

Landau-Lifshitz solutions of $N = 2$, $D = 5$ supergravity

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We study families of solutions depending only on one coordinate for the theories of $N = 2$, five-dimensional supergravity coupled to vector multiplets. Four families of solutions, valid for all space-time signatures, are obtained. The explicit metrics of the solutions depend on the Jordan normal forms of constrained matrices. Examples with three charges are given for the so-called STU model.

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I. INTRODUCTION

Many years ago, Kasner presented four-dimensional vacuum solutions depending on one variable [1], which when generalized to arbitrary space-time signature, take the form [2]

$$ds^2 = \epsilon_0 x^{2a_1} dx^2 + \epsilon_1 x^{2a_2} dy^2 + \epsilon_2 x^{2a_3} dz^2 + \epsilon_3 x^{2a_4} dw^2 \quad (1.1)$$

with the conditions

$$a_2 + a_3 + a_4 = a_1 + 1, \quad a_2^2 + a_3^2 + a_4^2 = (a_1 + 1)^2. \quad (1.2)$$

The cosmological synchronous Bianchi I metric [3],

$$ds^2 = -d\tau^2 + \tau^{2p_1} dx^2 + \tau^{2p_2} dy^2 + \tau^{2p_3} dz^2 \quad (1.3)$$

referred to as the Kasner metric, can be obtained from Lorentzian (1.1) through a change of variable or by setting $a_1 = 0$. Here the Kasner exponents p_1 , p_2 and p_3 satisfy

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1. \quad (1.4)$$

The metric (1.3) admits generalizations to arbitrary signature D -dimensional gravity, in the form

$$ds^2 = \epsilon_0 d\tau^2 + \epsilon_\mu \tau^{2p_\mu} (dx^\mu)^2 \quad (1.5)$$

with the exponents p_μ satisfying the conditions

$$\sum_{\mu=1}^{D-1} p_\mu = \sum_{\mu=1}^{D-1} p_\mu^2 = 1 \quad (1.6)$$

and ϵ_0 and ϵ_μ are constants taking the values ± 1 .

In the analysis of Landau and Lifshitz [4] of the four-dimensional gravitational solutions depending on one variable, one starts with the metric

$$ds^2 = \epsilon_0 d\tau^2 + g_{\mu\nu} dx^\mu dx^\nu \quad (1.7)$$

and obtains, in addition to the Bianchi I metric, non-diagonal vacuum solutions corresponding to $\omega = \sqrt{|g|} = \tau$.¹ As pointed out in [2], the cases with $a_1 = -1$ were not considered by Kasner. In the formulation of [4], these missed cases belong to solutions with $\omega = 1$.

Starting with the Kasner metric and the scalar-Kasner metric [6], generalized anisotropic Melvin cosmologies and flux tubes were obtained in [7,8]. This was achieved using solution generating techniques [9,10]. Generalizations of the Kasner metric to theories with form gauge fields and a coupled dilaton field in all space-time dimensions and signatures were considered in [11]. Also generalized Melvin metrics were found for the theories of $N = 2$ supergravity coupled to vector multiplets in four and five dimensions [12,13]. More recently, general solutions for D -dimensional Einstein-Maxwell theory and theories with form gauge fields and a coupled dilaton, depending on one variable, were obtained in [14,15] through an explicit analysis of the equations of motion.

The current work shall deal with the study of general solutions depending on one variable in $N = 2$, five-dimensional supergravity theories coupled to vector multiplets. Kasner-like solutions of the five-dimensional supergravity theory were considered in [12]. Depending on the choice of coordinates, the solutions obtained constitute

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¹The exact solutions of [4] were also presented in Appendix C of [5].

generalizations of Melvin/Rosen cosmologies, flux-branes, and domain walls [16]. They are of importance in addressing questions of gravitational physics and cosmology [5,17] within supergravity and string theory.

The Lagrangian of our models is given by

$$\begin{aligned} \mathcal{L}_5 = & \sqrt{|g|}(R - \mathcal{M}_{ij}(\phi)\partial_\alpha\phi^i\partial^\alpha\phi^j) \\ & - \frac{\varepsilon}{2} \left[\sqrt{|g|}G_{IJ}(\phi)\mathcal{F}_{\alpha\beta}^I\mathcal{F}^{J\alpha\beta} \right. \\ & \left. - \frac{1}{12}C_{IJK}\epsilon^{\alpha\beta\gamma\delta\eta}\mathcal{F}_{\alpha\beta}^I\mathcal{F}_{\gamma\delta}^J\mathcal{A}_\eta^K \right]. \end{aligned} \quad (1.8)$$

