

Landau singularities from Whitney stratifications

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We demonstrate that the complete and nonredundant set of Landau singularities of Feynman integrals may be explicitly obtained from the Whitney stratification of a certain map. As a proof of concept, we leverage recent theoretical and algorithmic advances in their computation in order to determine this set for nontrivial examples of two-loop integrals. Interestingly, different strata of the Whitney stratification describe not only the singularities of a given integral, but also those of integrals obtained from kinematic limits, e.g., by setting some of its masses or momenta to zero.

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At the heart of both cross-section calculations at the Large Hadron Collider and gravitational wave physics lie the evaluation of *Feynman integrals*. These integrals are multivalued meromorphic functions of their kinematic variables, and understanding their analytic structure has been an ongoing quest for theoretical physicists since the late 1950s [1].

Key information on the analytic structure of Feynman integrals is provided by the values of the kinematic variables for which they become singular, first studied in the pioneering work of Landau [2]. A virtue of these *Landau singularities* is that knowing them in advance may significantly aid the evaluation of the integrals, for example with the canonical differential equations approach [3]: They may constrain or even fully predict [4] the analytic building blocks or (symbol) *letters* [5] of this approach, thereby turning the determination of the differential equations from a symbolic, to a much simpler numerical problem. Thanks to their utility, also in other aspects of Feynman integration [6], Landau analysis is currently experiencing a revival with ever-increasing momentum, see for example [4,7–14].

A mathematically robust definition of what a Landau singularity is has been provided by Pham [15] as his *Landau variety*; roughly speaking it is characterized by the critical

values of a projection map from the variety describing the integrand in terms of all integration and kinematic variables, to the space of just the kinematic variables. Its direct calculation from this definition has proven quite challenging, and much of the recent effort has focused on developing alternative methods that more simply compute varieties that either contain it [7,8], or are contained in it [4,13,14]. In other words, these methods may in general either provide spurious candidates or miss certain Landau singularities entirely, and it is not known when these phenomena do not occur beyond certain special classes of integrals.

In this work, we demonstrate that the rigorous definition of Landau singularities *can* be used to obtain their complete set, and nothing but their complete set, in a practical manner. To this end, we apply recent advances on the computation of Whitney stratifications [16,17], which enter this definition. In particular, we leverage their implementation in the WHITNEYSTRATIFICATIONS2.27 package for the MACAULAY2 [18] software, available at [19], in order to compute the Landau singularities of nontrivial two-loop integrals, such as the two-mass hard slashed box and the parachute. While this computational route is currently not as efficient as that of other specialized software for sub- or supersets of Landau singularities [8,13,14], it provides a proof of concept that paves the way for their fast and accurate determination in the future.

Feynman integrals and their singularities. In this article we consider one-particle irreducible Feynman graphs $G := (E, V)$ with set of edges and vertices E and V , respectively, and loop number $L = n - |V| + 1$, where we abbreviate $n := |E|$. The vertex set V has the disjoint

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partition $V = V_{\text{ext}} \sqcup V_{\text{int}}$ where each vertex $v \in V_{\text{ext}}$ is assigned an external incoming four-dimensional momentum $p_v \in \mathbb{R}^{1,3}$, and each internal edge e is assigned a mass parameter m_e . Using dimensional regularization with $D := 4 - 2\epsilon$ and the *Lee-Pomeransky representation* [20], we assign the following integral to G :

$$\mathcal{I} = \int_{\mathbb{R}_+^n} \left(\prod_{i=1}^n \frac{x_i^{\nu_i} dx_i}{x_i \Gamma(\nu_i)} \right) \frac{1}{\mathcal{G}^{D/2}}, \quad \mathcal{G} = \mathcal{U} + \mathcal{F}, \quad (1)$$

where ν_i are propagator powers and $\mathcal{U}$, respectively $\mathcal{F}$, are the first and second *Symanzik polynomials*. These are homogeneous polynomials of degree L and $L + 1$ in the x_i , respectively, and may be easily obtained from graph-theoretic data, see e.g. [21]. The polynomial coefficients of $\mathcal{U}$ are just numbers, whereas those of $\mathcal{F}$ depend on the kinematic variables p_v , m_e , and are not necessarily algebraically independent.

