

Glueballs, constituent gluons, and instantons

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We present a constituent two-gluon description of the lowest-lying glueball states in pure Yang-Mills theory, calibrated against quenched lattice results. The framework incorporates an instanton-induced dynamical gluon mass, Casimir-scaled adjoint confinement, the short-distance adjoint Coulomb interaction, and instanton-induced central and tensor forces. The scalar 0^{++} glueball is found to be exceptionally compact, with a radius of order the instanton size, $\rho \sim \frac{1}{3}$ fm, consistent with lattice indications. By contrast, the tensor 2^{++} state remains spatially extended due to the centrifugal barrier. We also discuss the role of S – D mixing. A semiclassical analysis further supports Regge behavior for excited states, in agreement with lattice results.

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I. INTRODUCTION

A. Historic remarks

Glueballs are the color-singlet bound states of non-Abelian gauge theory, composed entirely of gluonic degrees of freedom. Their comparison with mesons provides a unique window into the mechanism of confinement, dynamical mass generation, and the role of topology in quantum chromodynamics (QCD).

One of the most obvious questions in QCD is why the observed hadrons are made of quarks and not of gluons. It appears that glueballs are much heavier than typical quark-model hadrons, and therefore they have large widths and/or complicated decay patterns, making them difficult to find. But why are glueballs so heavy? What are their masses, radii, and other parameters in a purely gluonic world, and how do they change if one includes light quarks?

In the pure Yang-Mills (“quenched”) theory, where quark degrees of freedom are absent, the glueball spectrum has been extensively determined using lattice gauge theory, which remains the most reliable nonperturbative approach to this problem [1–4]. Early high-precision lattice calculations, particularly those employing anisotropic lattices, established the ordering of the lowest-lying glueball states and provided quantitative benchmarks for their masses. These studies identified the scalar 0^{++} glueball as the

lightest state, with mass $m_{0^{++}} \approx 1.6$ GeV, followed by the tensor 2^{++} state with $m_{2^{++}} \approx 2.2$ GeV. The pseudoscalar 0^{-+} state appears at $m_{0^{-+}} \approx 2.4$ GeV, with higher-spin excitations occurring at significantly larger masses [1,3]. Subsequent studies employing improved actions, enlarged operator bases, and controlled continuum extrapolations have refined this picture and confirmed its robustness across lattice spacings and volumes. In particular, recent high-statistics calculations have provided a precise reference spectrum for pure SU(3) Yang-Mills theory, covering a wide range of J^{PC} channels [4].

While the quenched glueball spectrum is now comparatively well established, several important questions have gained renewed attention in recent years. One concerns the *internal structure* of glueballs in hadrons, including their spatial extent and form factors. Lattice studies of energy-momentum tensor matrix elements and related gravitational form factors have begun to probe the size and mass distribution of glueballs [5,6], with emerging evidence that the scalar 0^{++} state may be unusually compact compared to higher-spin excitations. Such results provide new, non-trivial constraints on phenomenological and microscopic models of glueballs. A recent review summarizes the current status of lattice calculations, experimental searches, and theoretical interpretations [7].

The nonperturbative interactions in the scalar, tensor, and pseudoscalar correlation functions at short distances were related, already three decades ago, to instanton-induced forces [8,9]; see Fig. 1. In particular, these studies showed that instantons generate a strong attractive interaction in the scalar ($J^{PC} = 0^{++}$) channel. As a consequence, it was predicted that the scalar glueball should be significantly more compact than typical hadrons, with a mean square radius $r_{\text{m.s.}} \approx 0.2$ fm. By contrast, the instanton-induced

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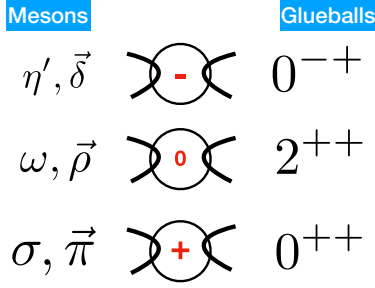


FIG. 1. Repulsive, neutral, and attractive channels induced by instanton-induced effects in Euclidean correlation functions, for mesons and glueballs.

interaction is repulsive in the pseudoscalar (0^{-+}) channel, while it is absent in the tensor (2^{++}) channel. Consequently, these three glueball channels exhibit a pattern analogous to that of the σ (scalar), η' (pseudoscalar), and ρ , ω (vector) mesons, respectively.

Correlation functions of scalar, pseudoscalar, and tensor glueball operators calculated in the instanton liquid model (ILM) in Ref. [8] are reproduced in Fig. 2. They clearly show that the interaction in these three channels is attractive, repulsive, and weak, respectively.

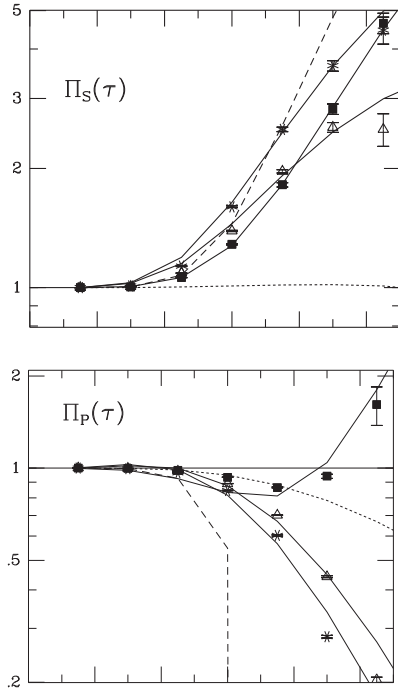


FIG. 2. Scalar G^2 and pseudoscalar $G\tilde{G}$ gluonic correlation functions normalized to the corresponding free correlators as functions of the Euclidean time separation, taken from [8]. Results in the random, quenched, and full ensembles are denoted by stars, open triangles, and solid squares, respectively. The solid lines show the parametrization described in the text, the dashed line the dilute instanton gas approximation, and the dotted line the QCD sum-rule calculation. The horizontal line in the second panel is added to guide the eye.

TABLE I. Masses and sizes of glueballs from fits to correlation functions and Bethe-Salpeter amplitudes, as predicted in [8].

	Scalar	Pseudoscalar	Tensor
M (GeV)	1.4 ± 0.2	> 3	?
$r_{\text{r.m.s.}}$ (fm)	0.21	?	0.61

Values for the masses and radii of the lowest glueball states, extracted from fits to the correlation functions and Bethe-Salpeter amplitudes 30 years ago in Ref. [8], are listed in Table I.

Let us now briefly review the current experimental status of these three glueballs. The scalar glueball mass, around $m_{0^{++}} \approx 1.6$ GeV, lies in a region populated by several scalar mesons. Its admixture with them remains a subject of ongoing investigation, with perhaps the largest share residing in $f_0(1710)$ [10]. The tensor state $M_{2^{++}} \approx 2.2$ GeV has perhaps been observed in “glue-rich” double-diffractive scattering processes; see also the analysis of the $f_2(1950)$ tensor-glueball candidate in Ref. [11].

Interestingly, the pseudoscalar glueball has received a boost following the BESIII announcement of the discovery of the pseudoscalar $X(2370)$ resonance, with mass and width $m = 2395$ MeV and $\Gamma = 188$ MeV, which fit well with expectations for the pseudoscalar glueball and related pseudoscalar-glueball phenomenology [12]. The reaction is

$$J/\psi \rightarrow \gamma gg, \quad gg \rightarrow X(2370) \rightarrow \eta' f_0(980).$$

“Holographic QCD” is a theoretical framework that combines mesons and glueballs within a common geometrical construction; see, e.g., [13]. It has significant predictive power: for example, the complicated meson-glueball mixing has been studied using this model in Ref. [14]. (This mixing defines the mutual interaction strength of the QCD flux tubes, which is important for event generators at colliders.)

Anomalous meson-glueball interactions have also been studied in a dilute-instanton treatment in Ref. [15].

Mixing of glueballs with mesons is, of course, also investigated on the lattice in full QCD. When light quarks are dynamical, pure-gluon operators can mix with flavor-singlet $q\bar{q}$ states, particularly in the scalar and pseudoscalar channels. This mixing with light mesons complicates the identification of experimental candidates and blurs the connection between quenched lattice spectra and observed isoscalar mesons.

Lattice calculations incorporating dynamical quarks have begun to address these issues directly, including studies of pseudoscalar glueball- η and η' mixing [16], as shown in Fig. 3.

In parallel with lattice developments, a variety of theoretical approaches have been pursued. Functional methods based on Dyson-Schwinger and Bethe-Salpeter equations reproduce many qualitative features of the

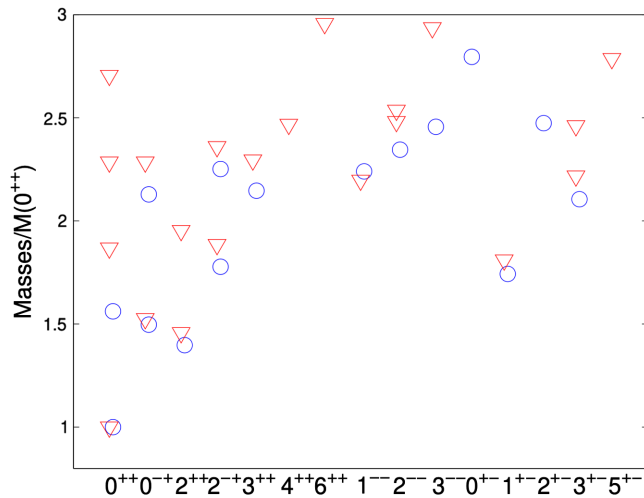


FIG. 3. Lattice data for glueball masses, normalized to lowest scalar, triangles from Meyer *et al.* and circles from Morningstar *et al.*

glueball spectrum [17–19]. Phenomenological models, including constituent-gluon Hamiltonians [20], flux tube [21] and bag models [22], as well as holographic constructions [23], provide intuitive pictures for the organization of glueball states and their Regge behavior. Earlier constituent descriptions employing massive gluons, transverse-helicity states, and Coulomb-gauge quasigluons were developed in Refs. [24–26].

We have already mentioned instanton-based contributions to Euclidean correlation functions [9], related to glueball masses and couplings via certain sum rules.

In this paper, however, we follow a different path, based on the notion of “effective gluons,” with a momentum-dependent dynamical gluon mass that induces channel-dependent short-range interactions [27,28]. Together with well-separated instantons, we also include contributions from instanton–anti-instanton “molecules.” While this approach also generates enhanced attraction for the 0^{++} glueball at small distances, its parametric strength (in this and other channels) turns out to be different.

B. Outline of this work

Having completed this brief overview, let us now explain the motivations of the present work. Note that two (out of three) glueballs mentioned— 0^{++} , 2^{++} , 0^{-+} —are “exceptional,” in the same sense as their mesonic analogues, σ , ρ , η' . In fact, only vector mesons are “normal,” in the sense that, for example, one can predict baryon masses as $M_{\text{baryons}} \approx (3/2)M_{\text{mesons}}$ using a simple additive count in a “constituent” model, in which the mean “interaction” part of the Hamiltonian is subleading due to partial cancellations.

What we attempt below is to construct a constituent two-gluon description of glueball states. In doing so, we do not focus on the few “exceptional” states discussed above, but

instead first aim to reproduce the “bulk” of glueball spectroscopy. Indeed, if quark models of mesons and baryons were to start with π , η' , they would not be successful either.

Our framework combines adjoint Coulomb interactions fixed by group theory, Casimir-scaled confinement with gluonic screening. Only after this is established do we add instanton-induced central and spin-dependent forces.

Further motivation for this work comes from our broader project of “bridging hadron spectroscopy with partonic observables,” which aims to translate quark models of mesons, baryons, and multiquark states from the rest frame to their corresponding light-front formulations. For completeness, it is clearly necessary to include glueballs and quark-gluon hybrids in the Fock components of hadrons. The renormalization group, in the form of an expanding set of admixed states in hadronic wave functions, would then replace the current “evolution equations,” which are defined for PDFs rather than wave functions.

To convince the reader that there exists a large set of glueball states with various quantum numbers, we reproduce Fig. 3 from [25], comparing results from two different lattice groups. Not only are there many states in total, but in the scalar case the celebrated ground state is supplemented by three excited states, and in several channels two states have been identified.

We first focus on a subset of excited states, shown in Table I. A surprising outcome of fitting these states is the rather heavy “constituent gluon mass,” $m_g \sim 900$ MeV, well above typical values in the literature. Only later do we return to the “exceptional” lowest states in the scalar and pseudoscalar sectors.

What we find is that these states can, in fact, be described by a Schrödinger equation without invoking any extreme assumptions. Moreover, the spectrum of “normal” glueballs turns out to lie between that of strange $\bar{s}s$ and charmed $\bar{c}c$ mesons [25,29].

Particular emphasis is placed on the scalar and tensor channels. While the 0^{++} state is dominated by short-distance dynamics and can become relatively compact, the 2^{++} glueball is affected by angular momentum barriers and significant S – D wave mixing. We solve the resulting coupled-channel problem and discuss the mass hierarchies and spatial structure.

The paper is organized as follows. In Sec. II we construct the constituent two-gluon Hamiltonian, including adjoint Coulomb interaction, Casimir-scaled confinement with screening, and short-distance nonperturbative forces induced by instantons. In Sec. III we solve the resulting Schrödinger equation and establish the spectrum of “normal” glueballs, focusing on radial and orbital excitations and their comparison to lattice results. The lowest scalar and pseudoscalar channels, where short-distance effects are dominant, are analyzed separately, including a semiclassical WKB (Wentzel-Kramers-Brillouin) baseline.

In Sec. IV we develop a relativistic treatment of the scalar 0^{++} glueball based on an instanton-induced four-gluon interaction and its reduction to an effective two-body kernel. This leads to a reduced Bethe-Salpeter (Salpeter) equation, from which we extract the mass and wave function of the compact scalar state. In Sec. V we assemble the full glueball spectrum, highlighting the interplay between confinement-driven Regge behavior and channel-dependent instanton dynamics across different J^{PC} sectors. Our conclusions are summarized in Sec. VI.

Technical details and complementary derivations are collected in the Appendixes. These include the WKB quantization procedure and semiclassical estimates, the construction and properties of the screened adjoint potential, details of instanton-induced interactions and spin-dependent forces, the derivation of the effective four-gluon vertex and its reduction to a two-body interaction, and the helicity and partial-wave projections relevant for the scalar and tensor channels.

II. THE HAMILTONIAN

A. Adjoint Coulomb

The Coulomb term follows from one-gluon exchange. Writing

$$V_C(r) = \frac{\alpha_s}{r} T_1 \cdot T_2, \quad T_1 \cdot T_2 = \frac{1}{2}(C_R - C_{R_1} - C_{R_2}), \quad (1)$$

one obtains the Casimir form

$$V_C(r) = -\frac{\alpha_s}{2r}(C_{R_1} + C_{R_2} - C_R). \quad (2)$$

For two gluons with $R_1 = R_2 = A$ and $C_A = 3$, coupled to a singlet $R = 1$ with $C_R = 0$, this reduces to

$$V_C^{(gg,1)}(r) = -\frac{3\alpha_s^{\text{eff}}}{r}, \quad (3)$$

which is attractive and $9/4$ times stronger than the $q\bar{q}$ singlet Coulomb coefficient at the same value of α_s^{eff} . We will meet rescaling by $9/4$ in other quantities, see below.