In these models, we have n scalar fields ϕ^i and $n+1$ Abelian gauge fields $\mathcal{A}_\alpha^I$ with field strengths $\mathcal{F}_{\alpha\beta}^I$. The $N=2$, five-dimensional supergravity theories coupled to vector multiplets, with Lorentzian signature and $\varepsilon=1$, were first considered by Gunaydin, Sierra, and Townsend in [18]. These theories can be described by the so-called very special geometry [19]. One defines the special coordinates $X^I = X^I(\phi)$ and their duals $X_I = \frac{1}{6}C_{IJK}X^JX^K$, which satisfy the conditions

$$X^IX_I = 1, \quad \mathcal{V} = \frac{1}{6}C_{IJK}X^IX^JX^K = 1. \quad (1.9)$$

The constants C_{IJK} are symmetric in all indices. For the five-dimensional theories obtained via Calabi-Yau compactification [20] of 11-dimensional supergravity [21], C_{IJK} represent the intersection numbers. The gauge coupling metric G_{IJ} can be expressed in terms of the special coordinates by

$$G_{IJ} = \frac{9}{2}X_IX_J - \frac{1}{2}C_{IJK}X^K. \quad (1.10)$$

In terms of the scalars X^I , the ϕ^i kinetic term is given by

$$\mathcal{M}_{ij}\partial_\alpha\phi^i\partial^\alpha\phi^j = G_{IJ}\partial_\alpha X^I\partial^\alpha X^J. \quad (1.11)$$

We also have the very special geometry relations

$$G_{IJ}X^J = \frac{3}{2}X_I, \quad dX_I = -\frac{2}{3}G_{IJ}dX^J, \quad X_IdX^I = 0. \quad (1.12)$$

Recently, the $N=2$, five-dimensional supergravity were considered for all space-time signatures [22–24]. A class of these supergravity theories can be obtained via Calabi-Yau three-folds compactification of the exotic 11-dimensional supergravity theories constructed by Hull [25]. Moreover, some of these theories come with the nonconventional sign of the gauge kinetic terms ($\varepsilon=-1$). For theories with (1,4), (3,2), and (5,0) signatures (odd numbers of time

dimensions) we have $\varepsilon=1$. For the mirror theories with signatures (4,1), (2,3), and (0,5), we have $\varepsilon=-1$.

We organize this work as follows. In the next section, the equations of motion obtained from (1.8) will be analyzed for general one coordinate dependent metrics. We consider metric solutions for two different choices for the gauge fields and obtain two sets of equations characterizing two classes of solutions. In Sec. three, four families of general solutions will be given. In Sec. four, examples will be presented for those families. We summarize our results in Sec. five.

II. ANALYSIS OF THE EQUATIONS OF MOTION WITH ONE COORDINATE DEPENDENCE

The Einstein, Maxwell, and scalar field equations of motion derived from (1.8) take the form

$$\begin{aligned} R_{\alpha\beta} - G_{IJ}\partial_\alpha X^I\partial_\beta X^J \\ - \varepsilon G_{IJ} \left(\mathcal{F}_{\alpha\gamma}^I\mathcal{F}_\beta^{J\gamma} - \frac{1}{6}g_{\alpha\beta}\mathcal{F}_{\delta\eta}^I\mathcal{F}^{J\delta\eta} \right) = 0, \end{aligned} \quad (2.1)$$

$$\partial_\beta(\sqrt{|g|}G_{IJ}\mathcal{F}^{J\alpha\beta}) - \frac{1}{16}C_{IJK}\epsilon^{\alpha\beta\gamma\delta\eta}\mathcal{F}_{\beta\gamma}^I\mathcal{F}_{\delta\eta}^J = 0, \quad (2.2)$$

$$\begin{aligned} \sqrt{|g|}\partial_i G_{IJ} \left(\frac{\varepsilon}{2}\mathcal{F}_{\alpha\beta}^I\mathcal{F}^{J\alpha\beta} + \partial_\alpha X^I\partial^\alpha X^J \right) \\ - 2\partial_\alpha(\sqrt{|g|}G_{IJ}\partial^\alpha X^J)\partial_i X^I = 0. \end{aligned} \quad (2.3)$$

A. Ansatz for Type 1 solutions

As a solution we start with the following metric

$$ds^2 = \epsilon_0 d\tau^2 + g_{\mu\nu}dx^\mu dx^\nu = \epsilon_0 d\tau^2 + g_{zz}dz^2 + g_{ab}dx^a dx^b \quad (2.4)$$

where $\epsilon_0 = \pm 1$ and $a, b = 1, 2, 3$. The space-time metric, the gauge, and the scalar fields are all functions of τ only. For the gauge fields, we first consider solutions with only $\mathcal{F}_{\tau z}^I$ nonvanishing. In this case, the equations of motion (2.2) give the solutions

$$G_{IJ}\mathcal{F}^{J\tau z} = \frac{q_I}{\omega}, \quad \omega = \sqrt{|g|} \quad (2.5)$$

where q_I are constants.

Using the variables $\kappa_{\mu\nu} = \dot{g}_{\mu\nu}$ introduced in [4] where the overdot denotes differentiation with respect to τ , the nonvanishing components of the Ricci tensor for the metric (2.4) can be written as

$$\begin{aligned} R^\tau_\tau &= -\frac{\epsilon_0}{2} \left(\dot{\kappa}^\mu_\mu + \frac{1}{2} \kappa^\mu_\nu \kappa^\nu_\mu \right), \\ R^\mu_\nu &= -\frac{\epsilon_0}{2} \left(\dot{\kappa}^\mu_\nu + \frac{\dot{\omega}}{\omega} \kappa^\mu_\nu \right), \end{aligned} \quad (2.6)$$

with

$$\kappa^\mu_\nu = g^{\mu\rho} \kappa_{\rho\nu}, \quad \kappa^\mu_\mu = 2 \frac{\dot{\omega}}{\omega}. \quad (2.7)$$

