)|^4]^{-1}$ diverges as $\Im\tau \rightarrow \infty$.

As discussed previously, the $|H(\tau)|^2$ factor can be extracted from $|\tilde{W}_M|^2$ for nonvanishing $H(\tau)$. Besides, $|\tilde{W}_M|^2$ vanishes when $H(\tau) = 0$ for choices of $P(j(\tau))$ in the form (4.51) and $m, n \neq 1$. Therefore, $|H(\tau)|^2 = 0$, which in general satisfies the extremum condition $V_\tau = 0$ for τ , can also in general satisfy the extremum conditions $V_\rho = V_\theta = 0$ for X_{nl} . So we can conclude that modulus VEVs that satisfy $|H(\tau)|^2 = 0$ for any values of X_{nl} are always local extrema of the scalar potential.

The extremum conditions for θ and ρ can also be satisfied in the following cases:

- (a) From (4.55), for $\sin \theta \neq 0$ (that corresponds to a nonreal X_{nl} with $\theta \neq 0, \pi$), we can get the condition for modulus τ

$$\frac{(2\Im\tau)^2}{3} |\tilde{W}_M|^2 - 2|H(\tau)|^2 = 0, \quad (4.57)$$

which is independent of the ρ value. After substituting this condition into (4.56), we can obtain the extremum condition for the ρ parameter:

$$\frac{\partial}{\partial \rho} V = 4\rho f_0^2 \frac{|H(\tau)|^2}{(2\Im\tau)^3 |\eta(\tau)|^{12}} = 0. \quad (4.58)$$

Such an extremum condition for ρ can be satisfied with

$$|H(\tau)|^2 = 0. \quad (4.59)$$

Thus, we have the extremum conditions for the case with $f_0, c_1 \neq 0$ and a nonvanishing

nonreal X_{nl} value,

$$|\tilde{W}_M|^2 = |H(\tau)|^2 = 0. \quad (4.60)$$

Substituting back into the scalar potential, we can see that the potential energy at this extremum vanishes.

Then, we would like to focus on the constraint

$$\tilde{W}_M \equiv H'(\tau) - \frac{3iH(\tau)}{2\pi} \hat{G}_2(\tau, \tau^*) = 0, \quad (4.61)$$

which satisfies $H(\tau) = 0$. Discussions on the solutions of $H'(\tau) = 0$ are given in Appendix C. For the fixed points $\tau = \omega, i$, they are always the local extrema of the scalar potential when $m, n > 1$. When $m < 1$ or $n < 1$, whether the fixed points $\tau = \omega, i$ are local extrema can easily be figured out case by case, with the relevant results agreeing with the single modulus case in [59]. The extrema conditions (4.60) and (4.39) of the scalar potential can also be satisfied for nonfixed points solutions of $H(\tau) = 0$ when $P(j(\tau))$ takes the form (4.45), and the corresponding vacua are always of Minkowski type. The Minkowski vacua determined by $H(\tau) = 0$ are independent of the specific value of X_{nl} . To identify the global minimum of the scalar potential, it is necessary to evaluate the potential energies for all extrema, including the case where X_{nl} is real.

- (b) Real values of X_{nl} with $\theta = 0$ or $\theta = \pi$.

In this case, the scalar potential reduces to

$$\begin{aligned} V(X_{nl}, X_{nl}^*, \tau, \tau^*) = & \frac{1}{(2\Im\tau)^3 |\eta(\tau)|^{12}} \left\{ \frac{(2\Im\tau)^2}{3} |\tilde{W}_M|^2 (f_0^2 \tilde{X}^2 + c_1^6 + 2f_0 \tilde{X} c_1^3) + |H(\tau)|^2 (f_0^2 + 2f_0^2 \tilde{X}^2 + 2f_0 c_1^3 \tilde{X} + \tilde{X}^2 c_1^6) \right. \\ & \left. - 3|H(\tau)|^2 (f_0^2 \tilde{X}^2 + c_1^6 + 2f_0 \tilde{X} c_1^3) \right\} + \frac{\tilde{X}^2}{(2\Im\tau)^3 |\eta(\tau)|^{12}} \left\{ \frac{(2\Im\tau)^2}{3} |\tilde{W}_M|^2 c_1^6 + |H(\tau)|^2 f_0^2 - 3c_1^6 |H(\tau)|^2 \right\}, \end{aligned} \quad (4.62)$$

for real values of X_{nl} , which we denote simply as $\tilde{X}$. It is a quadratic polynomial of $\tilde{X}$, which can be rewritten as

$$V(\tilde{X}, \tau, \tau^*) = \frac{1}{(2\Im\tau)^3 |\eta(\tau)|^{12}} [C_1 \tilde{X}^2 + C_2 \tilde{X} + C_3], \quad (4.63)$$

where

$$\begin{aligned} C_1 &= \frac{(2\Im\tau)^2}{3} |\tilde{W}_M|^2 (f_0^2 + c_1^6) - 2|H(\tau)|^2 c_1^6, \\ C_2 &= \frac{(2\Im\tau)^2}{3} |\tilde{W}_M|^2 2f_0 c_1^3 - 4|H(\tau)|^2 f_0 c_1^3, \\ C_3 &= \frac{(2\Im\tau)^2}{3} |\tilde{W}_M|^2 c_1^6 + |H(\tau)|^2 (f_0^2 - 3c_1^6). \end{aligned} \quad (4.64)$$

Obviously, the scalar potential is real for all (real) values of $\tilde{X}$. The scalar potential is bounded from below for $\tilde{X} \in \mathbb{R}$ when the coefficients of the quadratic polynomial in (4.63) satisfy $C_1 > 0$ (and $C_2 = 0$ when $C_1 = 0$, rendering the scalar potential finite).

For $C_1 > 0$, the scalar potential has a minimum with the corresponding $\tilde{X}$ value and the potential energy

$$V|_{\min; \tilde{X} = -\frac{C_2}{2C_1}} = C_3 - \frac{C_2^2}{4C_1}. \quad (4.65)$$

To determine the extrema of $V|_{\min}$, we need to determine

$$\frac{\partial}{\partial \tau} \left[C_3 - \frac{C_2^2}{4C_1} \right] = \frac{\partial}{\partial \tau^*} \left[C_3 - \frac{C_2^2}{4C_1} \right] = 0. \quad (4.66)$$

These extremum conditions, after factoring out a common nonvanishing $|H(\tau)|^2$ term, yield non-trivial equations for τ that generally require numerical solutions. For $C_1 = 0$, it can be checked that $C_2 = 0$ when stationary condition $H(\tau) = 0$ for modulus VEV τ holds, which corresponds to Minkowski vacua for all values of $\tilde{X}$. Therefore, in line with earlier discussions, VEVs of modulus field that satisfy $H(\tau) = 0$ for all values of X_{nl} (whether real or complex) represent local minima of the scalar potential.

The pseudomodulus X_{nl} , i.e., a continuous space of vacua with degenerate tree-level vacuum energies, can be lifted by SUSY breaking effects. Taking into account soft SUSY breaking mass term $m_X^2 |X_{nl}|^2$ from additional SUSY breaking sources with proper mediation mechanism (or from the one-loop Coleman-Weinberg potential of X_{nl} after including its couplings to matter fields, for example, of the O’Raifeartaigh type $[\lambda_{ij} X_{nl} + M_{ij}] \phi_i \phi_j$ in the superpotential), the scalar potential for X_{nl} taking the form $V(X_{nl}) \approx |\tilde{C}|^2 f_0^2 + m_X^2 |X_{nl}|^2$ can stabilize the pseudomodulus at $X_{nl} = 0$ in the flat SUSY limit, with $\tilde{C}$ some constant determined by the stabilized value of τ .

For $C_1 < 0$, the scalar potential, being a quadratic polynomial in $\tilde{X}$, is not bounded from below. From a physical perspective, the bounded-from-below requirement $C_1 > 0$ can be relaxed to allow $C_1 \leq 0$ because of the constraint $|\tilde{X}| \leq M_{\text{Pl}}$. In such cases, the minimum of the scalar potential at fixed τ is located at $\tilde{X} \sim M_{\text{Pl}}$ (or $\tilde{X} \sim -M_{\text{Pl}}$) when $C_1 \leq 0$. Higher-order

and quantum gravity effects can substantially modify the shape of the scalar potential for field values of $|\tilde{X}|$ close to M_{Pl} , rendering $\tilde{X} \sim \pm M_{\text{Pl}}$ local minima of the scalar potential. To determine the global minimum of the scalar potential, one must compare the potential energies for real X_{nl} with those of the Minkowski vacua determined by the condition $H(\tau) = 0$.

For the case $C_1 > 0$, the minimum (4.65) is always located at some value of $\tilde{X}$ with $\tilde{X} \ll M_{\text{Pl}}$. This minimum can either be the global minimum or merely a local minimum when the bounded-from-below condition is relaxed by allowing $C_1 \leq 0$.

We select some benchmark parameter points with various choices of f_0 , c_1 and calculate numerically the potential energies for all extrema to identify the global minimum. For many benchmark points, the global minimum is determined by the condition $H(\tau) = 0$ with vanishing vacuum energy, which is independent of the value of X_{nl} .¹ In Fig. 1, we show the energies of the scalar potential for modulus values within the fundamental domain and $X_{nl} = 0$, considering various parameter choices. In the limit $f_0 \rightarrow 0$, this scenario reduces to the one involving only a single modulus field τ , whose vacuum is generally not of Minkowski type and the corresponding vacuum energy can possibly be negative. Therefore, for tiny values of f_0 , the vacuum energy of the global minimum can become negative. Numerical results on the global minima of some benchmark points with nonvanishing vacuum energy are shown in Table I.