B. Adjoint confining potential

We model glueballs as two effective constituent gluons in their center-of-mass frame. The relative motion Hamiltonian is

$$H = 2m_g(\eta) + \frac{\mathbf{p}^2}{m_g(\eta)} + \Delta_0 + V_{\text{conf}}(r) + V_C(r) + V_{\text{inst}}(r) + V_{SS}(r) + V_T(r). \quad (4)$$

The confining term is taken as a screened adjoint (double) string,

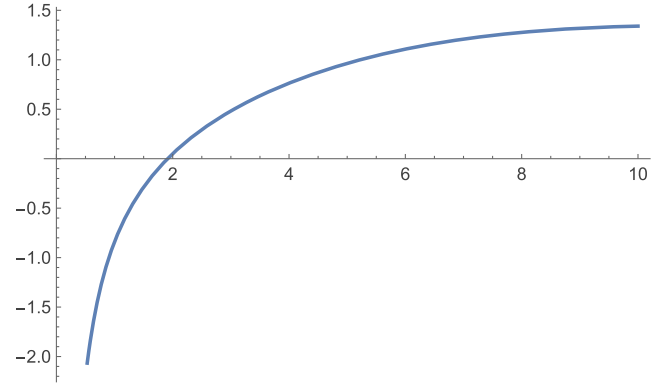


FIG. 4. Central potential $V_{\text{conf}}(r)$ (GeV) used versus r (GeV^{-1}).

$$\begin{aligned} V_{\text{conf}}(r) &= \sigma_8 r e^{-r/r_{\text{scr}}}, \\ \sigma_8 &= \frac{C_A}{C_F} \sigma_3 = \frac{9}{4} \sigma_3, \\ \sigma_3 &= 0.18 \text{ GeV}^2. \end{aligned} \quad (5)$$

The single constant spectral shift Δ_0 in Eq. (4) is not known *a priori*, but fits to the absolute spectrum yield a modest negative value. It is omitted when only level spacings are considered. The notation Δ_0 also distinguishes this spectral shift from the strength V_0 of the separable instanton kernel introduced below.

The choice of the adjoint screening length r_{scr} is a rather delicate issue. If r_{scr} is smaller than the typical size of the states, the potential acts merely as a barrier and is unable to support bound states of larger spatial extent, producing at most a few quasibound states, if any. To avoid this situation we place the states in a box of radius $R = 2$ fm and take $r_{\text{scr}} \sim R$, so that the potential inside the box does not decrease but instead saturates, as illustrated in Fig. 4. A more detailed discussion, as well as the relation of this potential shape to our Monte Carlo calculations with a single instanton or $\bar{II}$ molecule using numerically generated Wilson lines, is given in Appendix C.

C. The short-distance nonperturbative interactions

Let us start with simple estimates of instanton-induced interactions. Naively representing the gauge field as semiclassical $O(1/g)$ plus quantum fluctuations $O(1)$ $A = A_{\text{class}} + A_{\text{gluon}}$ yields a cross term of order $O(1/g)$. Yet it vanishes upon averaging over random instanton orientations. A correct estimate instead follows from the quartic term

$$g^2 A_{\text{class}}^2 A_{\text{gluon}}^2 \sim O(g^0) A_{\text{gluon}}^2,$$

which contributes to the effective gluon mass. Note that for light quarks the instanton-induced effective Lagrangian is built from fermion zero modes, and likewise carries no

explicit factor of g . So, parametrically we expect constituent masses to be similar.

Numerically, however, in glueballs this interaction is strong due to color Casimir factors. This explains why the scalar glueball is predicted to be unusually compact, with a size ~ 0.2 fm, whereas even the smallest meson, the pion, has a radius of order ~ 0.5 fm.

The perturbative spin-spin interaction is local $\sim \delta^{(3)}(r)$. The instanton-induced attraction is modeled by a Gaussian centered at the origin with range ρ ,

$$V_{\text{inst}}(r) = -G(\eta)e^{-r^2/2\rho^2}, \quad G(\eta) = G_0\eta, \quad (6)$$

with $\rho \simeq \frac{1}{3}$ fm. Because the (anti)instanton field is (anti)self-dual, the induced coupling derived in [30] [see Eq. (45) therein] involves the 't Hooft symbols $\eta, \bar{\eta}$ and projects most strongly onto the parity-even scalar combination $G_{\mu\nu}^a G_{\mu\nu}^a$, enhancing the 0^{++} channel, while the Levi-Civita structure governs the pseudoscalar $G\tilde{G}$ channel.

The spin-dependent interactions are taken in the standard form

$$V_{SS}(r) = \frac{C_{SS}(\eta)}{m_g^2(\eta)} \delta_\Lambda^{(3)}(\mathbf{r}) \mathbf{S}_1 \cdot \mathbf{S}_2, \\ V_T(r) = \frac{C_T(\eta)}{m_g^2(\eta)} \frac{1 - e^{-r^2/\rho^2}}{r^3} S_{12}, \quad (7)$$

with $S_{12} = 3(\mathbf{S}_1 \cdot \hat{\mathbf{r}})(\mathbf{S}_2 \cdot \hat{\mathbf{r}}) - \mathbf{S}_1 \cdot \mathbf{S}_2$, and a Gaussian-smeared contact regulator

$$\delta_\Lambda^{(3)}(\mathbf{r}) = \left(\frac{\Lambda^2}{\pi}\right)^{3/2} e^{-\Lambda^2 r^2}, \quad \Lambda \sim \rho^{-1}. \quad (8)$$

Dense ILM including instanton–anti-instanton correlated pairs or “molecules” use parameter $\eta \sim 7$ for their effects, as compared to dilute ILM with only instantons, and density $n_{\text{dilute}} \approx 1 \text{ fm}^{-4}$. This scaling is implemented through

$$C_{SS}(\eta) = C_{SS}^{(0)}\eta, \quad C_T(\eta) = C_T^{(0)}\eta, \quad (9)$$

with $C_T^{(0)}$ allowed to be negative, consistent with the ILM result of [31]. The dynamical constituent gluon mass is parametrized as

$$m_g(\eta) = m_{g0}\sqrt{\eta}, \quad m_{g0} \simeq 0.36 \text{ GeV}. \quad (10)$$

III. GLUEBALL SPECTRA FROM THE SCHRÖDINGER EQUATION

A. The $S=0, J=L$ sector

As already mentioned, since the effective gluon mass is $O(1 \text{ GeV})$, the accuracy of the nonrelativistic approximation is expected to be comparable to that for charmonium.

Of course, important differences arising from the distinct coefficients in the Hamiltonians.

We begin with the simplest case, with total spin $S=0$, so that the total and orbital angular momenta coincide, $J=L$. The calculated spectrum of several $J^{PC} = J^{++}$ channels is shown in Fig. 5. At this stage only the Coulomb and confining potentials are included, with no spin-dependent forces, as is commonly done, for example, in charmonium studies. The model parameters are chosen to reproduce the three higher states in the scalar channel, which, unlike the lowest $n=0$ state, have “normal” hadronic sizes (see the table of rms radii) and are therefore less sensitive to short-distance forces. The most important parameter obtained in this way is the effective gluon mass, fitted to be

$$m_g = 0.90 \text{ GeV}. \quad (11)$$

For comparison, the effective masses of the strange and charmed quarks are $m_s \approx 0.5 \text{ GeV}$ and $m_c \approx 1.5 \text{ GeV}$, respectively.

This implies a value of the parameter η —the enhancement of the gluon mass relative to the dilute ILM value due to instanton–anti-instanton molecules—of $\eta = 6.25$. This is close to the value $\eta \approx 7$ used in [31] to reproduce the central charmonium potential. That value was originally motivated by “gradient flow” cooling of lattice gauge configurations [31].

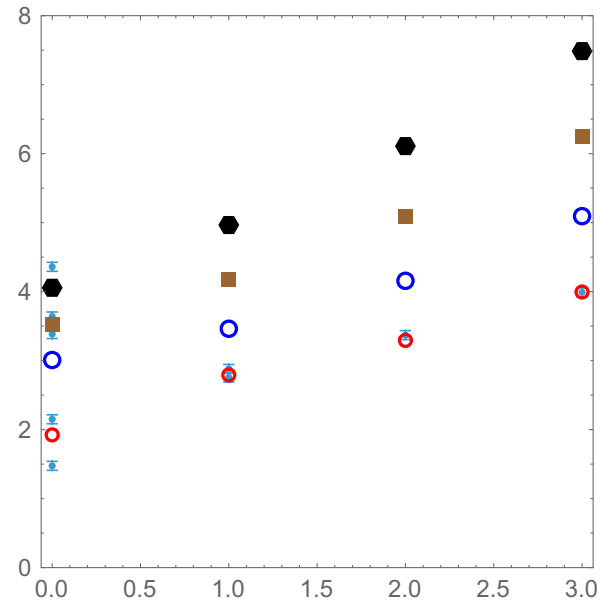
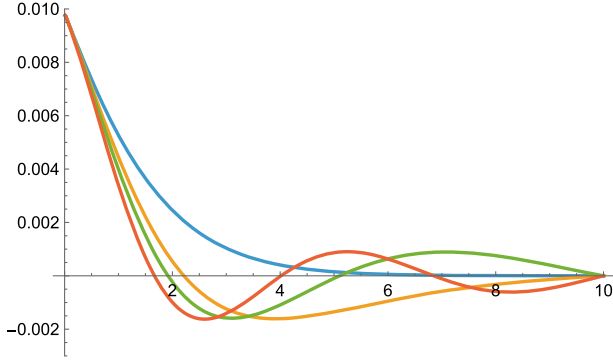


FIG. 5. The colored points show the calculated energies E_0, E_1, E_2, E_3 (GeV) of the four lowest states with $L=0, 2, 4, 6$. Small points with error bars are H. Meyer’s lattice simulations in pure SU(3) gauge theory [32] (see also Table II). We used J^{PC} lattice states $0^{++}, 2^{++}, 4^{++}$, and 6^{++} (from bottom to top) as a function of the principal quantum number n .

TABLE II. Energies (GeV) and error bars of all glueballs, for all $P = C = +$ states, from [32].

$J = 0$	1.475	(30)	(65)
0	2.755	(70)	(120)
0	3.370	(100)	(150)
0	3.990	(210)	(180)
2	2.150	(30)	(100)
2	2.880	(100)	(130)
3	3.385	(90)	(150)
4	3.640	(90)	(160)
6	4.360	(260)	(200)

FIG. 6. Wave functions (unnormalized) $\psi_n(r)$ for $n = 0, 1, 2, 3$ as functions of r (GeV^{-1}).

To quantify the sizes and shapes of these states, we display the corresponding wave functions in Fig. 6 and list the root-mean-square radii,

$$r_{\text{rms}} = \left[\frac{\int dr \psi_n^2(r) r^4}{\int dr \psi_n^2(r) r^2} \right]^{1/2}, \quad (12)$$

in Table III.

Fixing the model parameters from the masses of the “normal” 0^{++} states with $n = 1, 2, 3$, which have sizes $r_{\text{rms}} \sim 5 \text{ GeV}^{-1} \sim 1 \text{ fm}$, we can then consider other channels. The predicted masses for the higher- J states with $J = 4, 6$ turn out to be close to the lattice values. This is expected, since the large centrifugal barrier increases their spatial extent and further suppresses sensitivity to short-distance forces. When discussing these states one should keep in mind that, unlike for mesons, the confining potential for glueballs is expected to be screened at large distances, rendering sufficiently large states unstable, formally

TABLE III. Energies and rms radii of scalar glueballs obtained from the Schrödinger equation with the parameters defined in the text.

E_n (GeV)	1.92	2.79	3.29	4.0
r_{rms} (GeV^{-1})	2.03	5.0	6.26	6.25

corresponding to complex energies. This issue is avoided here by placing the system inside a spherical cavity: in our calculations we take a radius $R = 10 \text{ GeV}^{-1} \approx 2 \text{ fm}$ and impose Dirichlet boundary conditions.

B. The lowest 0^{++} scalar glueball

Here we turn to the lowest 0^{++} state, with $n = 0$. As discussed in the Introduction, this state has a long history, with an rms radius predicted in [8] to be $r_{\text{rms}}(0^{++}, n = 0) \approx 1 \text{ GeV}^{-1} \approx 0.2 \text{ fm}$, a result confirmed three decades later in [33]. As is clear from results of our preliminary calculations (Table III), neither the calculated mass nor the size of this state agrees with the lattice results. This discrepancy is expected, since those calculations have not yet included the short-distance attractive spin-spin forces, perturbative and instanton-induced. To estimate their combined effect via local interaction

$$V_{\text{local}}(r) = G\delta^3(r). \quad (13)$$

After doing so, we find that reproducing Meyer’s value $M_{0^{++}} = 1.475 \text{ GeV}$ requires

$$G \approx 38 \text{ GeV}^{-2}.$$

The corresponding rms radius is reduced, but only to $r_{\text{rms}}(0^{++}, n = 0) \approx 1.6 \text{ GeV}^{-1}$, which is still significantly larger than expected. We therefore conclude that the lowest scalar glueball is unlikely to be described reliably within the Schrödinger framework, and we will instead treat it below using a relativistic Bethe-Salpeter approach.

C. Pseudoscalar 0^{-+} glueballs

For two gluons coupled to a color singlet, the color wave function is symmetric. Their parity and charge conjugation are

$$P = (-1)^L, \quad C = +. \quad (14)$$

The pseudoscalar 0^{-+} state therefore has $L = S = 1$ and may be represented in the massive-spin basis as a 3P_0 configuration. It does not belong to the even- J positive-parity trajectory. In particular, it should not be identified with an odderon trajectory, since the odderon carries $C = -$. For two spin-one constituents, the spin wave function is symmetric for $S = 0, 2$ and antisymmetric for $S = 1$. Bose symmetry therefore requires L even for $S = 0, 2$ and L odd for $S = 1$. In particular, $(L, S) = (1, 1)$ is Bose allowed and contains $J = 0, 1, 2$. Consequently, Bose symmetry alone does not exclude a $J = 1$ state for generic massive spin-one constituents. The Landau-Yang exclusion of $J = 1$ follows only after restricting both gluons to physical transverse helicities [25]. The Schrödinger Hamiltonian used here is formulated in the

massive-spin LS basis; in particular, the tensor channel is treated in the $\{^5S_2, ^5D_2\}$ basis. It should therefore not be identified with a pure transverse-helicity construction. The transverse projection used below in the scalar instanton kernel is a channel-specific on-shell projection. We do not attempt a quantitative description of the $J = 1$ sector within the present two-body model. According to Ref. [32], the masses of the two lowest 0^{-+} states are

$$\begin{aligned} M_{0^{-+}}^0 &= 2.250(60)(100) \text{ GeV}, \\ M_{0^{-+}}^1 &= 3.370(150)(150) \text{ GeV}. \end{aligned} \quad (15)$$

Solving the Schrödinger equation with $L = 1$ and the shifted potential $V_{\text{conf}} + V_0$, as in the 0^{++} channel, we obtain

$$M_{0^{-+}}^0 = 2.65 \text{ GeV}, \quad M_{0^{-+}}^1 = 3.13 \text{ GeV}. \quad (16)$$

The excited state is reproduced reasonably well, while the lowest state shows a noticeable discrepancy. This discrepancy likely originates from short-distance effects, but the situation in this channel is subtle. The perturbative spin-spin interaction in the $S = 1$ channel is attractive, since $\mathbf{S}_1 \cdot \mathbf{S}_2 = -1$. In contrast, the instanton-induced interaction is repulsive, as illustrated in Fig. 2. Moreover, because $L = 1$, the wave function vanishes at the origin and the contact contribution depends sensitively on its finite-range regularization. A quantitative resolution of this competition is beyond the scope of the present treatment.