We will render the representation (1) of Feynman integrals projective, similarly to how this is done in the Feynman parameter representation [22], see also [23]. Namely, we insert $1 = \int x_0^{\nu_0} \delta(1 - x_0) dx_0$ in (1) and, dropping the gamma functions, we express it as an integral of the *homogenized Lee-Pomeransky polynomial* $\mathcal{G}_h$,

$$\mathcal{I} = \int_{\mathbb{P}_+^n} \frac{\prod_{i=0}^n x_i^{\nu_i}}{\mathcal{G}_h^{D/2}} \Omega, \quad \mathcal{G}_h := \mathcal{U}x_0 + \mathcal{F}, \quad (2)$$

over the *projective simplex* $\mathbb{P}_+^n = \{[x_0 : x_1 : \dots : x_n] \in \mathbb{RP}^n | x_i > 0, i = 0, 1, \dots, n\}$. In this equation, $\Omega = \sum_{i=0}^n (-1)^{n-1-i} dx_0/x_0 \wedge \dots \wedge \widehat{dx_i/x_i} \wedge \dots \wedge dx_n/x_n$, where the hatted term is omitted, and $\nu_0 = (L + 1)D/2 - \sum_{i=1}^n \nu_i$, ensuring projective invariance.

Singularities of Feynman integrals were initially formulated in terms of conditions for their contour of integration to become trapped between colliding poles of the integrand. These conditions are known as the Landau equations [2], which in the representation (2) are,

$$\mathcal{G}_h = 0 \quad \text{and} \quad x_i \frac{\partial \mathcal{G}_h}{\partial x_i} = 0 \quad \forall i = 0, 1, \dots, n. \quad (3)$$

Due to their generically nonlinear nature, directly and efficiently solving the above equations remains a challenge. When the polynomials $\mathcal{U}$ and $\mathcal{F}$ are taken to have generic coefficients, then Landau singularities are alternatively captured by the *principal A-determinant* [24], see also [9]. This is a polynomial in the kinematic variables, which vanishes whenever the equations (3) have a solution.

While the algorithmic nature of the principal A-determinant is appealing, an issue arises when specializing the polynomial coefficients back to their Feynman integral values: in this process certain factors of the principal A-determinant may vanish identically. Given the resulting

unphysical conclusion, i.e. that the Feynman integral is singular for all values of the kinematic variables, one may be tempted to define its Landau singularities by simply removing these vanishing factors. However this definition is incomplete, as it has been definitively shown to miss singularities [12]. An empirical refinement of this definition, based on sequential limits of the generic polynomial coefficients, has also been proposed and tested in many one- and some two-loop integrals [4], however a more extensive vetting and/or a proof would be desirable. Another closely related refinement, known as the *principal Landau determinant*, has also been recently defined and implemented in the Julia package PLD.JL [13,14]; unfortunately, this also misses singularities in certain cases.

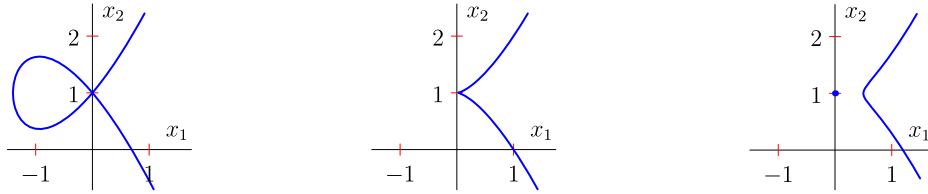
A different approach to singularities of integrals, due to Pham, boils down to considering the subspace of integration variables where the integrand denominator vanishes, and detecting for which values of the external parameters this variety changes topology. We illustrate the essence of this approach with the following simple example of an the integrand denominator,

$$f_z(x_1, x_2) = (x_2 - 1)^2 - (x_1 - z)x_1^2, \quad (4)$$

where x_1, x_2 are the integration variables and z the external (kinematic) parameter. The variety defined by $f_z(x_1, x_2) = 0$ is the curve shown in Fig. 1 for different values of z . Clearly, the topology of the curve changes at $z = 0$, and this is precisely where the integral develops a Landau singularity.