From a physical perspective, we are particularly interested in the local minimum with an *intermediate/small* value of $\tilde{X}$, rather than a global minimum situated at $\tilde{X} \sim M_{\text{Pl}}$ or $\tilde{X} \sim -M_{\text{Pl}}$. For typical benchmark parameter choices, an estimation of the lifetime for the local minimum with an *intermediate/small* value of $\tilde{X}$ suggests that such a local minimum can be metastable, with a lifetime exceeding the age of the universe. Therefore, we will concentrate on the global (or long-lived metastable) minimum with $|X_{nl}| \ll M_{\text{Pl}}$ in our subsequent discussions.

The lifetime of a vacuum can be evaluated semiclassically through a rate of bubble nucleation in unit volume and unit time [88,89]. The decay probability, which sets the decay rate, is given by $\Gamma/V = A \exp(-B)$, where B is the action of a Euclidean $O(4)$ -symmetric bounce solution, which smoothly interpolates between a true-vacuum configuration at its center and the false vacuum whose decay is being described. The prefactor A depends on the functional determinants and represents the loop correction of the fluctuations of the fields. The prefactor A can be

¹As noted previously, the pseudomodulus X_{nl} will be stabilized at $X_{nl} = 0$ after including the SUSY breaking contributions. So we choose the stabilized value $X_{nl} = 0$ for the extrema determined by the condition $H(\tau) = 0$.

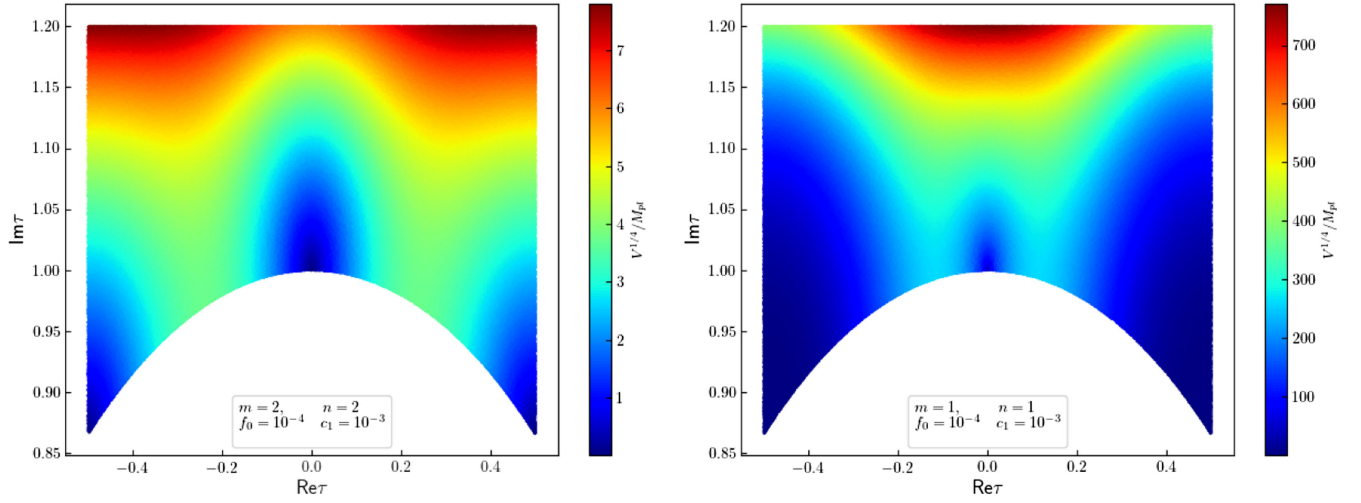


FIG. 1. Values of $V(X_{nl}, \bar{X}_{nl}, \tau, \tau^*)$ within the fundamental domain in the scenario with $k_X = 0$ and $X_{nl} = 0$ for various parameter choices when $f_0 \neq 0$ and $f_0 \gg c_1^2$. We adopt $\mathcal{P}(j(\tau)) = 1$ for the left panel and $\mathcal{P}(j(\tau)) = (j(\tau) - j(\tau_0))^2$ with $\tau_0 = e^{i\frac{2\pi}{5}(1-0.006)}$ for the right panel.

approximately given by $A \sim v^4$, where v is some characteristic mass scale of the theory under study. We adopt a conservative choice $v \sim \mathcal{O}(M_{\text{Pl}})$ for our estimation of lifetime.

The generalization to cases involving multiple fields entails determining the bounce configuration by solving a set of coupled ordinary differential equations, subject to the appropriate boundary conditions. For general multidimensional field spaces, there will be many bounce solutions with different actions, and the tunneling rate will be dominated by the solution with the lowest action. The field profile, which solves the Euclidean equations of motion, traverses a one-dimensional path in the field space. If one chooses a particular small subset of path in the field space and treats it as a one-field potential, solving the field equations for the one-dimensional potential along this path can give an estimation on the action, which can be close to the actual value of the action [90]. As the tunneling is expected intuitively to happen through the lowest possible barrier and shortest possible path in the field space, the one with the shortest possible path length, which is a straight line between the two minima (or the path with the lowest possible barrier in which a broken line goes through the

lowest ridge in the barrier), can act as the one-dimensional approximations of potential along such a path parametrized by the path length in the field space.

We estimate the decay lifetime with the bounce configuration via the two paths connecting the true and false vacuum for some benchmark parameter choices that can lead to false vacuums at *intermediate/small* value of $\tilde{X}$. We estimate the one-dimensional action for each case with the approximation of a triangular potential, whose bounce action is given approximately [91] by

$$B \approx \frac{32\pi^2}{3} \frac{(\Delta V_+)^2 (\Delta\phi)^4}{\epsilon^3}, \quad (4.67)$$

in the situation where the difference of potential energy between the true and false vacuum ϵ is very small. Here $\Delta V_{\pm} = V_{\text{Top}} - V_{\pm}$ denotes the height for false (and true) vacuum with respect to the top of the barrier and $\Delta\phi = (\phi_- - \phi_+)$ denotes the width of the barrier.

For a characteristic benchmark point, such as $f_0 = 10^{-7} M_{\text{Pl}}^2$ and $c_1 = -0.01 M_{\text{Pl}}$ with $m = 1$, $n = 1$ and $P(j(\tau)) = (j(\tau) - j(\tau_0))^2$ for $\tau_0 = 1.001 \times e^{i\frac{2\pi}{5}(1-\frac{1}{35})}$, we find that the

TABLE I. Global minima and the corresponding vacuum energies for some benchmark parameter points in the case $k_X = 0$, $f_0 \neq 0$ and $c_1 \neq 0$.

m	n	f_0/M_{Pl}^2	c_1/M_{Pl}	X_{nl}/M_{Pl}	τ	$P(j(\tau))$	$V_{\min}/M_{\text{Pl}}^4$
0	0	0.1	0.5	-0.6318	$0.0021 + 1.002i$	1	-0.0525
1	1	10^{-4}	0.1	0.9994	$0.19337 + 0.98275i$	1	-1.25
2	0	10^{-7}	-0.01	-0.9994	$0.4892 + 0.8733i$	1	-4.466×10^{-5}
1	2	1.128×10^{-6}	-0.01	-0.3322	$-0.1910 + 0.9830i$	1	-7.282×10^{-5}
2	1	2.974×10^{-7}	-0.01	0.2920	$0.2865 + 0.9612i$	1	-7.434×10^{-4}
1	3	2.337×10^{-6}	-0.01	-0.4126	$0.1588 + 0.9911i$	1	-7.103×10^{-4}

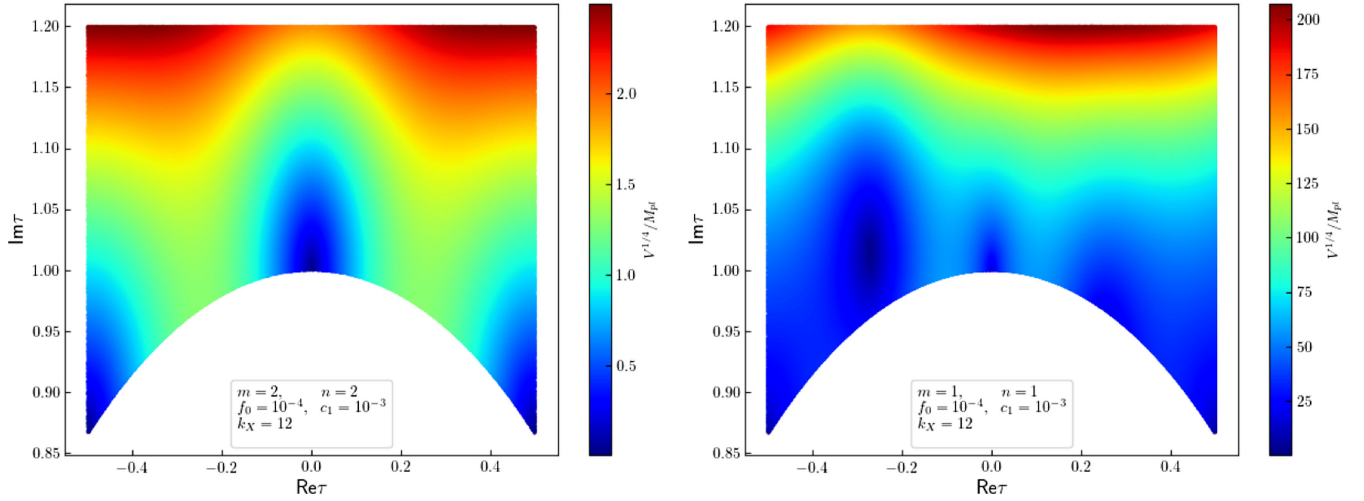


FIG. 2. Values of $V(X_{nl}, \bar{X}_{nl}, \tau, \tau^*)$ within the fundamental domain in the scenario with $k_X = 12$ and $X_{nl} = 0$ for various parameter choices when $f_0 \neq 0$ and $f_0 \gg c_1^2$. We adopt $\mathcal{P}(j) = 1$ for the left panel and $\mathcal{P}(j(\tau)) = (j(\tau) - j(\tau_0))^2$ with $\tau_0 = 1.05 \times e^{i\frac{\pi}{2}(1+\frac{1}{k})}$ for the right panel.