The Einstein equations of motion (2.1) then give

$$\begin{aligned} \frac{\ddot{\omega}}{\omega} - \frac{\dot{\omega}^2}{\omega^2} + \frac{1}{4} \kappa^\mu_\nu \kappa^\nu_\mu \\ + G_{IJ} \dot{X}^I \dot{X}^J + \frac{2\epsilon}{3\omega^2} g_{zz} G^{IJ} q_I q_J = 0, \end{aligned} \quad (2.8)$$

$$\dot{\kappa}^z_z + \frac{\dot{\omega}}{\omega} \kappa^z_z + \frac{4\epsilon}{3\omega^2} g_{zz} G^{IJ} q_I q_J = 0, \quad (2.9)$$

$$\dot{\kappa}^a_b + \frac{\dot{\omega}}{\omega} \kappa^a_b - \frac{2\epsilon}{3\omega^2} g_{zz} G^{IJ} q_I q_J \delta^a_b = 0. \quad (2.10)$$

Taking the trace of (2.10) and adding (2.9), we obtain

$$\ddot{\omega} = \frac{\epsilon}{3\omega} g_{zz} G^{IJ} q_I q_J. \quad (2.11)$$

Equation (2.11) when substituted back into Eqs. (2.8), (2.9), and (2.10) results in

$$3 \frac{\ddot{\omega}}{\omega} - \frac{\dot{\omega}^2}{\omega^2} + \frac{1}{4} \kappa^\mu_\nu \kappa^\nu_\mu + G_{IJ} \dot{X}^I \dot{X}^J = 0, \quad (2.12)$$

$$\dot{\kappa}^z_z + \frac{\dot{\omega}}{\omega} \kappa^z_z + 4 \frac{\ddot{\omega}}{\omega} = 0, \quad (2.13)$$

$$\dot{\kappa}^a_b + \frac{\dot{\omega}}{\omega} \kappa^a_b - 2 \frac{\ddot{\omega}}{\omega} \delta^a_b = 0. \quad (2.14)$$

The last two equations can be integrated to give

$$\begin{aligned} \kappa^z_z &= \frac{1}{\omega} (\theta^z_z - 4\dot{\omega}), \\ \kappa^a_b &= \frac{1}{\omega} (\theta^a_b + 2\dot{\omega} \delta^a_b), \end{aligned} \quad (2.15)$$

where θ^z_z and θ^a_b are constants satisfying the condition

$$\theta^z_z + \theta^a_a = 0. \quad (2.16)$$

For the metric components we obtain

$$\dot{g}_{zz} = \frac{1}{\omega} (\theta^z_z - 4\dot{\omega}) g_{zz}, \quad \dot{g}_{ab} = \frac{1}{\omega} (g_{ac} \theta^c_b + 2\dot{\omega} g_{ab}). \quad (2.17)$$

Substituting (2.15) back into (2.12), we obtain

$$\frac{1}{4} \theta^\mu_\nu \theta^\nu_\mu + 3(2\dot{\omega}^2 - \theta^z_z \dot{\omega} + \omega \ddot{\omega}) + \omega^2 G_{IJ} \dot{X}^I \dot{X}^J = 0. \quad (2.18)$$

The scalar equation of motion (2.3), for our solution, using very special geometry relations, reduces to

$$\partial_i G^{IJ} \left[\dot{\omega} X_J \dot{X}_I + \omega (X_J \ddot{X}_I - \dot{X}_I \dot{X}_J) - \frac{4}{9} \frac{\epsilon}{\omega} g_{zz} q_I q_J \right] = 0. \quad (2.19)$$

We conclude that our solutions are defined by Eqs. (2.11), (2.16), (2.17), (2.18), and (2.19).

B. Ansatz for Type 2 solutions

We also consider solutions with space-time metric

$$\begin{aligned} ds^2 &= \epsilon_0 d\tau^2 + g_{\mu\nu} dx^\mu dx^\nu \\ &= \epsilon_0 d\tau^2 + g_{xx} dx^2 + g_{yy} dy^2 + g_{ab} dx^a dx^b \end{aligned} \quad (2.20)$$

where $\epsilon_0 = \pm 1$ and $a, b = 1, 2$ and with gauge fields given by

$$\mathcal{F}^I_{xy} = p^I. \quad (2.21)$$

Here p^I are constants and the metric and scalar fields depend only on τ .

The analysis of the equations of motion (2.1) yields

$$\frac{3\ddot{\omega}}{2\omega} - \frac{\dot{\omega}^2}{\omega^2} + \frac{1}{4} \kappa^\mu_\nu \kappa^\nu_\mu + G_{IJ} \dot{X}^I \dot{X}^J = 0, \quad (2.22)$$

$$\dot{\kappa}^x_x + \frac{\dot{\omega}}{\omega} \kappa^x_x - 2 \frac{\ddot{\omega}}{\omega} = 0, \quad (2.23)$$

$$\dot{\kappa}^y_y + \frac{\dot{\omega}}{\omega} \kappa^y_y - 2 \frac{\ddot{\omega}}{\omega} = 0, \quad (2.24)$$

$$\dot{\kappa}^a_b + \frac{\dot{\omega}}{\omega} \kappa^a_b + \frac{\ddot{\omega}}{\omega} \delta^a_b = 0, \quad (2.25)$$

together with the relation

$$\ddot{\omega} = -\frac{2}{3} \epsilon \epsilon_0 g^{xx} g^{yy} G_{IJ} p^I p^J \omega. \quad (2.26)$$