IV. INSTANTANEOUS SEPARABLE QUASIPOTENTIAL TREATMENT

A. Emergent four-gluon interaction

We begin by sketching derivation of effective multigluon vertices induced by instantons. It starts by an exponential of the (LSZ (Lehmann–Symanzik–Zimmermann)-reduced) gluonic “tail emission” from the pseudoparticle, written compactly as a coupling to the field strength $G_{\mu\nu}^a$.

For the present purpose we isolate the purely gluonic source factor [30]

$$\begin{aligned} \mathcal{E}_I[G] &= \exp \left[-\kappa \rho^2 \int \frac{d^4 q}{(2\pi)^4} e^{iq \cdot x} \beta_{2g}(\rho|q|) R^{ab}(U) \bar{\eta}_{\mu\nu}^b G_{\mu\nu}^a(q) \right], \\ \mathcal{E}_A[G] &= \exp \left[-\kappa \rho^2 \int \frac{d^4 q}{(2\pi)^4} e^{iq \cdot x} \beta_{2g}(\rho|q|) R^{ab}(U) \eta_{\mu\nu}^b G_{\mu\nu}^a(q) \right]. \end{aligned} \quad (17)$$

For a single (anti)-instanton of size ρ centered at the origin, the field strength is

$$\mathbb{G}_{\mu\nu}^a(x) = \frac{4}{g} \bar{\eta}_{\mu\nu}^a \frac{\rho^2}{(x^2 + \rho^2)^2} \quad (18)$$

and the field-strength Fourier transform or form factor is

$$\mathbb{G}_{\mu\nu}^a(q) = \frac{4\pi^2 \rho^2}{g} \bar{\eta}_{\mu\nu}^a \beta_{2g}(t) \equiv \frac{4\pi^2 \rho^2}{g} \bar{\eta}_{\mu\nu}^a \left(\frac{t^2}{2} K_2(t) \right). \quad (19)$$

It is gauge invariant, satisfies $\beta_{2g}(0) = 1$, and decays exponentially for $t \gg 1$. An analogous expression holds for anti-instantons with $\bar{\eta} \rightarrow \eta$. The extra $\frac{1}{2}$ that appears in the exponential arises from the coupling to the background field

$$\begin{aligned} &\frac{1}{4} \int d^4 x 2 \mathbb{G}_{\mu\nu}^a(x) G_{\mu\nu}^a(x) \\ &\rightarrow \frac{1}{2} G_{\mu\nu}^a(q) \int d^4 x \mathbb{G}_{\mu\nu}^a(x) e^{-iq \cdot x} \end{aligned} \quad (20)$$

where the rightmost equation follows by LSZ reduction of the perturbative gluon. Note that this reconstruction of the exponent is fully gauge invariant.

The instanton-induced four-gluon operator arises from the fourth-order term in the expansion of the tail-emission functional. Writing

$$X_I[G] = \kappa \rho^2 \int \frac{d^4 q}{(2\pi)^4} e^{iq \cdot x} \beta_{2g}(\rho|q|) R^{ab}(U) \bar{\eta}_{\mu\nu}^b G_{\mu\nu}^a(q), \quad (21)$$

and similarly for $X_A[G]$, the quartic contribution is proportional to

$$n \left\langle \frac{X_I^4}{4!} + \frac{X_A^4}{4!} \right\rangle_U, \quad (22)$$

where n is the instanton density and $\langle \cdots \rangle_U$ denotes averaging over color orientations. Averaging over color orientation with the Haar measure projects onto singlets. For the adjoint dimension $d_A = N_c^2 - 1$, the fourth-group average yields the standard contraction structure

$$\begin{aligned} \langle R_{a_1 b_1} R_{a_2 b_2} R_{a_3 b_3} R_{a_4 b_4} \rangle_U &\Rightarrow \frac{3}{d_A(d_A + 1)} (\delta_{a_1 a_2} \delta_{b_1 b_2}) \\ &\times (\delta_{a_3 a_4} \delta_{b_3 b_4}) + \text{perm.} \end{aligned} \quad (23)$$

To proceed, we recall the identity

$$\bar{\eta}_{\mu\nu}^b \bar{\eta}_{\rho\sigma}^b = \delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho} - \epsilon_{\mu\nu\rho\sigma}, \quad (24)$$

which yields to the contractions

$$\bar{\eta}_{\mu\nu}^b \bar{\eta}_{\rho\sigma}^b G_{\mu\nu}^a G_{\rho\sigma}^a = 2 G_{\mu\nu}^a G_{\mu\nu}^a - 2 G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a. \quad (25)$$

Projecting onto the parity-even scalar channel retains only the $(G_{\mu\nu}^a G_{\mu\nu}^a)^2$ part, leading the induced and nonlocal scalar four-gluon interaction

$$\Delta\mathcal{L}_{4g} = n \frac{\kappa^4 \rho^8}{2d_A(d_A + 1)} \int \prod_{i=1}^4 \frac{d^4 q_i}{(2\pi)^4} \beta_{2g}(\rho|q_i|) (2\pi)^4 \delta^{(4)}\left(\sum_i q_i\right) [G_{\mu\nu}^a(q_1) G_{\mu\nu}^a(q_2)] [G_{\rho\sigma}^c(q_3) G_{\rho\sigma}^c(q_4)] + \text{perm.} \quad (26)$$

Each external gluon momentum carries its own form factor $\beta_{2g}(\rho|q_i|)$. Note the leg-by-leg factorization which will be key for the bound state equation to follow. In the local approximation (zero size) the four-gluon vertex is

$$\Delta\mathcal{L}_{4g}^{(0^{++})}(x) = n \frac{\kappa^4 \rho^8}{2d_A(d_A + 1)} (G_{\mu\nu}^a(x) G_{\mu\nu}^a(x))^2. \quad (27)$$

Equation (26) is the microscopic one-(anti)instanton contact vertex in the 0^{++} channel that we will now reduce to an instantaneous two-gluon potential.

B. Emergent two-gluon 0^{++} -potential

To extract the gluon two-body potential in the scalar 0^{++} channel, we will use on-shell in-out gluonic external states and the nonlocal operator (26). This procedure parallels the one discussed by us [31] for the constituent quark pair interactions.

For the on-shell one-gluon states in the CM (center of mass) frame, we use the normalization

$$\langle \vec{k}, \lambda, a | \vec{k}', \lambda', a' \rangle = (2\pi)^3 2\omega \delta^{(3)}(\vec{k} - \vec{k}') \delta_{\lambda\lambda'} \delta_{aa'}. \quad (28)$$

For a physical transverse polarization vector $\epsilon^\mu(k, \lambda)$, the field-strength matrix element is

$$\langle 0 | G_{\mu\nu}^a(0) | g^b(k, \lambda) \rangle = i\delta^{ab} (k_\mu \epsilon_\nu(k, \lambda) - k_\nu \epsilon_\mu(k, \lambda)), \quad (29)$$

and similarly for outgoing legs with $\epsilon \rightarrow \epsilon^*$. Define the gauge-invariant two-gluon contraction induced by $G_{\mu\nu}^a G_{\mu\nu}^a$,

$$\mathcal{F}(i, j) \equiv G_{\mu\nu}^a(i) G_{\mu\nu}^a(j) \Rightarrow -2\delta^{a_i a_j} [(k_i \cdot k_j)(\epsilon_i \cdot \epsilon_j) - (k_i \cdot \epsilon_j)(k_j \cdot \epsilon_i)]. \quad (30)$$

At Born level, (27) yields

$$i\mathcal{M}^{(4)}(1, 2 \rightarrow 3, 4) = in \frac{\kappa^4 \rho^8}{2d_A(d_A + 1)} \prod_{i=1}^4 \beta_{2g}(\rho|k_i|) \times [\mathcal{F}(1, 3)\mathcal{F}(2, 4) + \mathcal{F}(1, 4)\mathcal{F}(2, 3)]. \quad (31)$$

Projecting onto the two-gluon color singlet state $|1\rangle_{\text{color}} = \delta^{ab}|ab\rangle/\sqrt{d_A}$ sets the color factors in (31) to unity for both exchange structures.

The scalar 0^{++} projection is implemented by taking the $J_z = 0$ helicity combination of two transverse gluons,

$$|0^{++}\rangle_{\text{pol}} = \frac{1}{\sqrt{2}}(|\lambda_1 = +, \lambda_2 = -\rangle + |\lambda_1 = -, \lambda_2 = +\rangle), \quad (32)$$

and similarly for the outgoing state. In CM kinematics with scattering angle θ between $\vec{k}_1$ and $\vec{k}_3$, the detail contractions of the transverse helicities for massive on-shell gluons yield

$$\langle 0^{++} | [\mathcal{F}(1, 3)\mathcal{F}(2, 4) + \mathcal{F}(1, 4)\mathcal{F}(2, 3)] | 0^{++} \rangle = 2(\omega^2 + k^2)^2 (1 + \cos^2 \theta), \quad (33)$$

as detailed in Appendix G.

Using the Legendre polynomials,

$$1 + \cos^2 \theta = \frac{4}{3} P_0(\cos \theta) + \frac{2}{3} P_2(\cos \theta), \quad (34)$$

the decomposition apparently shows both $J = 0, 2$. If we regard (G12) as a helicity-0 object and performs a spinless partial-wave projection,

$$\mathcal{M}_J^{(4)} = \frac{2J+1}{2} \int_{-1}^1 dx P_J(x) \mathcal{M}^{(4)}(x), \quad x = \cos \theta, \quad (35)$$

then (34) implies the weights

$$\mathcal{K}_{0^{++}} = \frac{4}{3}, \quad \mathcal{K}_{2^{++}} = \frac{2}{3}, \quad (36)$$

hence

$$\frac{\mathcal{K}_{2^{++}}}{\mathcal{K}_{0^{++}}} = \frac{1}{2}.$$

However, this ratio reflects only the relative size of the P_2 and P_0 components in (34), with the latter essentially an S-channel amplitude. While $\mathcal{K}_{0^{++}}$ is the proper weight in the iteration of the 0^{++} , $\mathcal{K}_{2^{++}}$ does not carry the correct weight in the 2^{++} as detailed in Appendix G. Although the emergent instanton vertex contains a P_2 component with relative weight 1/2 in the helicity-0 decomposition, the *physical* 2^{++} interaction built from transverse gluons follows from helicity projection as we detail in Appendix G.

With this in mind, the in-out legs form factors in the S-channel can be restored as follows. In the glueball rest frame, the incoming legs carry relative momentum $\mathbf{p}$ and the outgoing legs carry $\mathbf{k}$. The four form factors combine as

$$\prod_{i=1}^4 \beta_{2g}(\rho|k_i|) = \beta_{2g}^2(\rho p) \beta_{2g}^2(\rho k). \quad (37)$$

Since the exchange is taking place inside an instanton (anti-instanton) this is appropriately described by an *instantaneous* scalar potential, hence

$$V(\mathbf{p}, \mathbf{k}) = -V_0 \beta_{2g}^2(\rho p) \beta_{2g}^2(\rho k), \quad (38)$$

which is separable. The strength V_0 is obtained by combining the quartic coefficient, the scalar projection factor, and the normalization required to convert an invariant amplitude into a three-dimensional potential. The latter introduces reduced-state factors $(2\omega)^{-1/2}$ per external leg. Matching these factors at the dominant instanton scale $p, k \sim \rho^{-1}$ yields

$$\omega_\rho = \sqrt{\rho^{-2} + m_g^2}, \quad (39)$$

and produces a factor $(2\omega_\rho)^{-2} = 1/(4\omega_\rho^2)$ in the effective coupling. Collecting all contributions gives

$$V_0 = n \frac{\kappa^4}{2d_A(d_A + 1)} \frac{16\mathcal{K}_{0^{++}}}{4\omega_\rho^2} \rho^4. \quad (40)$$

C. Reduced Bethe-Salpeter equation

Let $\Gamma(p; P)$ be the amputated Bethe-Salpeter (BS) vertex for two equal-mass constituent gluons of mass m_g , with total four-momentum P and relative four-momentum p . In the ladder approximation with an instantaneous kernel $V(\mathbf{p}, \mathbf{k})$, the reduced BS (Salpeter) equation for the scalar 0^{++} glueball is

$$\Gamma(p; P) = \int \frac{d^4 k}{(2\pi)^4} V(\mathbf{p}, \mathbf{k}) D\left(k + \frac{P}{2}\right) D\left(-k + \frac{P}{2}\right) \Gamma(k; P). \quad (41)$$

Here $-iD(q) = 1/(q^2 - m_g^2 + i0)$ denotes the scalar propagator denominator retained after the polarization projection. Equation (41) is an instantaneous separable Salpeter, or quasipotential, reduction rather than a full covariant Bethe-Salpeter equation. The interaction kernel is constructed using on-shell transverse gluons. Relative-energy dependence of the kernel, off-shell polarization components, and the full tensor structure of the dressed gluon propagator are not treated explicitly. These are limitations of the present reduction, although the underlying instanton-induced pair interaction is gauge invariant. Fully covariant continuum Bethe-Salpeter treatments retaining dressed gluon propagators and tensor amplitudes are discussed, for example, in Ref. [34].