false vacuum lies at $(X_{nl}, \tau) = (0, 1.001 \times e^{i\frac{2\pi}{3}\frac{35}{36}})$ with vanishing vacuum energy while the true minimum vacuum lies at $(X_{nl}, \tau) = (-0.99999M_{\text{Pl}}, -0.42825 + 0.91479i)$ with $V_{\text{min}} = -2.49896M_{\text{Pl}}^4$. The peak of the lowest barrier sits at the parameter point $(X_{nl}, \tau) = (-0.46285M_{\text{Pl}}, -0.40398 + 0.92017i)$ with $V_{L\text{-peak}} = 577.87447M_{\text{Pl}}^4$. We find that the bounce action of the broken line path through the lowest possible barrier leads to smaller action. It can be calculated with (4.67) that the corresponding bounce action $B \approx 2.28 \times 10^5$. Therefore, the lifetime of the metastable vacuum $\tau = (\Gamma/V)^{-1/4}$ is much larger than the universe. In fact, using the space-time volume of our past light cone [92], $(VT)_{LC} = 0.15/H_0^4 = 3.4 \times 10^{166} \text{ GeV}^{-4}$ with the Hubble constant $H_0 = 1.44 \times 10^{-42} \text{ GeV}$, the probability that we should have seen a bubble by now is $[(VT)_{LC}](\Gamma/V) \sim 10^{-9670}$, which is exponentially small.

B. The case with $k_X = 12$

When $k_X = 12$, interference terms in the scalar potential obscure the analysis of the extrema conditions. Therefore, we leave the most general discussions on this case to our subsequent studies. An important observation is that an $|H(\tau)|^2$ factor (for $H(\tau) \neq 0$) can be extracted from the scalar potential. According to the discussions in the case $k_X = 0$, for any value of X_{nl} , in addition to the fixed points $\tau = i, \omega$ for most choices of m, n , nonfixed point values of modulus VEVs satisfying $|H(\tau)|^2 = 0$ are local extrema of the scalar potential with vanishing vacuum energy when $\mathcal{P}(j(\tau))$ takes the form (4.45). Other local minima of the scalar potential can be determined numerically from the extrema conditions with respect to τ, X_{nl} and the corresponding Hessian matrix. After specifying

numerically all other extrema of the scalar potential, we can determine its global minimum or metastable minimum by evaluating the corresponding vacuum energies of various local minima (and the lifetime of the meta-stable minimum). We show in Fig. 2 the vacuum energy of the scalar potential for typical benchmark parameter choices with modulus values taken in the fundamental domain in the scenario with $k_X = 12$.

In our discussion, we assume the existence of modular symmetry in the SUSY breaking sector. The modular symmetry can be broken by SUSY breaking effects, depending on the transformation property of SUSY breaking source under the modular symmetry, as discussed in [93]. For example, in the moduli mediated SUSY breaking scenario [93], soft SUSY breaking scalar masses and trilinear terms are invariant under the modular symmetry when the F term is regarded as a spurion with the modular weight -2 . In our case, the nonvanishing F-term VEV of X_{nl} when $k_X \neq 0$ can possibly act as the SUSY breaking source and also break the modular symmetry, whose transformation should be compatible with the modular invariance of $K_{X_{nl}^\dagger X_{nl}} F_{X_{nl}}$, possibly leading to a soft SUSY breaking spectrum that is noninvariant under modular symmetry. The form of the low energy soft SUSY breaking spectrum depends on the concrete mediation mechanism of SUSY breaking that mediates the SUSY breaking effect from the hidden sector to the visible sector.

It should be noted that the modulus stabilization mechanism with nilpotent $\mathbf{X}_{nl}$ superfield can be fairly generic and ubiquitous, which is independent of the following discussions on the regeneration of small μ_{eff} term following modulus stabilization.

V. REGENERATION OF NATURALLY SMALL μ_{eff} FOLLOWING MODULUS STABILIZATION

In our proposal, the explicit μ term in the superpotential is prohibited by modular symmetry. The term involving the $H_u H_d$ bilinear and proper modular form will regenerate an effective μ_{eff} term after modular symmetry breaking by the VEV of the modulus field. It should be noted that the regenerated μ_{eff} scale is generally not much smaller than Λ_{SUSY}^2 unless the stabilized modulus VEV accidentally takes some specific value that leads to a large suppression factor. We know that naturalness requires μ_{eff} to lie at the electroweak scale or soft SUSY breaking scale. So, “generic” values of modulus VEVs from a modulus stabilization mechanism that lead to the full breaking of modular symmetry will in general not lead to a genuine solution of the SUSY μ problem. Therefore, one can consider some “nongeneric” values of modulus VEVs that do not break the modular symmetry completely or values of VEVs in the vicinity of such symmetric fixed points.

It is known that the fixed points $\tau_{FP} = i, \omega, i\infty$ for modular theories based on the Γ_N invariance can preserve Z_2^S, Z_3^{ST}, Z_N^T residual symmetries, respectively [94]. As noted previously, we can infer from the modular invariance of scalar potential that the finite fixed points should in general be the local extrema of the scalar potential, as they act nontrivially (with residual symmetries) on the derivatives of the potential and must vanish. Although they may not be the global minima, experiences from modulus stabilization in string theory indicate that the minima should always lie at/near $\tau_{FP} = i, \omega$ for the tree-level scalar potential, as the scalar potential generally diverges for $\tau \rightarrow i\infty$.

It was found in [57,94] that the hierarchies among the charged leptons and quarks mass can follow solely from the properties of the modular forms (strictly speaking, the representation of the mass term bilinear with respect to the residual group) in the vicinity of symmetric points, avoiding the fine-tuning of the constant parameters (or without the need to introduce extra weightonlike scalar fields). Similar ideas, augmented with additional suppression powers of $q^{1/24}$ (or $(2\Im\tau)^{-1}$), can be used to generate the hierarchy between Λ_{SUSY} and μ_{eff} .

²Strictly speaking, the regenerated μ_{eff} should lie generically at the scale

$$\mu_{\text{eff}} \sim \Lambda_{\text{SUSY}} F(\langle\tau\rangle) = \Lambda_{\text{SUSY}} F\left(\frac{\Lambda_{\mathcal{M}}}{\Lambda_0}\right), \quad (5.1)$$

where $\Lambda_{\mathcal{M}}$ is the modular symmetry breaking scale characterized by the modulus VEV $\langle\tau\rangle$, Λ_0 is the scale appearing in the modulus chiral superfield $\sigma = \Lambda_0\tau$ and $F(\tau)$ is some function approaching the involved modular forms. When $\Lambda_{\text{SUSY}} \sim \Lambda_0$ and $F(\tau) \sim \tau$, $\mu_{\text{eff}} \sim \Lambda_{\mathcal{M}}$.

Besides, although the scalar potential for modulus field τ in general diverges when $\tau \rightarrow i\infty$, the presence of X_{nl} contributions can alter the scalar potential, making it possible to stabilize τ at an intermediate value. Consequently, for an appropriate choice of k_3 , the effective μ_{eff} parameter can naturally be much smaller than the SUSY scale Λ_{SUSY} , due to a large suppression factor originating from both powers of $q^{1/24}$ [or $(2\Im\tau)^{-1}$] and typical suppression behavior of modular forms away from the fixed point $i\infty$ (at τ_0 with medium $\Im\tau_0$).

To regenerate the effective μ term with a stabilized modulus field, we need to construct a modular weight -3 bilinear for H_u and H_d using proper modular forms. For a modular invariant bilinear of the form $\psi_i^\dagger M_{ij}(\tau) \psi_j$, where $M_{ij}(\tau)$ is a modular form of level N and weight K , the zero entries of $M_{ij}(\tau)$ at a symmetric point in the mass matrices will generically become nonzero when τ deviates slightly from the symmetric point. The magnitudes of these residual symmetry breaking entries are controlled by the size of the departure ϵ from τ_{sym} and the field transformation properties under the residual symmetry group, which may depend on the modular weights [22,94,95]. That is, the entries of the mass matrices in the vicinity of the symmetric point are subjected to corrections of $\mathcal{O}(\epsilon^l)$, where the power l corresponding to the representations of the residual symmetry group depends only on how the representations of the fermion fields in the mass term bilinear decompose under the considered residual symmetry group. The degree of suppression, given by the integer l , can take values $l = 0, 1, \dots, N-1$ in the case of $\tau_{\text{sym}} = i\infty$, $l = 0, 1, 2$ in the case of $\tau_{\text{sym}} = \omega$, and $l = 0, 1$ in the case of $\tau_{\text{sym}} = i$. In [94], the authors systematically classify the residual symmetry representations for different choices of Γ_N representations of matter fields.