Equations (2.23), (2.24), and (2.25) can be integrated to give

$$\begin{aligned}\kappa^x_x &= \frac{1}{\omega}(\theta^x_x + 2\dot{\omega}), \\ \kappa^y_y &= \frac{1}{\omega}(\theta^y_y + 2\dot{\omega}), \\ \kappa^a_b &= \frac{1}{\omega}(\theta^a_b - \dot{\omega}\delta^a_b),\end{aligned}\quad (2.27)$$

with the condition

$$\theta^x_x + \theta^y_y + \theta^a_a = 0. \quad (2.28)$$

The metric components thus satisfy

$$\begin{aligned}\dot{g}_{xx} &= \frac{1}{\omega}(\theta^x_x + 2\dot{\omega})g_{xx}, \\ \dot{g}_{yy} &= \frac{1}{\omega}(\theta^y_y + 2\dot{\omega})g_{yy}, \\ \dot{g}_{ab} &= \frac{1}{\omega}(g_{ac}\theta^c_b - g_{ab}\dot{\omega}).\end{aligned}\quad (2.29)$$

Upon substituting (2.27) into (2.22), we obtain

$$\frac{1}{2}\theta^\mu_\nu\theta^\nu_\mu + 3(\dot{\omega}^2 - \dot{\omega}\theta^a_a + \omega\ddot{\omega}) + 2\omega^2 G_{IJ}\dot{X}^I\dot{X}^J = 0. \quad (2.30)$$

The scalar equation for our solution, with the help of very special geometry relation, takes the form

$$\partial_i G_{IJ} \left(\frac{\dot{\omega}}{\omega} \dot{X}^J X^I + X^I \ddot{X}^J - \dot{X}^I \dot{X}^J + \epsilon_0 \epsilon p^I p^J g^{xx} g^{yy} \right) = 0. \quad (2.31)$$

Equations (2.26), (2.28), (2.29), (2.30), and (2.31) completely fix this class of solutions.

III. FOUR FAMILIES OF GENERAL SOLUTIONS

In what follows, we will construct four families of solutions by solving the equations characterizing the two classes of solutions analyzed in the previous section.

A. Type 1A family

The first family of solutions is obtained by solving (2.11), (2.17), (2.18), and (2.19) using the following change of variables [14]

$$\frac{d\sigma}{d\tau} = \frac{1}{H(\sigma)}, \quad \omega = \sigma H(\sigma). \quad (3.1)$$

Defining the convenient parameters

$$\lambda^z_z = p = \frac{1}{2}(\theta^z_z - 4), \quad \lambda^a_b = \frac{1}{2}(\theta^a_b + 2\delta^a_b), \quad (3.2)$$

then the analysis of (2.17) gives the following metric solution

$$ds^2 = H^2(\epsilon_0 d\sigma^2 + h_{ac}(e^{2\lambda \log \sigma})^c_b dx^a dx^b) + h_{zz}\sigma^{2p}H^{-4}dz^2 \quad (3.3)$$

with $h_{zz} = \pm 1$ and h_{ac} as constants. Note that the condition (2.16) in terms of the new parameters becomes

$$p + \lambda^a_a = 1. \quad (3.4)$$

Equation (2.18), using variables (3.1) and (3.2), gives

$$\begin{aligned}G_{IJ} \frac{dX^I}{d\sigma} \frac{dX^J}{d\sigma} &= \frac{1}{\sigma^2}(1 - \lambda^\mu_\nu \lambda^\nu_\mu) - \frac{3}{H^2} \left[\left(\frac{dH}{d\sigma} \right)^2 \right. \\ &\quad \left. + \frac{(1-2p)}{\sigma} H \frac{dH}{d\sigma} + H \frac{d^2 H}{d\sigma^2} \right]\end{aligned}\quad (3.5)$$

and we obtain from (2.11) the condition

$$G^{IJ} q_I q_J = 3\epsilon h_{zz} \sigma^{1-2p} H^3 \left[\frac{dH}{d\sigma} + \sigma \frac{d^2 H}{d\sigma^2} - \frac{\sigma}{H} \left(\frac{dH}{d\sigma} \right)^2 \right]. \quad (3.6)$$

To proceed, we write

$$X_I = \frac{1}{3H^2} f_I(\sigma), \quad (3.7)$$

then with the help of very special geometry relations, the following equations can be derived

$$\begin{aligned}\frac{dH}{d\sigma} &= \frac{1}{6H} X^I \frac{df_I}{d\sigma}, \\ H \frac{d^2 H}{d\sigma^2} - \left(\frac{dH}{d\sigma} \right)^2 &= \frac{1}{6} X^I \frac{d^2 f_I}{d\sigma^2} - \frac{1}{12H^2} G^{IJ} \frac{df_I}{d\sigma} \frac{df_J}{d\sigma}, \\ G_{IJ} \frac{dX^I}{d\sigma} \frac{dX^J}{d\sigma} &= -\frac{3}{H} \frac{d^2 H}{d\sigma^2} - \frac{3}{H^2} \left(\frac{dH}{d\sigma} \right)^2 + \frac{1}{2H^2} X^I \frac{d^2 f_I}{d\sigma^2}.\end{aligned}\quad (3.8)$$