With this in mind and in the glueball rest frame

$$P^\mu = (M, \mathbf{0}), \quad p^\mu = (p_0, \mathbf{p}), \quad k^\mu = (k_0, \mathbf{k}), \quad (42)$$

we define the equal-time Salpeter amplitude

$$\phi(\mathbf{p}) \equiv \int \frac{dp_0}{2\pi} D\left(p + \frac{P}{2}\right) D\left(-p + \frac{P}{2}\right) \Gamma(p; P). \quad (43)$$

Since $V(\mathbf{p}, \mathbf{k})$ does not depend on p_0 or k_0 , we can multiply both sides of (41) by the product of propagators and integrate over p_0 as in (43). This yields

$$\begin{aligned} \phi(\mathbf{p}) &= \int \frac{d^3 k}{(2\pi)^3} V(\mathbf{p}, \mathbf{k}) \\ &\times \left[\int \frac{dp_0}{2\pi} D\left(p + \frac{P}{2}\right) D\left(-p + \frac{P}{2}\right) \right] \phi(\mathbf{k}). \end{aligned} \quad (44)$$

The remaining p_0 integral is elementary,

$$\begin{aligned} &\int \frac{dp_0}{2\pi} D\left(p + \frac{P}{2}\right) D\left(-p + \frac{P}{2}\right) \\ &= \int \frac{dp_0}{2\pi} \prod_{s=\pm} \frac{i}{(p_0 + sM/2)^2 - \omega_p^2 + i0} \\ &= \frac{1}{2\omega_p} \frac{1}{M^2 - 4\omega_p^2 + i0}, \end{aligned} \quad (45)$$

by contour integration, hence the reduced Bethe-Salpeter integral equation

$$\phi(\mathbf{p}) = \frac{1}{2\omega_p} \frac{1}{M^2 - 4\omega_p^2} \int \frac{d^3 k}{(2\pi)^3} V(\mathbf{p}, \mathbf{k}) \phi(\mathbf{k}). \quad (46)$$

D. The scalar glueball 0^{++} mass equation

Substituting the separable kernel yields the rank-one integral equation for the scalar glueball 0^{++} wave function in the CM frame

$$\phi(\mathbf{p}) = -V_0 \frac{\beta_{2g}^2(\rho p)}{2\omega_p(M^2 - 4\omega_p^2 + i0)} \int \frac{d^3 k}{(2\pi)^3} \beta_{2g}^2(\rho k) \phi(\mathbf{k}). \quad (47)$$

Eliminating the normalization constant leads to the root equation

$$1 = V_0 \int_0^\infty \frac{p^2 dp}{(2\pi)^2} \frac{\beta_{2g}^4(\rho p)}{2\omega_p(4\omega_p^2 - M^2)}. \quad (48)$$

For a bound state with $M < 2m_g$, one has $4\omega_p^2 - M^2 > 0$ for all p . The integral is therefore real and no principal-value prescription is required. Introducing $x = \rho p$, $\mu = m_g \rho$, and $\mathcal{M} = M\rho$ yields the dimensionless form

$$1 = \lambda_M \int_0^\infty dx \frac{x^2}{\sqrt{x^2 + \mu^2}} \frac{[\frac{x^2}{2} K_2(x)]^4}{4(x^2 + \mu^2) - \mathcal{M}^2}, \quad (49)$$

with

$$\lambda_M = \frac{V_0 \rho^2}{(2\pi)^2}.$$

Equation (49) is the explicit rank-one mass equation for the rescaled 0^{++} glueball mass.

The bound-state mass is fixed by the pole condition of the two-body Green function,

$$1 - V_0 \Pi(P^2) = 0, \quad (50)$$

which is analytic below threshold. For a real spacelike Euclidean momentum $P_E^2 \geq 0$, define $Q^2 = \rho^2 P_E^2$. The Euclidean continuation of the pole equation is then

$$1 = \lambda_M \int_0^\infty dx \frac{x^2}{\sqrt{x^2 + \mu^2}} \frac{[\frac{x^2}{2} K_2(x)]^4}{4(x^2 + \mu^2) + Q^2}. \quad (51)$$

The physical bound-state pole is reached by analytic continuation to $P_E^2 = -M^2$, or equivalently $Q^2 = -\mathcal{M}^2$. Equation (51) then reduces exactly to the Minkowski pole condition (49).

E. The scalar glueball wave function

In contrast, the reduced momentum-space wave functions in Minkowski and Euclidean space are different, yet related to each other. Indeed, in the equal-time (Minkowski) reduction the wave function in momentum space is explicitly given by

$$\phi_M(\mathbf{p}) = \mathcal{N}_M \frac{\beta_{2g}^2(\rho p)}{2\omega_p(M^2 - 4\omega_p^2 + i0)}. \quad (52)$$

In contrast, the Euclidean reduced Bethe-Salpeter amplitude is a four-dimensional function of the form

$$\chi_E(p_4, \mathbf{p}) \sim \frac{\beta_{2g}^2(\rho p)}{[(p_4 + \frac{p_4}{2})^2 + \omega_p^2][(p_4 - \frac{p_4}{2})^2 + \omega_p^2]}, \quad (53)$$

originating from the product of the two Euclidean propagators in the iterating kernel, with no additional factor $1/(2\omega_p)$. The equal-time (Salpeter) wave function follows by slicing over the relative energy,

$$\phi_M(\mathbf{p}) = \int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \chi_E(p_4, \mathbf{p}) \propto \frac{\beta_{2g}^2(\rho p)}{2\omega_p(M^2 - 4\omega_p^2 + i0)}. \quad (54)$$

The extra factor of $1/(2\omega_p)$ arises from the relative-energy integration, and is absent in the four-dimensional Euclidean amplitude.

F. Numerical results

A scalar bound state forms only when the instanton-induced coupling λ_M exceeds the critical value of the rank-one kernel. For a genuine bound state, its mass lies below the two-constituent-gluon threshold, $M < 2m_g$. Because $\lambda_M \sim \rho^{p+2}$, with $p+2 \gtrsim 6$, the binding is particularly sensitive to the instanton size. For example, retaining only the sixth-power dependence gives

$$\left(\frac{\rho}{0.32 \text{ fm}}\right)^6 \simeq 2.0 - 2.8 \quad \text{for } \rho = 0.36 - 0.38 \text{ fm}. \quad (55)$$

Higher powers contained in the complete coupling further enhance this dependence. Thus, the existence and quantitative properties of the compact solution should not be interpreted as insensitive to ρ . We solve the pole equation for a dense instanton ensemble using

$$\begin{aligned} \eta &= 6 - 7, & \rho &= 0.34 - 0.38 \text{ fm}, \\ m_g(\eta) &= m_{g0} \sqrt{\eta}, & m_{g0} &= 0.36 \text{ GeV}, \\ \alpha_s^{\text{eff}} &= \frac{g^2}{4\pi} = 0.35 - 0.45. \end{aligned} \quad (56)$$

Over this parameter window, after fixing the scalar interaction strength to the lattice scalar-mass region, the solution gives

$$M_{0^{++}} \simeq 1.4 - 1.5 \text{ GeV}, \quad (57)$$

well below the two-gluon threshold

$$2m_g(\eta) \simeq 1.8 - 1.9 \text{ GeV}. \quad (58)$$

The quoted mass interval is therefore a calibration range rather than a parameter-free prediction. The nontrivial output of the calculation is the corresponding wave function and its spatial extent. The 0^{++} rms radius is obtained from the normalized momentum-space eigenstate in Eq. (52) according to

$$\langle r^2 \rangle = \int \frac{d^3 p}{(2\pi)^3} \phi_M^*(\mathbf{p}) (-\nabla_{\mathbf{p}}^2) \phi_M(\mathbf{p}), \quad (59)$$

which yields

$$\langle r^2 \rangle_{0^{++}}^{1/2} \simeq 0.25 - 0.32 \text{ fm}. \quad (60)$$

The variation displayed in Fig. 7 shows that the state remains compact throughout the interval $\rho = 0.34 - 0.38 \text{ fm}$, although its detailed radius and binding are quantitatively

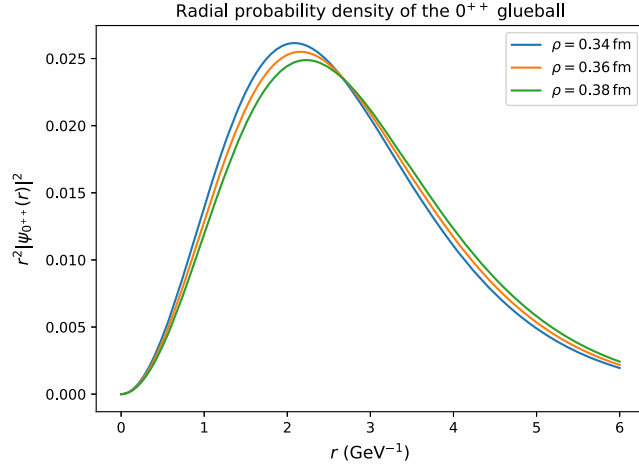


FIG. 7. Radial probability distribution $r^2|\psi_{0^{++}}(r)|^2$ for the scalar glueball obtained from the instantaneous separable Salpeter equation for instanton sizes $\rho = 0.34, 0.36$, and 0.38 fm. The scalar interaction is calibrated to the lattice mass region for the parameter window specified in the text.

sensitive to ρ . Varying $\alpha_s^{\text{eff}} = 0.35\text{--}0.45$ changes the relative contributions of the Coulomb and instanton-induced attractions but does not alter the qualitative conclusion that a bound scalar solution, when calibrated to the lattice mass, has an instanton-scale radius. We therefore claim only the existence of a compact solution within the stated parameter window, not parameter independence or absence of fine tuning. The spin-dependent coefficients C_{SS} and C_T do not enter the separable scalar pole equation considered in this subsection. Their effects are instead examined in the coupled $\{^5S_2, ^5D_2\}$ tensor calculation, where they control the scalar-tensor splitting and the D -wave admixture, respectively.

V. GLUEBALL SPECTRUM

The constituent two-gluon Hamiltonian organizes the glueball spectrum naturally into radial and orbital families, in close analogy with quarkonium spectroscopy. The parameter inputs and their calibration are summarized in Table VII. The compact scalar ground state is obtained from the instantaneous Salpeter equation, whereas the extended radial and orbital states are obtained from the Schrödinger Hamiltonian. For the tensor ground state we quote only the result of the coupled $\{^5S_2, ^5D_2\}$ calculation.

Radial excitations correspond to states with fixed total spin and parity but increasing number of nodes in the relative wave function. In the scalar channel, the ground state 0_0^{++} is strongly shifted downward by Coulomb and instanton-induced attraction and is compact, with a size of order the instanton radius. The first radial excitation 0_1^{++} is significantly less affected by short-distance dynamics because its wave function extends to larger radii; its mass therefore lies closer to the confinement-dominated WKB

prediction. This pattern reflects the rapid decoupling of instanton physics with increasing radial quantum number.

Orbital excitations instead form quasi-Regge towers at fixed radial quantum number. The tensor family $2^{++}, 4^{++}, 6^{++}, \dots$ provides the clearest example. Here the centrifugal barrier suppresses short-distance overlap already at the ground state, and $^5S_2\text{--}^5D_2$ mixing further redistributes probability toward larger radii. As a result, the tensor ground state is only moderately shifted from the WKB baseline, and higher- J states follow an approximately linear trajectory over the finite range of spins considered here, although the asymptotic WKB behavior is $M^2 \propto J^{4/3}$.

Figure 8 presents the glueball spectrum organized by parity and charge conjugation, following the standard lattice-spectroscopy layout. The comparison highlights the selective role of instanton-induced dynamics. In the $C = +$ scalar channel, coherent Coulomb, and instanton attraction produce a compact 0^{++} ground state well below the confinement baseline. For the tensor ground state, the final value shown is the lower eigenvalue of the coupled $\{^5S_2, ^5D_2\}$ Hamiltonian. The earlier unmixed value is not included in the final spectrum. No quantitative prediction is made here for the $J = 1$ channels.

VI. CONCLUSIONS

We have developed a constituent two-gluon Hamiltonian framework for glueballs that incorporates adjoint Coulomb interaction, instanton-induced short-range forces, and full tensor-driven $S\text{--}D$ mixing in the 2^{++} channel, with parameters calibrated directly against quenched lattice Yang-Mills results. Within this framework, several qualitative conclusions emerge that align naturally with current lattice spectroscopy.

At a qualitative level, the emerging picture is that of a rescaled theory of mesons. The effective gluon mass, $m_g \approx 0.9$ GeV, lies between the strange and charm quark masses. The interaction—both perturbative and confining, with the latter derived from adjoint instanton molecules—is enhanced by a factor of $9/4$. The bulk of the spectroscopy then follows from standard solutions of the Schrödinger equation, in close analogy with the case of charmonium.

There are, however, channels in which the short-distance forces are too strong to be treated reliably in this way. The main example is the scalar 0^{++} glueball, which is strongly compressed by the combined effect of the attractive adjoint Coulomb interaction and an instanton-induced core attraction. The resulting instanton-scale radius is consistent with lattice indications. Its quantitative value remains sensitive to ρ and to the short-distance coupling, as shown by the parameter variation discussed above. The calculation therefore supports a compact scalar solution within the stated parameter window, without claiming parameter independence.

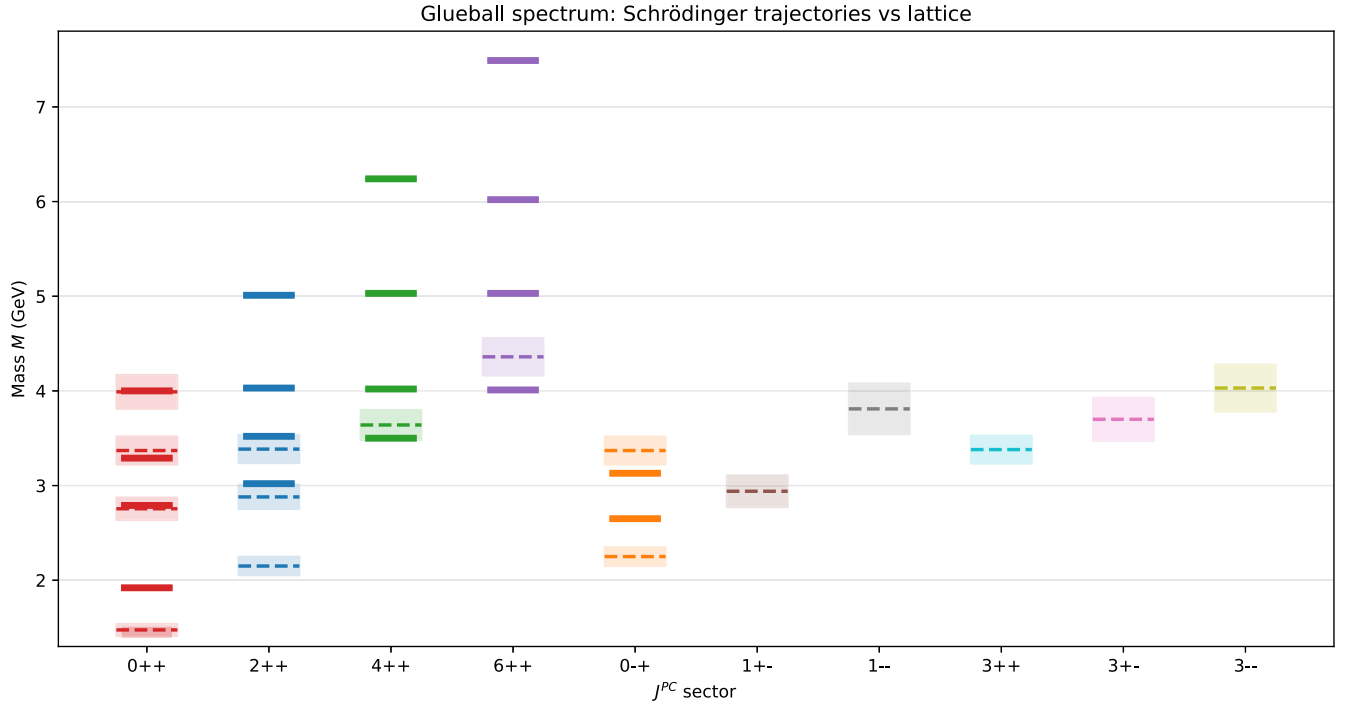


FIG. 8. Glueball mass spectrum organized by J^{PC} sectors, as listed in Table IV. The compact scalar point is obtained from the instantaneous Salpeter equation, the tensor ground state from the coupled 5S_2 – 5D_2 Hamiltonian, and the extended radial and orbital states from the Schrödinger calculation. Solid markers show this work; dashed markers and their spreads show the quenched SU(3) lattice results of Refs. [1,32].