Such a formalism can also be applied to the regeneration of small μ_{eff} for the $H_u H_d$ bilinear to achieve a desired suppression factor $\mathcal{O}(\epsilon^l)$ of the deviation parameter ϵ (from a given symmetric point). In our case, we need to specify the representations of the Higgs superfields H_u, H_d in MSSM under the residual symmetry group $\mathbb{Z}_N^T$ (with $N = 3$) for $\tau_{\text{sym}} = i\infty$ and the residual group $\mathbb{Z}_3^{ST}$ for $\tau_{\text{sym}} = \omega$. Obviously, H_u, H_d should always be singlets under the modular flavor symmetry Γ_N . They can only be assigned some singlet representations of Γ_N with nontrivial modular weights. The decompositions of the representation $(\mathbf{r}; k)$ multiplet of Γ_N' under the residual symmetry can be found in the Appendix of [94]. So, the desired suppression factor for the regenerated μ_{eff} can be directly obtained with proper choice of singlet representations and modular weights.

We should note that $\mu_{\text{eff}} \ll \Lambda_{\text{SUSY}}$ can also be naturally realized through a weightonlike mechanism [71] for $H_u H_d$ bilinear. Specifically, we can introduce a single weightonlike superfield, which is a singlet under both the Standard

Model and modular flavor symmetry, with a modular weight $k_\phi = 1$. By appropriately choosing the modular weights and representations under the finite modular group Γ_N for H_u and H_d , we can ensure that only the combination

$$W \supseteq \frac{Y_{\mathbf{r}}^{(k_\mu)}}{[\eta(\tau)]^{2k_3}} \left(\frac{\phi}{M_F} \right)^k H_u H_d, \quad (5.2)$$

is allowed by modular invariance, where $k = k_\mu - k_3 - k_{H_u} - k_{H_d} + 3$ is a positive integer, $2k_3 \in 24\mathbb{Z} + 6$ and M_F is a typical cutoff scale. When the weighton field ϕ acquires a VEV through the relevant superpotential, the electroweak-scale μ_{eff} can be naturally regenerated with a large suppression factor of order $\tilde{\phi}^k$, where $\tilde{\phi} \equiv \langle \phi \rangle / M_F \ll 1$. For example, the form of superpotential for weighton superfield can be chosen to be [71]

$$W \supseteq X \left(Y_1^4 \frac{\phi^4}{M_F^2} - M^2 \right), \quad (5.3)$$

with M a dimensional mass parameter and X an A_4 singlet. The F-flat condition $F_X^* = -W_X = 0$ drives the weighton VEV $\langle \phi \rangle \sim (MM_F)^{1/2}$, which leads to the suppression factor $\frac{\phi}{M_F} \sim (\frac{M}{M_F})^{1/2} \ll 1$ for $M \ll M_F$.

It should be noted that the H_u and H_d superfields, which transform nontrivially under modular symmetry, can also break the modular invariance after electroweak symmetry breaking. As the VEVs of H_u and H_d are much smaller than the typical scale of modulus stabilization, the extreme conditions for τ and X_{nl} will receive negligible corrections from H_u and H_d related terms in the limits $v_u, v_d \rightarrow 0$. Electroweak symmetry breaking, triggered by the VEVs of H_u and H_d , are also affected by the modular stabilization mechanism. However, additional contributions to the electroweak symmetry breaking sector from supergravity vanish in the $M_{\text{Pl}} \rightarrow \infty$ limit, leading to negligible contributions to the low energy MSSM scalar potential for H_u and H_d , a consequently negligible shift of the VEVs for H_u and H_d .

A. Natural μ_{eff} with modulus stabilization near the fixed point $\omega \equiv e^{i2\pi/3}$

In the vicinity of ω , the degree of suppression for a bilinear is proven [94] to be given by $\mathcal{O}(\epsilon^l)$ with l taking values $l = 0, 1, 2$. In this study, we choose to directly calculate the suppression factor and its coefficient for the bilinear $H_u H_d$ using explicit model construction involving appropriate modular forms. It should be emphasized again that the exponentiation power of deviation parameter in the suppression factor depends only on the $\mathbb{Z}_3^{ST}$ (or $\mathbb{Z}_3^T$ for modular VEVs near $i\infty$) charge of H_u, H_d instead of the explicit form of modular forms.

For modular A_4 symmetry, at the fixed point ω , the following modular forms satisfy

$$Y_1^{(4)}(\omega) = Y_1^{(8)}(\omega) = Y_{1'}^{(8)}(\omega) = 0. \quad (5.4)$$

The small deviation of modulus VEV away from ω , induced by the modulus stabilization mechanism, can act as the desired suppression parameter for μ_{eff} .

Expressions of modular forms with larger modular weights can be constructed from $Y_3^{(2)} \equiv (Y_1, Y_2, Y_3)$, for example,

$$\begin{aligned} Y_{1'}^{(8)} &= (Y_1^2 + 2Y_2Y_3)(Y_3^2 + 2Y_1Y_2), \\ Y_1^{(8)} &= (Y_1^2 + 2Y_2Y_3)^2. \end{aligned} \quad (5.5)$$

From the expansion [22] of modular functions at the point $z_0 = \omega + \epsilon$ (with ϵ depending on the choices of f_0 such that $|\epsilon| \ll 1$)

$$\begin{aligned} \frac{Y_2(z_0)}{Y_1(z_0)} &\simeq \omega(1 + 2.235i\epsilon - 2.95717\epsilon^2 + \dots), \\ \frac{Y_3(z_0)}{Y_1(z_0)} &\simeq -\frac{1}{2}\omega^2(1 + 4.47i\epsilon - 10.9096\epsilon^2 + \dots), \end{aligned} \quad (5.6)$$

we can obtain the expansion for $Y_1^{(8)}(z_0)$ etc. as

$$Y_1^{(8)}(\omega + \epsilon) \simeq -36.4138\epsilon^2 - 259.129i\epsilon^3 + \dots, \quad (5.7)$$

with $Y_1(\omega) \approx 0.948674$.

The bilinear term for $H_u H_d$, given in (3.1), takes the form

$$W \supseteq \alpha \Lambda_{\text{SUSY}} \frac{Y_{\mathbf{r}}^{(k_r)}(\tau)}{[\eta(\tau)]^{2k_3}} H_u H_d, \quad (5.8)$$

where $k_{H_u} + k_{H_d} = k_r - k_3 + 3$, $2k_3 \in 24\mathbb{Z} + 6$ and $\mathbf{r}(H_u) \otimes \mathbf{r}(H_d) \otimes \mathbf{r} \supset \mathbf{1}$. When the modulus field is stabilized close to the point ω , the effective μ term is given by

$$\begin{aligned} \mu_{\text{eff}} &= \alpha \Lambda_{\text{SUSY}} [\eta(\omega + \epsilon)]^{-2k_3} Y_{\mathbf{r}}^{(k_r)}(\omega + \epsilon) (2\Im \tau)^{(k_{H_u} + k_{H_d})/2} \\ &= \alpha \Lambda_{\text{SUSY}} [\eta(\omega)]^{-2k_3} Y_{\mathbf{r}}^{(k_r)}(\omega + \epsilon) (\sqrt{3})^{(k_r - k_3 + 3)/2} \\ &\approx \alpha \Lambda_{\text{SUSY}} [0.711506 \cdot e^{-0.523599i}]^{-k_3/2} Y_{\mathbf{r}}^{(k_r)} \\ &\quad \times (\omega + \epsilon) (\sqrt{3})^{k_r + 3/2}. \end{aligned} \quad (5.9)$$

To achieve a significantly suppressed μ_{eff} , we either need a very large negative value k_3 or a tiny $|\epsilon|$. However, calculations show that the suppression from k_3 power is not very efficient to obtain a tiny number. Therefore, we choose to use a small ϵ to suppress μ_{eff} with appropriate modular weights and representations for H_u, H_d .

For instance, we can choose

$$(r_{H_u}, k_{H_u}) \sim (\mathbf{1}', 4), \quad (r_{H_d}, k_{H_d}) \sim (\mathbf{1}'', 4), \quad (5.10)$$

and the modular forms $Y_1^{(8)}$ (or, for example, $Y_1^{(4)}$ with suitable k_{H_u} and k_{H_d}). Thus, we have

$$\begin{aligned} \mu_{\text{eff}} &= \frac{\alpha \Lambda_{\text{SUSY}}}{|\bar{N}_{Y_1^{(8)}}|} [\eta(\omega)]^{-6} Y_1^{(8)}(\omega + \epsilon) (\sqrt{3})^{(k_{H_u} + k_{H_d})/2}, \\ &\approx -0.927 \times 10^3 e^{i\frac{\pi}{4}} \alpha \Lambda_{\text{SUSY}} \epsilon^2. \end{aligned} \quad (5.11)$$

Here $|\bar{N}_{Y_1^{(8)}}| = 1.3423$ denotes the global normalization factor [96] for $Y_1^{(8)}$, which is independent of the modulus

VEV and defined by

$$\begin{aligned} (\bar{N}_Y^{(K)})^2 &= \int_D \int \left(N_Y^{(K)}(\tau, \tau^*) \right)^2 \frac{dx dy}{y^2} \\ &\equiv \int_D \int \sum_i |Y_{ri}^{(K)}(\tau)|^2 (2y)^K \frac{dx dy}{y^2}, \end{aligned} \quad (5.12)$$

with

$$N_Y^{(K)} = \left(\sum_i |Y_i^{(K)}|^2 (2 \text{Im}(\tau))^K \right)^{1/2}, \quad (5.13)$$

or

$$\begin{aligned} (\bar{N}_{Y_R}^{(K)})^2 &= \lim_{T \rightarrow \infty} \left(\int_{D_T} \int \left(N_Y^{(K)}(\tau, \tau^*) \right)^2 \frac{dx dy}{y^2} - |a_0|^2 \frac{2^K}{K-1} T^{K-1} \right) \\ &\equiv \lim_{T \rightarrow \infty} \left(\int_{D_T} \int \sum_i |Y_{ri}^{(K)}(\tau)|^2 (2y)^K \frac{dx dy}{y^2} - |a_0|^2 \frac{2^K}{K-1} T^{K-1} \right), \end{aligned} \quad (5.14)$$

for noncusp modular forms, where T is a real constant, D_T is the region below the line $y = T$ in the fundamental domain with $T > 1$, and $|a_0|^2$ is the constant q independent term in the q expansion of $\sum_i |Y_{ri}^{(K)}(\tau)|^2$. It should be noted that the normalization can not be fixed in the bottom-up approach to modular symmetry.