Using these relations in (3.5), we obtain

$$X^I \left(\frac{d^2 f_I}{d\sigma^2} + \frac{(1-2p)}{\sigma} \frac{df_I}{d\sigma} + \frac{2}{3\sigma^2} (\lambda^\mu_\nu \lambda^\nu_\mu - 1) f_I \right) = 0. \quad (3.9)$$

This admits the solutions

$$f_I = A_I \sigma^{(p-s)} + B_I \sigma^{(p+s)} \quad (3.10)$$

with

$$\lambda^\mu{}_\nu \lambda^\nu{}_\mu = 1 + \frac{3}{2}(p^2 - s^2) \quad (3.11)$$

and constant A_I and B_I .

We also obtain from (3.6) the condition

$$G^{IJ} \left(f_J \frac{d^2 f_I}{d\sigma^2} + \frac{1}{\sigma} f_J \frac{df_I}{d\sigma} - \frac{df_I}{d\sigma} \frac{df_J}{d\sigma} - 4\epsilon h_{zz} q_I q_J \sigma^{2p-2} \right) = 0. \quad (3.12)$$

Substituting our solution into (3.12), gives the algebraic condition

$$G^{IJ} (s^2 A_J B_I - \epsilon h_{zz} q_I q_J) = 0. \quad (3.13)$$

Equation (2.19) for our solutions, after some calculations, reduces to

$$\partial_i G^{IJ} (s^2 A_J B_I - \epsilon h_{zz} q_I q_J) = 0. \quad (3.14)$$

Finally Eqs. (1.9) and (1.12) imply

$$H^4 = \frac{1}{6} G^{IJ} f_I f_J. \quad (3.15)$$

Note that for $s = p$ and diagonal $\lambda^\mu{}_\nu$, we reproduce the solutions obtained in [12].

B. Type 1B family

We can also get a second family of solutions by making the following change of variables

$$\frac{d\rho}{d\tau} = \frac{1}{\omega(\rho)}. \quad (3.16)$$

Starting with Eq. (2.17), we obtain the metric solution

$$ds^2 = \omega^2 (\epsilon_0 d\rho^2 + h_{ac} (e^{\theta\rho})^c{}_b dx^a dx^b) + \frac{1}{\omega^4} h_{zz} e^{\theta z \rho} dz^2. \quad (3.17)$$

with $h_{zz} = \pm 1$, h_{ac} as constants and

$$\theta^z{}_z + \theta^a{}_a = 0. \quad (3.18)$$

To proceed, we write

$$X_I = \frac{1}{3\omega^2} g_I(\rho), \quad (3.19)$$

$$\omega^4 = \frac{1}{6} G^{IJ} g_I g_J. \quad (3.20)$$

Using the relations of very special geometry, then (2.18), (2.11), and (2.19), respectively, give

$$X^I \left(\frac{d^2 g_I}{d\rho^2} - \theta^z{}_z \frac{dg_I}{d\rho} + \frac{1}{6} \theta^\mu{}_\nu \theta^\nu{}_\mu g_I \right) = 0, \quad (3.21)$$

$$G^{IJ} \left(g_J \frac{d^2 g_I}{d\rho^2} - \frac{dg_I}{d\rho} \frac{dg_J}{d\rho} - 4\epsilon h_{zz} e^{\theta^z \rho} q_I q_J \right) = 0, \quad (3.22)$$

$$\partial_i G^{IJ} \left(g_J \frac{d^2 g_I}{d\rho^2} - \frac{dg_I}{d\rho} \frac{dg_J}{d\rho} - 4\epsilon h_{zz} e^{\theta^z \rho} q_I q_J \right) = 0. \quad (3.23)$$

Equation (3.21) admits the solution

$$g_I = A_I e^{\frac{1}{2}(\theta^z{}_z - s)\rho} + B_I e^{\frac{1}{2}(\theta^z{}_z + s)\rho} \quad (3.24)$$

with

$$\theta^a{}_b \theta^b{}_a = \frac{1}{2} (\theta^z{}_z)^2 - \frac{3}{2} s^2. \quad (3.25)$$

This solution when substituted in (3.22) and (3.23) gives the algebraic conditions

$$\begin{aligned} G^{IJ} (s^2 A_I B_J - 4\epsilon h_{zz} q_I q_J) \\ = \partial_i G^{IJ} (s^2 A_I B_J - 4\epsilon h_{zz} q_I q_J) = 0. \end{aligned} \quad (3.26)$$