In Pham's setting [15], one may equivalently consider the same variety but in the space of both integration variables and parameters, and examine the map projecting it to the space of parameters. The critical values where this projection map drops rank then define the Landau variety, capturing the aforementioned changes in topology. In the example (4), we apply the projection map $(x_1, x_2, z) \mapsto z$ from the surface $f_z(x_1, x_2) = 0$ inside the x_1, x_2, z space, this map's critical value is exactly $z = 0$.

Our work is motivated by the fact that no general recipe existed for directly computing the Landau variety. An upper bound for the latter is provided by Brown's polynomial reduction [7], also implemented in HyperInt [8]. It has subsequently been proposed that this superset may be reduced to the Landau variety by only keeping the components where the *Euler characteristic* of the parametric variety describing the integrand changes from its generic value [13,14]. Closely related to the number of handles of a surface, the Euler characteristic χ is a simple sufficient measure of changes in its topology, and the aforementioned proposal may be proven correct as a consequence of either [25] or [26] in the special case where the Landau variety is equal to the zero set of the principal A-determinant. More generally, however, it is not currently known if this proposal fails in cases where a change in topology leaves the Euler characteristic invariant.



(a) $z = -1$: Curve with one loop. (b) $z = 0$: Curve with a cusp. (c) $z = 1/2$: Smooth curve with two connected components.

FIG. 1. Example of how Landau singularities of integrals are captured by the changes of their integrand's topology. Here, the integrand (4) defines a curve in the integration variables x_i , whose topology changes when the external parameter takes the value $z = 0$. This precisely coincides with the integral's Landau singularity.

Here, we will instead show that the Landau variety may be computed directly and completely algorithmically, by means of the Whitney stratification of the projection map discussed two paragraphs above. This is summarized in Definition 1, and in what follows we introduce the necessary foundations.

From Whitney to Landau. Let $\mathbb{K}$ be either the field of real or complex numbers. In the discussion that follows it will be convenient to employ the following notation for the algebraic variety defined by polynomials $g_1, \dots, g_r$ in a polynomial ring in the variables $x = (x_1, \dots, x_n)$ over $\mathbb{K}$, $\mathbb{K}[x_1, \dots, x_n]$,

$$\mathbf{V}(g_1, \dots, g_r) := \{x \in \mathbb{K}^n | g_1(x) = \dots = g_r(x) = 0\}.$$

We will now proceed to define the Whitney stratification of a potentially singular variety, which loosely speaking decomposes it into smooth components such that the local geometry is the same for any two points of the same connected component. Consider an algebraic variety X of dimension d . Then a flag $X_\bullet$ of varieties $X_0 \subset \dots \subset X_d = X$ contained in one another is a *Whitney stratification* of X if the (open) sets $X_i - X_{i-1}$, consisting of all points in X_i which are not in X_{i-1} are smooth manifolds for all i , and *Whitney's condition B* holds for all pairs of connected components or *strata* of these manifolds. A pair of strata, M, N , whose closures obey $\bar{M} \subset \bar{N}$, satisfy *Whitney's condition B* at a point $x \in M$ with respect to N if for every sequence $\{p_\ell\} \subset M$ and $\{q_\ell\} \subset N$ with $\lim p_\ell = \lim q_\ell = x$, the limit of secant lines between p_ℓ, q_ℓ is contained in the limit of tangent planes to N at q_ℓ . Furthermore, the pair M, N satisfies condition B if condition B holds, with respect to N , at all points $x \in M$.