From the decompositions of Γ_N representations under the residual symmetry group in the Appendix of [94], it can be verified that the suppression factor of ϵ for the $H_u(\mathbf{1}', 4)$ and $H_d(\mathbf{1}'', 4)$ bilinear is given by $(k_{H_u} + k_{H_d} + 3) \bmod 3 = 2$, which is consistent with the result from our explicit construction.

If Λ_{SUSY} is naively taken to be the Planck scale, a tiny ϵ with $|\epsilon| \sim 10^{-10}$ or a tiny α is needed to achieve an electroweak-scale μ_{eff} . For a smaller Λ_{SUSY} scale, a smaller suppression factor is required to achieve an effective μ_{eff} at the electroweak scale. For instance, in the gravity mediated SUSY breaking mechanism, where the SUSY breaking scale Λ_{SUSY} is typically of the order $\sqrt{F} \sim \sqrt{m_{3/2} M_{\text{Pl}}} \sim 10^{10}$ GeV for $m_{3/2} \sim \mathcal{O}(10^2 \text{ GeV})$, a desired μ_{eff} at

electroweak scale can be regenerated for $|\epsilon| \sim 10^{-5}$ when $\alpha \sim \mathcal{O}(1)$. In the gauge mediated SUSY breaking mechanism where the SUSY breaking scale is as low as 10^5 GeV, the required suppression factor ϵ can be of the order $|\epsilon| \sim 10^{-2}$ for $\alpha \sim \mathcal{O}(1)$.

Note that the smallness of deviation parameter ϵ (away from ω) can be a consequence of SUSY breaking, which depends on the choice of f_0 . For example, if the parameter choice with $f_0 = 0$ leads to the global minimum at ω , SUSY breaking contributions may slightly shift the minimum to some point near ω .

The desired condition $|\epsilon| \ll 1$ can be achieved through typical modulus stabilization mechanisms, such as the one proposed in [75], which employs large values of m and $n = 0$. However, for the modulus stabilization mechanism discussed in the previous section, fine-tuning is required to obtain a minimum at $\omega + \epsilon$ with $|\epsilon| \ll 1$. This is particularly true when we consider the simplest realization $\mathcal{P}(j(\tau)) = (j(\tau) - c_p)^2$ with $c_p \ll 1$ for $H(\tau) = 0$. We provide in Table II some benchmark points that can achieve

TABLE II. Parameter choices for the scenario with $k_X = 0$, which can stabilize the modulus at the point $\omega + \epsilon$ (for $|\epsilon| \sim 10^{-5}$) with $X_{nl} = 0$.

m	n	f_0/M_{Pl}^2	c_1/M_{Pl}	$\epsilon/10^{-5}$	$\mathcal{P}(j(\tau))$
2	2	0.2	0.1	$(1.659 + 0.958i)$	$(j - 1.765 \times 10^{-9})^2$
0	3	0.2	0.1	$(5.441 + 3.141i)$	$(j - 2.861 \times 10^{-9})^2$
1	2	0.2	0.1	$(8.162 + 4.172i)$	$(j - 3.973 \times 10^{-8})^2$
0	2	0.2	0.1	$(6.802 + 3.927i)$	$(j - 280.469)(j - 1.008 \times 10^{-8})^2$
2	3	0.2	0.1	i	$(j - 1.398 \times 10^{-9})^2$

small suppression factors using the aforementioned modulus stabilization mechanism. It is worth noting that the fine-tuning needed here may arise from the specific choices of input parameters for our modulus stabilization mechanism. One could explore alternative modulus stabilization mechanisms to achieve a minimum very close to ω without large fine-tuning. Moreover, such fine-tuning is not necessary when employing a weighton-type mechanism for the regeneration of μ_{eff} .

B. Natural μ_{eff} with modulus stabilization at medium $\Im\tau$

When the modulus field is stabilized near $i\infty$, similar to the case where modulus VEV lies in the vicinity of ω , the exponentiation power l of deviation parameter ϵ in the suppression factor for the $H_u H_d$ bilinear can also be proven [94] to depend only on the $\mathbb{Z}_3^T$ charge of H_u, H_d , rather than the explicit form of modular forms. Since the coefficient of ϵ^l cannot be simply determined and may involve additional suppression factors, we opt to directly calculate the suppression factor and the coefficient for the bilinear $H_u H_d$ using an explicit model construction.

Although the scalar potential for the modulus generally diverges as $\tau \rightarrow i\infty$, the introduction of a nilpotent Goldstino $\mathbf{X}_{nl}$ superfield or multiple moduli/matter fields may alter the shape of the scalar potential at intermediate values of τ , making it possible to stabilize the modulus at τ_0 with an intermediate value of $\Im\tau_0$ (for example, $\Im\tau_0 \sim 5-10$). Detailed discussions on such a possibility will be given in our subsequent studies.³ In this work, we just concentrate on the implications of modulus stabilization at an intermediate value of $\Im\tau$ on the SUSY μ problem. Note that the stabilized VEV τ_0 (with an intermediate value $\Im\tau_0$) can be regarded as lying close to the $i\infty$ fixed point.

To obtain the suppressed effective μ parameter, in addition to a possible suppression factor from powers of $q^{1/24}$ [from $\eta(\tau)$] or $(2\Im\tau)^{-1}$ (from the normalization factor), we would also like to choose the modular function that vanishes at $\Im\tau = i\infty$. For example, the modular forms satisfy

$$Y_{\mathbf{1}}^{(4)}(i\infty) = Y_{\mathbf{1}'}^{(8)}(i\infty) = Y_{\mathbf{1}''}^{(8)}(i\infty) = 0, \quad (5.15)$$

as well as the vanishing singlet components from their direct products, such as the modular function $Y_{\mathbf{1}}^{(12)}(i\infty)$.

³For example, we can choose the form $\mathcal{P}(j(\tau)) = (j(\tau) - j(i\Im\tau_0))^2$ (or other polynomials that contain such a factor) so that $i\Im\tau_0$ is a solution of $\mathcal{P}(j(\tau)) = \mathcal{P}'(j(\tau)) = 0$, consequently being an extrema of the scalar potential. It is also possible to change the shape of scalar potential near $i\infty$ by introducing nonminimal gravitational couplings of modulus fields to the Ricci scalar. After transforming back from the Jordan frame to Einstein frame, the modulus field can be stabilized at intermediate values of τ more easily with the transformed scalar potential.

The asymptotic behavior of the previous modular forms near $i\infty$ within the fundamental domain can be calculated from [95] (for z_0 with $\Im z_0$ sufficiently large)

$$\begin{aligned} Y_{\mathbf{1}}^{(4)}(z_0) &= -12p\epsilon + 96p^4\epsilon^4 + \mathcal{O}(\epsilon^7), \\ Y_{\mathbf{1}'}^{(8)}(z_0) &= -6p\epsilon - 1230p^4\epsilon^4 + \mathcal{O}(\epsilon^7), \\ Y_{\mathbf{1}''}^{(8)}(z_0) &= 144p^2\epsilon^2 - 2304p^5\epsilon^5 + \mathcal{O}(\epsilon^8), \end{aligned} \quad (5.16)$$

with

$$\epsilon = \exp\left[-\frac{2\pi}{3}\Im z_0\right], \quad p = \exp\left[i\frac{2\pi}{3}\Re z_0\right], \quad (5.17)$$

where $\epsilon \ll 1$.

Let us suppose that the modulus field could be stabilized at the imaginary axis (within the fundamental domain) with $z_0 = it_0$ for medium large t_0 , when proper SUSY breaking contributions with nonvanishing f_0 are taken into account. Naively, if the desired suppression factor to regenerate μ_{eff} comes solely from ϵ^2 when Λ_{SUSY} is naively taken to be the Planck scale, very large t_0 (here $t_0 \gtrsim 10$) is needed. Such a large t_0 possibly causes difficulties for modulus stabilization realizations, as the scalar potential for the modulus field generally diverges for $\tau \rightarrow i\infty$. Therefore, it is desirable to introduce additional suppression factors other than ϵ so that medium large t_0 can lead to desired suppression factors.

The effective μ term from the bilinear term for $H_u H_d$ in (5.8), which satisfies $k_{H_u} + k_{H_d} = k_r - k_3 + 3$ for $2k_3 \in 24\mathbb{Z} + 6$, is given by

$$\begin{aligned} \mu_{\text{eff}} &= \alpha \Lambda_{\text{SUSY}} [\eta(it_0)]^{-2k_3} Y_{\mathbf{r}}^{(k_r)}(it_0) (2t_0)^{(k_{H_u} + k_{H_d})/2}, \\ &\approx \alpha \Lambda_{\text{SUSY}} e^{\frac{k_3 \pi t_0}{6}} (2t_0)^{-\frac{k_3}{2}} Y_{\mathbf{r}}^{(k_r)}(it_0) (2t_0)^{\frac{k_r + 3}{2}}, \end{aligned} \quad (5.18)$$

where proper normalization factors for H_u, H_d from the Kahler potential are taken into account.