C. Type 2A family

A third family of solutions is obtained by solving (2.26), (2.29), (2.30), and (2.31) using the change of variables given by (3.1). This gives for the metric solution

$$\begin{aligned} ds^2 = H^2 (\epsilon_0 d\tau^2 + h_{xx} \sigma^{2p} dx^2 + h_{yy} \sigma^{2r} dy^2) \\ + h_{ac} (e^{2\lambda \log \sigma})^c{}_b \frac{1}{H} dx^a dx^b, \end{aligned} \quad (3.27)$$

where we have defined

$$\begin{aligned} \lambda^x{}_x = p = \frac{1}{2} (\theta^x{}_x + 2), \quad \lambda^y{}_y = r = \frac{1}{2} (\theta^y{}_y + 2), \\ \lambda^a{}_b = \frac{1}{2} (\theta^a{}_b - \delta^a{}_b), \end{aligned} \quad (3.28)$$

and h_{xx} , h_{yy} take the values ± 1 and h_{ac} are constants. The condition (2.28) becomes

$$p + r + \lambda^a{}_a = 1. \quad (3.29)$$

To proceed, we set

$$X^I = \frac{1}{H} h^I(\sigma). \quad (3.30)$$

Again with the help of very special geometry, we obtain the following useful relations

$$\frac{dH}{d\sigma} = X_I \frac{dh^I}{d\sigma}, \quad (3.31)$$

$$-\frac{d^2H}{d\sigma^2} + \frac{1}{H} \left(\frac{dH}{d\sigma} \right)^2 + X_I \frac{d^2h^I}{d\sigma^2} = \frac{2}{3H} G_{IJ} \frac{dh^I}{d\sigma} \frac{dh^J}{d\sigma}, \quad (3.32)$$

$$G_{IJ} \frac{dX^I}{d\sigma} \frac{dX^J}{d\sigma} = \frac{3}{2H} \left(-\frac{d^2H}{d\sigma^2} + X_I \frac{d^2h^I}{d\sigma^2} \right). \quad (3.33)$$

These relations, when used in Eqs. (2.30), (2.26), and (2.31), give

$$X_I \left[\frac{2}{3} (\lambda^\mu{}_\nu \lambda^\nu{}_\mu - 1) h^I + (1 - 2l) \sigma \frac{dh^I}{d\sigma} + \sigma^2 \frac{d^2h^I}{d\sigma^2} \right] = 0, \quad (3.34)$$

and

$$G_{IJ} \left[h^J \left(\frac{1}{\sigma} \frac{dh^I}{d\sigma} + \frac{d^2h^I}{d\sigma^2} \right) - \frac{dh^I}{d\sigma} \frac{dh^J}{d\sigma} + \sigma^{2l-2} \epsilon \epsilon_0 h_{xx} h_{yy} p^J p^I \right] = 0, \quad (3.35)$$

$$\partial_i G_{IJ} \left[h^J \left(\frac{1}{\sigma} \frac{dh^I}{d\sigma} + \frac{d^2h^I}{d\sigma^2} \right) - \frac{dh^I}{d\sigma} \frac{dh^J}{d\sigma} + \sigma^{2l-2} \epsilon \epsilon_0 h_{xx} h_{yy} p^J p^I \right] = 0, \quad (3.36)$$

where we have defined $l = \lambda^a{}_a$. Equation (3.34) admits the solution

$$h^I = C^I \sigma^{l-s} + D^I \sigma^{l+s}, \quad (3.37)$$

with constant C^I and D^I and

$$\lambda^\mu{}_\nu \lambda^\nu{}_\mu = p^2 + r^2 + \lambda^a{}_b \lambda^b{}_a = 1 + \frac{3}{2} (l^2 - s^2).$$

Substituting this solution into (3.35) and (3.36), we obtain the algebraic conditions

$$G_{IJ} (4s^2 C^I D^J + \epsilon \epsilon_0 h_{xx} h_{yy} p^J p^I) = \partial_i G_{IJ} (4s^2 C^I D^J + \epsilon \epsilon_0 h_{xx} h_{yy} p^J p^I) = 0. \quad (3.38)$$

Employing the condition (1.9) fixes H and we obtain

$$H^3 = \frac{1}{6} C_{IJK} h^I h^J h^K. \quad (3.39)$$

D. Type 2B family

Finally, a fourth family of solutions can be obtained by solving (2.26), (2.29), (2.30), and (2.31) using the change of variables given by (3.16). Solving for the metric we obtain

$$ds^2 = \omega^2 (\epsilon_0 d\rho^2 + h_{xx} e^{\theta^x{}_\rho} dx^2 + h_{yy} e^{\theta^y{}_\rho} dy^2) + h_{ac} \frac{1}{\omega} (e^{\theta^\rho}{}^c{}_b dx^a dx^b), \quad (3.40)$$

with h_{xx} and h_{yy} taking the values ± 1 , h_{ac} are constants, and

$$\theta^x{}_x + \theta^y{}_y + \theta^a{}_a = 0. \quad (3.41)$$

To proceed we write

$$X^I = \frac{1}{\omega} h^I(\rho), \quad (3.42)$$

$$\omega^3 = \frac{1}{6} C_{IJK} h^I h^J h^K. \quad (3.43)$$

We obtain, after some calculation using very special geometry relations, the following equations

$$X_I \left[\frac{1}{2} \theta^\mu{}_\nu \theta^\nu{}_\mu h^I + 3 \left(\frac{d^2h^I}{d\sigma^2} - l \frac{dh^I}{d\sigma} \right) \right] = 0, \quad (3.44)$$

$$G_{IJ} \left(h^J \frac{d^2h^I}{d\sigma^2} - \frac{dh^I}{d\sigma} \frac{dh^J}{d\sigma} + \epsilon \epsilon_0 h_{xx} h_{yy} e^{-\rho(\theta^x{}_x + \theta^y{}_y)} p^J p^I \right) = 0, \quad (3.45)$$