A concrete example illustrating these definitions is the Whitney stratification of the Whitney cusp,

$$X = \mathbf{V}(x^2 + z^3 - y^2 z^2),$$

shown in Fig. 2, where the color-coding used here and below matches that of the figure. Notice that while this

surface would still be decomposed into smooth pieces if we took the entire y axis $\mathbf{V}(x, z)$ as a component. However, the local geometry at the origin $\{(0, 0, 0)\} = \mathbf{V}(x, y, z)$ is different, as the loop in the (x, z) plane contracts to a point. Whitney's condition B captures this, as two sequences approaching the origin, one on the negative y axis and one on the component above it, have a limiting secant line that does not lie on the limiting tangent plane of the latter component. Therefore the origin needs to be placed in a separate stratum. Together with the two half-infinite lines of the y axis and the four two-dimensional strata of the Whitney cusp, the full Whitney stratification in the field of reals is thus

$$X \supset \mathbf{V}(x, z) \supset \{(0, 0, 0)\}.$$

Note that a Whitney stratification of a variety is not unique, however it was shown in [27] that when X is any complex algebraic (or analytic) variety there exists a unique minimal (or coarsest) Whitney stratification, such that all other Whitney stratifications are obtained by adding strata inside the strata of the minimal one, see also [28].

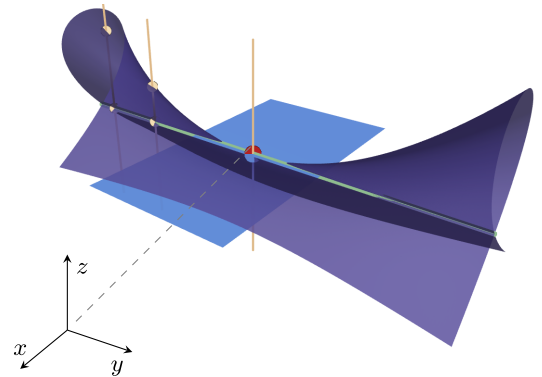


FIG. 2. Whitney stratification of the *Whitney cusp* $X = \mathbf{V}(x^2 + z^3 - y^2 z^2)$ is $X \supset \mathbf{V}(x, z) \supset \{(0, 0, 0)\}$. The origin must be a separate stratum since the limiting tangent plane of a sequence approaching it from the top left does not contain the limiting secant line between this and the sequence on the negative y axis.

Next, we consider an algebraic map $f: X \rightarrow Y$ between varieties X and Y . A Whitney stratification of the map f is a pair $(X_\bullet, Y_\bullet)$ where $X_\bullet$ is a Whitney stratification of X , $Y_\bullet$ is a Whitney stratification of Y and for each strata M of X there is a stratum N of Y with $f(M) \subset N$ such that the map $f|_M: M \rightarrow N$ is a *submersion*, i.e. the differential $df|_M$ is surjective. It follows from the existence of the minimal Whitney stratification of a variety and, e.g. the algorithm of [17, §2], that there exists a unique minimal stratification of a map as well.

Given the defining equations of varieties X , Y and of a map f between them, the stratification $(X_\bullet, Y_\bullet)$ may be obtained explicitly using the algorithm of [16, 17] as implemented in the `WHITNEYSTRATIFICATIONS` `MACAULAY2` package. Let us apply these to our toy example (4), where $X = \mathbf{V}(f_z(x_1, x_2)) \subset \mathbb{R}^3$ and the map $f: X \rightarrow Y = \mathbb{R}$ is given by the projection $(x_1, x_2, z) \mapsto z$. We find that the stratification of $Y = \mathbb{R}$ is given by $\mathbf{V}(z) \subset Y$, hence in this example it correctly captures the change in topology at $z = 0$. More generally, this is guaranteed thanks to *Thom's First Isotopy Lemma* [29]: If the map f is proper (i.e. if the inverse image of any compact subset is compact) then the topology of the inverse image or fiber $f^{-1}(q)$ is constant over all points $q \in N$ for any strata N of Y .