For fixed k_r , we require that the factor

$$\left[e^{\frac{\pi t_0}{3}} (2t_0)^{-1}\right]^{\frac{k_3}{2}} \ll 1, \quad (5.19)$$

is tiny to act as an additional suppression factor other than ϵ . As the factor

$$e^{\frac{\pi t_0}{3}} (2t_0)^{-1} > 1, \quad (5.20)$$

for all $t_0 > 0$, the value of k_3 needs to be negative.

We can choose the following modular weights and representations

$$(r_{H_u}, k_{H_u}) \sim (\mathbf{1}, 16), \quad (r_{H_d}, k_{H_d}) \sim (\mathbf{1}', 16), \quad (5.21)$$

and the choice of modular form $Y_{\mathbf{1}''}^{(8)}$ (or, for example, $Y_{\mathbf{1}}^{(12)}$, $Y_{\mathbf{1}'}^{(16)}$ etc. with proper k_{H_u} and k_{H_d}) with $k_3 = -21$ to

generate a suppressed μ_{eff} parameter. In this case, the effective μ_{eff} parameter is

$$\begin{aligned}\mu_{\text{eff}} &= \frac{\alpha}{\bar{N}_{Y_{1''}^{(8)}}} \Lambda_{\text{SUSY}} [e^{\frac{\pi i_0}{3}} (2t_0)^{-1}]^{-21/2} 144 e^{-\frac{4\pi i_0}{3}} (2t_0)^{11/2}, \\ &= 7.05 \alpha \Lambda_{\text{SUSY}} e^{-29\pi i_0/6} (2t_0)^{16},\end{aligned}\quad (5.22)$$

where $\bar{N}_{Y_{1''}^{(8)}} = 20.4152$ is the global normalization factor for the modular form $Y_{1''}^{(8)}$.

Calculations show that the value $\mu_{\text{eff}} \lesssim 10^{-17} \Lambda_{\text{SUSY}}$ can be achieved for $t_0 \gtrsim 5.01$, if Λ_{SUSY} is naively taken to be the Planck scale. For Λ_{SUSY} typically of the order $\sqrt{F} \sim \sqrt{m_{3/2} M_{\text{Pl}}} \sim 10^{10}$ GeV, it is sufficient to choose $t_0 \gtrsim 3.2$ to satisfy $\mu_{\text{eff}} \lesssim 10^{-8} \Lambda_{\text{SUSY}}$. For a smaller Λ_{SUSY} scale, a smaller suppression factor is needed to achieve an effective μ_{eff} at the electroweak scale, requiring a smaller value of t_0 . The tiny value of μ_{eff} results from the combined effect of suppression by powers of q (or $(2\Im\tau)^{-1}$) and the asymptotic suppression behavior of typical modular forms away from the fixed point $i\infty$.

It can also be verified that the suppression power of $\epsilon = \exp[-\frac{2\pi}{3}\Im z_0]$ for $H_u(\mathbf{1}, 16)$ and $H_d(\mathbf{1}', 16)$ bilinear is given by the power factor l of $(\rho_i^c \rho_j)^* = \zeta^l$ for $\zeta = \exp(2\pi i/3)$, which can be read out from [94] to be $(1 \cdot \zeta)^* = \zeta^2$ and predicted as $l = 2$. This power factor for ϵ is consistent with the result from our explicit construction.

VI. CONCLUSIONS

In this study, we propose a novel solution to the SUSY μ problem within the framework of modular flavor symmetry. The explicit μ term is prohibited by modular symmetry after assigning nonvanishing modular weights and/or non-trivial singlet representations to the H_u , H_d superfields. The effective μ term is then regenerated following the stabilization of the modulus field, with the corresponding modulus VEV breaking the modular symmetry partially (at the fixed points) or completely (in the vicinity of the fixed points). Since naturalness requires μ_{eff} to lie at the electroweak scale or soft SUSY breaking scale, a large suppression factor leading to $\mu_{\text{eff}} \ll \Lambda_{\text{SUSY}}$ is required when the modulus field is stabilized. Such a large suppression factor can naturally arise from a modulus VEV in the vicinity of certain fixed points or by the suppression factor in a weightonlike mechanism.

An ordinary single modulus stabilization mechanism can be affected by SUSY breaking contributions. The Goldstino, which is necessarily present as a consequence of UV SUSY breaking and mixes with the gravitino, can be promoted into the constrained nilpotent superfield $\mathbf{X}_{nl}$ and provides a way for an arbitrary nonsupersymmetric low-energy theory to realize SUSY nonlinearly. We discuss the stabilization mechanism of a single modulus field in the

presence of SUSY breaking contributions, described by the nonlinear SUSY realization scheme with a nilpotent Goldstino $\mathbf{X}_{nl}$.

The natural value of μ_{eff} , which is significantly suppressed relative to the SUSY scale, can be achieved from the expansion of typical modular forms using a tiny deviation parameter near the fixed points. This is similar to the mechanism that can naturally generate mass hierarchies for quarks and leptons in the vicinity of certain fixed points. The exponentiation power of the deviation parameter in the suppression factor, in fact, depends solely on the $\mathbb{Z}_3^{ST}$ charge (for fixed point ω), or $\mathbb{Z}_3^T$ charge (for fixed point $i\infty$) of H_u , H_d , rather than the explicit form of modular forms.

For a modulus VEV with an intermediate value of $\Im\tau$, the significantly suppressed μ_{eff} results from the combined effects of suppression by powers of $q^{1/24}$ [or $(2\Im\tau)^{-1}$], where the exponentiation power is determined by the modular weights of H_u and H_d , along with the asymptotic suppression behavior of typical modular forms away from the fixed point $i\infty$, taking the form of appropriate power of the tiny deviation parameter.

The modulus stabilization mechanism with the nilpotent $\mathbf{X}_{nl}$ superfield can be fairly generic and ubiquitous. The regeneration of a small μ_{eff} term following modulus stabilization, which just needs a modulus VEV near some fixed points or through a weightonlike mechanism for the $H_u H_d$ bilinear, is independent of the concrete modulus stabilization mechanism.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: PROPERTIES OF MODULAR A_4

The finite modular group $A_4 \cong \Gamma_3$ can be generated by

$$S^2 = (ST)^3 = T^3 = 1. \quad (A1)$$

There are four inequivalent irreducible representations of A_4 : three singlets $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ and a triplet $\mathbf{3}$.

The triplet representation $\mathbf{3}$ (in the basis where T is diagonal) is given by

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \quad (\text{A2})$$

The decompositions of the direct product of A_4 representations are

$$\begin{aligned} \mathbf{1}' \otimes \mathbf{1}' &= \mathbf{1}'', & \mathbf{1}' \otimes \mathbf{1}'' &= \mathbf{1}, & \mathbf{1}'' \otimes \mathbf{1}'' &= \mathbf{1}', \\ \mathbf{3} \otimes \mathbf{3} &= \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3}_S \oplus \mathbf{3}_A, \end{aligned} \quad (\text{A3})$$

where $\mathbf{3}_{S(A)}$ denotes the symmetric (antisymmetric) combinations, respectively.

Given two A_4 triplet representations $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$, the irreducible representations obtained

from their direct products are,

$$\begin{aligned} \mathbf{1} &= \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2, \\ \mathbf{1}' &= \alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1, \\ \mathbf{1}'' &= \alpha_2\beta_2 + \alpha_1\beta_3 + \alpha_3\beta_1, \\ \mathbf{3}_S &= (2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2, 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1, 2\alpha_2\beta_2 \\ &\quad - \alpha_1\beta_3 - \alpha_3\beta_1), \\ \mathbf{3}_A &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_1\beta_2 - \alpha_2\beta_1, \alpha_3\beta_1 - \alpha_1\beta_3). \end{aligned} \quad (\text{A4})$$

Modular weight $k = 2$ and the representation $\mathbf{r} = \mathbf{3}$ are given as $Y_3^{(2)} = (Y_1, Y_2, Y_3)^T$ with

$$\begin{aligned} Y_1(\tau) &= \frac{i}{2\pi} \left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right], \\ Y_2(\tau) &= \frac{-i}{\pi} \left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2 \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} \right], \\ Y_3(\tau) &= \frac{-i}{\pi} \left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2 \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} \right], \end{aligned} \quad (\text{A5})$$

where $\eta(\tau)$ is the Dedekind eta function,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i \tau}. \quad (\text{A6})$$

The q expansions of $Y_{1,2,3}(\tau)$ are given as

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + 84q^4 + 72q^5 + \dots, \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + 18q^3 + 14q^4 + \dots), \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + 4q^3 + 8q^4 + \dots). \end{aligned} \quad (\text{A7})$$

APPENDIX B: EINSTEIN SERIES

The Eisenstein series $G_k(\tau)$, defined by

$$\begin{aligned} G_k(\tau) &= \sum_{m,n \in \mathbb{Z}}' \frac{1}{(m+n\tau)^k} = \zeta(k) \sum_{\substack{ac,d \in \mathbb{Z}; \\ \gcd(c,d)=1}} \frac{1}{(c\tau+d)^k}, \\ &= 1 \frac{1}{(c\tau+d)^k}, \end{aligned} \quad (\text{B1})$$

is a modular form of weight k for $\tau \in \mathcal{H}$ and has the following Fourier decomposition

$$G_k(\tau) = 2\zeta(k) + 2 \frac{(2\pi i)^k}{\Gamma(k)} \sum_{N=1}^{\infty} \sigma_{k-1}(N) q^N, \quad (\text{B2})$$

with $\sigma_\alpha(N)$ the sums of divisors functions, defined for any $\alpha \in \mathbb{C}$ by

$$\sigma_\alpha(N) = \sum_{l|N} l^\alpha. \quad (\text{B3})$$

It can be proven that $G_4(\rho) = G_6(i) = 0$.