TABLE IV. Glueball masses in pure SU(3) gauge theory. The adopted scalar ground-state result is obtained from the instantaneous Salpeter equation, while the adopted tensor ground-state result is obtained from the coupled 5S_2 – 5D_2 calculation. The higher scalar and orbital levels are obtained from the Schrödinger calculation. Lattice results are from Refs. [1,32].

J^{PC}	This work	Lattice [1,32]
0^{++}	1.4–1.5 (Salpeter)	1.475(30)(65)
0^{++}	2.79	2.755(70)(120)
0^{++}	3.29	3.370(100)(150)
0^{++}	4.00	3.990(210)(180)
2^{++}	2.40 (coupled S – D)	2.150(30)(100)
4^{++}	3.52	3.640(90)(160)
4^{++}	4.18	...
4^{++}	5.10	...
4^{++}	6.24	...
6^{++}	4.06	4.360(260)(200)
6^{++}	4.97	...
6^{++}	6.11	...
6^{++}	7.49	...
0^{-+}	2.65	2.250(60)(100)
0^{-+}	3.13	3.370(150)(150)
1^{-+}	-	2.94(17)
1^{-+}	-	3.81(27)
3^{++}	-	3.38(15)
3^{+-}	-	3.70(23)
3^{--}	-	4.03(25)

The tensor 2^{++} glueball remains spatially extended. Centrifugal suppression and moderate S – D wave mixing driven by the tensor interaction reduce short-distance overlap and weaken instanton effects. As a result, the tensor mass is shifted only moderately relative to the confinement baseline, and its radius remains considerably larger than that of the scalar, consistent with lattice mass hierarchies and emerging lattice probes of glueball spatial structure.

The semiclassical WKB analysis provides analytic insight into the lattice-observed organization of glueball excitations. The nonrelativistic linear-potential model yields a quasi-Regge behavior $M^2 \propto J^{4/3}$ rather than a constant asymptotic Regge slope. Adjoint confinement fixes the scale and curvature of this trajectory, which may appear approximately linear over the finite range of spins currently accessible on the lattice. Coulomb and instanton-induced interactions predominantly modify the low- J masses and the local curvature near the trajectory intercept.

For generic massive spin-one constituents, Bose symmetry alone does not exclude $J = 1$ two-gluon states. A Landau-Yang exclusion requires a transverse-helicity formulation, whereas the present Hamiltonian is written in the massive-spin LS basis. We therefore draw no quantitative conclusion about the vector and pseudovector sectors within the present two-body treatment.

Overall, the present analysis provides a coherent constituent-gluon interpretation of lattice glueball spectroscopy, linking mass hierarchies, spatial sizes, and quasi-Regge organization to a small set of physically motivated nonperturbative mechanisms. This framework offers a useful bridge between lattice calculations and phenomenological modeling, and can be systematically extended to explore glueball form factors and mixing with quarkonia in full QCD.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: GLUON MASS AND DENSITY SCALING

Musakhanov and Egamberdiev [27] derived the gluon polarization operator in the instanton medium and extracted a momentum-dependent dynamical mass. In their ILM setup the scalar “gluon” mass $M_s(q)$ is generated by rescattering on instantons, and the physical transverse gluon mass satisfies $M_g^2(q) = 2M_s^2(q)$, implying at $q = 0$ the standard-ILM value $M_g(0) \simeq 0.36$ GeV [27]. The key scaling with density follows from the structure of the self-energy in a random instanton background: to leading order in the density expansion the polarization operator is proportional to the instanton density n , and the mass is obtained from the infrared limit of the propagator denominator, schematically $D^{-1}(q) \sim q^2 + \Pi(q)$ with $\Pi(0) \propto n$. This implies $m_g^2 \propto n$ and therefore

$$m_g(\eta) = m_{g0}\sqrt{\eta}, \quad \eta \equiv \frac{n_{\text{eff}}}{n_0}, \quad (\text{A1})$$

where $n_0 \simeq 1 \text{ fm}^{-4}$ is the traditional ILM density used to define m_{g0} . In the dense-ILM-ensemble [35], η is naturally taken as a limit of gradient flow time going to zero. It is relatively uncertain from original studies; we use $\eta = 7$.

APPENDIX B: SEMICLASSICAL (WKB) SPECTRUM

A complementary analytic description follows from semiclassical quantization. For the linear potential $V(r) = \sigma_8 r$, with reduced mass $\mu = m_g/2$, the WKB spectrum with

the Langer modification yields

$$E_{nL}^{\text{WKB}}(\eta) = \left[\frac{3\pi}{2} \left(n + \frac{L}{2} + \frac{3}{4} \right) \right]^{2/3} \frac{\sigma_8^{2/3}}{m_g^{1/3}(\eta)},$$

$$M_{nL}^{\text{WKB}}(\eta) = 2m_g(\eta) + E_{nL}^{\text{WKB}}(\eta). \quad (\text{B1})$$

At fixed radial quantum number and large orbital angular momentum, $J \simeq L$, the excitation energy dominates the constant constituent mass term. Equation (B1) then gives

$$M_{nJ}^2 \simeq \left(\frac{3\pi}{4} \right)^{4/3} \frac{\sigma_8^{4/3}}{m_g^{2/3}} J^{4/3}, \quad J \gg 1. \quad (\text{B2})$$

Consequently,

$$\frac{dM_{nJ}^2}{dJ} \simeq \frac{4}{3} \left(\frac{3\pi}{4} \right)^{4/3} \frac{\sigma_8^{4/3}}{m_g^{2/3}} J^{1/3}, \quad (\text{B3})$$

which is not a constant asymptotic Regge slope. The nonrelativistic linear-potential model therefore produces a power-law “quasi-Regge” trajectory, $M^2 \propto J^{4/3}$, rather than a strictly linear Regge trajectory. Adjoint confinement fixes the overall scale and curvature of this trajectory. Over the moderate range of spins currently accessible in lattice calculations, the curvature is mild enough that $M^2(J)$ may appear approximately linear.

The size of relativistic kinetic corrections can be assessed by comparing the nonrelativistic two-body kinetic energy used in Eq. (B1),

$$T_{\text{NR}}(p) = 2m_g + \frac{p^2}{m_g}, \quad (\text{B4})$$

with the semirelativistic expression

$$T_{\text{SR}}(p) = 2\sqrt{p^2 + m_g^2} = 2m_g + \frac{p^2}{m_g} - \frac{p^4}{4m_g^3} + \mathcal{O}\left(\frac{p^6}{m_g^5}\right). \quad (\text{B5})$$

Thus, at fixed momentum, the nonrelativistic approximation overestimates the kinetic energy, with the first correction of relative order p^2/m_g^2 . This correction is expected to be moderate for the spatially extended orbital and radial excitations, for which the characteristic momentum is smaller than m_g . It is not parametrically controlled for the compact scalar state, where momenta of order p^{-1} contribute substantially. The WKB construction is therefore used only as a confinement-dominated baseline for extended states; the compact scalar is treated separately with the instantaneous Salpeter reduction.

For clarity, we recall below the unshifted WKB baseline, and list the shifted results for comparison in Table V. Additive low- J shifts adjusted to reproduce the scalar or tensor masses are calibrations rather than predictions and

TABLE V. Smoothly shifted trajectory points at $\eta = 7$. The unshifted WKB baseline is computed from adjoint linear confinement. The shift is anchored at the physical scalar intercept $M_{0^{++}}(J = 0) = 1.73$ GeV and decreases smoothly with J .

J	M^{WKB} (GeV)	$\Delta M(J)$ (GeV)	$M^{\text{WKB}} + \Delta M$ (GeV)	$(M^{\text{WKB}})^2$ (GeV ²)	$(M^{\text{WKB}} + \Delta M)^2$ (GeV ²)
0	3.196	-1.466	1.730	10.213	2.993
2	4.176	-0.681	3.495	17.438	12.215
4	4.974	-0.481	4.493	24.745	20.190
6	5.680	-0.383	5.296	32.257	28.049

are not used to define an asymptotic Regge slope. Coulomb, instanton-induced, and spin-dependent interactions modify primarily the low- J region and generate channel-dependent changes in the intercept and local curvature.

A convenient WKB size measure is the outer turning point $r_+ = E/\sigma_8$. For $L = 0$, the WKB radial probability density gives closed expressions for its moments. With

$$p(r) = \sqrt{2\mu(E - \sigma_8 r)} \quad (\text{B6})$$

and

$$|\psi|^2 d^3 r \propto P(r) dr \propto \frac{dr}{p(r)}, \quad (\text{B7})$$

one obtains

$$P(r) = \frac{1}{2\sqrt{r_+}} \frac{1}{\sqrt{r_+ - r}}, \quad r_+ = \frac{E}{\sigma_8}. \quad (\text{B8})$$

In particular,

$$\begin{aligned} \langle r \rangle_{L=0}^{\text{WKB}} &= \frac{2}{3} r_+, \\ \langle r^2 \rangle_{L=0}^{\text{WKB}} &= \frac{8}{15} r_+^2, \\ r_{\text{rms}, L=0}^{\text{WKB}} &= \sqrt{\frac{8}{15}} r_+. \end{aligned} \quad (\text{B9})$$

The moments follow from beta-function integrals, giving Eq. (B9). The same turning-point definition applies at general (n, L) , with $r_+ = E_{nL}^{\text{WKB}}/\sigma_8$, and provides a useful measure of the spatial scale of excited states.

For $L > 0$, the centrifugal term modifies the radial probability distribution, so the $L = 0$ relation $r_{\text{rms}} = \sqrt{8/15} r_+$ is not exact. In Table VI we use it only as a uniform semiclassical size estimate for comparing different levels; it should not be interpreted as a precise higher- L rms radius.

Table VI gives the WKB masses and size estimates at $\eta = 7$ for low-lying $L = 0$ scalarlike and $L = 2$ tensorlike

TABLE VI. Unshifted WKB masses and semiclassical size estimates at $\eta = 7$ for the linear adjoint potential. For the orbital sequence, J is identified with L . The listed values define a confinement-dominated quasi-Regge baseline. Coulomb and instanton-induced interactions are omitted and therefore no fitted low- J mass shifts are included. For $L > 0$, the quoted $r_{\text{rms}}^{\text{WKB}}$ values use the $L = 0$ relation as a uniform size estimate rather than an exact higher-orbital result.

n	L	J	M_{nL}^{WKB} (GeV)	r_+ (fm)	$r_{\text{rms}}^{\text{WKB}}$ (fm)
0	0	0	3.196	0.628	0.459
1	0	0	4.176	1.105	0.807
2	0	0	4.974	1.493	1.090
0	2	2	4.176	1.105	0.807
1	2	2	4.974	1.493	1.090
0	4	4	4.974	1.493	1.090
0	6	6	5.680	1.836	1.341

orbital levels, together with several higher orbital excitations. The conversion $1 \text{ GeV}^{-1} = 0.197 \text{ fm}$ is used.

APPENDIX C: STATIC POTENTIALS FOR FUNDAMENTAL AND ADJOINT CHARGES FROM INSTANTONS AND $I\bar{I}$ MOLECULES

Wilson loops involve path-ordered exponents

$$W_C = P \exp \left[i \sum \Delta x_\mu A_\mu^a T^a \right], \quad (\text{C1})$$

taken over closed contours C , usually of rectangular shape. The sum runs over infinitesimal elements Δx_μ along the loop, and A_μ^a are vacuum gauge fields. Since the contour is closed, W_C is gauge invariant.

For static color charges corresponding to quarks, the color generators are in the fundamental $SU(N_c)$ representation, $T^a = \lambda^a/2$, with Pauli or Gell-Mann matrices. In several of our earlier publications we have numerically calculated the corresponding static confining potentials from ensembles of instantons or $I\bar{I}$ molecules, and applied them to quarkonia, baryons, and tetraquarks [36]. The upper panel of Fig. 9 shows an example of such a Monte-Carlo simulation. (The Wilson line locations and orientations are randomized.)

Since a “constituent gluon” belongs to the adjoint color representation, the only modification in the Wilson loop is that the generators are now adjoint:

$$(T^a)_{bc} = -i\epsilon^a_{bc}. \quad (\text{C2})$$

Unlike the case of Pauli matrices, for which a compact closed form for the exponent exists, no such expression is available here. The practical method we employ is based on a Taylor expansion of the exponent to second order in Δx_μ , assumed to be small. The adjoint potential shown in the

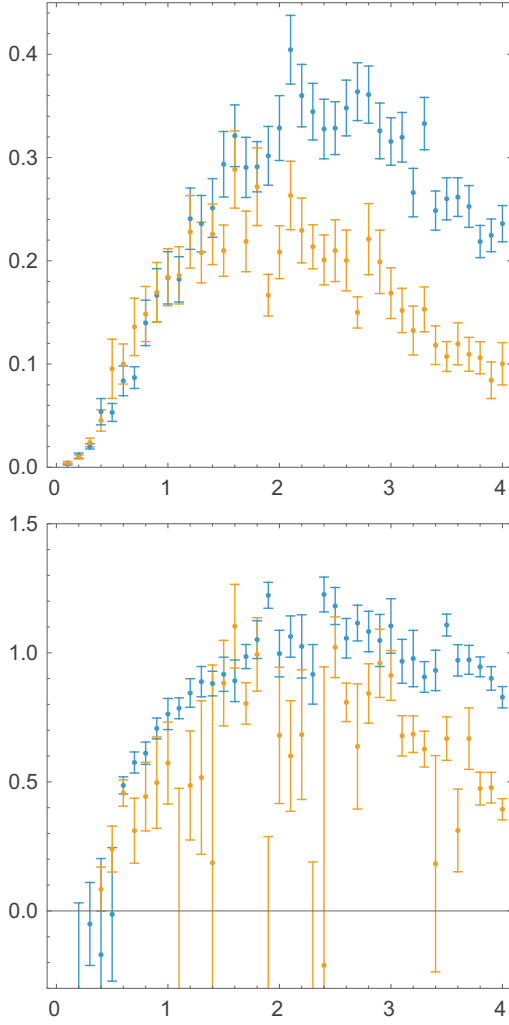


FIG. 9. Static potentials $V_{\text{conf}}(r)$ in GeV as functions of r in GeV^{-1} , calculated from Wilson lines for static sources in the fundamental representation (upper panel) and the adjoint representation (lower panel). Blue points correspond to an $\bar{I}\bar{I}$ molecule and red points to a single instanton. The potentials must subsequently be rescaled according to the density of pseudo-particles.

lower panel of Fig. 9 was calculated in this way, using the same gauge-field configurations and the same set of Wilson lines as in the upper panel.

First, note that the vertical scales of the two plots are different, and that, crudely, the ratio of the two potentials is approximately 9/4, as suggested by the ratio of the corresponding color Casimir operators.