Building on our earlier definitions and conventions, we now consider a Feynman integral specified by the homogenized Lee-Pomeransky polynomial $\mathcal{G}_h$ in the ring $\mathbb{C}[z][x] := \mathbb{C}[z_1, \dots, z_m][x_0, \dots, x_n]$, where the z_i are external parameters (e.g. masses, momenta) and the x_i are the projective integration variables. We make no assumptions on the independence of the parameters, and in particular allow there to be algebraic relations between them. We then seek to describe the *Landau variety*, which is the locus in the parameter space $\mathbb{C}_z^m$ where the Feynman integral is singular; the following definition is (a minor rephrasing of) that of Pham [15, §IV.5] for Feynman integrals in Lee-Pomeransky form,

Definition 1 (Landau variety). Consider a Feynman integral in homogenized Lee-Pomeransky form (2). Set $X = \mathbf{V}(x_0 \cdots x_n \mathcal{G}_h) \subset \mathbb{P}_x^n \times \mathbb{C}_z^m$ and consider the projection map $\pi: X \rightarrow \mathbb{C}_z^m$ to the space of kinematics. The Landau variety is the variety Y_{m-1} appearing in the minimal Whitney stratification $(X_\bullet, Y_\bullet)$ of the map π .

The virtue of the definition is that it renders the calculation of the Landau variety fully algorithmic. The relevant algorithms have been implemented in the `WHITNEYSTRATIFICATIONS` package, and include both obtaining the Whitney stratification of varieties and maps, and coarsening a Whitney stratification of a variety to the minimal one [30].

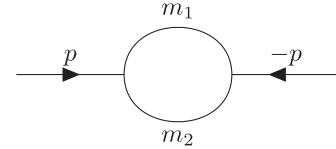
It is also interesting to note that the lower dimensional strata in the stratification of Definition 1 also have a physical significance: As we will see in an example shortly, they contain the Landau variety of integrals obtained as kinematic limits of the original one, e.g. by setting some of its

masses or momenta to zero. More generally, the stratification in Definition 1 describes all regions of the kinematic space such that any choice of parameters z in a given region yields a variety $\pi^{-1}(z)$ in $\mathbb{P}_x^n \times \mathbb{C}_z^m$ with constant topology. This follows from the fact that the map π in Definition 1 is proper by construction, as the fibers are (always compact) projective varieties and the aforementioned Thom's isotopy lemma always holds.

This observation has further implications when coupled with the result of [31], that the Euler characteristic $|n + 1 - \chi(\mathbf{V}(x_0 x_1 \cdots x_n \mathcal{G}_h))|$ counts the number of basis or master integrals [32] in the integral family (2). Given the relation of lower strata with limits of the original integral, from their Euler characteristic computation we may obtain the number of master integrals also in these limits.

Example computations. We now demonstrate how the Landau singularities of several Feynman integrals may be calculated from Whitney stratifications of maps via Definition 1, and compare our results with existing approaches in the literature. More details on how these were carried out with the help of the `WHITNEYSTRATIFICATIONS` package may be found in Supplemental Material [33].