Factoring out $2\zeta(k)$, and using the relation between Bernoulli numbers B_k and $\zeta(k)$ for even k , the normalized Eisenstein series E_k can be defined by

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 - \frac{2k}{B_k} \sum_{N=1}^{\infty} \sigma_{k-1}(N) q^N, \quad (\text{B4})$$

with some of the $\nu_k \equiv -2k/B_k$ values given by

$$\nu_4 = 240, \quad \nu_6 = 504, \quad \nu_8 = 480. \quad (\text{B5})$$

The quasimodular transformation property of $E_2(\tau)$

$$E_2(\gamma\tau) = (c\tau+d)^2 E_2(\tau) + \frac{12}{2\pi i} c(c\tau+d), \quad (\text{B6})$$

allows one to evaluate $E_2(\tau)$ at the points $\tau = i$ and $\tau = \omega$.

The values of $E_k(\tau)$ are listed below [70]

$$\begin{aligned} E_2(i) &= \frac{3}{\pi}, & E_2(\omega) &= \frac{2\sqrt{3}}{\pi}, & E_4(i) &= \frac{48[\Gamma(\frac{5}{4})]^4}{\pi^2[\Gamma(\frac{3}{4})]^4}, \\ E_4(\omega) &= 0, & E_6(i) &= 0, & E_6(\omega) &= \frac{729[\Gamma(\frac{4}{3})]^6}{2\pi^3[\Gamma(\frac{2}{3})]^6}. \end{aligned} \quad (\text{B7})$$

The second derivative of $j(\tau)$ is given by

$$j''(\tau) = \frac{4\pi^2}{[\eta(\tau)]^{24}} \left\{ \frac{1}{6} E_2(\tau) [E_4(\tau)]^2 E_6(\tau) - \frac{2}{3} E_4(\tau) [E_6(\tau)]^2 - \frac{1}{2} [E_4(\tau)]^4 \right\}, \quad (\text{B8})$$

from Ramanujan's formulas for derivatives of Eisenstein series, which gives

$$j''(\omega) = 0, \quad j''(i) = -\frac{2\pi^2}{[\eta(i)]^{24}} [E_4(i)]^4. \quad (\text{B9})$$

APPENDIX C: SOLUTIONS OF $H'(\tau) = 0$

Such a constraint reduces to

$$H'(\tau) = \{j(\tau)^{n/3} [j(\tau) - 1728]^{m/2} \mathcal{P}(j(\tau))\}' = 0, \quad (\text{C1})$$

or equivalently,

$$H(\tau) j'(\tau) \left[\frac{n}{3} \frac{1}{j(\tau)} + \frac{m}{2} \frac{1}{j(\tau) - 1728} + \frac{1}{\mathcal{P}(j(\tau))} \frac{\partial \mathcal{P}(j(\tau))}{\partial j(\tau)} \right] = 0. \quad (\text{C2})$$

Note that the expression

$$H(\tau) \left[\frac{n}{3} \frac{1}{j(\tau)} + \frac{m}{2} \frac{1}{j(\tau) - 1728} + \frac{1}{\mathcal{P}(j(\tau))} \frac{\partial \mathcal{P}(j(\tau))}{\partial j(\tau)} \right], \quad (\text{C3})$$

can be singular at certain solutions of $H(\tau) = 0$. Such singular behavior would be eliminated by the multiplication of $j'(\tau)$ to render $H'(\tau)$ finite at such points, which can be seen in (4.23).

First, we would like to discuss the case $H(\tau) \neq 0$. From Eq. (C3), it is obvious that either

$$\left[\frac{n}{3} \frac{1}{j(\tau)} + \frac{m}{2} \frac{1}{j(\tau) - 1728} + \frac{1}{\mathcal{P}(j(\tau))} \frac{\partial \mathcal{P}(j(\tau))}{\partial j(\tau)} \right] = 0, \quad (\text{C4})$$

or $j'(\tau) = 0$.

When $H(\tau) \neq 0$, for non-negative integers m and n [by the definition of $H(\tau)$], we have the following discussions for Eq. (C4) after eliminating the $H(\tau)$ factor:

- (i) $m + n > 0$:
- (a) When $n \neq 0$ and $m = 0$, condition (C4) reduces to

$$\frac{d\mathcal{P}(j(\tau))}{\mathcal{P}(j(\tau))} = -\frac{n}{3} \frac{dj(\tau)}{j(\tau)}, \quad (\text{C5})$$

which gives

$$\mathcal{P}(j(\tau)) = C_0 [j(\tau)]^{-n/3}. \quad (\text{C6})$$

So, $\mathcal{P}(j(\tau))$ is not a polynomial of $j(\tau)$ unless $n \in 3\mathbb{Z}$ is negative, contradicting the assumption that n is a non-negative integer.

- (b) When $m \neq 0$ and $n = 0$, it can also be deduced that $\mathcal{P}(j(\tau))$ is a polynomial of $j(\tau)$ unless $m \in 2\mathbb{Z}$ is negative, contradicting the assumption that m is a non-negative integer.
- (c) For $m \neq 0$ and $n \neq 0$, similar deduction gives

$$\mathcal{P}(j(\tau)) = C'_0 [j(\tau)]^{-n/3} [j(\tau) - 1728]^{-m/2}. \quad (\text{C7})$$

So, $\mathcal{P}(j(\tau))$ is not a polynomial of $j(\tau)$ unless both $m \in 2\mathbb{Z}$ and $n \in 3\mathbb{Z}$ are negative. As m, n are defined to be non-negative integers, this condition requires $m = n = 0$, contradicting the assumptions that $m, n \neq 0$.

Therefore, $m + n > 0$ can not hold.

- (ii) $m = 0$ and $n = 0$:

With the constraints $m = n = 0$, the condition (C3) reduces to

$$\frac{\partial \mathcal{P}(j(\tau))}{\partial j(\tau)} = 0, \quad (\text{C8})$$

which gives

$$\mathcal{P}(j(\tau)) = \text{Constant}. \quad (\text{C9})$$

Since $H(\tau) \neq 0$, this constant must therefore be nonzero.

When $H(\tau) \neq 0$, for the condition $j'(\tau) = 0$, the derivative of the Klein j function is given by

$$\frac{\partial}{\partial \tau} j(\tau) = -2\pi i \frac{E_6(\tau) E_4^2(\tau)}{[\eta(\tau)]^{24}} = 0. \quad (\text{C10})$$

It can be checked that its solutions are ω and i , contradicting the assumption $H(\tau) \neq 0$ with $m = n = 0$. Therefore, the condition $H'(\tau) = 0$ with $H(\tau) \neq 0$ implies that $H(\tau)$ is a nonzero constant.

The solutions of $H'(\tau) = 0$ for the case $H(\tau) = 0$ are addressed in the paragraphs containing Eqs. (4.43)–(4.45).