Second, since the Wilson lines and the resulting potentials contain nonlinear terms (higher powers of the gauge field), such scaling is not expected to hold exactly. Indeed, the shapes of the two potentials differ somewhat. Nevertheless, all potentials reach a maximum at $r \sim 2.5 \text{ GeV}^{-1} \sim 0.5 \text{ fm}$ and then decrease slightly. This feature is likely an artifact of the setup used here, in which a single instanton (or molecule) is surrounded by a Wilson

loop. In an ensemble of such objects, the potential is expected to saturate at large r to a constant, equal to twice the effective mass of the static charges.

For heavy quarks interacting via instantons, this asymptotic value is

$$V_{\text{conf}}(r \rightarrow \infty) \approx 2\eta \times (0.070 \text{ GeV}) \approx 1 \text{ GeV}, \quad (\text{C3})$$

for $\eta \sim 7$, corresponding to a rescaling from the dilute instanton liquid model to a dense ensemble of molecules. In physical QCD such an energy is sufficient to produce an additional $\bar{q}q$ pair and split quarkonium $\bar{Q}Q$ into two $\bar{Q}q$ mesons.

For “constituent gluons” in pure gauge theory, the large-distance splitting instead corresponds to $gg \rightarrow (gg)(gg)$, so that $V_{\text{conf}}(r \rightarrow \infty)$ must be at least of order $M_{0^{++}} \sim 1.5 \text{ GeV}$. Guided by these considerations, we model our potential in the form shown in Fig. 4.

APPENDIX D: DETAILS OF THE ${}^5S_2 - {}^5D_2$ MIXING IN THE TENSOR CHANNEL

For two spin-1 constituents, the tensor glueball 2^{++} is dominated by total spin $S = 2$ and total $J = 2$, with orbital components $L = 0$ and $L = 2$ mixed by the tensor operator S_{12} . In the coupled basis $\{|{}^5S_2\rangle, |{}^5D_2\rangle\}$, the tensor operator has the standard reduced matrix elements

$$\begin{aligned} \langle {}^5S_2 | S_{12} | {}^5S_2 \rangle &= 0, & \langle {}^5D_2 | S_{12} | {}^5D_2 \rangle &= -\frac{2}{7}, \\ \langle {}^5S_2 | S_{12} | {}^5D_2 \rangle &= \sqrt{\frac{8}{7}}, \end{aligned} \quad (\text{D1})$$

up to phase conventions.

Using the Gaussian trial functions (E1) and (E3), the D -wave normalization is

$$\mathcal{N}_D = \left(\frac{8}{15}\right)^{1/2} \left(\frac{\beta_D^2}{\pi}\right)^{3/4} \beta_D^2, \quad (\text{D2})$$

and the D -wave kinetic and linear moments are

$$\langle \mathbf{p}^2 \rangle_D = \frac{7}{2} \beta_D^2, \quad \langle r \rangle_D = \frac{16}{3\sqrt{\pi}} \frac{1}{\beta_D}, \quad \langle e^{-r^2/\rho^2} \rangle_D = \left(\frac{\beta_D^2}{\beta_D^2 + \rho^{-2}} \right)^{7/2}. \quad (\text{D3})$$

The corresponding spin-independent functional is

$$\begin{aligned} E_D^{(0)}(\beta_D; \eta) &= 2m_g(\eta) + \frac{7\beta_D^2}{2m_g(\eta)} + \sigma_8 \frac{16}{3\sqrt{\pi}\beta_D} \\ &\quad - \frac{6\alpha_s^{\text{eff}}}{\sqrt{\pi}} \beta_D \kappa_D - G(\eta) \left(\frac{\beta_D^2}{\beta_D^2 + \rho^{-2}} \right)^{7/2}, \end{aligned} \quad (\text{D4})$$

where κ_D encodes the reduced D -wave expectation of $1/r$ relative to the S -wave [it is an $\mathcal{O}(1)$ number obtained by a straightforward radial integral; for compactness we keep it symbolic here since the tensor state is typically

controlled more by confinement and mixing than by the Coulomb core.]

The tensor-sector 2×2 Hamiltonian is

$$\mathbb{H}_{J=2} = \begin{pmatrix} E_S^{(0)}(\beta_S; \eta) + \Delta_{SS}^{(S=2)}(\eta) & V_{SD}(\eta) \\ V_{SD}(\eta) & E_D^{(0)}(\beta_D; \eta) + \Delta_{SS,D}^{(S=2)}(\eta) + \Delta_{T,DD}(\eta) \end{pmatrix}, \quad (\text{D5})$$

with

$$\Delta_{SS}^{(S=2)}(\eta) = \frac{C_{SS}(\eta)}{m_g^2(\eta)} \langle \delta_\Lambda^{(3)} \rangle_S, \quad \Delta_{SS,D}^{(S=2)}(\eta) = \frac{C_{SS}(\eta)}{m_g^2(\eta)} \langle \delta_\Lambda^{(3)} \rangle_D, \quad (\text{D6})$$

and tensor terms

$$V_{SD}(\eta) = \frac{C_T(\eta)}{m_g^2(\eta)} \sqrt{\frac{8}{7}} \int_0^\infty dr r^2 R_S(r) R_D(r) \frac{1 - e^{-r^2/\rho^2}}{r^3}, \quad (\text{D7})$$

$$\Delta_{T,DD}(\eta) = \frac{C_T(\eta)}{m_g^2(\eta)} \left(-\frac{2}{7} \right) \int_0^\infty dr r^2 |R_D(r)|^2 \frac{1 - e^{-r^2/\rho^2}}{r^3}. \quad (\text{D8})$$

Diagonalizing (D5) gives the physical tensor mass as the lower eigenvalue and defines a mixing angle θ by

$$\tan 2\theta = \frac{2V_{SD}}{H_{DD} - H_{SS}}. \quad (\text{D9})$$

The tensor rms radius is then computed from the mixed state,

$$\langle r^2 \rangle_{2++} = \cos^2 \theta \langle r^2 \rangle_S + \sin^2 \theta \langle r^2 \rangle_D, \quad (\text{D10})$$

with $r_{2++} = \sqrt{\langle r^2 \rangle_{2++}}$. The interference term vanishes because r^2 is purely radial and orthogonal between $L = 0$ and $L = 2$.

APPENDIX E: DETAILS OF THE RADIAL INTEGRATIONS FOR TENSOR MIXING

This Appendix provides the explicit analytic evaluation of all radial integrals entering the ${}^5S_2 - {}^5D_2$ tensor mixing problem and the corresponding expectation values of the Hamiltonian terms.

1. Normalized trial wave functions

For simplicity, we will use variational estimates of matrix elements using simplified wave functions. The S -wave Gaussian trial function is

$$\psi_S(\mathbf{r}) = \left(\frac{\beta_S^2}{\pi} \right)^{3/4} e^{-\beta_S^2 r^2/2}, \quad (\text{E1})$$

with radial part

$$R_S(r) = \left(\frac{2\beta_S^3}{\sqrt{\pi}} \right)^{1/2} e^{-\beta_S^2 r^2/2}. \quad (\text{E2})$$

The D -wave trial function is

$$\psi_{D,m}(\mathbf{r}) = \mathcal{N}_D r^2 e^{-\beta_D^2 r^2/2} Y_{2m}(\hat{\mathbf{r}}), \quad (\text{E3})$$

with normalization fixed by

$$1 = \int d^3r |\psi_{D,m}(\mathbf{r})|^2 = \mathcal{N}_D^2 \int_0^\infty dr r^6 e^{-\beta_D^2 r^2/2}, \quad (\text{E4})$$

which yields

$$\mathcal{N}_D = \left(\frac{8}{15} \right)^{1/2} \left(\frac{\beta_D^2}{\pi} \right)^{3/4} \beta_D^2. \quad (\text{E5})$$

The radial function is therefore

$$R_D(r) = \left(\frac{8}{15} \right)^{1/2} \left(\frac{\beta_D^2}{\pi} \right)^{3/4} \beta_D^2 r^2 e^{-\beta_D^2 r^2/2}. \quad (\text{E6})$$

2. Kinetic and confining expectation values

Using standard Gaussian integrals,

$$\langle \mathbf{p}^2 \rangle_S = \frac{3}{2} \beta_S^2, \quad \langle \mathbf{p}^2 \rangle_D = \frac{7}{2} \beta_D^2. \quad (\text{E7})$$

The linear confinement expectation values are

$$\langle r \rangle_S = \frac{2}{\sqrt{\pi}} \frac{1}{\beta_S}, \quad \langle r \rangle_D = \frac{16}{3\sqrt{\pi}} \frac{1}{\beta_D}. \quad (\text{E8})$$

The corresponding rms radii are

$$\langle r^2 \rangle_S = \frac{3}{2\beta_S^2}, \quad \langle r^2 \rangle_D = \frac{7}{2\beta_D^2}. \quad (\text{E9})$$

3. Coulomb expectation values

For the Coulomb potential $V_C(r) = -3\alpha_s^{\text{eff}}/r$, one finds

$$\left\langle \frac{1}{r} \right\rangle_S = \frac{2\beta_S}{\sqrt{\pi}}, \quad (\text{E10})$$

and for the D -wave,

$$\left\langle \frac{1}{r} \right\rangle_D = \frac{8}{3\sqrt{\pi}}\beta_D. \quad (\text{E11})$$

Thus the Coulomb term is parametrically suppressed in the tensor channel by angular momentum.

4. Instanton central attraction

The scalar instanton term yields

$$\langle e^{-r^2/\rho^2} \rangle_S = \left(\frac{\beta_S^2}{\beta_S^2 + \rho^{-2}} \right)^{3/2}. \quad (\text{E12})$$

For the D -wave,

$$\langle e^{-r^2/\rho^2} \rangle_D = \left(\frac{\beta_D^2}{\beta_D^2 + \rho^{-2}} \right)^{7/2}, \quad (\text{E13})$$

showing that instanton attraction is strongly suppressed for higher partial waves.

5. Spin-spin contact term

With the Gaussian regulator

$$\delta_\Lambda^{(3)}(\mathbf{r}) = \left(\frac{\Lambda^2}{\pi} \right)^{3/2} e^{-\Lambda^2 r^2}, \quad (\text{E14})$$

the contact expectation values are

$$\langle \delta_\Lambda^{(3)} \rangle_S = \left(\frac{\Lambda^2}{\Lambda^2 + \beta_S^2} \right)^{3/2} \left(\frac{\beta_S^2}{\pi} \right)^{3/2}, \quad (\text{E15})$$

and

$$\langle \delta_\Lambda^{(3)} \rangle_D = \left(\frac{\Lambda^2}{\Lambda^2 + \beta_D^2} \right)^{7/2} \left(\frac{\beta_D^2}{\pi} \right)^{3/2} \frac{15}{8}. \quad (\text{E16})$$

6. Tensor matrix elements

The tensor operator enters through

$$V_T(r) = \frac{C_T(\eta)}{m_g^2(\eta)} \frac{1 - e^{-r^2/\rho^2}}{r^3} S_{12}. \quad (\text{E17})$$

The reduced angular matrix elements in the $\{^5S_2, ^5D_2\}$ basis are

$$\begin{aligned} \langle ^5S_2 | S_{12} | ^5S_2 \rangle &= 0, & \langle ^5D_2 | S_{12} | ^5D_2 \rangle &= -\frac{2}{7}, \\ \langle ^5S_2 | S_{12} | ^5D_2 \rangle &= \sqrt{\frac{8}{7}}. \end{aligned} \quad (\text{E18})$$

The radial mixing integral is

$$I_{SD} = \int_0^\infty dr R_S(r) R_D(r) \frac{1 - e^{-r^2/\rho^2}}{r}, \quad (\text{E19})$$

which yields

$$I_{SD} = C \left[\frac{1}{(\beta_S^2 + \beta_D^2)^{1/2}} - \frac{1}{(\beta_S^2 + \beta_D^2 + \rho^{-2})^{1/2}} \right], \quad (\text{E20})$$

with the normalization factor

$$C = \left(\frac{16}{15\pi} \right)^{1/2} \beta_S^{3/2} \beta_D^{7/2}. \quad (\text{E21})$$

The diagonal D -wave tensor term is

$$I_{DD} = \int_0^\infty dr R_D^2(r) \frac{1 - e^{-r^2/\rho^2}}{r}, \quad (\text{E22})$$

which yields

$$I_{DD} = \frac{4}{7} \left[\beta_D - \frac{\beta_D^2}{(\beta_D^2 + \rho^{-2})^{1/2}} \right]. \quad (\text{E23})$$

7. Explicit $^5S_2 - ^5D_2$ tensor Hamiltonian

In the coupled basis $\{|^5S_2\rangle, |^5D_2\rangle\}$ the tensor glueball Hamiltonian takes the explicit 2×2 form

$$\mathbb{H}_{2++}(\eta) = \begin{pmatrix} H_{SS}(\eta) & H_{SD}(\eta) \\ H_{SD}(\eta) & H_{DD}(\eta) \end{pmatrix}, \quad (\text{E24})$$

with the diagonal S -wave entry

$$H_{SS}(\eta) = E_S^{(0)}(\beta_S; \eta) + \frac{C_{SS}(\eta)}{m_g^2(\eta)} \langle \delta_\Lambda^{(3)} \rangle_S, \quad (\text{E25})$$

where $\langle \delta_\Lambda^{(3)} \rangle_S$ is given in Eq. (E15).

The diagonal D -wave entry reads

$$H_{DD}(\eta) = E_D^{(0)}(\beta_D; \eta) + \frac{C_{SD}(\eta)}{m_g^2(\eta)} \langle \delta_\Lambda^{(3)} \rangle_D - \frac{2}{7} \frac{C_T(\eta)}{m_g^2(\eta)} I_{DD}, \quad (\text{E26})$$

where the factor $-2/7$ is the reduced angular matrix element $\langle ^5D_2 | S_{12} | ^5D_2 \rangle$ and I_{DD} is given in Eq. (E23).

The off-diagonal mixing term is

$$H_{SD}(\eta) = \sqrt{\frac{8}{7}} \frac{C_T(\eta)}{m_g^2(\eta)} I_{SD}, \quad (\text{E27})$$

where $\sqrt{8/7}$ is the reduced matrix element $\langle {}^5S_2 | S_{12} | {}^5D_2 \rangle$ and I_{SD} is given in Eq. (E20).