One-loop bubble. We begin with this simple example,



illustrating that the Whitney stratification captures this integral's singularities not only for generic m_1 , m_2 , p^2 , but also in the configurations where any of these kinematic variables vanish. Here, we have that $\mathcal{G}_h = x_0(x_1 + x_2) + (m_1^2 + m_2^2 - p^2)x_1x_2 + m_1^2x_1^2 + m_2^2x_2^2$, $X = \mathbf{V}(x_0x_1x_2\mathcal{G}_h) \subset \mathbb{P}^2 \times \mathbb{C}^3$ and $Y = \mathbb{C}^3$ is the space of kinematics. Calculating the (minimal) Whitney stratification $(X_\bullet, Y_\bullet)$ of the projection map $\pi: X \rightarrow \mathbb{C}^3$ yields the following the strata of $Y_\bullet$ below $Y_3 = Y$,

$$\begin{aligned} Y_2 &= \mathbf{V}(m_1^2) \cup \mathbf{V}(m_2^2) \cup \mathbf{V}(p^2) \\ &\quad \cup \mathbf{V}(p^4 + m_1^4 + m_2^4 - 2p^2m_1^2 - 2p^2m_2^2 - 2m_1^2m_2^2), \\ Y_1 &= \mathbf{V}(p^2, m_1^2 - m_2^2) \cup \mathbf{V}(m_2^2 - p^2, m_1^2) \cup \mathbf{V}(m_2^2, m_1^2 - p^2) \\ &\quad \cup \mathbf{V}(p^2, m_1^2) \cup \mathbf{V}(p^2, m_2^2) \cup \mathbf{V}(m_2^2, m_1^2), \\ Y_0 &= \mathbf{V}(p^2, m_1^2, m_2^2). \end{aligned} \tag{5}$$

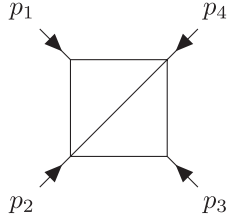
Per Definition 1, the codimension-one components, Y_2 , constitute the Landau variety. As already mentioned, the lower dimensional strata contain the Landau variety at certain limits. In the limit $m_1 \rightarrow 0$, the Landau variety is read off from the three components in Y_1 containing a single m_1^2 , and is in particular equal to

$\mathbf{V}(m_2^2 - p^2) \cup \mathbf{V}(p^2) \cup \mathbf{V}(m_2^2)$. In this example the result is the same as substituting $m_1^2 = 0$ in the original Landau variety, but unlike the stratification the latter is not always guaranteed to work. Multiple limits can be treated in a similar fashion, e.g. components containing both m_1^2, m_2^2 yield the Landau variety of the bubble with massless propagators, $\mathbf{V}(p^2)$.

Our results are in complete agreement with the known Landau singularities of the generic one-loop bubble graph and its kinematic limits, see e.g. [4], sections IV.1, 3.4 as well as the accompanying ancillary file of the latter reference. As also explained in the latter reference, Landau singularities precisely coincide with the rational letters of polylogarithmic integrals, whereas square-root letters are obtained by refactorizing products thereof. The precise refactorization rule is only known at one loop, with current attempts for a higher-loop generalization known to be incomplete.

The Landau variety gives information of the full analytic structure of the meromorphic function defined by the integral. However the methods in [17] are capable of calculating Whitney stratifications of real semialgebraic sets, potentially allowing direct access to singularities specifically in the physical region; we leave this fascinating exploration to a future work.

Slashed box. We consider this first two-loop example,



in the “two-mass hard” configuration with $p_3^2 \neq 0, p_4^2 \neq 0$ and everything else massless. In this example, $X = \mathbf{V}(x_0 \cdots x_5 \mathcal{G}_h) \subset \mathbb{P}^5 \times \mathbb{C}^4$, with $Y = \mathbb{C}^4$ spanned by $p_3^2, p_4^2, s = (p_1 + p_2)^2, t = (p_2 + p_3)^2$. The Whitney stratification (X, Y) of the projection map $\pi: X \rightarrow \mathbb{C}^4$ gives the following expression for the Landau variety,