- [1] ATLAS Collaboration, *Phys. Lett. B* **710**, 49 (2012).
- [2] CMS Collaboration, *Phys. Lett. B* **710**, 26 (2012).
- [3] J. E. Kim and H. P. Nilles, *Phys. Lett.* **138B**, 150 (1984).
- [4] G. F. Giudice and A. Masiero, *Phys. Lett. B* **206**, 480 (1988).
- [5] K. J. Bae, H. Baer, V. Barger, and D. Sengupta, *Phys. Rev. D* **99**, 115027 (2019).
- [6] A. Strominger, *Commun. Math. Phys.* **133**, 163 (1990).
- [7] P. Candelas and X. de la Ossa, *Nucl. Phys.* **B355**, 455 (1991).
- [8] F. Feruglio, [arXiv:1706.08749](#).
- [9] F. Feruglio and A. Romanino, *Rev. Mod. Phys.* **93**, 015007 (2021).
- [10] T. Kobayashi and M. Tanimoto, *Int. J. Mod. Phys. A* **39**, 2441012 (2024).
- [11] G. J. Ding and S. F. King, *Rep. Prog. Phys.* **87**, 084201 (2024).
- [12] G. J. Ding and J. W. F. Valle, *Phys. Rep.* **1109**, 1 (2025).
- [13] T. Kobayashi, K. Tanaka, and T. H. Tatsuishi, *Phys. Rev. D* **98**, 016004 (2018).
- [14] H. Okada and Y. Orikasa, *Phys. Rev. D* **100**, 115037 (2019).
- [15] X. Du and F. Wang, *J. High Energy Phys.* 02 (2021) 221.
- [16] S. T. Petcov and A. V. Titov, *Phys. Rev. D* **97**, 115045 (2018).
- [17] J. C. Criado and F. Feruglio, *SciPost Phys.* **5**, 042 (2018).
- [18] T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, *J. High Energy Phys.* 11 (2018) 196.
- [19] H. Okada and M. Tanimoto, *Phys. Lett. B* **791**, 54 (2019).
- [20] P. P. Novichkov, S. T. Petcov, and M. Tanimoto, *Phys. Lett. B* **793**, 247 (2019).
- [21] H. Okada and M. Tanimoto, *Phys. Rev. D* **103**, 015005 (2021).
- [22] S. T. Petcov and M. Tanimoto, *Eur. Phys. J. C* **83**, 579 (2023).
- [23] S. Centelles Chuliá, R. Kumar, O. Popov, and R. Srivastava, *Phys. Rev. D* **109**, 035016 (2024).
- [24] R. Kumar, P. Mishra, M. K. Behera, R. Mohanta, and R. Srivastava, *Phys. Lett. B* **853**, 138635 (2024).
- [25] T. Nomura and H. Okada, [arXiv:2409.10912](#).
- [26] G. Pathak, P. Das, and M. K. Das, *Eur. Phys. J. C* **85**, 569 (2025).
- [27] J. T. Penedo and S. T. Petcov, *Nucl. Phys.* **B939**, 292 (2019).
- [28] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, *J. High Energy Phys.* 04 (2019) 005.
- [29] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, *Phys. Rev. D* **100**, 115045 (2019); **101**, 039904(E) (2020).
- [30] X. Wang, *Nucl. Phys.* **B962**, 115247 (2021).
- [31] B. Y. Qu, X. G. Liu, P. T. Chen, and G. J. Ding, *Phys. Rev. D* **104**, 076001 (2021).
- [32] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, *J. High Energy Phys.* 04 (2019) 174.
- [33] G. J. Ding, S. F. King, and X. G. Liu, *Phys. Rev. D* **100**, 115005 (2019).
- [34] C. Y. Yao, X. G. Liu, and G. J. Ding, *Phys. Rev. D* **103**, 095013 (2021).
- [35] I. de Medeiros Varzielas and J. Lourenço, *Nucl. Phys.* **B984**, 115974 (2022).
- [36] M. A. Abbas, *Lett. High Energy Phys.* **2024**, 545 (2024).
- [37] B. Y. Qu and G. J. Ding, *J. High Energy Phys.* 08 (2024) 136.
- [38] T. Nomura and H. Okada, *Phys. Lett. B* **868**, 139763 (2025).
- [39] G. J. Ding, J. N. Lu, S. T. Petcov, and B. Y. Qu, *J. High Energy Phys.* 01 (2025) 191.
- [40] C. C. Li, J. N. Lu, and G. J. Ding, *J. High Energy Phys.* 12 (2024) 189.
- [41] F. Feruglio, A. Strumia, and A. Titov, *J. High Energy Phys.* 07 (2023) 027.
- [42] F. Feruglio, M. Parriciatu, A. Strumia, and A. Titov, *J. High Energy Phys.* 08 (2024) 214.
- [43] F. J. de Anda, S. F. King, and E. Perdomo, *Phys. Rev. D* **101**, 015028 (2020).
- [44] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, *Prog. Theor. Exp. Phys.* **2020**, 053B05 (2020).
- [45] P. Chen, G. J. Ding, and S. F. King, *J. High Energy Phys.* 04 (2021) 239.
- [46] Y. Zhao and H. H. Zhang, *J. High Energy Phys.* 03 (2021) 002.
- [47] S. F. King and Y. L. Zhou, *J. High Energy Phys.* 04 (2021) 291.
- [48] G. J. Ding, S. F. King, and C. Y. Yao, *Phys. Rev. D* **104**, 055034 (2021).
- [49] G. J. Ding, S. F. King, and J. N. Lu, *J. High Energy Phys.* 11 (2021) 007.
- [50] G. J. Ding, S. F. King, J. N. Lu, and B. Y. Qu, *J. High Energy Phys.* 10 (2022) 071.
- [51] G. Charalampous, S. F. King, G. K. Leontaris, and Y. L. Zhou, *Phys. Rev. D* **104**, 115015 (2021).
- [52] X. K. Du and F. Wang, *J. High Energy Phys.* 01 (2023) 036.
- [53] S. F. King, G. K. Leontaris, L. Marsili, and Y. L. Zhou, *J. High Energy Phys.* 12 (2024) 211.
- [54] A. Baur, H. P. Nilles, A. Trautner, and P. K. S. Vaudrevange, *Phys. Lett. B* **795**, 7 (2019).
- [55] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, *J. High Energy Phys.* 07 (2019) 165.
- [56] K. Ishiguro, T. Kobayashi, and H. Otsuka, *Nucl. Phys.* **B973**, 115598 (2021).
- [57] F. Feruglio, V. Gherardi, A. Romanino, and A. Titov, *J. High Energy Phys.* 05 (2021) 242.
- [58] M. Cvetič, A. Font, L. E. Ibanez, D. Lust, and F. Quevedo, *Nucl. Phys.* **B361**, 194 (1991).
- [59] J. M. Leedom, N. Righi, and A. Westphal, *J. High Energy Phys.* 02 (2023) 209.
- [60] E. Gonzalo, L. E. Ibáñez, and Á. M. Uranga, *J. High Energy Phys.* 05 (2019) 105.
- [61] Z. Komargodski and N. Seiberg, *J. High Energy Phys.* 09 (2009) 066.
- [62] C. P. Burgess and F. Quevedo, *Fortschr. Phys.* **70**, 2200077 (2022).
- [63] L. Aparicio, F. Quevedo, and R. Valandro, *J. High Energy Phys.* 03 (2016) 036.
- [64] S. M. Kuzenko and S. J. Tyler, *J. High Energy Phys.* 04 (2011) 057.
- [65] F. Farakos and A. Kehagias, *Phys. Lett. B* **724**, 322 (2013).
- [66] E. Dudas, S. Ferrara, A. Kehagias, and A. Sagnotti, *J. High Energy Phys.* 09 (2015) 217.
- [67] R. Kallosh and T. Wrase, *Phys. Rev. D* **92**, 105010 (2015),

- [68] R. Kallosh, A. Karlsson, and D. Murli, *Phys. Rev. D* **93**, 025012 (2016).
- [69] S. Ferrara, R. Kallosh, A. Van Proeyen, and T. Wrase, *J. High Energy Phys.* 04 (2016) 065.
- [70] E. D'Hoker and J. Kaidi, [arXiv:2208.07242](#).
- [71] S. J. D. King and S. F. King, *J. High Energy Phys.* 09 (2020) 043.
- [72] T. Kobayashi and H. Otsuka, *Phys. Rev. D* **101**, 106017 (2020).
- [73] T. Kobayashi and H. Otsuka, *Phys. Rev. D* **102**, 026004 (2020).
- [74] K. Ishiguro, T. Kobayashi, and H. Otsuka, *J. High Energy Phys.* 03 (2021) 161.
- [75] P. P. Novichkov, J. T. Penedo, and S. T. Petcov, *J. High Energy Phys.* 03 (2022) 149.
- [76] V. Knapp-Perez, X.-G. Liu, H. P. Nilles, S. Ramos-Sanchez, and M. Ratz, *Phys. Lett. B* **844**, 138106 (2023).
- [77] S. F. King and X. Wang, *J. Cosmol. Astropart. Phys.* 07 (2024) 073.
- [78] G. J. Ding, S. Y. Jiang, and W. Zhao, *J. Cosmol. Astropart. Phys.* 10 (2024) 016.
- [79] T. Higaki, T. Kobayashi, K. Nasu, and H. Otsuka, *J. High Energy Phys.* 09 (2024) 024.
- [80] T. Higaki, J. Kawamura, and T. Kobayashi, *J. High Energy Phys.* 04 (2024) 147.
- [81] T. Kobayashi, K. Nasu, R. Sakuma, and Y. Yamada, *Phys. Rev. D* **108**, 115038 (2023).
- [82] S. Funakoshi, J. Kawamura, T. Kobayashi, K. Nasu, and H. Otsuka, *J. High Energy Phys.* 03 (2025) 093.
- [83] S. F. King and X. Wang, *Phys. Rev. D* **110**, 076026 (2024).
- [84] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Graduate Texts in Mathematics (Springer, New York, NY, 1990).
- [85] J. M. Maldacena and C. Nunez, *Int. J. Mod. Phys. A* **16**, 822 (2001).
- [86] S. Kachru, R. Kallosh, A. D. Linde, and S. P. Trivedi, *Phys. Rev. D* **68**, 046005 (2003).
- [87] J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton University Press, Princeton, NJ, 1992).
- [88] S. R. Coleman, *Phys. Rev. D* **15**, 2929 (1977).
- [89] C. G. Callan and S. R. Coleman, *Phys. Rev. D* **16**, 1762 (1977).
- [90] A. Masoumi, K. D. Olum, and J. M. Wachter, *J. Cosmol. Astropart. Phys.* 10 (2017) 022.
- [91] M. J. Duncan and L. G. Jensen, *Phys. Lett. B* **291**, 109 (1992).
- [92] D. Buttazzo, G. Degrandi, P. P. Giardino, G. F. Giudice, F. Sala, A. Salvio, and A. Strumia, *J. High Energy Phys.* 12 (2013) 089.
- [93] S. Kikuchi, T. Kobayashi, K. Nasu, H. Otsuka, S. Takada, and H. Uchida, *Prog. Theor. Exp. Phys.* **2022**, 123B02 (2022).
- [94] P. P. Novichkov, J. T. Penedo, and S. T. Petcov, *J. High Energy Phys.* 04 (2021) 206.
- [95] S. T. Petcov and M. Tanimoto, *J. High Energy Phys.* 08 (2023) 086.
- [96] S. T. Petcov, *Phys. Lett. B* **850**, 138540 (2024).