The physical tensor glueball mass is the lower eigenvalue

$$M_{2^{++}}(\eta) = \frac{1}{2} (H_{SS} + H_{DD}) - \frac{1}{2} \sqrt{(H_{SS} - H_{DD})^2 + 4H_{SD}^2}, \quad (\text{E28})$$

and the $S - D$ mixing angle θ is defined by

$$\tan 2\theta(\eta) = \frac{2H_{SD}(\eta)}{H_{DD}(\eta) - H_{SS}(\eta)}. \quad (\text{E29})$$

The tensor rms radius follows directly as

$$\begin{aligned} \langle r^2 \rangle_{2^{++}} &= \cos^2 \theta \langle r^2 \rangle_S + \sin^2 \theta \langle r^2 \rangle_D, \\ r_{2^{++}} &= \sqrt{\langle r^2 \rangle_{2^{++}}}. \end{aligned} \quad (\text{E30})$$

APPENDIX F: MODEL PARAMETERS

This Appendix summarizes the parameters of the model and distinguishes external inputs, fitted quantities, and regulator or model choices. The classification is important because the lowest scalar and tensor masses, their radii, and the $S - D$ admixture are not parameter-free predictions. Once the reference parameter set has been fixed, the

masses of states not included in the calibration and the corresponding wave function observables are conditional predictions of the model. The parameters used in the calculation are summarized in Table VII. The fundamental string tension is taken to be

$$\sigma_3 = 0.18 \text{ GeV}^2, \quad (\text{F1})$$

consistent with lattice determinations in pure Yang-Mills theory. Assuming Casimir scaling for adjoint sources gives

$$\sigma_8 = \frac{C_A}{C_F} \sigma_3 = \frac{9}{4} \sigma_3 \simeq 0.405 \text{ GeV}^2. \quad (\text{F2})$$

This input fixes the confinement scale and the curvature of the confinement-dominated quasi-Regge trajectories. The instanton size is taken as

$$\rho = \frac{1}{3} \text{ fm} \simeq 1.67 \text{ GeV}^{-1}, \quad (\text{F3})$$

in accordance with instanton-vacuum phenomenology and lattice determinations of the instanton-size distribution. It sets both the range of the instanton-induced interactions and the contact regulator scale. The dependence of the compact scalar solution on ρ is examined separately in the sensitivity analysis. The effective gluon mass at unit instanton density is

$$m_{g0} \simeq 0.36 \text{ GeV}, \quad (\text{F4})$$

as obtained from the infrared gluon propagator in the instanton vacuum. Its density dependence is parametrized by

TABLE VII. Model parameters, their reference values, and their status in the calculation. “External input” denotes a quantity taken from lattice or instanton-vacuum phenomenology; “fitted parameter” denotes a quantity adjusted using selected glueball masses; and “model choice” or “regulator choice” denotes a quantity introduced in the Hamiltonian reduction or numerical implementation.

Parameter	Reference value or range	Status	Primary role
σ_3	0.18 GeV ²	External input	Fundamental string tension
σ_8	$\frac{9}{4} \sigma_3 = 0.405 \text{ GeV}^2$	Derived input	Adjoint confinement
ρ	$\frac{1}{3} \text{ fm} \simeq 1.67 \text{ GeV}^{-1}$	External ILM input	Range of instanton forces
m_{g0}	0.36 GeV	External ILM input	Gluon mass at $\eta = 1$
η	Specified for each calculation	Density parameter	fixes $m_g(\eta)$ and instanton strength
α_s^{eff}	0.35–0.45	Phenomenological input	Adjoint Coulomb Attraction
G_0	2–2.5 GeV	Fitted parameter	Scalar instanton attraction
$C_{SS}^{(0)}$	3.0	Fitted parameter	Scalar-tensor spin splitting
$C_T^{(0)}$	−1.5	Phenomenological input	Tensor mixing
Λ	ρ^{-1}	Regulator choice	Contact-interaction range
R	2 fm	Finite-volume regulator	Numerical boundary condition
r_{scr}	approximately R	Model choice	Adjoint screening scale
V_0	fixed from the absolute spectrum	Fitted energy shift	Common spectral offset

$$m_g(\eta) = m_{g0}\sqrt{\eta}, \quad (\text{F5})$$

reflecting the proportionality of the polarization operator to the instanton density [27]. The value of η , and hence of m_g , is stated explicitly for each numerical calculation and is not counted as an independent prediction. The effective Coulomb coupling is treated phenomenologically in the range

$$\alpha_s^{\text{eff}} = 0.35 - 0.45. \quad (\text{F6})$$

It controls the short-distance adjoint Coulomb attraction and therefore affects both the mass and radius of the compact scalar state. Results obtained by varying α_s^{eff} within this interval are included in the sensitivity analysis. The instanton-induced scalar attraction is parametrized as

$$G(\eta) = G_0\eta, \quad (\text{F7})$$

where G_0 is adjusted, together with the Coulomb contribution, to the lowest scalar mass. Typical fitted values are

$$G_0 \simeq 2 - 2.5 \text{ GeV}. \quad (\text{F8})$$

Consequently, the scalar mass used in this calibration is not a prediction. Its radius is a derived model result, conditional on ρ , α_s^{eff} , and the fitted value of G_0 . The spin-spin interaction is written as

$$C_{SS}(\eta) = C_{SS}^{(0)}\eta. \quad (\text{F9})$$

Using the optimized S -wave width and the regulated contact matrix element derived in Appendix E, we use

$$C_{SS}^{(0)} = 3.0. \quad (\text{F10})$$

This coefficient is fitted to the scalar-tensor splitting and is therefore not an independent prediction of the model. The tensor coupling is parametrized by

$$C_T(\eta) = C_T^{(0)}\eta. \quad (\text{F11})$$

The instanton analysis of Ref. [37] indicates a negative tensor contribution, partially cancelling the perturbative one. We therefore use

$$C_T^{(0)} = -1.5, \quad (\text{F12})$$

corresponding to one-half the magnitude of the spin-spin coupling with the opposite sign. This is a phenomenological choice rather than a fitted prediction. The resulting D -wave probability is consequently a model-dependent output. For the reference calculation at $\eta = 1$,

$$P_D = \sin^2 \theta \simeq 0.15. \quad (\text{F13})$$

The contact regulator is fixed by the instanton size,

$$\Lambda = \rho^{-1}. \quad (\text{F14})$$

The numerical Schrödinger problem is solved inside a spherical cavity of radius

$$R = 2 \text{ fm}, \quad (\text{F15})$$

with the screening scale chosen as $r_{\text{scr}} \simeq R$. These are regulator and model choices rather than physical predictions. Their effects on the lowest extended states are checked by varying the box and screening scales. At fixed η , the spin-independent energies are minimized independently with respect to the widths β_S and β_D . Spin-dependent matrix elements are then evaluated with the optimized wave functions, and the tensor Hamiltonian is diagonalized in the $\{^5S_2, ^5D_2\}$ basis. The WKB calculation uses the same σ_8 and $m_g(\eta)$ but omits Coulomb and instanton interactions at leading order. It therefore provides a confinement-dominated quasi-Regge baseline rather than an independent fit to the complete spectrum.

APPENDIX G: $0^{++}, 2^{++}$ BS-KERNELS FROM ILM

The instanton-induced local operator derived in Sec. IV, may be written schematically as

$$\Delta\mathcal{L}_{4g}(x) \propto (G_{\mu\nu}^a(x)G_{\mu\nu}^a(x))(G_{\rho\sigma}^b(x)G_{\rho\sigma}^b(x)), \quad (\text{G1})$$

so the $gg \rightarrow gg$ Born amplitude is obtained by evaluating

$$\langle g_3 g_4 | \int d^4x \Delta\mathcal{L}_{4g}(x) | g_1 g_2 \rangle.$$

Using LSZ for external gluons,

$$\langle 0 | G_{\mu\nu}^a(x) | g^b(k, \lambda) \rangle = i\delta^{ab}(k_\mu \epsilon_\nu(k, \lambda) - k_\nu \epsilon_\mu(k, \lambda))e^{-ik \cdot x}, \quad (\text{G2})$$

with $k \cdot \epsilon(k, \lambda) = 0$, we can reduce each field strength to its on-shell transverse wave function. We now define the basic gauge-invariant contraction for two external legs i, j ,

$$\begin{aligned} F(i, j) &\equiv G_{\mu\nu}^a(i)G_{\mu\nu}^a(j) \\ &= -\delta^{a_i a_j}(k_{i\mu}\epsilon_{i\nu} - k_{i\nu}\epsilon_{i\mu})(k_{j\mu}\epsilon_{j\nu} - k_{j\nu}\epsilon_{j\mu}) \\ &= -2\delta^{a_i a_j}[(k_i \cdot k_j)(\epsilon_i \cdot \epsilon_j) - (k_i \cdot \epsilon_j)(k_j \cdot \epsilon_i)]. \end{aligned} \quad (\text{G3})$$

The instanton-induced four-gluon amplitude is then a sum of pairings as in IV,

$$\mathcal{M}^{(4)}(1, 2 \rightarrow 3, 4) \propto F(1, 3)F(2, 4) + F(1, 4)F(2, 3), \quad (\text{G4})$$

up to channel-independent prefactors and form factors β_{2g} on each leg.

1. CM kinematics for massive transverse polarizations

The effective gluons carry a mass m_g . In the CM frame, the two transverse polarizations read

$$\begin{aligned} k_1^\mu &= (\omega, 0, 0, +k), \\ k_2^\mu &= (\omega, 0, 0, -k), \\ k_3^\mu &= (\omega, k \sin \theta, 0, k \cos \theta), \\ k_4^\mu &= (\omega, -k \sin \theta, 0, -k \cos \theta), \end{aligned} \quad (\text{G5})$$

with the on-shell constituent gluon energy $\omega \equiv \sqrt{k^2 + m_g^2}$. Consider now only the transverse polarizations in the LSZ reduction of the in-out gluons

$$\begin{aligned} \epsilon_\pm^\mu(k_1) &= \frac{1}{\sqrt{2}}(0, 1, \pm i, 0), \\ \epsilon_\pm^\mu(k_2) &= \frac{1}{\sqrt{2}}(0, -1, \pm i, 0), \\ \epsilon_\pm^\mu(k_3) &= \frac{1}{\sqrt{2}}(0, \cos \theta, \pm i, -\sin \theta), \\ \epsilon_\pm^\mu(k_4) &= \frac{1}{\sqrt{2}}(0, -\cos \theta, \pm i, +\sin \theta), \end{aligned} \quad (\text{G6})$$

so that $k_i \cdot \epsilon_i = 0$ for each leg.

In the CM kinematics above, the contractions in (G3) give

$$\begin{aligned} k_1 \cdot k_3 &= \omega^2 - k^2 x, & k_1 \cdot k_4 &= \omega^2 + k^2 x, \\ k_1 \cdot \epsilon_\pm(k_3) &= +\frac{k \sin \theta}{\sqrt{2}}, & k_1 \cdot \epsilon_\pm(k_4) &= -\frac{k \sin \theta}{\sqrt{2}}, \end{aligned} \quad (\text{G7})$$

and similarly with $1 \leftrightarrow 2$. With the polarization choice (G6) we have the identities

$$\begin{aligned} \epsilon_\pm(k_1) \cdot \epsilon_\pm(k_3) &= \frac{1-x}{2}, & \epsilon_\pm(k_1) \cdot \epsilon_\mp(k_3) &= -\frac{1+x}{2}, \\ k_3 \cdot \epsilon_\pm(k_1) &= -\frac{k \sin \theta}{\sqrt{2}}, & k_4 \cdot \epsilon_\pm(k_1) &= +\frac{k \sin \theta}{\sqrt{2}}, \end{aligned} \quad (\text{G8})$$

and similarly for the contractions (2,4) and (2,3).

2. Instanton vertex contractions

Using these kinematics and polarizations into (G3), yield the instanton vertex contractions

$$\begin{aligned} F(1,3)|_{++\rightarrow++} &= -(1-x)(\omega^2 + k^2), \\ F(1,3)|_{+-\rightarrow+-} &= -(1+x)(\omega^2 + k^2), \\ F(1,4)|_{++\rightarrow++} &= -(1+x)(\omega^2 + k^2), \\ F(1,4)|_{+-\rightarrow+-} &= -(1-x)(\omega^2 + k^2), \end{aligned} \quad (\text{G9})$$

with $x \equiv \cos \theta$, modulo diagonal color factors. The remaining contractions $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, follow similarly. The key simplification is the cancellation of the mixed terms proportional to $k^2 x$, leaving the universal factor $(\omega^2 + k^2)$.

For the 0^{++} scalar glueball we use the $J_z = 0$ polarization combination

$$|0^{++}, 0\rangle_{\text{pol}} = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle).$$

Using (G9), we obtain

$$\begin{aligned} \mathcal{M}_{0^{++}}^{(4)}(\theta) &\propto (\omega^2 + k^2)^2[(1+x)^2 + (1-x)^2] \\ &= 2(\omega^2 + k^2)^2(1+x^2), \end{aligned} \quad (\text{G10})$$

as noted in Eq. (33). Similarly, for the tensor glueball

$$|2^{++}, \pm 2\rangle_{\text{pol}} = \frac{1}{\sqrt{2}}(|++\rangle \pm |--\rangle),$$

we obtain

$$\begin{aligned} \mathcal{M}_{2^{++}}^{(4)}(\theta) &\propto (\omega^2 + k^2)^2[(1+x)^2 + (1-x)^2] \\ &= 2(\omega^2 + k^2)^2(1+x^2), \end{aligned} \quad (\text{G11})$$

so the instanton-induced on-shell helicity amplitudes in both 0^{++} and 2^{++} channels share the same $(1 + \cos^2 \theta)$ dependence, with either massive or massless gluons.

3. Scalar and tensor projections

The scalar and tensor channel differences enter only through the subsequent J -projection used to construct the CM kernel, which is then iterated in the Bethe-Salpeter equation. For two on-shell particles with definite helicities, the partial-wave decomposition of a helicity amplitude $\mathcal{M}_{\lambda\lambda'}(\theta)$ is given by the helicity projection formula

$$V_{\lambda\lambda'}^J(k, k') = 2\pi \int_{-1}^1 d(\cos \theta) d_{\lambda\lambda'}^J(\theta) \mathcal{M}_{\lambda\lambda'}(\theta), \quad (\text{G12})$$

where $d_{\lambda\lambda'}^J(\theta)$ is the Wigner d -function and $\lambda = \lambda_1 - \lambda_2$, $\lambda' = \lambda_3 - \lambda_4$. For the scalar and tensor channels

$$d_{00}^0(\theta) = 1(0^{++}),$$

$$d_{22}^2(\theta) = \frac{(1 + \cos \theta)^2}{4} \quad (2^{++}, |\lambda| = 2). \quad (\text{G13})$$

Inserting (G13) in (G12) yields the relevant angular integrals

$$I_{0^{++}} = \int_{-1}^1 dx (1 + x^2) = \frac{8}{3}, \quad (\text{G14})$$

$$I_{2^{++}} = \int_{-1}^1 dx (1 + x^2) \frac{(1 + x)^2}{4} = \frac{14}{15}, \quad (\text{G15})$$

hence the physical suppression factor

$$\frac{I_{2^{++}}}{I_{0^{++}}} = \frac{14/15}{8/3} = \frac{7}{20} \simeq \frac{1}{3} \quad (\text{G16})$$

in comparison to the $\frac{1}{2}$ obtained using the Legendre polynomial projection in the main text. Note that in the scalar glueball channel both the Legendre and the helicity projection method yield the same factor $\mathcal{K}_{0^{++}}$ in the iterated Bethe-Salpeter kernel.

4. Consequences for binding

After the instantaneous reduction, the tensor channel in the Bethe-Salpeter derivation inherits the same separable structure as the scalar channel,

$$V_{2^{++}}(\mathbf{p}, \mathbf{k}) = -V_{0,2} \beta_{2g}^2(\rho p) \beta_{2g}^2(\rho k), \quad (\text{G17})$$

but with a reduced strength

$$V_{0,2} = \left(\frac{7}{20}\right) V_{0,0}, \quad (\text{G18})$$

relative to the scalar 0^{++} channel. In addition, the tensor state carries an $L = 2$ centrifugal barrier and is spatially more extended, further reducing its sensitivity to a short-range interaction.