$$\partial_i G_{IJ} \left(h^J \frac{d^2h^I}{d\sigma^2} - \frac{dh^I}{d\sigma} \frac{dh^J}{d\sigma} + \epsilon \epsilon_0 h_{xx} h_{yy} e^{-\rho(\theta^x{}_x + \theta^y{}_y)} p^J p^I \right) = 0, \quad (3.46)$$

with $l = \theta^a{}_a$. Equation (3.44) can be solved by

$$h^I = C^I e^{\frac{1}{2}(l-s)\rho} + D^I e^{\frac{1}{2}(l+s)\rho} \quad (3.47)$$

with

$$\theta^\mu{}_\nu \theta^\nu{}_\mu = (\theta^x{}_x)^2 + (\theta^y{}_y)^2 + \theta^a{}_b \theta^b{}_a = \frac{3}{2} (l^2 - s^2). \quad (3.48)$$

Upon substituting the solution in (3.45) and (3.46), we obtain the algebraic conditions

$$G_{IJ} (s^2 C^I D^J + \epsilon \epsilon_0 h_{xx} h_{yy} p^J p^I) = \partial_i G_{IJ} (s^2 C^I D^J + \epsilon \epsilon_0 h_{xx} h_{yy} p^J p^I) = 0. \quad (3.49)$$

IV. EXAMPLES AS SOLUTIONS FOR THE STU MODEL

following solutions

We construct solutions with three charges for the STU supergravity model with the prepotential $\mathcal{V} = X^1 X^2 X^3 = 1$. This theory can be obtained via the reduction of 11-dimensional supergravity on T^6 . For this model, the gauge coupling matrix with element G_{IJ} , is given by

$$G = \frac{1}{2} \begin{pmatrix} (X^1)^{-2} & 0 & 0 \\ 0 & (X^2)^{-2} & 0 \\ 0 & 0 & (X^3)^{-2} \end{pmatrix}. \quad (4.1) \quad \text{with}$$

$$\begin{aligned} X^2 X^3 &= \frac{1}{H^2} f_1, \\ X^1 X^3 &= \frac{1}{H^2} f_2, \\ X^1 X^2 &= \frac{1}{H^2} f_3, \end{aligned} \quad (4.2)$$

A. Type 1A example

We start with the first family of solutions where the metric is given by (3.3). Using (3.7) we obtain the

$$f_I = \sigma^{(p-s)} (1 + B_I \sigma^{2s}). \quad (4.3)$$

The scalars and gauge fields are given by

$$\begin{aligned} X^1 &= \frac{(f_1 f_2 f_3)^{\frac{1}{3}}}{f_1}, & X^2 &= \frac{(f_1 f_2 f_3)^{\frac{1}{3}}}{f_2}, & X^3 &= \frac{(f_1 f_2 f_3)^{\frac{1}{3}}}{f_3}, \\ \mathcal{F}^{1\sigma z} &= 2q_1 \frac{(f_1 f_2 f_3)^{\frac{1}{3}}}{\sigma f_1^2}, & \mathcal{F}^{2\sigma z} &= 2q_2 \frac{(f_1 f_2 f_3)^{\frac{1}{3}}}{\sigma f_2^2}, & \mathcal{F}^{3\sigma z} &= 2q_3 \frac{(f_1 f_2 f_3)^{\frac{1}{3}}}{\sigma f_3^2}. \end{aligned} \quad (4.4)$$

Our solutions have three independent charges q_I with

$$B_I = \frac{1}{s^2} \varepsilon h_{zz} q_I^2. \quad (4.5)$$

The metric, using (3.15), is given by

$$ds^2 = (f_1 f_2 f_3)^{\frac{1}{3}} (\epsilon_0 d\sigma^2 + h_{ac} (e^{2\lambda \log \sigma})^c_b dx^a dx^b) + h_{zz} \sigma^{2p} \frac{1}{(f_1 f_2 f_3)^{\frac{2}{3}}} dz^2 \quad (4.6)$$

with the conditions

$$\begin{aligned} \lambda^a_a &= 1 - p, \\ \lambda^a_b \lambda^b_a &= 1 - \frac{3}{2} s^2 + \frac{1}{2} p^2. \end{aligned} \quad (4.7)$$

B. Type 1B example

For the second family of solutions described by the metric (3.17), we obtain

$$ds^2 = (g_1 g_2 g_3)^{\frac{1}{3}} (\epsilon_0 d\rho^2 + h_{ac} (e^{\theta \rho})^c_b dx^a dx^b) + \frac{1}{(g_1 g_2 g_3)^{\frac{2}{3}}} h_{zz} e^{\theta \rho} dz^2, \quad (4.8)$$

and

$$\begin{aligned} X^1 &= \frac{(g_1 g_2 g_3)^{\frac{1}{3}}}{g_1}, & X^2 &= \frac{(g_1 g_2 g_3)^{\frac{1}{3}}}{g_2}, & X^3 &= \frac{(g_1 g_2 g_3)^{\frac{1}{3}}}{g_3}, \\ \mathcal{F}^{1\rho z} &= 2q_1 \frac{(g_1 g_2 g_3)^{\frac{1}{3}}}{g_1^2}, & \mathcal{F}^{2\rho z} &= 2q_2 \frac{(g_1 g_2 g_3)^{\frac{1}{3}}}{g_2^2}, & \mathcal{F}^{3\rho z} &= 2q_3 \frac{(g_1 g_2 g_3)^{\frac{1}{3}}}{g_3^2} \end{aligned} \quad (4.9)$$