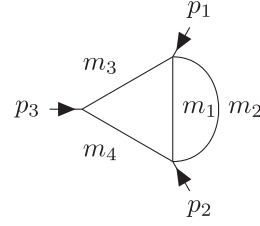
$$\begin{aligned} Y_3 = & \mathbf{V}(p_3^2) \cup \mathbf{V}(s) \cup \mathbf{V}(st + t^2 - tp_3^2 - tp_4^2 + p_3^2 p_4^2) \\ & \cup \mathbf{V}(t - p_4^2) \cup \mathbf{V}(s^2 - 2sp_3^2 + p_3^4 - 2sp_4^2 - 2p_3^2 p_4^2 + p_4^4) \\ & \cup \mathbf{V}(t - p_3^2) \cup \mathbf{V}(t) \cup \mathbf{V}(p_4^2) \cup \mathbf{V}(p_4^2 - s - t). \end{aligned} \quad (6)$$

This is in agreement with HyperInt, whereas PLD misses the component $p_4^2 - s - t = 0$. That this is a true singularity of the integral may be inferred e.g. by its explicit analytic expression [34]; therefore this nontrivial example strongly supports the completeness of our approach for computing Landau singularities.

We note that the empirical procedure proposed in [4] for the calculation of Landau singularities also reproduces (6).

It would be very interesting to investigate if its coincidence with our rigorous Definition 1 holds more generally.

Parachute integral. This is another two-loop example,



whose singularities are not entirely captured by either the naïve principal A-determinant [35], or its principal Landau determinant refinement. In particular, these approaches miss the singularity where

$$\begin{aligned} & p_3^2(m_4^2 - p_2^2)(m_3^2 - p_1^2) + (m_3^2 - m_4^2 - p_1^2 + p_2^2) \\ & \times (m_3^2 p_2^2 - m_4^2 p_1^2) \end{aligned} \quad (7)$$

vanishes, first computed in [12]. The authors of this reference define the Landau variety by demanding that the union of the (integration) coordinate hyperplanes $x_i = 0$ with the hypersurfaces $\mathcal{U} = 0$ and $\mathcal{F} = 0$ is *transverse*, otherwise *blowing up* this manifold in order to ensure this property; it turns out that the parachute requires additional blowups beyond what was previously considered sufficient for Feynman integrals. We note that the method of blowups is an equivalent alternative method to calculate the Whitney stratification in our context, since in the transversal case the Whitney stratification has a simple closed form expression.

The stratification of the map $\pi: \mathbb{P}^4 \times \mathbb{C}^7 \rightarrow \mathbb{C}^7$ is currently too taxing for the WHITNEYSTRATIFICATIONS package, however we will validate our approach by restricting the kinematics to $\{m_1^2 = 1, m_2^2 = 0, m_3^2 = 0, m_4^2 = 2, p_1^2 = -1, p_2^2 = 0\}$, keeping only p_3^2 free. In this limit, the singularity (7) reduces to $p_3^2 - 1$, and direct integration confirms its presence as a letter of the integral [36]. We confirm that this is also contained in our Whitney stratification,

$$\begin{aligned} Y_0 = & \mathbf{V}(p_3^2) \cup \mathbf{V}(p_3^2 + 2) \cup \mathbf{V}(p_3^2 - 2) \\ & \cup \mathbf{V}(p_3^2 + 1) \cup \mathbf{V}(p_3^2 - 1); \end{aligned}$$

for comparison, PLD appears to predict only the first three of these singularities.

Conclusions and outlook. We presented an algorithm to calculate the Landau variety of an integral, from the Whitney stratification of the projection map on its integrand. Applicable to any parametrized integral with a rational integrand, this provides the complete and nonredundant set of Landau singularities of not only the original integral, but also of those arising as limits of its parameters.

Leveraging the software implementation of Whitney stratifications, we focused on Feynman integrals and validated our method in several examples through two loops, including cases where singularities are missed by other approaches. It would be exciting to improve our method's efficiency by taking advantage of the special structure of Feynman integrals. Given the relation between Landau singularities and symbol letters established in [4], could it further be used to predict the latter, and prove the remarkable cluster-algebraic structures they sometimes exhibit [37]?

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