For the same parameter set that produces a moderately bound scalar glueball ($M_{0^{++}} \simeq 1.4\text{--}1.5$ GeV), the reduced coupling (G18) is insufficient to overcome the centrifugal suppression, and the reduced Bethe-Salpeter equation admits no tensor bound state below the two-gluon threshold. The 2^{++} channel therefore remains unbound within the single-instanton, rank-one instantaneous approximation. Its larger size makes it more sensitive to the confining potential, as we discussed in the main text. It is confined with a mass above the constitutive two-gluon threshold.

APPENDIX H: 0^{-+} SUPPRESSION

In the instantaneous reduction of the BS approach in the ILM, the emergent four-gluon coupling from the expanded

exponent in (17) vanishes in the 0^{-+} channel while it is enhanced in the 0^{++} channel owing to self- and anti-self-duality of the 't Hooft symbols,

$$\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \eta_{\alpha\beta}^a = +\eta_{\mu\nu}^a, \quad \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \bar{\eta}_{\alpha\beta}^a = -\bar{\eta}_{\mu\nu}^a. \quad (\text{H1})$$

Expanding the exponentials in (17) to fourth order in $G_{\mu\nu}$ and averaging over color orientations produce the gauge-invariant four-gluon kernel $\langle X^4 \rangle$ quoted earlier. Schematically, the Lorentz structure induced by a single instanton I is built from products of anti-self-dual tensors $\bar{\eta}_{\mu\nu}^b$, while the anti-instanton A produces the same structure with $\eta_{\mu\nu}^b$,

$$K_I^{(4)} \propto (R^{ab} \bar{\eta}_{\mu\nu}^b G_{\mu\nu}^a) (R^{a'b'} \bar{\eta}_{\alpha\beta}^{b'} G_{\alpha\beta}^{a'}) \\ \times (R^{a''b''} \bar{\eta}_{\rho\sigma}^{b''} G_{\rho\sigma}^{a''}) (R^{a''b''} \bar{\eta}_{\lambda\delta}^{b''} G_{\lambda\delta}^{a''}), \quad (\text{H2})$$

$$K_A^{(4)} \propto (R^{ab} \eta_{\mu\nu}^b G_{\mu\nu}^a) (R^{a'b'} \eta_{\alpha\beta}^{b'} G_{\alpha\beta}^{a'}) (R^{a''b''} \eta_{\rho\sigma}^{b''} G_{\rho\sigma}^{a''}) \\ \times (R^{a''b''} \eta_{\lambda\delta}^{b''} G_{\lambda\delta}^{a''}). \quad (\text{H3})$$

For each of the channels under consideration, it is enough to track the Lorentz duality structure. The color-orientation average produces the same adjoint Kronecker contractions in both cases.

The 0^{-+} glueball couples to the pseudoscalar operator

$$\mathcal{O}_{0^{-+}}(x) = G_{\mu\nu}^a(x) \tilde{G}_{\mu\nu}^a(x), \quad \tilde{G}_{\mu\nu}^a \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} G_{\alpha\beta}^a. \quad (\text{H4})$$

To see the cancellation, it is sufficient to study the two-gluon irreducible kernel induced by one pseudoparticle in the ladder approximation, i.e., the part of $\langle X^4 \rangle$ that contracts two external field strengths into the pseudoscalar bilinear. More specifically,

$$\mathcal{C}_I \equiv \bar{\eta}_{\mu\nu}^b \bar{\eta}_{\alpha\beta}^b G_{\mu\nu} \tilde{G}_{\alpha\beta}, \quad \mathcal{C}_A \equiv \eta_{\mu\nu}^b \eta_{\alpha\beta}^b G_{\mu\nu} \tilde{G}_{\alpha\beta}, \quad (\text{H5})$$

where we suppressed color indices on G for clarity.

Using $\tilde{G}_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\rho\sigma} G_{\rho\sigma}$ and the duality relations (H1), we have

$$\mathcal{C}_I = \bar{\eta}_{\mu\nu}^b \bar{\eta}_{\alpha\beta}^b G_{\mu\nu} \frac{1}{2} \epsilon_{\alpha\beta\rho\sigma} G_{\rho\sigma} = \bar{\eta}_{\mu\nu}^b \left(\frac{1}{2} \epsilon_{\alpha\beta\rho\sigma} \bar{\eta}_{\alpha\beta}^b \right) G_{\mu\nu} G_{\rho\sigma} \\ = \bar{\eta}_{\mu\nu}^b (-\bar{\eta}_{\rho\sigma}^b) G_{\mu\nu} G_{\rho\sigma} = -(\bar{\eta}_{\mu\nu}^b \bar{\eta}_{\rho\sigma}^b) G_{\mu\nu} G_{\rho\sigma}, \quad (\text{H6})$$

$$\mathcal{C}_A = \eta_{\mu\nu}^b \eta_{\alpha\beta}^b G_{\mu\nu} \frac{1}{2} \epsilon_{\alpha\beta\rho\sigma} G_{\rho\sigma} = \eta_{\mu\nu}^b \left(\frac{1}{2} \epsilon_{\alpha\beta\rho\sigma} \eta_{\alpha\beta}^b \right) G_{\mu\nu} G_{\rho\sigma} \\ = \eta_{\mu\nu}^b (+\eta_{\rho\sigma}^b) G_{\mu\nu} G_{\rho\sigma} = +(\eta_{\mu\nu}^b \eta_{\rho\sigma}^b) G_{\mu\nu} G_{\rho\sigma}. \quad (\text{H7})$$

Thus, the pseudoscalar contraction picks up an *opposite sign* for instantons and anti-instantons.

In a CP -even dense ILM ensemble with equal instanton and anti-instanton weights, the leading contribution to the pseudoscalar ladder kernel is proportional to the sum of these contractions,

$$C_I + C_A = -(\bar{\eta}_{\mu\nu}^b \bar{\eta}_{\rho\sigma}^b) G_{\mu\nu} G_{\rho\sigma} + (\eta_{\mu\nu}^b \eta_{\rho\sigma}^b) G_{\mu\nu} G_{\rho\sigma}. \quad (\text{H8})$$

After the orientation average, the purely algebraic identity $\eta_{\mu\nu}^b \eta_{\rho\sigma}^b = \bar{\eta}_{\mu\nu}^b \bar{\eta}_{\rho\sigma}^b$ holds for the symmetric part that survives in the CP -even ensemble, since both are equal to the same transverse tensor built from Kronecker deltas plus an ϵ -term of opposite sign,

$$\begin{aligned} \eta_{\mu\nu}^a \eta_{\rho\sigma}^a &= \delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma}, \\ \bar{\eta}_{\mu\nu}^a \bar{\eta}_{\rho\sigma}^a &= \delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho} - \epsilon_{\mu\nu\rho\sigma}. \end{aligned} \quad (\text{H9})$$

Inserting (H9) into (H8), the $\delta\delta$ terms cancel because of the relative sign in (H6) and (H7), and the remaining ϵ -terms cancel because they appear with opposite signs in (H9). Therefore

$$C_I + C_A = 0, \quad (\text{H10})$$

which shows explicitly that the leading one-pseudoparticle induced four-gluon kernel does not generate a net ladder attraction in the 0^{-+} channel in a CP -even ensemble.

Finally, we note that these arguments also show that the 0^{++} channel is in contrast enhanced. For the scalar contraction $G_{\mu\nu} G_{\alpha\beta}$ (with no epsilon tensor) both instanton and anti-instanton contributions add, since both $\eta\eta$ and $\bar{\eta}\bar{\eta}$ have the same $\delta\delta$ part and the ϵ part drops out after contraction with the symmetric scalar projector. Hence the ILM produces a maximally attractive parity-even scalar core but a vanishing leading parity-odd core, in agreement with the qualitative discussion in the main text.

APPENDIX I: 0^{++} ON THE LIGHT-FRONT

Our conventions for the light-front (LF) coordinates are $x^\pm = x^0 \pm x^3$ and $k^\pm = k^0 \pm k^3$, with transverse components $\mathbf{k}_\perp = (k^1, k^2)$. With this in mind, and in the two-gluon Fock space approximation, the scalar glueball LF state reads

$$|\Psi_{0^{++}}(P^+)\rangle = \frac{1}{\sqrt{2}} \sum_{\lambda_1, \lambda_2} \int_0^1 \frac{dx}{\sqrt{x(1-x)}} \int \frac{d^2 k_\perp}{(2\pi)^3} \Psi_{0^{++}}^{\lambda_1 \lambda_2}(x, \mathbf{k}_\perp) \frac{\delta^{a_1 a_2}}{\sqrt{d_A}} |k_1, \lambda_1, a_1; k_2, \lambda_2, a_2\rangle, \quad (\text{I1})$$

where $k_1^+ = xP^+$, $k_2^+ = (1-x)P^+$, and in the transverse rest frame $\mathbf{k}_{1\perp} = +\mathbf{k}_\perp$, $\mathbf{k}_{2\perp} = -\mathbf{k}_\perp$. The 0^{++} helicity structure in (I1) is the even combination of opposite helicities:

$$\Psi_{0^{++}}^{\lambda_1 \lambda_2}(x, \mathbf{k}_\perp) = \frac{1}{\sqrt{2}} [\delta_{\lambda_1, +} \delta_{\lambda_2, -} + \delta_{\lambda_1, -} \delta_{\lambda_2, +}] \psi_{0^{++}}(x, \mathbf{k}_\perp), \quad (\text{I2})$$

with the 1-particle LF states normalized as

$$\langle k', \lambda', a' | k, \lambda, a \rangle = 2k^+ (2\pi)^3 \delta(k'^+ - k^+) \delta^{(2)}(\mathbf{k}'_\perp - \mathbf{k}_\perp) \delta_{\lambda'\lambda} \delta_{a'a}. \quad (\text{I3})$$

A glueball eigenstate satisfies

$$(P^+ P^- - \mathbf{P}_\perp^2) |\Psi\rangle = M^2 |\Psi\rangle.$$

In the transverse rest frame $\mathbf{P}_\perp = \mathbf{0}$ this becomes the M^2 eigenvalue problem.

In terms of the free two-body invariant mass

$$\mathcal{M}_0^2(x, \mathbf{k}_\perp) = \frac{k_\perp^2 + m_g^2}{x(1-x)}, \quad (\text{I4})$$

the two-body 0^{++} LF eigenvalue equation reads

$$\begin{aligned} M^2 \psi_{0^{++}}(x, \mathbf{k}_\perp) &= \mathcal{M}_0^2(x, \mathbf{k}_\perp) \psi_{0^{++}}(x, \mathbf{k}_\perp) \\ &+ \int_0^1 dx' \int \frac{d^2 k'_\perp}{(2\pi)^3} \mathcal{V}_{\text{LF}}^{0^{++}}(x, \mathbf{k}_\perp; x', \mathbf{k}'_\perp) \\ &\times \psi_{0^{++}}(x', \mathbf{k}'_\perp). \end{aligned} \quad (\text{I5})$$

To embed the same singular-gauge instanton form factor, we identify the internal momentum modulus $p = \sqrt{k_\perp^2 + k_z^2}$ with [38]

$$\begin{aligned} k_z &= \left(x - \frac{1}{2}\right) \mathcal{M}_0(x, \mathbf{k}_\perp), \\ p^2(x, \mathbf{k}_\perp) &= k_\perp^2 + \left(x - \frac{1}{2}\right)^2 \mathcal{M}_0^2(x, \mathbf{k}_\perp) \end{aligned} \quad (\text{I6})$$

and define

$$f(x, \mathbf{k}_\perp) = \beta_{2g}(\rho p(x, \mathbf{k}_\perp)). \quad (\text{I7})$$

The LF scalar instanton interaction has rank-one form

$$\mathcal{V}_{\text{LF}}^{0^{++}}(x, \mathbf{k}_\perp; x', \mathbf{k}'_\perp) = -g_{\text{LF}} f^2(x, \mathbf{k}_\perp) f^2(x', \mathbf{k}'_\perp), \quad (\text{I8})$$

hence the explicit LF wave function

$$\psi_{0^{++}}(x, k_{\perp}) = -g_{\text{LF}} C_{\text{LF}} \frac{f^2(x, k_{\perp})}{M^2 - \mathcal{M}_0^2(x, k_{\perp})}, \quad (\text{I9})$$

with

$$C_{\text{LF}} = \int_0^1 dx \int \frac{d^2 k_{\perp}}{(2\pi)^3} f^2(x, k_{\perp}) \psi_{0^{++}}(x, k_{\perp}).$$

Eliminating C_{LF} gives the LF scalar root equation

$$1 = -g_{\text{LF}} \int_0^1 dx \int \frac{d^2 k_{\perp}}{(2\pi)^3} \frac{f^4(x, k_{\perp})}{M^2 - \mathcal{M}_0^2(x, k_{\perp})}. \quad (\text{I10})$$

We now show that $g_{\text{LF}} = V_0$.

APPENDIX J: LF FROM CM

The LF mass root equation and in general the LF wave function detailed in Appendix I, are in one-to-one correspondence with the CM mass root equation and wave function. Indeed, using (I6) and the Jacobian

$$dx d^2 k_{\perp} = d^3 p \frac{2x(1-x)}{\mathcal{M}_0(x, k_{\perp})}, \quad (\text{J1})$$

together with $f(x, k_{\perp}) = \beta_{2g}(\rho p)$, the scalar glueball root mass Eq. (I10) maps onto

$$1 = -V_0 \int \frac{d^3 p}{(2\pi)^3} \frac{\beta_{2g}^4(\rho p)}{2\omega_p(M^2 - 4\omega_p^2)}, \quad (\text{J2})$$

with the identification $g_{\text{LF}} \equiv V_0$, up to normalization conventions.

Also, recall that the Bethe-Salpeter reduction in the rest-frame, yields the equal-time wave function

$$\phi(\mathbf{p}) = \mathcal{N}_M \frac{\beta_{2g}^2(\rho p)}{2\omega_p(4\omega_p^2 - M^2)}. \quad (\text{J3})$$

Using (I6), we can map (J3) onto the LF wave function by

$$\psi_{0^{++}}(x, \mathbf{k}_{\perp}) = \sqrt{\frac{\partial k_z}{\partial x}} \phi(\mathbf{p}(x, \mathbf{k}_{\perp})). \quad (\text{J4})$$

Differentiating $k_z = (x - \frac{1}{2})\mathcal{M}_0$ yields

$$\frac{\partial k_z}{\partial x} = \frac{\mathcal{M}_0}{2x(1-x)}. \quad (\text{J5})$$

Using $\mathcal{M}_0^2 = 4\omega_p^2$ in the equal-mass case yields

$$\psi_{0^{++}}(x, \mathbf{k}_{\perp}) = \mathcal{N}_{\text{LF}} \frac{\beta_{2g}^2(\rho p(x, k_{\perp}))}{M^2 - \mathcal{M}_0^2(x, k_{\perp})}. \quad (\text{J6})$$

This coincides with the functional dependence of the LF solution obtained dynamically from the rank-one LF eigenvalue equation, with the identification

$$f(x, k_{\perp}) = \beta_{2g}(\rho p(x, k_{\perp})).$$

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