with

$$g_I = e^{\frac{1}{2}(\theta_z^z - s)\rho} \left(1 + \frac{4\epsilon h_{zz}(q_I)^2}{s^2} e^{s\rho} \right) \quad (4.10)$$

and

$$\theta_z^z + \theta_a^a = 0, \quad \theta_a^a \theta_b^b = \frac{1}{2}(\theta_z^z)^2 - \frac{3}{2}s^2. \quad (4.11)$$

C. Type 2A example

Similarly one can construct solutions of the third family with $\mathcal{F}_{xy}^I = p^I$ and obtain

$$\begin{aligned} ds^2 &= (h_1 h_2 h_3)^2 (\epsilon_0 d\tau^2 + h_{xx} \sigma^{2p} dx^2 + h_{yy} \sigma^{2r} dy^2) + h_{ac} (e^{2\lambda \log \sigma})^c_b \frac{1}{(h_1 h_2 h_3)} dx^a dx^b, \\ X^1 &= \frac{h^1}{(h^1 h^2 h^3)^{\frac{1}{3}}}, \quad X^2 = \frac{h^2}{(h^1 h^2 h^3)^{\frac{1}{3}}}, \quad X^3 = \frac{h^3}{(h^1 h^2 h^3)^{\frac{1}{3}}}, \\ h^I &= \sigma^{l-s} \left(1 - \frac{1}{4s^2} \epsilon \epsilon_0 h_{xx} h_{yy} (p^I)^2 \sigma^{2s} \right), \end{aligned} \quad (4.12)$$

with

$$p + r + \lambda_a^a = 1, \quad l = \lambda_a^a, \quad p^2 + r^2 + \lambda_a^a \lambda_b^b = 1 + \frac{3}{2}(l^2 - s^2). \quad (4.13)$$

D. Type 2B example

For the fourth family of solutions we obtain

$$\begin{aligned} ds^2 &= (h_1 h_2 h_3)^2 (\epsilon_0 d\rho^2 + h_{xx} e^{\theta_x^x \rho} dx^2 + h_{yy} e^{\theta_y^y \rho} dy^2) + h_{ac} \frac{1}{(h_1 h_2 h_3)^2} (e^{\theta \rho})^c_b dx^a dx^b, \\ X^1 &= \frac{h^1}{(h^1 h^2 h^3)^{\frac{1}{3}}}, \quad X^2 = \frac{h^2}{(h^1 h^2 h^3)^{\frac{1}{3}}}, \quad X^3 = \frac{h^3}{(h^1 h^2 h^3)^{\frac{1}{3}}}, \\ h^I &= e^{\frac{\ell}{2}(l-s)} \left(1 - \frac{1}{s^2} \epsilon \epsilon_0 h_{xx} h_{yy} (p^I)^2 e^{s\rho} \right) \end{aligned} \quad (4.14)$$

with

$$\theta_x^x + \theta_y^y + \theta_a^a = 0, \quad l = \theta_a^a, \quad (\theta_x^x)^2 + (\theta_y^y)^2 + \theta_a^a \theta_b^b = \frac{3}{2}(l^2 - s^2). \quad (4.15)$$

V. SUMMARY

Four families of solutions depending only on one coordinate are obtained for $N = 2$ five-dimensional supergravity theories. These solutions have nontrivial scalar and gauge fields and are valid for all space-time signatures. As examples we constructed explicit solutions with three charges for the so-called STU model. The constructed solutions can be lifted up to solutions of eleven-dimensional supergravity as well as type IIB supergravity (see for example Sec. 4.1 of [26]).

The various metric solutions are expressed in terms of constrained matrices. Their explicit form will depend on the various Jordan normal forms of the constrained matrices

appearing in the metric. One can consider the following choices for λ^μ_ν and θ^μ_ν

$$\begin{aligned} &\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & a + ib & 0 \\ 0 & 0 & 0 & a - ib \end{pmatrix}, \\ &\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & \alpha \end{pmatrix}. \end{aligned} \quad (5.1)$$

All these choices can be used to construct solutions for the families represented by (3.3) and (3.17). For the families represented by (3.27) and (3.40) only the first three choices are valid. In working out the explicit solutions, one should take into account that the metric components are symmetric and that $\omega = \sqrt{|g|}$. This implies that the matrix with elements $h_{\mu\nu}$ appearing in our solutions is symmetric and satisfies the condition $h_{\mu\rho}M^\rho{}_\nu = h_{\nu\rho}M^\rho{}_\mu$ where M is either λ or θ .

The results of this paper should be extended to other supergravity theories and in particular $N = 2$

four-dimensional supergravity. An interesting direction for further investigation is to generalize our results to supergravity theories with a cosmological constant or a scalar potential. Solutions of such theories should have applications to (A)dS/CFT correspondence. We hope to report on this in our future work.

DATA AVAILABILITY

No data were created or analyzed in this study.

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